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Abstract. We study strategic games where players' preferences are weak orders which need not admit utility representations. First of all, we extend Voorneveld's concept of best-response potential from cardinal to ordinal games and derive the analogue of his characterization result: An ordinal game is a best-response potential game if and only if it does not have a best-response cycle. Further, Milgrom and Shannon's concept of quasi-supermodularity is extended from cardinal games to ordinal games. We find that under certain compactness and semicontinuity assumptions, the ordinal Nash equilibria of a quasi-supermodular game form a nonempty complete lattice. Finally, we extend several set-valued solution concepts from cardinal to ordinal games in our sense.

Keywords: Ordinal Games, Potential Games, Quasi-Supermodularity, Rationalizable Sets, Sets Closed under Behavior Correspondences

JEL Classification: C72

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1 Introduction

The purpose of this paper is to derive general results for certain classes of ordinal games. It is motivated by the increasingly important role that ordinal properties have played in game-theoretic analysis in recent years. First of all, the introduction of supermodular games by Topkis (1979) and the subsequent analysis by Bulow, Geanakoplos and Klemperer (1985), Vives (1990), and Milgrom and Roberts (1990) gave rise to an important new strand of literature in economics and game theory. But strategic complementarity, the key ingredient of supermodularity, is essentially an ordinal concept and much of the theory of supermodular games can be reformulated in ordinal terms. Second, since the seminal paper by Monderer and Shapley (1996), a sizeable strand of literature on potential games has emerged. Monderer and Shapley already distinguish between exact, weighted, and ordinal potentials for cardinal games. Kukushkin (1999) and Norde and Patrone (2001) have introduced the concept of ordinal potential for ordinal games.

Ordinality in strategic games stands for two different, not mutually exclusive concepts. On the one hand, within the confines of traditional game theory, an ordinal perspective abstracts from particular utility representations (payoff functions). It considers invariant properties with respect to utility representations. It identifies games having the same game form and identical ordinal preferences or identical best response correspondences. More generally, it investigates isomorphisms and equivalence classes of games. For the ordinal perspective of games, see the contributions of Thompson (1952), Mertens (1987, 2003), Vermeulen and Jansen (2000), and Morris and Ui (2004). On the other hand, the concept of ordinal games transcends traditional game theory and allows for players' preferences which do not admit utility representations. Many contributions to demand theory and general equilibrium theory consider incomplete or intransitive preferences. Sonnenschein (1971), Shafer (1974), Kim and Richter (1986) made pioneering contributions to demand theory with incomplete or intransitive preferences. Schmeidler (1969), Shafer and Sonnenschein (1975), Bergstrom (1976), Borglin and Keiding (1976), Shafer (1976) belong to the early contributors to general equilibrium theory with incomplete or intransitive preferences. To the extent that the work of these and subsequent authors deals with abstract economies (generalized games, pseudo-games), it applies to ordinal games as well.

Incomplete or intransitive preferences constitute an important, but not the only class of preferences without utility representations. Specifically, the present paper deals with ordinal games where players' preference relations are weak orders: Players' preferences are complete and transitive, yet need not admit utility representations. An example is the following **public project proposal game**:

> Consider the problem of locating a finite number of identical public projects, say libraries, on a street represented by the unit interval. An outcome of this problem is a list of locations. Ehlers (2002, 2003) suggests that a library patron will visit his second choice library if a book he wants to borrow is unavailable at the first choice library. Thus, the patron's preference for locations induces a "lexicographic" preference relation for outcomes (lists of locations). Further consider two patrons, each with single-peaked preferences for locations, both evaluating lists of locations by means of the respective induced "lexicographic exten-

sion" à la Ehlers. Each patron proposes an outcome (list of locations). The average of the two proposed outcomes gets implemented. The procedure gives rise to an ordinal game where players' preference relations are weak orders without utility representations. The game proves to be a best-response potential game in the sense of subsection 3.1. For a formal description and elaborate analysis of the public project proposal game, see section 6.

As mentioned above, both potential games and supermodular games lend themselves to ordinal analysis. Notice that in contrast to arbitrary finite games, both finite potential games and finite supermodular games always possess a Nash equilibrium in pure strategies. In both cases, a purely ordinal approach can be taken. The arguments differ from the standard equilibrium existence proofs by means of the Brouwer, Kakutani, or Fan-Glicksberg fixed point theorem, which require a topological vector space structure and continuity in that topology.

To begin with, we introduce the concept of ordinal Nash equilibrium for ordinal games. In the case of potential games, no fixed point theorem is needed. We extend Voorneveld's (2000) concept of best-response potential from cardinal games to ordinal games in our sense and derive the analogue of his characterization result: An ordinal game is a best-response potential game if and only if it does not have a best-response cycle. In the case of supermodular games, one can resort to the lattice-theoretic, non-topological fixed point theorem of Zhou (1994). Next Milgrom and Shannon's (1994) concept of quasi-supermodularity is extended from cardinal games to ordinal games in our sense. We find that under certain compactness and semicontinuity assumptions, the ordinal Nash equilibria of a quasi-supermodular game form a nonempty complete lattice. As an immediate corollary, one obtains that the ordinal Nash equilibria of a finite quasi-supermodular game form a nonempty complete lattice.

Finally, we extend several set-valued solution concepts from cardinal to ordinal games in our sense. The definition of rationalizability follows Pearce (1984), with attention confined to pointwise beliefs, but not restricted to subsets of Euclidean spaces. We prove the existence of a nonempty and compact subset of rationalizable joint strategies in ordinal games where each individual strategy set is a compact Hausdorff space and all preference relations are continuous. The definition of a closed set under a behavior relation is an adaptation of Ritzberger and Weibull's (1995) concept of a closed set under a behavior correspondence, again without the restriction to subsets of Euclidean spaces. We demonstrate the existence of a minimal closed set under a behavior correspondence for the class of ordinal games where each strategy set is a compact Hausdorff space. We show a similar result for minimal prep sets, a concept adapted from Voorneveld (2004, 2005).

In sum, the contribution of this paper is two-fold: As a methodological advance, all concepts, assertions, and derivations are formulated in purely ordinal terms. Moreover, we generalize several previous results by relaxing the restrictions imposed in the literature: In addition to weaker assumptions regarding strategy spaces and preferences in some instances, finiteness of the player set is not assumed in Theorems 1, 3–5.

The next section contains the basic definitions regarding weak orders and ordinal games. We also elaborate on the fact that our assumptions on strategy sets and preferences are more general than the stated assumptions for cardinal games. Section 3 is devoted to potential games. Section 4 is about quasi-supermodular games. Section 5 is about set-valued concepts. Section 6 provides a formal description and elaborate analysis of the public project proposal game. Section 7 offers final remarks.

2 Preliminaries

We first collect some definitions and properties pertaining to preference relations, in particular weak orders. We then define ordinal games and related concepts, in particular ordinal Nash equilibria. For two nonempty sets S and Y, $\psi : S \twoheadrightarrow Y$ denotes a relation from S to Y, that is a mapping $\psi : S \to 2^Y$ that assigns to each $s \in S$ a subset $\psi(s)$ of Y. The relation ψ is called a correspondence if $\psi(s) \neq \emptyset$ for all $s \in S$.

2.1 Weak Orders

Let X be a nonempty set. A binary relation $\succeq$ on X is called a **weak order** if it is transitive and strongly complete. The latter means that $x \succeq y$ or $y \succeq x$ for all $x, y \in X$. For a weak order $\succeq$ on X, its asymmetric part $\succ$, defined by

$$x \succ z :\Longleftrightarrow [x \succeq z \,\&\, \neg(z \succeq x)]$$

for all $x, z \in X$, is irreflexive and transitive, and its symmetric part $\sim$, defined by

$$x \sim z :\Longleftrightarrow [x \succeq z \,\&\, z \succeq x]$$

for all $x, z \in X$, is an equivalence relation, that is reflexive, symmetric and transitive. A weak order $\succeq$ is called a **total order** if it is antisymmetric: $x \sim z \Longrightarrow x = z$ for all $x, z \in X$. The weak order $\succeq$ has or admits a utility representation if there exists a function $u : X \to \mathbb{R}$ such that $x \succeq y \iff u(x) \geq u(y)$ for $x, y \in X$. Then u is called a utility or payoff function representing $\succeq$ or a **utility representation** of $\succeq$.

In case X is endowed with a topology τ, we say that $\succeq$ is **upper semicontinuous** if the sets $\{x \in X | x \prec z\}, z \in X$, are open and $\succeq$ is **lower semicontinuous** if the sets $\{x \in X | x \succ z\}, z \in X$, are open. $\succeq$ is **continuous** if it is both upper and lower semicontinuous. The **order topology** on X induced by $\succeq$ has the sets $\{x \in X | x \succ z\}, z \in X$, and $\{x \in X | x \prec z\}, z \in X$, as a subbase of open sets. By definition, $\succeq$ is continuous in its order topology.

An element $z \in X$ is called a **maximal element** of the binary relation $\succeq$ on X if $\{x \in X | x \succ z\} = \emptyset$. It is called a **greatest element** if $z \succeq x$ for all $x \in X$. In the case of a weak order, maximal and greatest elements coincide. For convenient reference, we state the following well known fact.

Lemma 1 *Let (X, τ) be a compact topological space and $\succeq$ be an upper semicontinuous weak order on X. Then the set of maximal elements of $\succeq$ is nonempty and compact.*

GENERALITY OF ASSUMPTIONS. It is important to note that the lemma applies in instances where $\succeq$ does not admit a utility representation, like in the following example.

Example 1. Let $X = [0,1] \times \{0,1\}$ be endowed with the following total order $\succeq$ which is the restriction of the lexicographic order on $\mathbb{R}^2$ to $[0,1] \times \{0,1\}$:

For $x, y \in [0,1]$, with $x > y$, $(x,1) \succ (x,0) \succ (y,1) \succ (y,0)$.

Let τ be the order topology induced by $\succeq$. Then the following hold:

(i) (X, τ) is a compact Hausdorff space.

(ii) $\succeq$ is continuous with respect to τ.

(iii) $\succeq$ does not have a utility representation.

PROOF. (i) For $(x,a),(y,b) \in X$ with $x > y$, $\{\chi \in X | \chi \succ ((x+y)/2, 0)\}$ and $\{\chi \in X | \chi \prec ((x+y)/2, 0)\}$ are disjoint open sets with $(x,a) \in \{\chi \in X | \chi \succ ((x+y)/2, 0)\}$ and $(y,b) \in \{\chi \in X | \chi \prec ((x+y)/2, 0)\}$. For $(x,1),(x,0) \in X$, $\{\chi \in X | \chi \succ (x,0)\}$ and $\{\chi \in X | \chi \prec (x,1)\}$ are disjoint open sets with $(x,1) \in \{\chi \in X | \chi \succ (x,0)\}$ and $(x,0) \in \{\chi \in X | \chi \prec (x,1)\}$. This shows that (X, τ) is Hausdorff.

If $(X, \succeq)$ is order-complete, i.e. if every non-empty subset of X with an upper bound has a supremum, then every closed and bounded subset of X is compact in the order topology. See Problem 5.C in Kelley (1955). Now every subset of X is bounded. Thus, if we can show that every non-empty subset of X has a supremum, then compactness of (X, τ) is demonstrated.

For a non-empty subset A of X, let $A_1 = \{x \in [0,1] : (x,0) \in A \text{ or } (x,1) \in A\}$. In case $(\sup A_1, 1) \in A$, $(\sup A_1, 1)$ is the supremum of A. In case $(\sup A_1, 1) \notin A$, $(\sup A_1, 0)$ is the supremum of A. In any case, A has a supremum. This shows that $(X, \succeq)$ is order-complete and, consequently, (X, τ) is compact.

(ii) By definition, $\succeq$ is continuous in its order topology.

(iii) $\succeq$ has a continuum of "gaps" of the form $((x,0),(x,1))$, $x \in [0,1]$. Therefore, it does not have any (continuous or discontinuous) utility representation. ■■

Amoz Kats has suggested the following interpretation of the example. A voter has preferences over pairs (x, c) where $x \in [0, 1]$ is the platform chosen by a political candidate and $c \in \{0, 1\}$ is one of the two candidates. If the voter has the choice between two different platforms, the identity of the candidates does not matter. If both candidates offer the same platform, then the voter has a preference for candidate 1.

The relation $\succeq$ in Example 1 is the restriction of the lexicographic order on $\mathbb{R}^2$ to X. The restriction of the lexicographic order to the unit square $[0, 1]^2$ creates a similar example. A further example with properties (a)-(c) can be generated by means of the well ordering principle.

Example 2. Namely, let $\succeq$ be a well order on $\mathbb{R}$, i.e. $\succeq$ is a total order such that every nonempty subset of $\mathbb{R}$ has a minimum. For each $r \in \mathbb{R}$, let $I_r = \{x \in \mathbb{R} : r \succ x\}$. Let $R = \{r \in \mathbb{R} : I_r \text{ is uncountable}\}$. If $R = \emptyset$, set $\Omega = \mathbb{R}$. If $R \neq \emptyset$, set $\Omega = I_{\min R}$. Then $\succeq$ induces a well order on Ω which does not have a utility representation. Choose an element $\omega^* \notin \Omega$, set $X = \Omega \cup \{\omega^*\}$ and extend the well order to X by postulating $\omega^* \succ \omega$ for all $\omega \in \Omega$. A well order $\succeq$ renders $(X, \succeq)$ order-complete so that one can follow the pattern of proof of Example 1. ■■

The literature has been mostly concerned with the existence of continuous or upper semi-continuous utility representations for continuous or upper semi-continuous weak orders. The key results of the seminal contributions of Eilenberg (1941), Debreu (1954), Rader (1963) can be summarized as follows: *Let X be a topological space which is second countable (has a countable base of open sets) or is separable and connected. If $\succeq$ is a continuous weak* *order on X then $\succeq$ has a continuous utility representation.*[1] Beardon *et al.* (2002) provide a classification of total orders which do not admit a utility representation. The lexicographic order is a prominent example of the so-called "planar type". Example 2 is of the so-called "long type". Estévez Toranzo and Hervés Beloso (1995) show that if $X \neq \emptyset$ is a non-separable metric space, then there exists a continuous weak order on X which cannot be represented by a utility function. However, this result becomes obsolete under the compactness assumption of Lemma 1: If X is a compact metric space then it is separable and second countable.

2.2 Ordinal Games

Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ denote an **ordinal non-cooperative game** with the following interpretation and properties:

- $N \neq \emptyset$ denotes the set of players.
- Every player $i \in N$ has a nonempty set X_i of strategies. The product set $X = \prod_{i \in N} X_i$ represents the set of joint strategies or strategy profiles.
- Every player $i \in N$ has a binary relation $\succeq_i$ over the joint strategy set X, which reflects his preferences over the outcomes of the game G. Each of the binary relations $\succeq_i$ is assumed to be a weak order.

We denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$. For a player $i \in N$ and a joint strategy $x = (x_j)_{j \in N} \in X$, we write $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$ and, with slight abuse of notation, $x = (x_i, x_{-i}) \in X$. For every player $i \in N$ and every joint strategy

[1] For generalizations and variations, see Rader (1963), Monteiro (1987), Candeal, Hervés, and Induráin (1998), Bosi and Mehta (2002).

of his opponents $x_{-i} \in X_{-i}$,

$$M_i(x_{-i}) = \{x_i \in X_i \mid \nexists z_i \in X_i : (z_i, x_{-i}) \succ_i (x_i, x_{-i})\}$$

is the set of **best responses**, that is the set of i's maximal strategies against x_{-i} under $\succeq_i$. Recall that every maximal element is a greatest element because $\succeq_i$ is a weak order. For every player $i \in N$ and every joint strategy $x = (x_i, x_{-i}) \in X_i \times X_{-i}$, we denote by

$$B_i(x) = \{z_i \in X_i \mid (z_i, x_{-i}) \succeq_i (x_i, x_{-i})\}$$

the **set of better responses** or **upper contour set**.

Let $B : X \twoheadrightarrow X$, $x \mapsto \prod_{i \in N} B_i(x)$ be the joint better-response relation. Let $M : X \twoheadrightarrow X$, $x \mapsto \prod_{i \in N} M_i(x_{-i})$ be the joint best-response relation which maps each joint strategy to its joint best-responses. The set of **ordinal Nash equilibria** of G is defined by

$$\mathcal{N}(G) = \{x \in X \mid x \in M(x)\}.$$

When appropriate, we shall consider each strategy set X_i endowed with a topology. For the remainder of this paragraph, suppose each individual strategy space X_i is endowed with a topology τ_i and $X = \prod_i X_i$ is endowed with the corresponding product topology. We say that $\succeq_i$ is **upper semicontinuous** on X_i for every $x_{-i} \in X_{-i}$ if the set of better responses $B_i(x_i, x_{-i})$ is a closed subset of X_i for every $x = (x_i, x_{-i}) \in X$. Clearly, if a preference $\succeq_i$ is continuous on X then it is upper semicontinuous on X and upper semicontinuous on X_i for every $x_{-i} \in X_{-i}$.

3 Ordinal Potential Games

Monderer and Shapley (1996) develop the concept of potential for cardinal games $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ in both cardinal and ordinal versions. Monderer and Shapley (1996), Voorneveld and Norde (1997) and Ui (2000) characterize several classes of potential games. However, their approach is biased towards the cardinal side since it requires the potential to be a real-valued function, while from a strictly ordinal viewpoint, the potential provides an order. Consequently, Kukushkin (1999) and Norde and Patrone (2001) have introduced the concept of ordinal potential for ordinal games. An ordinal game has a potential if there exists a **quasi-order** on X, that is a reflexive and transitive binary relation $\succeq$, containing the preferences of all players: $(x_i, x_{-i}) \succeq_i (z_i, x_{-i}) \iff (x_i, x_{-i}) \succeq (z_i, x_{-i})$ for all $i \in N$, $x_i, z_i \in X_i$, $x_{-i} \in X_{-i}$.

Voorneveld (2000) introduces and studies best-response potential games, a new class of potential games. A game $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$, is a best-response potential game if there exists a real-valued function $P : X \to \mathbb{R}$ such that for every $i \in N$, $x_{-i} \in X_{-i}$, we have

$$\arg\max_{x_i\in X_i} u_i(x_i, x_{-i}) = \arg\max_{x_i\in X_i} P(x_i, x_{-i}).$$

Here, we adapt his definition for ordinal games. An ordinal game $G = (N, (X_i)_{i\in N}, (\succeq_i)_{i\in N})$ is a **best-response potential game** if there exists a quasi-order $\trianglerighteq$ on X, such that for every $i \in N$, $x_{-i} \in X_{-i}$, we have

$$M_i(x_{-i}) = M_{\trianglerighteq}(x_{-i})$$

where $M_{\trianglerighteq}(x_{-i})$ denotes the set of greatest elements of $\trianglerighteq$ over X_i given $x_{-i} \in X_{-i}$. Obviously, the definitions imply that if an ordinal game is a potential game, then it is a best-response potential game. It is clear from the definition of a best-response potential that (i) each greatest element of $\trianglerighteq$ is an ordinal Nash equilibrium of G; (ii) the set of ordinal Nash equilibria of G coincides with the set of ordinal Nash equilibria of the ordinal game $(N, (X_i)_{i\in N}, (\trianglerighteq)_{i\in N})$ provided $\trianglerighteq$ is a weak order on X.

3.1 Characterization of best-response potentials

Voorneveld (2000) provides a characterization of best-response potential games. We implement in the ordinal setting the ideas introduced in Voorneveld (2000) and Norde and Patrone (2001). First we need the following definition.

Let $G = (N, (X_i)_{i\in N}, (\succeq_i)_{i\in N})$ be an ordinal game. A **path** in the set of joint strategies X is a sequence $(x^1, x^2, \ldots)$ of elements $x^k \in X$ such that for all $k = 1, 2, \ldots$, the joint strategies x^k and x^{k+1} differ in exactly one, say the $i(k)^{th}$, component. A path is **best-response compatible** if the deviating player moves to a best response, that is

$$\forall k = 1, 2, \ldots : \; x^{k+1}_{i(k)} \in M_{i(k)}(x^k_{-i(k)}).$$

By definition the trivial path (x^1) consisting of a single joint strategy $x^1 \in X$ is best-response compatible. A finite path $(x^1, x^2, \ldots, x^m)$ is called a **best-response cycle** if it is best-response compatible, $x^1 = x^m$, and $x^{k+1} = (x^{k+1}_{i(k)}, x^k_{-i(k)}) \succ_{i(k)} x^k$ for some $k \in \{1, \ldots, m-1\}$.

Theorem 1 *An ordinal game $G = (N, (X_i)_{i\in N}, (\succeq_i)_{i\in N})$ is a best-response potential game if and only if there is no best-response cycle.*

PROOF. ($\Rightarrow$) Assume that G is an ordinal game with best-response potential

$\trianglerighteq$. Suppose that $(x^1, x^2, \ldots, x^m)$ is a best-response cycle. Define by $\triangleright$ the asymmetric part of $\trianglerighteq$ on X, that is for all $x, z \in X$,

$$x \triangleright z :\Longleftrightarrow [x \trianglerighteq z \,\&\, \neg(z \trianglerighteq x)] \,.$$

By definition, $x_{i(k)}^{k+1} \in M_{i(k)}(x_{-i(k)}^k)$ for all $k \in \{1, \ldots, m-1\}$ and $x^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k) \succ_{i(k)} x^k$ for at least one such k. So $x^{k+1} \trianglerighteq x^k$ for all $k \in \{1, \ldots, m-1\}$ and $x^m \trianglerighteq x^1$ by transitivity. But since there is at least one k such that $x^{k+1} \succ_{i(k)} x^k$ we necessarily have $x^{k+1} \triangleright x^k$ for this k by definition of a best-response potential. It follows that $x^m \trianglerighteq x^{k+1} \triangleright x^k \trianglerighteq x^1$ which in turn implies that $x^m \triangleright x^1$, contradicting the fact that $x^1 = x^m$.

($\Leftarrow$) Suppose that G has no best-response cycle. Define the binary relation $(X, \trianglerighteq)$ as follows:

$$(\forall x, z \in X) : z \trianglerighteq x :\Longleftrightarrow [\exists \text{ a best-response compatible path from } x \text{ to } z]$$

First note that the binary relation $\trianglerighteq$ on X is reflexive and transitive; i.e., it is a quasi-order. We have to show that $M_i(x_{-i}) = M_{\trianglerighteq}(x_{-i})$ for every $i \in N$ and $x_{-i} \in X_{-i}$. Let $i \in N, x_{-i} \in X_{-i}$.

(a) Pick $z_i \in M_i(x_{-i})$. Since $\succeq_i$ is strongly complete, for all $x_i \in X_i$ we have that $(z_i, x_{-i}) \succeq_i (x_i, x_{-i})$, hence the path $((x_i, x_{-i}), (z_i, x_{-i}))$ is best-response compatible and $(z_i, x_{-i}) \trianglerighteq (x_i, x_{-i})$. Therefore, $z_i \in M_{\trianglerighteq}(x_{-i})$. This observation implies that $M_i(x_{-i}) \subseteq M_{\trianglerighteq}(x_{-i})$.

(b) Pick $z_i \in M_{\trianglerighteq}(x_{-i})$. Suppose $z_i \notin M_i(x_{-i})$. Then there exists $x_i \in X_i$ such that $(x_i, x_{-i}) \succ_i (z_i, x_{-i})$. By the absence of a best-response cycle, it cannot be the case that $(z_i, x_{-i}) \trianglerighteq (x_i, x_{-i})$, contradicting $z_i \in M_{\trianglerighteq}(x_{-i})$. We conclude that $z_i \in M_i(x_{-i})$. From this observation we get $M_i(x_{-i}) \supseteq M_{\trianglerighteq}(x_{-i})$.

The assertion that $(X, \trianglerighteq)$ is a best-response potential for G follows from (a) and (b). ■■

Voorneveld (2000, Theorem 3.1) proves in a cardinal setting that a best-response potential exists if and only if: (i) X contains no best-response cycles and (ii) the quotient of $(X, \trianglerighteq)$ can be represented by a real-valued function. Theorem 1 states that from a purely ordinal perspective, condition (ii) has nothing to do with best-response potential games. A similar remark has been made by Norde and Patrone (2001) for the class of ordinal potential games.

Finally note that if $(x^1, x^2, \ldots, x^m)$ is a best-response cycle, then it is a weak improvement cycle in the sense of Norde and Patrone (2001), that is $x^{k+1} = (x^{k+1}_{i(k)}, x^k_{-i(k)}) \succeq_{i(k)} x^k$ for all $k \in \{1, \ldots, m-1\}$, $x^1 = x^m$, and $x^{k+1} = (x^{k+1}_{i(k)}, x^k_{-i(k)}) \succ_{i(k)} x^k$ for some $k \in \{1, \ldots, m-1\}$. A weak improvement path is thus a path such that at each iteration one player is drawn from the population to play a better response against the current joint strategy of his opponents. Theorem 2.2 in Norde and Patrone (2001) states that an ordinal game G is a potential game if and only if G contains no weak improvement cycle.

3.2 Existence of Ordinal Nash Equilibria

If a finite ordinal game has a best-response potential, then it has an ordinal Nash equilibrium. If we consider best-response potential games in which all but one player have a finite set of strategies, and if we equip the only infinite strategy set with a topology we obtain the following result.

Theorem 2 *Let* $G = (N, (X_j)_{j\in N}, (\succeq_j)_{j\in N})$ *be a best-response potential game with* N *finite. If for some* $i \in N$*,* X_j *is a finite set for every* $j \neq i$*,* X_i *is a compact topological space and* $\succeq_i$ *is upper semicontinuous on* X_i *for each* $x_{-i} \in X_{-i}$*, then* $\mathcal{N}(G)$ *is nonempty.*

PROOF. Suppose that $\mathcal{N}(G)$ is empty. Let i satisfy the hypothesis. Because of the compactness and upper semicontinuity conditions, $M_i(x_{-i})$ is nonempty and compact for every $x_{-i} \in X_{-i}$, by Lemma 1. For every $j \neq i$, $M_j(x_{-j})$ is nonempty for each $x_{-j} \in X_{-j}$ by the finiteness of X_j. Pick any selection $m_i(\cdot)$ from $M_i(\cdot)$ and any x_{-i} in X_{-i}. Construct a best-response compatible path $(x^1, x^2, \ldots)$ as follows: $x^1 = (m_i(x_{-i}), x_{-i})$ and for $k = 2, 3, \ldots$, if $x_i^k \notin M_i(x_{-i}^k)$, then $x^{k+1} = (m_i(x_{-i}^k), x_{-i}^k)$; otherwise $x^{k+1} = (z_{j(k)}^{k+1}, x_{-j(k)}^k)$ for some player $j(k) \neq i$ such that $z_{j(k)}^{k+1} \in M_{j(k)}(x_{-j(k)}^k)$ and $(z_{j(k)}^{k+1}, x_{-j(k)}^k) \succ_{j(k)} x^k$. Note that such a player $j(k)$ exists by the presumed emptiness of $\mathcal{N}(G)$. Since X_{-i} is a finite set and player i uses only the finite set of strategies $m_i(X_{-i})$, there exist $k, l \in \mathbb{N}$ such that $x^k = x^{k+l}$. Hence $(x^k, x^{k+1}, \ldots, x^{k+l})$ is a best-response cycle which by Theorem 1 contradicts the premise that G is a best-response potential game. We conclude that G possesses at least one Nash equilibrium. ■■

This existence result extends earlier results by Voorneveld (1997) and Norde and Tijs (1998) which were obtained in a cardinal setting for exact potential games and generalized ordinal potential games, respectively.

4 Quasi-Supermodular Ordinal Games (Games with Strategic Complementarities)

Let X be a partially ordered set, with the reflexive, antisymmetric and transitive binary relation $\geq$. Given elements x and z in X, denote by $x \vee z$ the least upper bound or **join** of x and z in X, provided it exists, and $x \wedge z$ the greatest lower bound or **meet** of x and z in X, provided it exists. A partially ordered set X that contains the join and the meet of each pair of its elements is called a **lattice**. A lattice in which each nonempty subset has a supremum and an infimum is **complete**. In particular, a finite lattice is complete. If Y is a subset of a lattice X and Y contains the join and the meet with respect to X of each pair of elements of Y, then is Y is a **sublattice** of X. A sublattice Y of a lattice X in which each nonempty subset has a supremum and an infimum with respect to X that are contained in Y is a **subcomplete sublattice** of X. Any finite sublattice of a lattice is subcomplete.

We now define an order on the subsets of a lattice. We use the **strong set order** $\geq_s$ introduced by Milgrom and Shannon (1994). Let X be a lattice and let Y and Z be two subsets of X. We say that $Y \geq_s Z$ if for every $y \in Y$ and every $z \in Z$, $y \vee z \in Y$ and $y \wedge z \in Z$. We say that a relation $\rho : X \twoheadrightarrow Y$ from a lattice X to a lattice Y is **increasing** in x on X if for every $x \in X$, $\rho(x)$ is a sublattice of Y and if for $x \geq z$, $\rho(x) \geq_s \rho(z)$.

If X is a lattice partially ordered by the relation $\geq$, then subsets of the form $[a, b] = \{x \in X : b \geq x \geq a\}$, $[a, \infty) = \{x \in X : x \geq a\}$, or $(-\infty, b] = \{x \in X : b \geq x\}$ are sublattices of X for all $a, b \in X$. These sets and X are the **closed intervals** in X. We say that a lattice X is equipped with the **interval topology** when each closed set can be represented as the intersection of sets that are finite unions of closed intervals in X, including the empty set as the empty union of sets. In other words, the closed intervals constitute a subbase of closed sets of the interval topology.

Next consider an ordinal game. Suppose that each individual strategy set X_i is a lattice partially ordered by the relation $\geq_i$. Then the product sets X and X_{-i}, $i \in N$, are also lattices with respect to the canonical partial orders $\geq$ and $\geq_{-i}$, $i \in N$, respectively. For instance, $x \geq y \Leftrightarrow \forall i : x_i \geq_i y_i$ for $x, y \in X$. We say that the preference $\succeq_i$ is **quasi-supermodular** on X_i for each $x_{-i} \in X_{-i}$ if for every $x_i, z_i \in X_i$, $x_i \neq z_i$, $x_{-i} \in X_{-i}$,

(i) $(x_i, x_{-i}) \succeq_i (x_i \wedge z_i, x_{-i}) \Longrightarrow (x_i \vee z_i, x_{-i}) \succeq_i (z_i, x_{-i})$;

(ii) $(x_i, x_{-i}) \succ_i (x_i \wedge z_i, x_{-i}) \Longrightarrow (x_i \vee z_i, x_{-i}) \succ_i (z_i, x_{-i})$.

We say that a preference $\succeq_i$ satisfies the **strategic complement property** in (x_i, x_{-i}) on $X_i \times X_{-i}$ if for every $x_i, z_i \in X_i$ and $x_{-i}, z_{-i} \in X_{-i}$ with $x_i \geq_i z_i$, $x_i \neq z_i$ and $x_{-i} \geq_{-i} z_{-i}$, $x_{-i} \neq z_{-i}$,

(iii) $(x_i, z_{-i}) \succeq_i (z_i, z_{-i}) \Longrightarrow (x_i, x_{-i}) \succeq_i (z_i, x_{-i})$;

(iv) $(x_i, z_{-i}) \succ_i (z_i, z_{-i}) \Longrightarrow (x_i, x_{-i}) \succ_i (z_i, x_{-i})$.

An **ordinal game** $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ is **quasi-supermodular** if for each $i \in N$, X_i is a lattice, $\succeq_i$ is quasi-supermodular on X_i for each $x_{-i} \in X_{-i}$, and $\succeq_i$ satisfies the strategic complement property on $X_i \times X_{-i}$.

Quasi-supermodular games in a cardinal setting were introduced by Migrom and Shannon (1994). Note that a (quasi-supermodular) game $\Gamma =$

$(N, (X_i)_{i\in N}, (u_i)_{i\in N})$ with payoff functions $u_i : X \to \mathbb{R}$, $i \in N$, uniquely determines a (quasi-supermodular) ordinal game G. But not every ordinal quasi-supermodular game has a cardinal representation. For instance, take the pair $(X, \succeq)$ from Example 1 as the strategy space and the preference relation of the player in a one-person game. Moreover, let $X = [0, 1] \times \{0, 1\}$ be partially ordered according to the canonical partial order on $\mathbb{R}^2$. Then the one-player game is quasi-supermodular and does not have a cardinal representation.

Lemma 2 *Let $G = (N, (X_i)_{i\in N}, (\succeq_i)_{i\in N})$ be a quasi-supermodular game. Then for each $i \in N$, $M_i(x_{-i})$ is increasing in x_{-i} on X_{-i}.*

PROOF. Pick $x_{-i} \geq_{-i} z_{-i}$ and $x_i \in M_i(x_{-i})$, $z_i \in M_i(z_{-i})$. The elements $x_i \wedge z_i$ and $x_i \vee z_i$ are in X_i since X_i is a lattice. We have $(z_i, z_{-i}) \succeq_i (x_i \wedge z_i, z_{-i})$. Then $(z_i \vee x_i, z_{-i}) \succeq_i (x_i, z_{-i})$ by quasi-supermodularity and, consequently, $(x_i \vee z_i, x_{-i}) \succeq_i (x_i, x_{-i})$ by the strategic complement property. Since $x_i \in M_i(x_{-i})$, it cannot be the case that $(x_i \vee z_i, x_{-i}) \succ_i (x_i, x_{-i})$. Therefore, $(x_i \vee z_i, x_{-i}) \sim_i (x_i, x_{-i})$, which implies $z_i \vee x_i \in M_i(x_{-i})$. Now suppose $(z_i, z_{-i}) \succ_i (x_i \wedge z_i, z_{-i})$. Then $(z_i \vee x_i, z_{-i}) \succ_i (x_i, z_{-i})$ by quasi-supermodularity and hence $(x_i \vee z_i, x_{-i}) \succ_i (x_i, x_{-i})$ by the strategic complement property. But this contradicts $(x_i \vee z_i, x_{-i}) \sim_i (x_i, x_{-i})$ (and $x_i \in M_i(x_{-i})$). Therefore, $(z_i, z_{-i}) \sim_i (x_i \wedge z_i, z_{-i})$, which implies $x_i \wedge z_i \in M_i(z_{-i})$. We have shown $M_i(x_{-i}) \geq_s M_i(z_{-i})$. Finally, setting $x_{-i} = z_{-i}$ yields $x_i \vee z_i \in M_i(x_{-i})$ and $x_i \wedge z_i \in M_i(x_{-i})$ for $x_i, z_i \in M_i(x_{-i})$ — which means that $M_i(x_{-i})$ is a sublattice of X_i. This completes the proof. ■■

Theorem 3 *Let the game $G = (N, (X_i)_{i\in N}, (\succeq_i)_{i\in N})$ be quasi-supermodular and each individual strategy set X_i be equipped with the interval topology.*

If for all $i \in N$, the set X_i is compact and the preference $\succeq_i$ is upper semi-continuous on X_i for every $x_{-i} \in X_{-i}$, then the set of ordinal Nash equilibria of G, $\mathcal{N}(G)$, is a nonempty complete lattice.

PROOF. Birkhoff's theorem (1967, Theorem X.20) states that a lattice is compact in its interval topology if and only if it is complete.[2] Hence each X_i is a complete lattice and, therefore, has a lower bound ℓ_i and an upper bound o_i; consequently, the closed intervals of the form $[a_i, b_i]$ constitute a subbase of closed sets for the interval topology. The product $X = \prod_{i \in N} X_i$ is compact in the product topology as the product of compact spaces. X has lower bound $\ell = (\ell_i)_{i \in N}$ and upper bound $o = (o_i)_{i \in N}$ and the cylinder sets of the form $[a_i, b_i] \times \prod_{j \neq i} X_j$, $i \in N$, with $X_j = [\ell_j, o_j]$ for $j \in N$, constitute a subbase $\mathcal{B}$ of closed sets for the product topology. The closed intervals of the form $[a, b] = \prod_{i \in N}[a_i, b_i]$ constitute a subbase $\mathcal{B}'$ of closed sets for the interval topology on X. On the one hand, $\mathcal{B} \subseteq \mathcal{B}'$. On the other hand, each $[a, b] \in \mathcal{B}'$ is a closed set in the product topology, for

$$[a, b] = \prod_{i \in N}[a_i, b_i] = \bigcap_{i \in N} \left([a_i, b_i] \times \prod_{j \neq i} X_j \right).$$

This shows that $\mathcal{B}'$ is a subbase of closed sets for both the product topology and the interval topology on X. Hence the assertion of Frink (1942, Theorem 4) holds: The product and the interval topology on X coincide. Thus, X is compact in its interval topology. By Birkhoff's theorem, X is a complete lattice.

Now let $i \in N$. The best-response relation $M_i : X_{-i} \twoheadrightarrow X_i$ is nonempty- and compact-valued by Lemma 1 and is increasing with respect to the strong

[2] We adopt Frink's (1942) and Topkis' (1998) definition of the interval topology, which in contrast to Birkhoff's does not presume existence of a priori universal lower and upper bounds. Birkhoff's proof still applies.

set order by Lemma 2. Therefore, the joint best-response relation M from X to itself, with $M(x) = \prod_{i \in N} M_i(x_{-i})$, is a compact-valued correspondence as the product of compact-valued correspondences and is increasing as the product of increasing correspondences.

Next pick any $x \in X$. Since $M(x)$ is compact in X, it is also compact as a subspace of X when X is endowed with the product topology. Hence $M(x)$ is also compact as a subspace of X when X is endowed with the interval topology because the product and the interval topology on X coincide. Consider any nonempty subset A of $M(x)$. Since X is a complete lattice, $\sup_X A$, the supremum of A in X exists. We claim $\sup_X A \in M(x)$. As observed above, the closed intervals of the form $[a, b]$ form a subbase of closed sets for both topologies on X. Therefore, the sets $M(x) \cap [a, b], a, b \in X$, form a subbase of closed sets for the topological subspace $M(x)$ of X. Now consider the family of sets $M(x) \cap [a, \sup_X A], a \in A$. Let F be a finite nonempty subset of A. Since $M(x)$ is a sublattice of X, $\sup_X F \in M(x)$ and $a \leq \sup_X F \leq \sup_X A$ for $a \in F$. Hence $\sup_X F$ belongs to the intersection of the sets $M(x) \cap [a, \sup_X A], a \in F$. Since the family of closed sets $M(x) \cap [a, \sup_X A], a \in A$, has nonempty finite intersections and $M(x)$ is compact, the entire family has a nonempty intersection. Let b belong to this intersection. Then $b \in M(x)$ and $a \leq b \leq \sup_X A$ for all $a \in A$. Hence $b \in M(x)$ and $b = \sup_X A$, which shows our claim that $\sup_X A \in M(x)$. In an analogous way, one proves $\inf_X A \in M(x)$. Since A was an arbitrary nonempty subset, $M(x)$ is a subcomplete sublattice of X.[3]

To summarize, the joint best-response relation M is an increasing corre-

[3]Topkis (1998, p. 31), referring to Topkis (1977), states that when X is a lattice with the interval topology and X' is a sublattice of X, then X' is subcomplete if and only if X' is compact in the relative topology. We have demonstrated the "if" part in case X is a complete lattice.

spondence from the complete lattice X to itself and $M(x)$ is a nonempty and subcomplete sublattice for each $x \in X$. The assertion of the theorem follows from Zhou's fixed-point theorem (1994, Theorem 1, p. 297). ■■

Corollary 1 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be a finite quasi-supermodular game. Then the set of ordinal Nash equilibria of G, $\mathcal{N}(G)$, is a nonempty complete lattice.*

5 Set-Valued Concepts

Set-valued concepts have proved to have many desirable properties in large classes of cardinal games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$. Here we consider two set-valued concepts: the set of rationalizable joint strategies introduced independently by Bernheim (1984) and Pearce (1984) and the concept of minimal closed set under some behavior correspondence introduced by Ritzberger and Weibull (1995). We are going to extend the field of applications of these set-valued concepts from cardinal games to ordinal games. However, in contrast to most of the literature, our definitions involve only pure strategies. The reason is that there is no straightforward and commonly agreed upon extension of ordinal preferences from pure to mixed strategies. The modified concepts have similar properties as the original ones and may be of interest on their own. In fact, Basu (1992) and Pruzhansky (2003) work with pointwise beliefs or conjectures like us.

A strategy for a player is rationalizable if it survives iterated removal of strategies that are never a best response. Rationalizability is a concept that generalizes that of Nash equilibrium. Minimal closed set under some behavior correspondence is a concept that generalizes that of strict Nash equilibrium. More precisely, Basu and Weibull (1991) introduce first the concept of closed set under rational behavior (curb), a set-valued extension of the strict Nash equilibrium concept for cardinal games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$. A product set of pure strategies is closed under rational behavior if it is non-empty and compact and contains the image under the best-response correspondence of every mixed joint strategy with support in this set. A curb set is minimal if it does not contain any proper subset which is curb. As noted by Basu and Weibull (1991), the set of rationalizable joint strategies is the largest tight curb set. Thus, a minimal curb set and the set of rationalizable joint strategies can be viewed as the two ends of a spectrum. Ritzberger and Weibull (1995) generalize the concept of curb set to a very large class of behavior correspondences. A product set of pure strategies is closed under some behavior correspondence if it is non-empty and compact and contains the image under the particular correspondence of every mixed joint strategy with support in this set. The class of behavior correspondences considered by the authors includes the better-response correspondence and the best-response correspondence. All these set-valued concepts have proved to be very useful to characterize stable sets of dynamic strategy adjustments (Ritzberger and Weibull, 1995; Young, 1998; Matros and Josephson, 2004).

Pearce (1984) considers only finite games. Bernheim (1984) considers games where each strategy space X_i is a compact subset of some Euclidean space and every $u_i : X \to \mathbb{R}$ is continuous. Similarly, Basu and Weibull (1991) prove the existence of minimal curb sets for the mixed extension of games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ where each X_i is a compact set in some Euclidean space and every payoff function $u_i : X \to \mathbb{R}$ is continuous. Ritzberger and Weibull (1995) focus on finite games. We consider a more general setting: In Theorem 4 on the existence of rationalizable joint strategies, each strategy set is endowed with a compact Hausdorff topology and each player's preference $\succeq_i$ is continuous on X. In Theorem 5 on the existence of minimal closed sets under a behavior relation, each individual strategy space is compact and Hausdorff. Theorem 5 (iv) states an analogous result for minimal prep sets à la Voorneveld (2004, 2005). Notice that our specification cannot be reduced to the class of games $G = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ considered by Bernheim (1984) and Basu and Weibull (1991). The reason is that the classical utility representation theorems of Eilenberg (1941) and Debreu (1954, 1964) apply only to separable topological spaces that is topological spaces which include a countable dense subset.[4]

In the following subsections, we prove (i) the existence of a nonempty and compact subset of rationalizable joint strategies in ordinal games where each X_i is a compact Hausdorff space and each preference $\succeq_i$ is continuous on X; (ii) the existence of at least one minimal closed set under a behavior relation for the class of ordinal games where each X_i is a compact Hausdorff space.

5.1 Rationalizable strategies

We shall define rationalizability via the method used by Pearce (1984). In addition, attention is confined to pointwise beliefs. For each $i \in N$, construct

[4] A more recent result due to Monteiro (1987) requires that the underlying topological space is arc-connected and the preferences $\succeq_i$ are continuous and countably bounded. Recall that the weak order $\succeq_i$ on X is countably bounded if there is a countable subset Z of X such that for every $x \in X$, there exist $z^1, z^2 \in Z$ such that $z^1 \succeq_i x \succeq_i z^2$.

a sequence $H_i^k, k \in \mathbb{N}$, of subsets of X_i as follows: Let $H_i^1 = X_i$ and define H_i^k inductively for $k = 2, 3, \dots$ by

$$H_i^k = \left\{ x_i \in H_i^{k-1} \;\middle|\; \exists\, x_{-i} \in \prod_{j \in N \setminus \{i\}} H_j^{k-1} : x_i \in M_i(x_{-i}) \right\}.$$

The set of **rationalizable strategies** of player $i \in N$ is defined as

$$R_i = \bigcap_{k \in \mathbb{N}} H_i^k$$

and a **joint strategy** $x \in X$ is **rationalizable** if $x \in R := \prod_{i \in N} R_i$.

Theorem 4 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be an ordinal game. Suppose that for each $i \in N$, the set of strategies X_i is equipped with a compact Hausdorff topology and the preference $\succeq_i$ is continuous on X. Then the set of joint rationalizable strategies $R \subseteq X$ is nonempty and compact.*

PROOF. We shall show by induction on k that $H_i^k, k \in \mathbb{N}$, is a nested sequence of nonempty and closed subsets of the compact space X_i for all $i \in N$. Therefore, the intersection R_i is nonempty and closed. Consequently, R_i is compact as a closed subset of a compact space.

THE INDUCTION ARGUMENT: For each $i \in N$, $H_i^1 = X_i$ is closed by definition. Consider any $k \in \mathbb{N}$ and suppose that $\prod_{i \in N} H_i^k$ is the Cartesian product of nonempty and closed sets. Then each H_i^k is a nonempty and compact subset of the compact space X_i and $\prod_{i \in N} H_i^k$ is a nonempty and compact subspace of X. We have to show that each H_i^{k+1} is also nonempty and closed. Since for $i \in N$, the preference $\succeq_i$ is continuous on X, its restriction to $\prod_{j \in N} H_j^k$ is continuous as well. For every $x_{-i} \in H_{-i}^k = \prod_{j \neq i} H_j^k$, the restriction of $\succeq_i$ to $H_i^k \times \{x_{-i}\}$ is also continuous. Hence by Lemma 1, the set $M_i(x_{-i}|H_i^k) = \{x_i \in H_i^k | (x_i, x_{-i}) \succeq_i (z_i, x_{-i}) \text{ for all } z_i \in H_i^k\}$ is nonempty and compact. Hence H_i^{k+1} is nonempty as the union of the sets $M_i(x_{-i}|H_i^k)$, $x_{-i} \in H_{-i}^k$.

It remains to show that H_i^{k+1} is closed. To this end, it suffices to show that for any net in H_i^{k+1} converging to a point x_i^* in X_i, it follows that x_i^* belongs to H_i^{k+1}. Suppose that $(D, \gg)$ is a directed set, that $x_i^\delta, \delta \in D$, is a net in H_i^{k+1} and that the net $x_i^\delta, \delta \in D$, converges to $x_i^* \in X_i$. For each $\delta \in D$, let us select an $x_{-i}^\delta \in H_{-i}^k$ such that $x_i^\delta \in M_i(x_{-i}^\delta | H_i^k)$. Then $(x_i^\delta, x_{-i}^\delta), \delta \in D$, is a net in $\prod_{j \in N} H_j^k$. Since $\prod_{j \in N} H_j^k$ is compact, there exists a subnet of $(x_i^\delta, x_{-i}^\delta), \delta \in D$, convergent to some $(x_i', x_{-i}') \in X$. Without restriction, we may assume that $(x_i^\delta, x_{-i}^\delta), \delta \in D$, is such a subnet. Then the net $x_i^\delta, \delta \in D$, converges to both x_i' and x_i^*. Since X_i is Hausdorff, $x_i' = x_i^*$. Because of continuity of the restriction of $\succeq_i$ to $\prod_{j \in N} H_j^k$, the set $\mathfrak{H}_i^k = \{(z', z'') \in \prod_{j \in N} H_j^k \times \prod_{j \in N} H_j^k \mid z' \succeq_i z''\}$ is a closed subset of $\prod_{j \in N} H_j^k \times \prod_{j \in N} H_j^k$; see Bridges and Mehta (1995, Proposition 1.6.2). Now let $z_i \in H_i^k$. For each $\delta \in D$, we have

$$(x_i^\delta, x_{-i}^\delta) \succeq_i (z_i, x_{-i}^\delta),$$

that is $((x_i^\delta, x_{-i}^\delta), (z_i, x_{-i}^\delta)) \in \mathfrak{H}_i^k$. Therefore, $((x_i', x_{-i}'), (z_i, x_{-i}')) \in \mathfrak{H}_i^k$. Consequently, $x_{-i}' \in H_{-i}^k$ and $(x_i', x_{-i}') \succeq_i (z_i, x_{-i}')$. Since the latter holds for arbitrary $z_i \in H_i^k$, we obtain $x_{-i}' \in H_{-i}^k$ and $x_i' \in M_i(x_{-i}'|H_i^k)$. This implies $x_i^* = x_i' \in H_i^{k+1}$ as desired. ■■

5.2 Closed sets under a behavior relation

Assume that each set of strategies X_i, $i \in N$, is a compact Hausdorff space. For the sake of convenience, we shall take X as the space of beliefs of each player $i \in N$, so that the beliefs for player i include his own strategy. In particular we shall denote the best response relation by $M_i(x)$ — though it is functionally independent of the component $x_i \in X_i$. Let Φ be the class of **behavior relations**: $\phi \in \Phi$ if $\phi = \prod_{i \in N} \phi_i : X \twoheadrightarrow X$ such that $M(x) \subseteq \phi(x)$ for every $x \in X$. More precisely, for each $i \in N$, the individual behavior relation $\phi_i : X \twoheadrightarrow X_i$ maps each joint strategy $x \in X$ to the superset $\phi_i(x)$ of player i's best responses $M_i(x)$.[5] For any behavior relation $\phi : X \twoheadrightarrow X$ and any nonempty product set $Z \subseteq X$, $\phi(Z)$ denotes the union of all images $\phi(z)$, $z \in Z$, i.e.

$$\phi(Z) = \bigcup_{z \in Z} \phi(z).$$

Given any behavior relation $\phi \in \Phi$, a **closed set under** ϕ is a product set $Z = \prod_{i \in N} Z_i \subseteq X$ such that

(i) for each $i \in N$, $Z_i \subseteq X_i$ is a nonempty compact set of strategies;

(ii) for each $i \in N$, and each belief $z \in Z$ of player i, the set Z_i contains all best responses of player i against his belief: $\phi_i(Z) \subseteq Z_i$.

A **prep set** under ϕ is a product set of strategies $Z = \prod_{i \in N} Z_i \subseteq X$ that satisfies (i) and

(iii) for each $i \in N$, and each belief $z \in Z$ of player i such that $\phi_i(z) \neq \emptyset$, the set Z_i contains at least one best response of player i against his belief: $\forall i \in N$, $\forall z \in Z$ such that $\phi_i(z) \neq \emptyset$, it holds that $\phi_i(z) \cap Z_i \neq \emptyset$.

[5] In the context of Basu and Weibull (1991) and Ritzberger and Weibull (1995), behavior relations are correspondences.

A closed set $Z \subseteq X$ under $\phi \in \Phi$ is called **minimal** if Z does not contain a proper subset which is closed under ϕ. A prep set $Z \subseteq X$ under ϕ is called **minimal** if no prep set is a proper subset of Z.

Remark 1 (i) Basu and Weibull (1991) call a compact product set $Z \subseteq X$ closed under rational behavior (curb) if Z is closed under the combined best-response relation $M \in \Phi$. The concept of a minimal curb set generalizes the notion of strict Nash equilibrium. Indeed, consider any $x \in X$. The singleton set $\{x\}$ is a strict Nash equilibrium if and only if it is closed under M, i.e. if $M(\{x\}) = \{x\}$.

(ii) A product set is a prep set in the sense of Voorneveld (2004, 2005) if it contains at least one best response (but not necessarily all best responses) to any consistent belief that a player may have about the strategic behavior of his opponents. The concept of a minimal prep set under M generalizes the notion of Nash equilibrium. Indeed, consider any $x \in X$. The singleton set $\{x\}$ is a Nash equilibrium if and only if it is a prep set under M. While the minimal prep sets and the minimal curb sets of a game can differ, they coincide in generic finite games.

(iii) Ritzberger and Weibull (1995) call a compact product set $Z \subseteq X$ closed under better responses (cubr) if Z is closed under the combined better-response relation $B \in \Phi$. Because of the finiteness of X, the compactness requirement on Z is not restrictive.

Theorem 5 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be an ordinal game. Suppose that for each $i \in N$, the set of strategies X_i is equipped with a compact Hausdorff topology. Then:*

(i) *for every $\phi \in \Phi$, there exists a minimal closed set;*

(ii) *for every $\phi \in \Phi$, minimal closed sets are pairwise disjoint;*

(iii) *if $Z \subseteq X$ is a minimal closed set under $\phi \in \Phi$ and $\phi(Z)$ is a compact and nonempty subset of X, then $\phi(Z) = Z$.*

(iv) *for every $\phi \in \Phi$ with closed-valued components ϕ_i, $i \in N$, there exists a minimal prep set.*

PROOF. Part (i). Let $\phi \in \Phi$ and let Q_ϕ be the collection of all sets which are closed under ϕ in G. Obviously, $X \in Q_\phi$. Hence Q_ϕ is nonempty. Consider the partially ordered set $(Q_\phi, \subseteq)$. By Hausdorff's maximum principle, Q_ϕ contains a maximal totally ordered subset, say $(Q^*_\phi, \subseteq)$. Set

$$\tilde{Z} = \bigcap_{Z \in Q^*_\phi} Z \text{ and } \tilde{Z}_i = \bigcap_{Z \in Q^*_\phi} Z_i$$

for $i \in N$. Observe that each $Z \in Q^*_\phi$ is a closed subset of the compact Hausdorff space X. By construction, the collection of closed sets Q^*_ϕ is totally ordered by the relation $\subseteq$. Therefore, every nonempty finite subcollection of Q^*_ϕ has a nonempty intersection. Since X is compact, the finite intersection property holds and thus $\tilde{Z}$ is nonempty. As the intersection of closed sets, $\tilde{Z}$ is closed. As a closed subset of the compact set X, $\tilde{Z}$ is compact. Moreover, it is a product set, $\tilde{Z} = \prod_{i \in N} \tilde{Z}_i$. By definition, $\phi(\tilde{Z}) \subseteq \phi(Z) \subseteq Z$ for each $Z \in Q^*_\phi$ and, consequently, $\phi(\tilde{Z}) \subseteq \tilde{Z}$. This shows that $\tilde{Z}$ is a closed set under ϕ. $\tilde{Z}$ is necessarily minimal. For otherwise, $(Q^*_\phi, \subseteq)$ would not be maximal.

Part (ii). Suppose that $Z, Y \subseteq X$ are two arbitrary distinct minimal closed sets under some $\phi \in \Phi$, but $Z \cap Y \neq \emptyset$. Let $C = Z \cap Y$. By definition, $\phi(C) \subseteq C$ which contradicts that both Y and Z are minimal.

Part (iii). $\phi(Z) \subseteq Z$ implies $\phi(\phi(Z)) \subseteq \phi(Z)$. Compactness and nonemptiness of $\phi(Z)$ then implies that $\phi(Z)$ is closed with respect to ϕ. Hence $\phi(Z) \subsetneq Z$ contradicts the minimal closedness of Z with respect to ϕ.

Part (iv). Modify the proof of part (i) and let Q_ϕ rather be the collection of all prep sets under ϕ in G. Construct as before a nonempty and compact product set $\tilde{Z} = \prod_{i \in N} \tilde{Z}_i$, starting from a maximal totally ordered subset $(Q^*_\phi, \subseteq)$ of Q_ϕ. Pick any $i \in N$, $z \in \tilde{Z}$ such that $\phi_i(z) \neq \emptyset$. To show: $\phi_i(z) \cap \tilde{Z}_i \neq \emptyset$. Note that

$$\tilde{Z}_i \cap \phi_i(z) = \left(\bigcap_{Z \in Q^*_\phi} Z_i \right) \cap \phi_i(z) = \bigcap_{Z \in Q^*_\phi} \Big(Z_i \cap \phi_i(z) \Big).$$

For each $Z \in Q^*_\phi$, the set $Z_i \cap \phi_i(z)$ is nonempty since $z \in \tilde{Z} \subseteq Z$, $\phi_i(z) \neq \emptyset$ and Z is a prep set. Moreover, it is closed as the intersection of Z_i, a closed subset of X_i, and $\phi_i(z)$ a closed set by assumption. Finally, the collection of closed sets $\{Z_i \cap \phi_i(z)\}$ is nested, since Q^*_ϕ is totally ordered. Because X_i is compact, the finite intersection property holds and we obtain

$$\bigcap_{Z \in Q^*_\phi} \Big(Z_i \cap \phi_i(z) \Big) \neq \emptyset.$$

So $\tilde{Z}$ is a prep set under ϕ. It is necessarily minimal. For otherwise, $(Q^*_\phi, \subseteq)$ would not be maximal. ■■

Remark 2 (i) A closed set under ϕ is also a prep set under ϕ. Hence, if each ϕ_i is closed-valued, part (iv) of the foregoing proof can be reiterated to show that every closed set under ϕ contains a minimal prep set under ϕ.

(ii) Notice that if the relation ϕ is compact-valued and upper hemicontinuous, then for any compact subset Z of X, $\phi(Z)$ is compact.[6] In par-

[6]See Hildenbrand (1974, B.III, Proposition 3). The argument holds for relations as well.

ticular, finiteness of N, continuity of each $\succeq_i$ on X plus nonemptiness and compactness of each X_i imply that $M \in \Phi$ is compact-valued and upper hemicontinuous.

(iii) Also notice that in the case of a compact Hausdorff space X, a subset of X is compact if and only if it is topologically closed. Therefore, "compact" can be replaced by "topologically closed" in the definition of a closed set under ϕ and the assertions of the theorem.

(iv) Further observe that one can define closedness of a product set $Z \subseteq X$ under an arbitrary relation $\phi : X \twoheadrightarrow X$. Then the analogue of Theorem 5 still holds true.

Finally, we obtain the analogue of an observation by Basu and Weibull (1991): The rationalizable strategies form the largest "tight" curb set. If ϕ is a behavior relation and $Z \subseteq X$ is a nonempty compact product set with $\phi(Z) = Z$, then Z is called a **tight set closed under** ϕ.

Corollary 2 *Under the hypothesis of Theorem 4,*

(i) *the set of joint rationalizable strategies $R \subseteq X$ is a tight set closed under the best response correspondence M;*

(ii) *if $Z \subseteq X$ is a tight set closed under M then $Z \subseteq R$.*

PROOF. (i) By construction, R is a product set. By Theorem 4, R is nonempty and compact. By construction, $R = \bigcap_k M^k(X)$ where $k = 0, 1, 2, \ldots$ and $M^0(S) = S$ for $S \subseteq X$. Hence $M(R) = \bigcap_k M^{k+1}(X) = \bigcap_k M^k(X) = R$, since $M^0(X) = X$. This shows that R is a tight set closed under M.

(ii) Suppose $Z \subseteq X$ is a tight set closed under M. Then Z is a compact product set with $M(Z) = Z$. Therefore, $Z = \bigcap_k M^k(Z) \subseteq \bigcap_k M^k(X) = R$. ■■

In general, R is not a minimal set closed under M. For instance, if the game G has two strict ordinal Nash equilibria x and x', then $\{x\}$ and $\{x'\}$ are two disjoint minimal tight sets closed under M.

6 Public Project Proposal Game

In this section, we provide a formal description and elaborate analysis of the public project proposal game highlighted in the introduction.

Consider the problem of locating a finite number $p \geq 2$ of identical public projects, say libraries, on a street represented by the unit interval $[0, 1]$. An outcome of this problem is a list of p locations, $y = (y_1, \ldots, y_p) \in [0, 1]^p$. For $p = 2$ and a single-peaked preference relation R on the unit interval, Ehlers (2002, 2003) introduces the "lexicographic extension" of R, a preference relation P on $[0, 1]^p$. The rationale is that a library patron will visit his second choice library if a book he wants to borrow is unavailable at the first choice library. Therefore, the patron's preference for locations induces a "lexicographic" preference relation for outcomes (lists of locations). Ehlers' "lexicographic extension" can be constructed for any $p \geq 2$:

> A preference relation (weak order) R_i on $[0, 1]$ induces a "lexicographic" preference relation P_i on $[0, 1]^p$ as follows. Given two alternatives $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p) \in [0, 1]^p$ such that (possibly after rearranging the order in each sequence)

$a_1 R_i a_2 R_i \ldots R_i a_p$ and $b_1 R_i b_2 R_i \ldots R_i b_p$. An agent i — with preference relations R_i and P_i — prefers a to b if [i prefers a_1 to b_1] OR [i is indifferent between a_1 and b_1 and prefers a_2 to b_2] OR [i is indifferent between a_1 and b_1, is indifferent between a_2 and b_2 and prefers a_3 to b_3], etc. Agent i is indifferent between a and b if i is indifferent between a_1 and b_1, a_2 and b_2, ... , a_p and b_p .

Next consider two patrons, $i = 1, 2$, each with a single-peaked preference relation R_i for locations in $[0, 1]$, with a peak at $\hat{x}_i$. Let P_i denote the "lexicographic extension" of R_i. Each patron proposes an outcome $x_i \in [0, 1]^p$. The pair of proposals determines the outcome $y = F(x_1, x_2) = (x_1 + x_2)/2$. The **public project proposal game**, a particular ordinal game $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ where players' preference relations are weak orders without utility representations is given by:

- The player set $N = \{1, 2\}$;
- strategy sets $X_1 = X_2 = [0, 1]^p$;
- preference relations $\succeq_1, \succeq_2$ on $X_1 \times X_2$, defined by

$$(x_1, x_2) \succeq_i (z_1, z_2) :\Longleftrightarrow F(x_1, x_2) P_i F(z_1, z_2)$$

  for $(x_1, x_2), (z_1, z_2) \in X_1 \times X_2$.

One obtains:

Proposition 1 *Suppose the players' peaks satisfy* $0 < \hat{x}_1 < \hat{x}_2 < 1$. *Then:*

(I) *The game G is a best-response potential game.*

(II) *The game G has a unique Nash equilibrium.*

(III) *Each best-response compatible path of G converges to the Nash equilibrium in finitely many steps.*

Note that property (I) holds for arbitrary $\hat{x}_1$ and $\hat{x}_2$. Also note that in case $0 < \hat{x}_1 = \hat{x}_2 < 1$, there exists a continuum of Nash equilibria, all resulting in the outcome $y = \hat{x}_1 = \hat{x}_2$. Further observe that for a finite game G, (I) would have as immediate consequences (II') existence of a Nash equilibrium and (III') convergence of any best-response compatible path to a Nash equilibrium in finitely many steps. It follows from the proof of (III) that a best-response compatible path reaches the Nash equilibrium in at most $6 + \lceil 1/(\hat{x}_2 - \hat{x}_1) \rceil$ steps, where for a real number r, the symbol $\lceil r \rceil$ means "the smallest integer not smaller than r".

PROOF:

PART I. By Theorem 1, it suffices to show that G has no best-response cycle.

Note that in this context, a patrons's optimal proposal against his opponent's proposal with respect to $\succeq_i$ is quite simple. It suffices to observe that F is symmetric and each $F_l(x_1, x_2)$ only depends on the l-th element of each proposal. Thus, a best response $M_i(x_{-i})$ for player $i \in N$ against x_{-i} can be decomposed into p independent best responses $M_{i,l}(x_{-i,l})$, $l \in A = \{1, \dots, p\}$, as follows:

$$M_{i,l}(x_{-i,l}) = \begin{cases} 0 & \text{if } \; x_{-i,l} > 2\hat{x}_i \\ 2\hat{x}_i - x_{-i,l} & \text{if } \; 2\hat{x}_i - 1 \leq x_{-i,l} \leq 2\hat{x}_i \\ 1 & \text{if } \; x_{-i,l} < 2\hat{x}_i - 1 \end{cases} \tag{1}$$

For the sake of contradiction assume that G has a best-response cycle $(x^1, \dots, x^m)$. Define the function $W : X \longrightarrow \mathbb{R}^p$ which assigns to each pair of proposals (x_1, x_2) the vector of real numbers $W(x) = (W_l(x))_{l \in A}$ where for each $l \in A$, $W_l(x) = x_{1,l} + 1 - x_{2,l}$. Pick any $l \in A$ and $k \in \{1, \dots, m-1\}$. We distinguish between different cases according to the positions of $x^k_{i,l}$ and $\hat{x}_i$, $i \in N$.

(a) Suppose that $x^k_{1,l} \geq \hat{x}_1$ and $x^k_{1,l} + x^k_{2,l} > 2\hat{x}_1$. By definition of a best-response cycle, $x^k_{i(k-1),l} = M_{i(k-1),l}(x^{k-1}_{-i(k-1),l})$. Observe that the deviating player $i(k-1)$ cannot be player 1. Because the best-response correspondence is single-valued, it is the turn of player 1 to play his best response at step k. Obviously, he makes a proposal such that $x^{k+1}_{1,l} < x^k_{1,l}$ in order to obtain

$$\hat{x}_1 \leq \frac{x^{k+1}_{1,l} + x^{k+1}_{2,l}}{2} < \frac{x^k_{1,l} + x^k_{2,l}}{2}.$$

It follows that $W_l(x^k) > W_l(x^{k+1})$.

(b) Suppose that $x^k_{1,l} \geq \hat{x}_1$ and $x^k_{1,l} + x^k_{2,l} = 2\hat{x}_1$. Player $i(k-1)$ cannot be player 2: because $\hat{x}_1 < \hat{x}_2 < 1$ and $x^k_{1,l} + x^k_{2,l} = 2\hat{x}_1$, player 2's best response is such that $x^k_{2,l} = 1$, which implies $x^k_{1,l} + 1 > 2\hat{x}_1$. Consequently, it is the turn of player 2 to play his best response at step k and he makes a proposal such that $x^{k+1}_{2,l} > x^k_{2,l}$ in order to obtain

$$\frac{x^k_{1,l} + x^k_{2,l}}{2} < \frac{x^{k+1}_{1,l} + x^{k+1}_{2,l}}{2} \leq \hat{x}_2.$$

It follows that $W_l(x^k) > W_l(x^{k+1})$.

(c) Suppose that $x^k_{1,l} \geq \hat{x}_1$ and $x^k_{1,l} + x^k_{2,l} < 2\hat{x}_1$. As in (b), player $i(k-1)$ can not be player 2. Thus, player $i(k-1)$ is player 1 and $x^k_{1,l} = 1$. But $1 + x^k_{2,l} < 2\hat{x}_1$ and $\hat{x}_1 < \hat{x}_2$ imply that $x^k_{2,l}$ cannot be part of player 2's best response whatever the choice of location by player 1. Thus, since x^k belongs to a best-response cycle, x^k cannot satisfy $x^k_{1,l} \geq \hat{x}_1$ and $x^k_{1,l} + x^k_{2,l} < 2\hat{x}_1$.

(d) $x^k_{1,l} < \hat{x}_1$ and $x^k_{1,l} + x^k_{2,l} < 2\hat{x}_1$. This case is similar to (c). Player $i(k-1)$ cannot be player 1. It follows that player $i(k-1)$ is player 2 and $x^k_{2,l} = 1$. But $x^k_{1,l} + 1 < 2\hat{x}_1$ implies that $x^k_{1,l}$ cannot be part of player 1's best response against any choice of location by player 2. Thus, since x^k belongs to a best-response cycle, x^k cannot satisfy $x_{1,l}^k < \hat{x}_1$ and $x_{1,l}^k + x_{2,l}^k < 2\hat{x}_1$.

(e) $x_{1,l}^k < \hat{x}_1$ and $x_{1,l}^k + x_{2,l}^k = 2\hat{x}_1$. Note that player $i(k-1)$ can be either player 1 or player 2. We distinguish between two cases. Firstly, assume that player $i(k)$ is player 1. Then, he chooses $x_{1,l}^{k+1} = x_{1,l}^k$. It follows that $W_l(x^k) = W_l(x^{k+1})$. Secondly, assume that player $i(k)$ is player 2. Then, he makes a proposal such that $x_{2,l}^{k+1} \geq x_{2,l}^k$ (equality appears when $x_{2,l}^k = 1$) in order to obtain

$$\frac{x_{1,l}^k + x_{2,l}^k}{2} \leq \frac{x_{1,l}^{k+1} + x_{2,l}^{k+1}}{2} \leq \hat{x}_2.$$

It follows that $W_l(x^k) \geq W_l(x^{k+1})$.

(f) $x_{1,l}^k < \hat{x}_1$ and $2\hat{x}_1 < x_{1,l}^k + x_{2,l}^k < 2\hat{x}_2$. Note that player $i(k-1)$ can be either player 1 or player 2. We distinguish between these two cases. Firstly, assume that player $i(k-1)$ is player 1. Then, it is the turn of player 2 to play his best response at step k and he makes a proposal such that $x_{2,l}^{k+1} \geq x_{2,l}^k$ in order to obtain

$$\frac{x_{1,l}^k + x_{2,l}^k}{2} \leq \frac{x_{1,l}^{k+1} + x_{2,l}^{k+1}}{2} \leq \hat{x}_2.$$

It follows that $W_l(x^k) \geq W_l(x^{k+1})$. Secondly, assume that player $i(k-1)$ is player 2. Then, it is the turn of player 1 to play his best response at step k and he makes a proposal such that $x_{1,l}^{k+1} \leq x_{1,l}^k$ in order to obtain

$$\hat{x}_1 \leq \frac{x_{1,l}^{k+1} + x_{2,l}^{k+1}}{2} \leq \frac{x_{1,l}^k + x_{2,l}^k}{2}.$$

It follows that $W_l(x^k) \geq W_l(x^{k+1})$.

(g) $x_{1,l}^k < \hat{x}_1$ and $x_{1,l}^k + x_{2,l}^k = 2\hat{x}_2$. This case is similar to (e). Note that player $i(k-1)$ can be either player 1 or player 2. We distinguish between two cases. Firstly, assume that player $i(k)$ is player 2. Then, he makes a proposal such that $x_{2,l}^{k+1} = x_{2,l}^{k}$. It follows that $W_l(x^k) = W_l(x^{k+1})$. Secondly, assume that player $i(k)$ is player 1. Then, he makes a proposal $x_{1,l}^{k+1} \leq x_{1,l}^{k}$ in order to obtain

$$\frac{x_{1,l}^{k+1} + x_{2,l}^{k+1}}{2} \leq \frac{x_{1,l}^{k} + x_{2,l}^{k}}{2} \leq \hat{x}_2.$$

It follows that $W_l(x^k) \geq W_l(x^{k+1})$.

(h) $x_{1,l}^{k} < \hat{x}_1$ and $2\hat{x}_2 < x_{1,l}^{k} + x_{2,l}^{k}$. This case is similar to (c). Player $i(k-1)$ cannot be player 2. Then, player $i(k-1)$ is player 1 and $x_{1,l}^{k} = 0$. But, $0 + x_{2,l}^{k} > 2\hat{x}_2$ implies that $x_{2,l}^{k}$ cannot be part of player 2's best response against any choice of location by player 1. Thus, since x^k belongs to a best-response cycle, x^k cannot satisfy $x_{1,l}^{k} < \hat{x}_1$ and $x_{1,l}^{k} + x_{2,l}^{k} > 2\hat{x}_2$.

Notice that in cases (e), (f) and (g), $W_l(x^k) = W_l(x^{k+1})$ obtains only if $x_{i(k),l}^{k+1} = x_{i(k),l}^{k}$. Moreover, by the definition of a best-response cycle, $x^1 = x^m$, and $x^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^{k}) \succ_{i(k)} x^k$ for some $k \in \{1, \dots, m-1\}$. Hence for some $l \in A$, $W_l(x^1) > W_l(x^m) = W_l(x^1)$, a contradiction. Hence contrary to the above assumption, G does not have a best-response cycle.

PART II. We distinguish between three cases.

(a) The preference profile $R = (R_1, R_2)$ is such that $\hat{x}_1 < \hat{x}_2 \leq 1/2$. In such a case, consider the strategy profile $x^* = (x_1^*, x_2^*)$ such that, for any $l \in A$, $x_{1,l}^* = 0$ and $x_{2,l}^* = 2\hat{x}_2$. We claim that x^* is the only Nash equilibrium of G.

Note that $x_{2,l}^* = 2\hat{x}_2$, $l \in A$, is the best response for player 2 against $x_{1,l}^* = 0$, $l \in A$, since $x_{2,l}^* = 2\hat{x}_2 - x_{1,l}^*$ and $2\hat{x}_2 - 1 \leq x_{1,l}^* \leq 2\hat{x}_2$. And, $x_{1,l}^* = 0$, $l \in A$, is the best response for player 1 against $x_{2,l}^* = 2\hat{x}_2$, $l \in A$, since

$x^*_{2,l} > 2\hat{x}_1$ — which shows that x^* is a Nash equilibrium of G.

It remains to check that x^* is the unique Nash equilibrium of the game. We proceed by demonstrating that a strategy profile $z = (z_1, z_2)$, where for some $l \in A$, $z_{1,l} \neq 0$ or $z_{2,l} \neq 2\hat{x}_2$, cannot be a Nash equilibrium of G. First observe that if $z_{1,l} = z_{2,l}$ for some $l \in A$, then z cannot be a Nash equilibrium. Because $\hat{x}_1 < \hat{x}_2$, there exists at least one player $i \in \{1, 2\}$ such that $z_{i,l} \neq \hat{x}_i$. This means that player i does not play a best response against z_{-i}. Secondly, suppose that $z_{1,l}, z_{2,l} \in]0, 1[$ and $z_{1,l} \neq z_{2,l}$ for some $l \in A$. Then, there exists at least one player $i \in \{1, 2\}$ such that $z_{1,l} + z_{2,l} \neq 2\hat{x}_i$. If $z_{1,l} + z_{2,l} < 2\hat{x}_i$, then $M_{i,l}(z_{-i,l}) > z_{i,l}$. And if $z_{1,l}+z_{2,l} > 2\hat{x}_i$, then $M_{i,l}(z_{-i,l}) < z_{i,l}$. Therefore, the profile z cannot be a Nash equilibrium. Thirdly, suppose that $z_{1,l} = 1$ and $z_{2,l} = 0$ for some $l \in A$. It follows that $z_{1,l} + z_{2,l} > 2\hat{x}_1$, $M_{1,l}(z_{2,l}) < z_{1,l}$, and so z is not a Nash equilibrium. Fourthly, suppose that $z_{2,l} = 1$ and $z_{1,l} \in [0, 1[$ for some $l \in A$. If $z_{1,l} = 0$ and $\hat{x}_2 = 1/2$, then $z = x^*$. Otherwise, $M_{2,l}(0) < 1$ means that $(z_{1,l}, z_{2,l}) = (0, 1)$ cannot be part of a Nash equilibrium. But if $z_{1,l} \in]0, 1[$, then player 1 does not play a best response against the strategy played by player 2 since $M_{1,l}(1) = 0$. This means that z is not a Nash equilibrium. We conclude that x^* is the only Nash equilibrium of G.

(b) The preference profile $R = (R_1, R_2)$ is such that $1/2 \leq \hat{x}_1 < \hat{x}_2$. In such a case, consider the strategy profile $x^* = (x^*_1, x^*_2)$ such that, for any $l \in A$, $x^*_{1,l} = 2\hat{x}_1 - 1$ and $x^*_{2,l} = 1$. Then x^* is a Nash equilibrium of G. To see this, note that $x^*_{1,l} = 2\hat{x}_1 - 1$, $l \in A$, is the best response for player 1 against $x^*_{2,l} = 1$, $l \in A$, since $x^*_{1,l} = 2\hat{x}_1 - x^*_{2,l}$ and $2\hat{x}_1 - 1 \leq x^*_{2,l} \leq 2\hat{x}_1$. And $x^*_{2,l} = 1$ is the best response for player 2 against $x^*_{1,l} = 2\hat{x}_1 - 1$ since $x^*_{1,l} < 2\hat{x}_2 - 1$. We can prove uniqueness of the Nash equilibrium as in (a).

(c) The preference profile $R = (R_1, R_2)$ is such that $\hat{x}_1 < 1/2 < \hat{x}_2$. In such a case, consider the strategy profile $x^* = (x_1^*, x_2^*)$ such that, for any $l \in A$, $x_{1,l}^* = 0$ and $x_{2,l}^* = 1$. Then x^* is a Nash equilibrium of G. To see this, note that $x_{1,l}^* = 0$, $l \in A$, is the best response for player 1 against $x_{2,l}^* = 1$, $l \in A$, since $x_{2,l}^* > 2\hat{x}_1$. And $x_{2,l}^* = 1$ is the best response for player 2 against $x_{1,l}^* = 0$, $l \in A$, since $x_{1,l}^* < 2\hat{x}_2 - 1$. Uniqueness of the Nash equilibrium can be proven as in (a).

PART III. We prove that each best-response compatible path $x^1, x^2, x^3, \ldots$ converges in finite time to the Nash equilibrium x^* of G. Without loss of generality, we may assume $p = 1$, since best responses can be determined componentwise. We provide the complete argument for the case $\hat{x}_1 < \hat{x}_2 \leq 1/2$, $x^* = (0, 2\hat{x}_2)$. Suppose the last adjustment was made by player 2 resulting in (x_1^k, x_2^k).

(A) $x_2^k > 2\hat{x}_1$: Then $x_1^{k+1} = x_1^{k+2} = 0$ and $x^{k+2} = x^*$.

(B) $x_2^k \leq 2\hat{x}_1$ (and $2\hat{x}_1 - 1 < 0 < x_2^k$): Then $x_1^{k+1} = 2\hat{x}_1 - x_2^k < 2\hat{x}_2$.

(B.1) $x_1^{k+1} < 2\hat{x}_2 - 1$: Then $x_2^{k+2} = 1, x_1^{k+3} = 0, x^{k+4} = x^*$.

(B.2) $x_1^{k+1} \geq 2\hat{x}_2 - 1$: Then

$$x_2^{k+2} = 2\hat{x}_2 - x_1^{k+1} = 2\hat{x}_2 - (2\hat{x}_1 - x_2^k) = x_2^k + 2(\hat{x}_2 - \hat{x}_1).$$

In case (B.2), we repeat the loop, starting with x_2^{k+2} instead of x_2^k. If we end up in (A) or (B.1), x^* is reached in at most four steps. Whenever we end up in (B.2), $x_2^{k+2\ell+2} = x_2^{k+2\ell} + 2(\hat{x}_2 - \hat{x}_1)$ for $\ell \geq 1$ which can only happen finitely many times.

By symmetry, a similar argument can be made in the case $1/2 \leq \hat{x}_1 < \hat{x}_2$. In the case $\hat{x}_1 < 1/2 < \hat{x}_2$, a best response by player 1 is always less than

1 and a best response by player 2 always exceeds 0. If the best-response compatible path $x^1, x^2, x^3, \ldots$ did not converge to x^* in finitely many steps, then $x_1^k, x_2^k \in (0,1)$ for all k and, therefore, $x_2^{k+2\ell+2} = x_2^{k+2\ell} + 2(\hat{x}_2 - \hat{x}_1)$ and $x_1^{k+2\ell+2} = x_1^{k+2\ell} - 2(\hat{x}_2 - \hat{x}_1)$ for all $k \geq 1, \ell \geq 1$, a contradiction. Hence, to the contrary, the path converges to x^* in finitely many steps. ■■

7 Final Remarks

The focus of this paper lies on games with players' preferences which are weak orders. Within this broad category of games, we analyze games with ordinal best-response potentials and quasi-supermodular games. We further provide sufficient conditions for the existence of a nonempty set of rationalizable joint strategies and of a closed set under a behavior relation.

In the context of cardinal games, it is frequently assumed that the mixed extension of a game exists, that is each cardinal utility representation can be extended to an expected utility functional on the set of joint mixed strategies. Consequently, the notion of a Nash equilibrium in mixed strategies can be adopted. Furthermore, the definitions of rationalizable joint strategies and of closed sets under a behavior correspondence may include best responses against mixed strategies. There is no straightforward and commonly agreed upon extension of ordinal preferences from pure to mixed strategies. Therefore our analysis and definitions are confined to pure strategies. In lieu of expected utility comparisons, Fishburn (1978) and Perea *et al.* (2006) apply first-order stochastic dominance (induced by the ordinal preferences) to joint mixed strategies. This defines a partial order on joint mixed strategies.

As mentioned in the introduction, there exists a sizeable literature on generalized games with incomplete or intransitive preferences. Extensions of our analysis to generalized games and/or games with incomplete or intransitive preferences are left to future research.

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