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# TIME TO RUIN, INSOLVENCY PENALTIES AND DIVIDENDS IN A MARKOV-MODULATED MULTI-RISK MODEL WITH COMMON SHOCKS

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# TIME TO RUIN, INSOLVENCY PENALTIES AND DIVIDENDS IN A MARKOV-MODULATED MULTI-RISK MODEL WITH COMMON SHOCKS.

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## ABSTRACT

We consider a main insurance company with $K$ subcompanies (or lines of business). The joint evolution of the surpluses of these lines of business is modeled by a Markov-modulated multivariate compound Poisson model with Poisson common shocks, modified by interactions between the lines of business and paiement of dividends. We assume that the financial situation of the subcompanies has an impact on the other companies, for example because they have part of their surplus invested in one another. If a line of business is in the red, the others have to pay a penalty, which is traduced by a decrease of the premium received by unit of time, or by a lost of dividends for the shareholders if the other line of business is "doing well". Conversely, a line of business with a high surplus level may increase the premium by unit of time of the others as they receive part of the dividends. In this paper, we focus on a particular line of business, and provide an approximation for expected time to ruin, and the expected amounts of dividends paid to the shareholders, and used to pay penalty due to insolvency of some subcompany. The method is to discretize claim amounts and to approximate the multidimensional surplus process of the subcompanies with a continuous time Markov process with finite state space. A technique of Frostig (2005) and Kella and Whitt (1992) enables us to get approximates, which are shown to converge to the desired values. It is possible to compare the behavior of the main company with and without the other subcompanies, which could provide a tool to help making consortium building decision.

## I INTRODUCTION

In this paper, some risk and dividend-related problems are defined and studied for a $K$-dimensional process $R(t)$ that models the joint evolution of the surpluses of the $K \geq 1$ lines of business of an insurance company. The model we propose takes into account dependence between lines of business both for the multivariate claim process, and for the premium incomes and dividends, and is based on the models developed in loisel (2004) and loisel (2005).

We consider that line of business 1 behaves slightly differently from the other ones (it might correspond for example to a main company with $K-1$ subcompanies). To take dependence between claim arrivals and amounts of the $K$ lines of business, $R(t)$ is assumed to be a $K$-dimensional process based on a compound Poisson process with Poisson common shocks in Markovian environment. More precisely, conditionally on the state of the environment, the multivariate claim process is supposed to be compound Poisson with common shocks. As for the environment, it is modeled by a Markov process with finite state space. Claim amounts, taking values in $(\mathbb{R}^+)^K$, are supposed to have exponentially distributed marginals.

Besides, the interaction between lines of business is also traduced by penalties and dividends which modify the vector of premium income rates, which becomes a function of the position of the $K$-dimensional surplus process. Informally,

- If one or several lines of business become insolvent, this may trigger a penalty for the other lines of business, modeled by a decrease of their premium income rates.
- The surplus level of each line of business is limited to an upper barrier. The excess is transformed into dividends for the shareholders if no line of business is in the red. Otherwise, the excess is used to pay a penalty due to the insolvency of some other lines of business, for example because of financial consequences, or as a way to help the other lines borrow some money in order to recover.
- Line of business 1 may benefit from part or totality of the excesses of other lines of business if they are at their maximum wealth level. This is modeled by an increase of the premium income rate of line of business 1 when another line of business reaches its maximum level.

Typical questions that arise are:

- what proportion of the excess of the surplus of line of business 1 is in average lost for the shareholders due to the insolvency of some other lines of business? This represents for the shareholders a lost of dividends that they would not have undergone if the subcompanies were completely separate.
- Does the expected time to ruin of a line of business increase or decrease due to the possible financial support or penalty coming from the impact of the surpluses of other lines of business ?
- What is the probability for a line of business to get recovered after its ruin ?

To provide a clue for the last question, we shall derive the distribution of the joint surplus process for the $K$ lines of business at the time to ruin of one line of business.

All these questions seem very tricky, because they involve the *simultaneous* behavior of the different lines of business. The first approach is to try to generalize results of Asmussen and Kella (2000) for multidimensional martingales to multidimensional processes. In Section II, we precise the motivations for the model, which we formally present. We show that it is possible to obtain the Laplace transform and a martingale for the multidimensional process.

In Section III, we try to answer the three questions mentioned earlier. We show how to approximate the model with a Markov-modulated risk model in which the evolution of $K-1$ lines of business are now contained in the Markovian environment process. We apply the methods of Asmussen and Kella (2000) and Frostig (2005), and compute the expected amounts of dividends paid by line of business 1 to the shareholders until its ruin, and the expected amount paid to the other lines of business. We also derive the distribution of the $K$-dimensional surplus process at the time to ruin of a line of business.

In Section IV, we give other ideas of applications, including reinsurance reliability, and adding quasi-default states.

## II THE MULTI-DIMENSIONAL MODEL

### 1 Why this model

Consider first the process modeling the wealth of the $K$ lines of business of an insurance company, without notions of dividends and impact of insolvency of one line of business on the other one.

Typical lines of business are driving insurance, house insurance, health, incapacity, death, liability,... They may also be subcompanies of a main insurance company, which may be a line of business or not. They may also face risks of the same nature, but in different countries or regions of the world.

Two main kinds of phenomena may generate dependence between the aggregated claim amounts of these lines.

- Firstly, in some cases, claims for different lines of business may come from a common event: for example, a car accident may cause a claim for driving insurance, liability and disablement insurance. Hurricanes might cause losses in different countries. This should correspond to simultaneous jumps for the multivariate process. The most common tool to take this into account is the Poisson common shock model.
- Secondly, there exist other sources of dependence, for example the influence of the weather on health insurance and on agriculture insurance. In this case, claims seem to outcome independently for each branch, depending on the weather. This seems to correspond rather to models with modulation by a Markov process which describes the evolution of the state of the environment.

Another example is the influence of police controls. Recently, in France, the development of speed controls downed the number of accidents, and the number of severely injured people on the road. The frequency of controls may vary over time and create time-correlation. This is another aspect of Markov-modulation, which may generate over-dispersion for some lines of business.

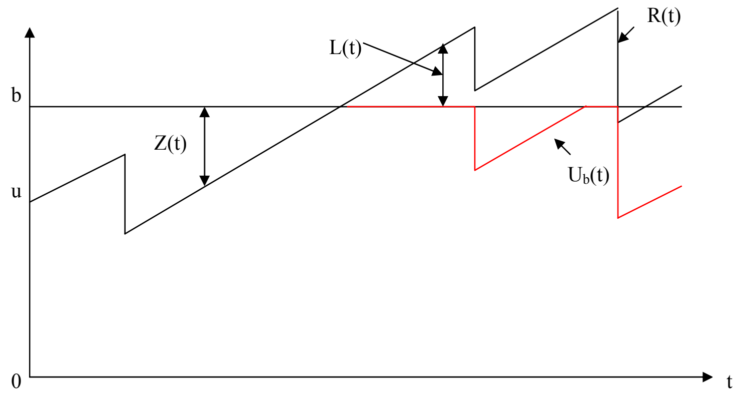


Figure 1: Explanation for $U_b(t)$, $Z(t)$ and $L(t)$.

Each jump may be specific to a line of business, or occur at the same instant as a jump of the other line. Here, amounts are positively correlated, which can be seen on the graph.

To illustrate the effect of a Markovian environment on a multidimensional claim process, Figure 2 shows a sample path of the surpluses of 3 lines of business of an insurance company, in a Markovian environment, but without common shock. The set of states of the environment has cardinality three. State 3 is the most favorable for the company, almost no claim occurs for lines 1 and 2 in this state. State 1 is the least favorable state for the company, claim frequencies and severities are higher for lines 1 and 2.

Events for the third line of business are independent from the state of the environment. One can see the strong positive dependence between lines 1 and 2, but also their independence conditionally to the environment state. At some moment, the two curves for lines 1 and 2 evolve differently because of this conditional independence.

We also assume that the surplus process of each line of business is limited by an upper horizontal barrier. The excess of surplus is instantaneously transformed into dividends paid to the shareholders, or used to pay penalty due to insolvency of another line. Besides, when a line of business is in the red (its surplus process is below zero), the premium income rate is supposed to be decreased for the other lines of business. If line 1 is the main company, and the other ones its subcompanies, this may be understood because of the surplus of the main company which may be partly invested in some subcompanies.

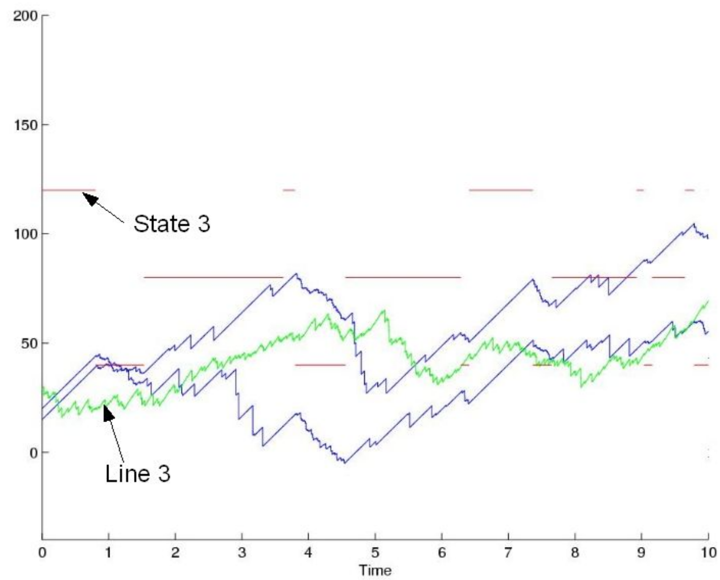


Figure 2: Sample path for three lines of business: Line 3 does not depend on the state of the environment. The two other lines of business have identical parameters, and are independent conditionally on the environmental state. Occupation periods for environment states are characterized by horizontal lines.

## 2 The model

We first define the multidimensional claim process $X(t)$. Then we define the multidimensional surplus process $R(t)$.

### 2.1 Claim process

Let $n$ be the number of states of the environment and $K$ the number of lines of business. Assume that the evolution of the state of the environment is modelled by the Markov process $J(t)$ with initial distribution $\pi_0$ and rate transition matrix $Q$. To define the risk process with $K$ lines of business, consider for $1 \leq i \leq n$ a sequence of i.i.d. random vectors $(W_m^i)_{m \geq 1}$ taking values in $(\mathbb{R}^+)^K$, with distribution function $F_{W^i}$ and exponentially distributed marginals, and independent from a Poisson process $N^i(t)$ with parameter $\lambda^i$, and define the $n$ independent $K$-dimensional Lévy processes

$$X^i(t) = \sum_{l=1}^{N^i(t)} W_m^i - c^i t$$

whose Lévy exponents are denoted by $\varphi^i(\alpha_1, \ldots, \alpha_K)$ (the $N^i$ and the $W_m^i$ are independent from one another), where $c^i = (c_1^i, \ldots, c_K^i)$ is the vector of the premium income rates for all lines of business when the environment state is $i$. Then, define $X(t) = (X_1(t), \ldots, X_K(t))$ as follows:
let $T_p$ be the instant of the $p^{\text{th}}$ jump of the process $J_t$, and

$$\forall k \leq K, \quad X(t) - X(0) = \sum_{p \geq 1} \sum_{1 \leq i \leq n} (X^i(T_p) - X^i(T_{p-1})\mathbf{1}_{\{J_{T_{p-1}}=i, T_p \leq t\}}$$
$$+ \sum_{p \geq 1} \sum_{1 \leq i \leq n} (X^i(t) - X^i(T_{p-1})\mathbf{1}_{\{J_{T_{p-1}}=i, T_{p-1} \leq t < T_p\}}.$$

Define then

$$F(\alpha_1, \ldots, \alpha_K) = Q + \operatorname{diag}(\varphi^1(\alpha_1, \ldots, \alpha_K), \ldots, \varphi^n(\alpha_1, \ldots, \alpha_K)).$$

Note that in case of no common shock, when the environment state is $i$, each claim only attains one single line of business $k$, with probability $\lambda_k^i/\lambda^i$, and severity exponentially distributed with parameter $1/\mu_k^i$. In this case, each $\varphi^i(\alpha_1, \ldots, \alpha_k)$ simplifies into:

$$\varphi^i(\alpha_1, \ldots, \alpha_K) = \sum_{k=1}^{K} \varphi_k^i(\alpha_k),$$

where for $1 \leq i \leq n$ and $1 \leq k \leq K$,

$$\varphi_k^i(\alpha_k) = -c_k^i \alpha_k + \lambda_k^i \frac{1}{1 - \alpha_k \mu_k^i} - \lambda_k^i.$$

**Theorem 1**

$$M'(t,\alpha) = e^{<\alpha,X(t)>}\widetilde{\underline{\mathbf{1}}}_{J_t} e^{-F(\alpha_1,\ldots,\alpha_K)t}$$

*is a n-dimensional martingale (n is the environment state space size) for all* $\alpha \in \mathbb{C}^K$ *such that the* $\varphi_k^i(\alpha_k)$ *all exist, and for all distribution of* $(X(0), J_0)$.
*If* $h(\alpha_1,\ldots,\alpha_K)$ *is a right eigenvector of* $F(\alpha_1,\ldots,\alpha_K)$ *with eigenvalue* $\lambda(\alpha_1,\ldots,\alpha_K)$*, then*

$$N'(t,\alpha) = e^{<\alpha,X(t)>-\lambda(\alpha_1,\ldots,\alpha_K)t} h_{J_t}(\alpha_1,\ldots,\alpha_K)$$

*is a martingale.*

This martingale cannot really be used in this form. Looking forward to apply Doob's optimal stopping theorem, we would not be able to get the position of the multidimensional process at the considered stopping time, as often in a multidimensional setting. Nevertheless, the Laplace transform of $X(t)$ may be used to make recursive computations after discretizing time and space. The corresponding algorithm involves computation of generalized Appell functionals. However, if the state space is not small, or if the number of lines of business is large, it may take too long to run the algorithm. For more information on this, see LOISEL (2005), and PICARD, LEFÈVRE and COULIBALY (2003) for the case without Markovian environment.

## 3 Surplus process modified by the barriers and interactions

Let

$$Y(t) = (Y_1(t),\ldots,Y_K(t))$$

be the $n$ dimensional surplus process defined by:

$$\text{for } 1 \le k \le K, \quad Y_k(t) = u_k - X_k(t),$$

where $u_k$ is the initial reserve level for line of business $k$. Let

$$U(t) = (U_1(t),\ldots,U_K(t))$$

correspond to the process $Y(t)$ modified with the barrier strategy $b = (b_1,\ldots,b_K)$, as in FROSTIG (2005). For $1 \le k \le K$, define first

$$L_k(t) = -\inf{}_{0\le s\le t}\{b_k - u_k + X_k(s)\}^-,$$

where $x^- = \min\,(x,0)$, and

$$Z_k(t) = b_k - u_k + X_k(t) + L_k(t).$$

Then,

$$U_k(t) = b_k - Z_k(t).$$

This defines the surplus process modified by the barrier strategy $(b_1,\ldots,b_K)$: when a line of business $k$ reaches the level $b_k$, all the premium income is paid as dividends until the next claim for line of business $k$.

Define now the new process $R(t) = (R_1(t), \dots, R_K(t))$ as the modification of $U(t)$ induced by the fact that the $c_k^i$ in the $\varphi^i(\alpha_1, \dots, \alpha_K)$ now vary in time, and are actually random processes

$$c_k^i(t) = g_k^i(s_1(t), \dots, s_{k-1}(t), s_{k+1}(t), \dots, s_K(t)),$$

where the $g_k^i$ are nondecreasing functions from $(\{-1, 0, 1\})^{K-1}$ to $]0, +\infty[$, and for $1 \leq k \leq K$,

$$\begin{aligned} s_k(t) &= -1 && \text{if } \mathrm{R}_k(t) < 0, \\ &= +1 && \text{if } \mathrm{R}_k(t) = b_k, \\ &= 0 && \text{otherwise.} \end{aligned}$$

This is illustrated by Figure 3. In this example, line 2 becomes insolvent at $t_1$, which decreases the premium income rate for line 1. Then, line 2 recovers at $t_2$ and reaches its maximum $b_2$ at $t_3$, which increases the premium income rate for line 1.

## III EXPECTED VALUES OF TIME TO RUIN, DIVIDENDS AND INSOLVENCY PENALTIES

Let $\tau$ be the time to ruin of one of the lines of business (called line of business zero). We are interested in knowing the expected value of $\tau$, the distribution $\mathbb{P}_{X_1,\dots,X_K}(\tau)$ of the surpluses of all lines of business at time $\tau$ (to have an idea on possible recovery), and the expected value of the amount of dividends $L_0(\tau)$ paid to the shareholders until $\tau$, and those of penalties paid due to insolvency of at least line of business $k$, denoted by $L_k(\tau)$.

### 1 Outline of the method

We shall now focus on one line of business, which we can choose as line of business 1, and the idea is the following. First, we will approximate the process modeling the evolution of lines of business from 2 to K with a finite state space continuous time Markov process. Then, we will incorporate the position of this $(K-1)$-dimensional process into the environment space. Definition and the way to obtain parameters of this new Markovian environment process are explained in Subsection III.2. We have to make the assumption that all common shocks involving line of business 1 have an exponential marginal distribution for line of business 1, independent from the other severities, but with a probability that may depend on the other severities. This is necessary to use a method of Frostig (2005), slightly modified to add common shocks into the model. We state and prove our main result in Subsection III.3: we answer the three questions mentioned in the introduction in the approximate model, and we prove the convergence of the expected time to ruin of line of business 1, and

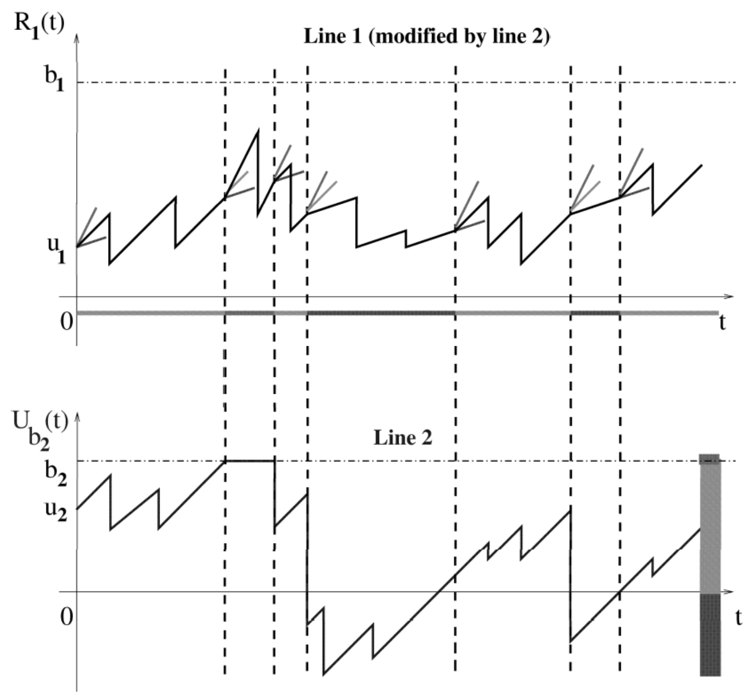


Figure 3: Sample path for two lines of business, and changes in the premium income rates.

the expected values of the dividends paid to the shareholders and of the penalty paid because of insolvency of some other line(s) of business, as well as the joint distribution of the surplus of the $K$ lines of business at the time to ruin of line of business 1.

## 2 Construction of the approximating process

We have to incorporate the position of a $K-1$ dimensional process into a finite state space. We first try to explain the approximation procedure for the simple case of a compound Poisson process with drift on a finite time interval.

To approximate a compound Poisson process with drift by a continuous Markov process with finite state space, the idea is to consider a Poisson process of parameter $a$, and a discretization of the space with a step $d$, and to define the paths of the approximated process from the ones of the real process. We will let the approximated process jump upwards if no claim occurred since its previous jump, and jump downwards with the same jump size as the real process with a $d$ approximation. We let $ad = c$, $d$ tends to zero and $a \to +\infty$.

To ensure a finite number of states at the end, we can make the following approximation: consider a common state for all positions less than some lower level.

Let us now build the approximated process from a generalization of the previous idea. The Markov modulation is reproduced from the impact of the environment component of the whole modulating process on the transition probabilities of the surpluses of the subcompanies. To model common shocks, we may allow the process to jump at jump instants of the environment process as in Asmussen and Kella (2000). However, we must remain able to stop the process at the time of ruin of the main company. We must thus know the severity at ruin conditionally on the state of the environment at ruin. Therefore, we assume that the following hypothesis is satisfied:
(H1): the jump distribution $G_{i,j}$ should be equal to $p_{i,j}F_j$, where $p_{i,j}$ represents the probability that the jump occurs, and $F_j$ is the exponential distribution function with same parameter as for the jumps of $X(t)$ in state $j$.

Choosing the $p_{i,j}$ enables us to introduce dependence between claim amounts for the different lines of business for common shocks. (H1) may be relaxed to hypothesis (H2), under which the claim distribution for line of business 1 depends only on the state of the original Markovian environment, and not of the position of the other lines of business. As announced in the introduction, we consider here the case where hypothesis (H2) is satisfied, but the results obtained hereafter may be generalized to the case where only (H1) is satisfied, as explained in section **??**.

Following the previous statements, we want now to define a new environment state

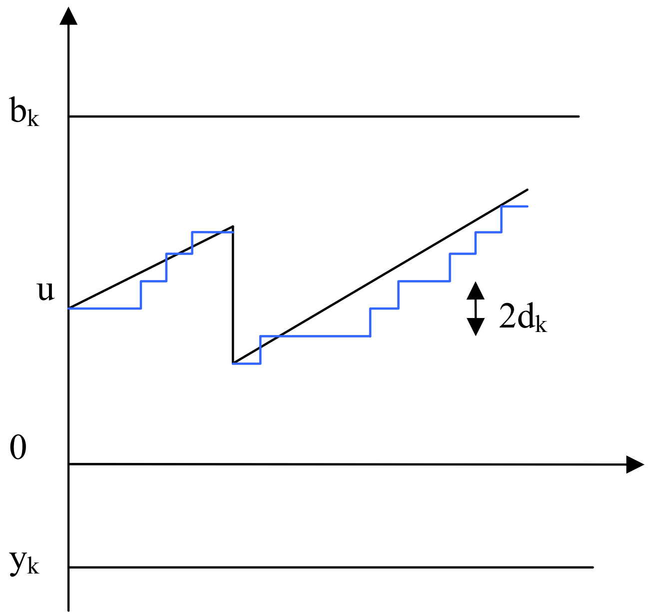


Figure 4: Sample path of the approximated process, and interpretation of the $d_k$, $a_k^i$ and $n_k$ for one line of business.

space

$$\mathcal{J} = \mathcal{J}^1 \times \cdots \times \mathcal{J}^K$$

with

$$n = n_1. \prod_{2 \le k \le K} (n_k + 1)$$

states, where $\mathcal{J}^1$ is the original environment state space($n$ becomes $n_1$, and $\mathcal{J}$ becomes $\mathcal{J}^1$).

Let $y_2, \ldots, y_K < 0$ be the lower bounds for each line of business, and $n_2, \ldots, n_K \geq 0$ the numbers of subdivisions, and $d_k = \frac{b_k + y_k}{n_k}$ the corresponding discretization steps. Let $a_k^i$ be the Poisson parameter of the process which determines the upwards jumps of the approximating process for line $k$ when the original environment state is $i$. The $d_k$ should satisfy $c_k^i / a_k^i = d_k$ to have only common upwards jumps. These parameters and the corresponding approximation are illustrated by Figure 4.

Let $Q_1$ be the rate transition matrix of the original environment (with state space $\mathcal{J}_1$). We shall use the notation $\underline{n} = (n_2, \ldots, n_K)$, $\overline{a} = (a_2^1, \ldots, a_2^{n_1}, \ldots, a_K^1, \ldots, a_K^{n_1})$

and $\underline{y} = (y_2, \dots, y_K)$.

Denote also $\underline{j} = (j_2, \dots, j_K)$ and $\underline{l} = (l_2, \dots, l_K)$ for any $j_2, \dots, j_K, l_2, \dots, l_K$. Let

$$Q^{\underline{n}, \underline{\overline{a}}, \underline{d}}$$

be the transition rate matrix of the new environment for these parameters.

For an instant, let us fix $\underline{n}$, $\underline{\overline{a}}$ and $\underline{d}$ and abbreviate objects like $Q^{\underline{n}, \underline{\overline{a}}, \underline{d}}$ as $Q$.

We choose to number the states as follows:
for each line $k \in [2, K]$, $j_k = 0$ corresponds to the minimum level $y_k$,
$j_k = n_k$ corresponds to the top level $b_k$,
and more generally the state

$$(j_1, j_2, \dots, j_K)$$

corresponds to the case where the original environment state is $j_1$, and the position of the $(K-1)$-dimensional surplus level of the $K-1$ last lines of business is

$$(y_2 + j_2 d_2, \dots, y_K + j_K d_K).$$

For $\underline{d}$, and $1 \le j_1 \le n_1$, let $f_{j_1}(\underline{l})$ (resp. $F_{j_1}(\underline{l})$) be the probability mass (resp. distribution) function of the discretized distribution of $F_{W^{j_1}}$ restricted to the $K-1$ last lines of business. The $f_{j_1}(\underline{l})$ and the $F_{j_1}(\underline{l})$ may be obtained by a procedure generalizing the one described in De Vylder (1999), to preserve mathematical expectations for each line of business.

We can now describe the transition rate matrix $Q$. Denote $\underline{\widetilde{\mathbf{1}}} = (1, \dots, 1)$ and $\underline{\widetilde{\mathbf{0}}} = (0, \dots, 0) \in \mathbb{R}^{K-1}$.
For $j_1 \neq j_2$, $Q_{(j_1, \underline{j}), (j_2, \underline{l})} = Q_{1(j_1, j_2)}$ if $\underline{j} = \underline{l}$ and 0 otherwise.

This corresponds to jumps of the original environment process $J_1(t)$, and comes from the fact that, in a small interval of length $dt$, the probability to have a change for $J_1(t)$ and a claim for a line of business is of order $(dt)^2$.

For $\underline{j} = (j_2, \dots, j_K)$, define

$$A(\underline{j}) = \{k \in [2, K], \quad j_k \notin \{0, n_k\}\}.$$

For $\underline{j}$ such that $A(\underline{j}) \neq \emptyset$, for $k \in A(\underline{j})$ and $1 \le j_1 \le n_1$,

$$Q_{(j_1, \underline{j}), (j_1, \underline{j} + \underline{\widetilde{\mathbf{1}}}_k)} = a_k^{j_1}.$$

This corresponds to the case where lines of business which are neither at their top level, nor at their minimum level increase because they receive some premium income. Lines of business which are at their top level than remain at their top level. Lines of business at the minimum level must wait longer to jump upwards, because of the severity of the last claim, which may have sent the wealth of this line further down. This is why we treat this case separately.

Denote by $\underline{j} \geq \underline{l}$ the fact that

$$\forall k \in [2, K], j_k \geq l_k,$$

and by $\underline{j} > \underline{l}$ the fact that

$$\underline{j} \geq \underline{l} \quad \text{and that} \quad \exists k \in [2, K], j_k > l_k.$$

For each $\underline{l}$, define the subset $\mathcal{K}(\underline{l})$ of $[2, K]$ of the indices of the lines of business which are at their minimum level when the state of the environment is $(l_1, \underline{l})$ for any state of the original environment $l_1 \in [1, n_1]$:

$$\mathcal{K}(\underline{l}) = \{k \in [2, K], l_k = 0\}.$$

For each subset $\mathcal{K}$ of $[2, K]$, denote by

$$\widetilde{\underline{\mathbf{1}}}_{\mathcal{K}}(+\infty)$$

the vector whose $k^{\text{th}}$ entry is $+\infty$ if $k \in \mathcal{K}$, and 0 otherwise.

For $\underline{j} > \underline{l}$, and $1 \leq j_1 \leq n_1$,

$$Q_{(j_1,\underline{j}),(j_1,\underline{l})} = \lambda^{j_1} f_{j_1,\underline{l}}(\underline{j} - \underline{l}),$$

where

$$f_{j_1,\underline{l}}(\underline{j} - \underline{l}) = f_{j_1}(\underline{j} - \underline{l}) + F_{j_1}(\underline{j} - \underline{l} + \widetilde{\underline{\mathbf{1}}}_{\mathcal{K}(\underline{l})}(+\infty)) - F_{j_1}(\underline{j} - \underline{l}). \tag{1}$$

This corresponds to the case where a multivariate claim, concerning some lines of business (between 2 and $K$) makes the position of the joint surplus of the $K-1$ last lines of business change, provoking a change in the environment from state $(j_1, \underline{j})$ to state $(j_1, \underline{l})$. We already explained that there cannot be a change of $j_1$ and a claim at the same time, whence $j_1$ cannot change here. Besides, the real surplus of some lines of business $k$ may fall in fact to a lower level than the minimum level $y_k$ in this approximated model. Thus, we have to incorporate the probability of these events into the transition rates from any state to states $(j_1, \underline{l})$ for which at least one line of business is at its minimum level, *id est* for which $\mathcal{K}(\underline{l}) \neq \emptyset$. To do this, we need to take differences of distribution functions and limits of distribution functions in (1).

Now, we have to compensate this by adapting the transition rates for which one line of business exits its minimum level.
For all $k \in [2, K]$, and for all $j_1, j_2, j_{k-1}, 0, j_{k+1}, \ldots, j_K$, define

$$Q_{(j_1,j_2,j_{k-1},0,j_{k+1},\ldots,j_K),(j_1,j_2,j_{k-1},1,j_{k+1},\ldots,j_K)} = \frac{c_k^{j_1} - \lambda_k^i \mu_k^i}{\mu_k^i + d_k}$$

in order to respect the average time the process of line $k$ would asymptotically take to reach $y_k + d_k$ from $y_k - W_k^{j_1}$ at an average increase rate $c_k^{j_1} - \lambda_k^i \mu_k^i$ if the case of frequent, small claims.

If

$$\text{Card }\left(\{k \in [2,K], (\, j_k = 0 \text{ and } l_k = 1\,)\}\right) \geq 2,$$

then for any $1 \leq j_1 \leq n_1$,

$$Q_{(j_1,\underline{j}),(l_1,\underline{l})} = 0.$$

This is because we allow only one line to exit its minimum level state at a time.

For all $\underline{l}$ and $\underline{j}$ such that

$$\exists k \in [2,K], l_k \geq j_k + 2,$$

as we only allow small upwards jumps,

$$Q_{(j_1,\underline{j}),(l_1,\underline{l})} = 0.$$

As usual, define the diagonal terms of the transition rate matrix for all $j_1, \underline{j}$ as

$$Q_{(j_1,\underline{j}),(j_1,\underline{j})} = - \sum_{(l_1,\underline{l}) \neq (j_1,\underline{j})} Q_{(j_1,\underline{j}),(l_1,\underline{l})}.$$

Consider now the parameter $\lambda_{j_1,\underline{j}} = \lambda_{j_1}$ of the Poisson process modelling the jumps which only concern line of business 1, and the parameter $1/\mu_{j_1,\underline{j}} = 1/\mu_{j_1}$ of the exponential distribution of these claim amounts. These parameters only depend on the state $j_1$ of the original environment, and may be easily obtained from the common shock distribution $F_W$ and the common shock intensity $\lambda$. Usually, the common shock distribution will be built from these parameters first, as they can most often be estimated with the data of insurance companies.

Now, we have to incorporate common shocks which involve line of business 1 and some other lines of business. We thus consider also jumps of the surplus process of line of business 1 at instants of change of the environment from state $i$ to state $j$. For $i = (j_1, \underline{j})$ and $j = (j_1, \underline{l})$, under (H2), these jumps (with size $U_{i,j}$) have a probability $p_{i,j}$ to occur at each transition from $i$ to $j$. If they occur, the conditional jump size distribution is exponentially distributed with parameter $1/\mu_{j_1}$. Let $G(\alpha)$ be the corresponding $n \times n$ Laplace transform matrix defined by

$$G_{(j_1,\underline{j}),(j_2,\underline{l})}(\alpha) = p_{(j_1,\underline{j}),(j_2,\underline{l})} E\left[e^{\alpha U_{(j_1,\underline{j}),(j_2,\underline{l})}}\right] = p_{(j_1,\underline{j}),(j_2,\underline{l})} \frac{1}{1 - \alpha \mu^{j_2}}.$$

Assumption (H2) may be relaxed to $(H1)$ as explained in Section IV. Now we are ready to apply the formalism of Frostig (2005) with the adding of common shocks. Define

$$c_{j_1,\underline{j}} = g_1^{j_1}(s_2(j_2), \ldots, s_K(j_k))$$

the premium income rate effectively received by line of business 1 when the state of the new environment is $(j_1, \underline{j})$.

## 3 Main result

The idea is now to consider line of business 1, in the new, global environment

$$\mathcal{J} = J^1 \times \cdots \times J^K.$$

First, consider some fixed $\underline{n}$, $\underline{d}$ and $\overline{a}$, and omit superscripts dedicated to these quantities. Define for $\alpha \in \mathbb{C}$

$$K(\alpha) = Q \circ G(\alpha) - \alpha \text{ diag } (c) + \text{ diag } (\lambda \frac{1}{1-\alpha\mu}) - \text{ diag } (\lambda_{j_1,\underline{j}}),$$

where for two $n \times n$ matrices, $M \circ N$ denotes the $n \times n$ matrix defined by

$$(M \circ N)_{ij} = M_{ij} N_{ij},$$

and for a vector

$$x = (x_{j_1,\underline{j}})_{1 \le j_1 \le n_1, 0 \le j_k \le n_k \text{ for } k \in [2,K]},$$

diag $(x)$ is $n \times n$ diagonal matrix whose entries are

$$\text{diag } (x)_{j_1,\underline{j} l_1,\underline{l}} = x_{j_1,\underline{j}} \mathbf{1}_{(j_1,\underline{j})=(l_1,\underline{l})}.$$

From lemma 2.1 of Asmussen and Kella (2000),

$$M^W(t,\alpha) = e^{\alpha X(t)} \widetilde{\underline{\mathbf{1}}}_{J_t} e^{-K(\alpha)t}$$

is a $n$-dimensional martingale for all $\alpha \in \mathbb{C}$ such that all the $\varphi^i(\alpha)$ exist and for all distribution of $(X(0), J_0)$. If $h(\alpha)$ is a right eigenvector of $K(\alpha)$ for eigenvalue $\kappa(\alpha)$, then

$$N(t,\alpha) = e^{\alpha X(t) - \kappa(\alpha)t} h_{J_t}(\alpha)$$

is a martingale.

Consider the dividend processes

$$L_j^{\underline{n},\underline{d},\overline{a}}(t) = \int_0^t \mathbf{1}_{\{J(s)=j\}} dL_1(s),$$

with $L_1(t)$ as in the previous section.

For $j = (j_1, \underline{j})$, it is possible to obtain the $l_j = \mathbb{E} L_j^{\underline{n},\underline{d},\overline{a}}(\tau)$, the $p_j = p_{j_1,\underline{j}}^{\underline{n},\underline{d},\overline{a}}(\tau) = \mathbb{P}\left[J(\tau) = (j_1, \underline{j})\right]$ and the expected time to ruin $\mathbb{E}\tau^{\underline{n},\underline{d},\overline{a}}$ from a slight modification of the method of Frostig (2005). Denote by $\pi$ the stationary distribution of $Q$. Note that $K(\alpha)$ has a real eigenvalue $\kappa(\alpha)$ with maximum real part. Define $h(\alpha)$ the corresponding right eigenvector with positive components satisfying $\pi h(\alpha) = 1$. Let $k$ be the derivative of $h(\alpha)$ at $\alpha = 0$, and $m = \kappa'(0)$. As in page 12 of Frostig (2005),

$$\mathbb{E}X(\tau) = m\mathbb{E}\tau + \mathbb{E}\left[k_{J(0)}\right] - \mathbb{E}\left[k_{J(\tau)}\right],$$

but

$$m = \kappa'(0) = \pi.\left[-\text{ diag }(c - \lambda\mu) + Q \circ G'(0)\right]\widetilde{\underline{\mathbf{1}}}$$

has here a different expression. The adapted end of the proof shows that

$$k = (Q - \widetilde{\underline{\mathbf{1}}}\pi)^{-1}\left(m\widetilde{\underline{\mathbf{1}}} + [\text{ diag }(c - \lambda\mu) + Q \circ G'(0)]\,\widetilde{\underline{\mathbf{1}}}\right).$$

**Theorem 2** *(Theorem 2.1,* Asmussen and Kella (2000)) The multidimensional process

$$M(t,\alpha) = \int_0^t e^{\alpha Z(t)}\widetilde{\underline{\mathbf{1}}}_{J(s)}ds K(\alpha) + e^{\alpha Z(0)}\widetilde{\underline{\mathbf{1}}}_{J(0)} - e^{\alpha Z(t)}\widetilde{\underline{\mathbf{1}}}_{J(t)} + \alpha\int_0^t \widetilde{\underline{\mathbf{1}}}_{J(s)}dL(s) \quad (2)$$

is a $n$-dimensional martingale for all $\alpha \in \mathbb{C}$ such that the $\varphi_k^i(\alpha)$ exist and for all distributions of $(X(0), J_0)$.

Note that det $(K(\alpha))$ may be written as a quotient of two polynomials where the numerator is of degree $2n$. Assume that the numerator has $2n$ distincts roots $\alpha_1, \ldots, \alpha_{2n}$. Let $h^j(\alpha)$ be a column vector such that $K(\alpha_j)h^j(\alpha) = 0$. By multiplying (2) by $h^j(\alpha)$, we get the following system of $2n$ equations for the $p_j =$ and $l_j$: for $1 \le j \le 2n$,

$$\mathbb{E}\left[e^{\alpha_j Z(0)}h^j_{J(0)}(\alpha_j)\right] - \sum_{i=1}^n p_i e^{\alpha_j b}\frac{1}{1-\alpha_j\mu_i}h_i^j(\alpha_j) + \alpha_j\sum_{i=1}^n l_i h_i^j(\alpha_j) = 0. \quad (3)$$

Then, using $\mathbb{E}X(\tau) = \mathbb{E}Z(\tau) - b + u - \mathbb{E}L(\tau)$, and $\mathbb{E}Z(\tau) = \sum_{i=1}^n p_i(b + \mu_i)$,

$$\mathbb{E}\tau^{\underline{n},\underline{d},\overline{\underline{a}}} = \frac{1}{m}\left(\sum_{i=1}^n\left(\mathbb{P}[J(0) = i] - p_i\right)k_i\right) + \sum_{i=1}^n p_i(b+\mu_i) - \sum_{i=1}^n l_i + u - b. \quad (4)$$

**Theorem 3** *As all components of $\overline{\underline{a}}$ and $\underline{d}$ tend to zero (satisfying $c_k^i = d_k a_k^i$ for all $k$ and $i$), and all components of $\underline{n}$ tend to $+\infty$, for all $i \in \mathcal{J}$ and $\mathcal{K} \subset [2, K]$,*

$$\tau^{\underline{n},\underline{d},\overline{\underline{a}}} \to \tau$$

*and*

$$\sum_{j_k > n_k - [b_k/d_k], k\in\mathcal{K}, 0\le j_l\le n_l, l\notin\mathcal{K}} \mathbb{E}L^{\underline{n},\underline{d},\overline{\underline{a}}}_{j_1,\underline{j}}(\tau^{\underline{n},\underline{d},\overline{\underline{a}}}) \to \mathbb{E}L_1^{j_1\mathcal{K}}(\tau),$$

*where $\tau$ is the time to ruin in the original model, $L_1^{j_1\mathcal{K}}(t)$ is the part of dividends used by line 1 to pay some penalty because of insolvency of at least all lines $k \in \mathcal{K}$, and where $L^{\underline{n},\underline{d},\overline{\underline{a}}}_{j_1,\underline{j}}(t)$ and $\tau^{\underline{n},\underline{d},\overline{\underline{a}}}$ are respectively defined by (3) and (4).*

*Let $F^{\underline{n},\underline{d},\overline{a}}(\tau^{\underline{n},\underline{d},\overline{a}})$ be the joint distribution of the K-dimensional surplus process at time $(\tau^{\underline{n},\underline{d},\overline{a}})$ (it is directly obtained by (3) and the memoryless property satisfied by exponentials distributions). As all components of $\overline{a}$and $\underline{d}$ tend to zero, and all components of $\underline{n}$ tend to $+\infty$,*

$$F^{\underline{n},\underline{d},\overline{a}}(\tau^{\underline{n},\underline{d},\overline{a}})$$

*converges pointwisely to $F(\tau)$, the joint distribution of the multidimensional surplus in the original model.*

The discretized process converges almost surely as all components of $\overline{a}$and $\underline{d}$ tend to zero (satisfying $c_k^i = d_k a_k^i$ for all $k$ and $i$), and all components of $\underline{n}$ tend to $+\infty$. To show this, let us start with classical convergence results in $L^\infty([0,1])$. The following Theorem is the analogue of Schilder's Theorem for Poisson processes.

**Theorem 4** *(See* DEMBO and ZEITOUNI (1998))

Let $\mu_\epsilon$ be the probability measures induced on $L^\infty([0,1])$ by $\epsilon\hat{N}(t/\epsilon)$, where $\hat{N}(.)$ is a Poisson process on $\mathbb{R}^+$ of intensity one. The $\{\mu_\epsilon\}$ satisfy the large deviation principle with good rate function :

$$I_{\hat{N}}(\varphi) = \int_0^1 [\dot{\varphi}(t)\ln\dot{\varphi}(t) - \dot{\varphi}(t) + 1]\,dt$$

if $\varphi$ is absolutely continuous and increasing with $\varphi(0) = 0$, and

$$I_{\hat{N}}(\varphi) = \infty \quad \text{otherwise.}$$

The previous Theorem gives the almost sure convergence of the modified Poisson process to the deterministic linear process in the Skorohod space $D([0,1])$ equipped with the topology of uniform convergence.

Consider first a classical risk process

$$R(t) = u + ct - \sum_{k=1}^{N(t)} U_k,$$

where $N(t)$ is a Poisson process with parameter $\lambda$. Define then the approximating process

$$R^{\mu,d}(t)$$

sample path by sample path, from the paths of $R(t)$: the downwards jumps of $R^{\mu,d}(t)$ occur at the same instants as those of $R(t)$, and have severity $d\,[x/d]$, where $x$ is the severity of the corresponding jump for $R(t)$. Upwards jumps are described by a Poisson process with parameter $\mu$, and their size is deterministic, and equal to $\eta d$ (which prescribes $\eta d = c/\mu$). Let $[0,T]$ be a fixed time interval. Sample path by sample path, downwards jumps are the same up to the discretization step $d$, and we superimpose a compound Poisson process with parameter $c/d$ and deterministic jump size $+d$. From Theorem 4, we can restrict our attention to the downwards jumps part, as for all $p \geq 1$,

$$\mathbb{P}\left(\forall \epsilon > 0, \exists m \geq 1, \forall 0 \leq k \leq p, \quad \| R(kT/p) - R^{m,1/m}(kT/p) \| \leq \epsilon\right) = 1.$$

So for all $p \geq 1$,

$$\mathbb{P}\left(\forall \epsilon > 0, \exists m \geq 1, \forall 0 \leq k \leq p, \quad \| R(t) - R^{m,1/m}(t) \| \leq 2\epsilon + cT/p\right) = 1.$$

By choosing $\epsilon'$ and $p$ such that $2\epsilon + cT/p \leq \epsilon$, we obtain the almost sure convergence of $R^{m,1/m}(t)$ to $R(t)$ in the space of càdlàg functions on $[0, T]$ endowed with the norm of the uniform convergence. The sojourn and hitting times until $T$ thus converge almost surely. From the dominated convergence theorem, the expected sojourn times and hitting times until $\tau$ converge to those of the continuous model.

For the multidimensional Markov modulated process with common shocks, it is possible to do a similar reasoning. Path by path, the approximating strategy consists in decomposing the path into a finite number (almost surely) of smaller paths on a partition of $[0, T]$ given by the instants of change of the environment, and then doing the same as previously. We must just be careful and respect the relations

$$c_k^i = a_k^i d_k.$$

We thus obtain the almost sure uniform convergence (see also Dembo, Gantert and Zeitouni (2004) for large deviations in a Markovian environment, and Dembo and Zajic (1995)).

As all hitting times and sojourn times before T in the approximating model converge almost surely to the corresponding random variables in the continuous model, the approximated multidimensional process taking into account impact of the position of the other lines of business on the premium income rate converges also almost surely to the continuous corresponding process.

The sojourn times and hitting times until the time to ruin of line of business 1 thus converge almost surely. Besides, we know that in the most favorable case (other lines of business always at their top level), $\mathbb{E}\tau < +\infty$ from the standard model. Thus, by the dominated convergence theorem, the mathematical expectations of the sojourn times and of hitting times until time to ruin of line of business 1 in the approximate model converge to the corresponding times in the continuous model, which ends the proof.

## IV EXTENSIONS, LIMITS AND OTHER IDEAS OF APPLICATIONS

We could also introduce a quasi-default state for each secondary line of business, modeled by a state with a very high exit time (different from $+\infty$ to ensure existence and uniqueness of the stationary distribution), and with a severe jump distribution for the main process at the arrival time to this state.

The $c_k^i(t)$ might also be general increasing functions of the $X_k(t)$.

The jump marginals for lines of business $k \geq 2$ do not have to be exponentially distributed. Besides, Hypothesis (H2) may be relaxed to (H1).

The main numerical problems are the degree of the polynomial whose roots have to be found and the size of the system to solve. This limits the size of the space discretization step. The time discretization step has to be chosen in quite an optimal way, to maintain a balance between how fine the time discretization is and the global error (proportional to the number of jumps). More numerical analysis would be needed to study the impact of discretization errors, as well as calibration issues.

However, the discretized model may be interesting by itself, for example for M& A or reinsurance default problems. Consider a company which has the opportunity to buy another one. It may be easier to obtain data about that company to model its surplus evolution with a rating-like model than with a compound Poisson process. Besides, it could be easier to model the impact of the subcompanies on the risk premium by unit of time received by the company only through their signatures (from AAA to D). These ideas are left for future research work.

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