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Repeated games with asymmetric information and random price fluctuations at finance markets: the case of countable state space [1]

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Abstract. This paper is concerned with multistage bidding models introduced by De Meyer and Moussa Saley (2002) to analyze the evolution of the price system at finance markets with asymmetric information. The zero-sum repeated games with incomplete information are considered modeling the bidding with countable sets of possible prices and admissible bids.
It is shown that, if the liquidation price of a share has a finite variance, then the sequence of values of n-step games is bounded and converges to the value of the game with infinite number of steps. We construct explicitly the optimal strategies for this game.
The optimal strategy of Player 1 (the insider) generates a symmetric random walk of posterior mathematical expectations of liquidation price with absorption. The expected duration of this random walk is equal to the initial variance of liquidation price. The guaranteed total gain of Player 1 (the value of the game) is equal to this expected duration multiplied with the fixed gain per step.

Keywords: multistage bidding, asymmetric information, repeated games, optimal strategy.

1. Introduction

Random fluctuations of stock market prices are usually explained by the effect of multiple exogenous factors subjected to accidental variations. The work of De Meyer and Saley (2002) proposes a different strategic motivation for these phenomena. The authors assert that the Brownian component in the evolution of prices on the stock market may originate from asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of the oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a model of multistage bidding between two agents for risky assets (shares). A liquidation price of shares depends on a random "state of nature". Before the bidding starts a chance move determines the "state of nature" and therefore the liquidation value of shares once for all. Player 1 is informed on the "state of nature", Player 2 is not. Both players know probabilities of chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, ..., n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

In this model the uninformed Player 2 should use the history of the informed Player 1 moves to update his beliefs about the state of nature. In fact, at each step Player 2 may use the Bayes rule to re-estimate the posterior probabilities of chance move outcome, or, at least, the posterior mathematical expectations of liquidation value of a share. Player 1 could control these posterior probabilities.

Thus Player 1 faces a problem of how best to use his private information without revealing it to Player 2. Using a myopic policy – bid the high price if the liquidation value is high, the low price if this value is low – is not optimal for Player 1, as it fully reveals the state of nature to Player 2. On the other hand, a strategy that does not depend on the state of nature reveals no information to Player 2, but does not allow Player 1 to take any advantage of his superior knowledge. Thus Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider the model where a liquidation price of a share takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann, Maschler (1995), but with continual action sets. De Meyer and Saley show that these n-stage games have the values (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values and the optimal strategies of players. As n tends to infinity, the values infinitely grow up with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

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It is more natural to assume that players may assign only discrete bids proportional to a minimal currency unit. In our previous papers (Domansky, 2007), (Domansky and Kreps, 2007) we investigate the model with two possible values of liquidation price and discrete admissible bids. We show that, unlike the model (De Meyer and Saley, 2002), as n tends to ∞, the sequence of guaranteed gains of insider is bounded from above and converges. It makes reasonable to consider the bidding with infinite number of steps. We construct the optimal strategies for corresponding infinite games. We write out explicitly the random process formed by the prices of transactions at sequential steps. The transaction prices perform a symmetric random walk over the admissible bids between two possible values of liquidation price with absorbing extreme points. The absorption of transaction prices means revealing of the true value of share by Player 2.

Here we consider the model where any integer non-negative bids are admissible. The liquidation price of a share $C_{\mathbf{p}}$ may take any nonnegative integer values $k = 0, 1, 2, \ldots$ according to a probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$. This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 and with countable state and action spaces. The games considered in (Domansky, 2007), (Domansky and Kreps, 2007) represent a particular case of these games corresponding to probability distributions with two-point supports.

We show that if the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}]$, then the values $V_n(\mathbf{p})$ of n-stage games $G_n(\mathbf{p})$ exist (i.e. the guaranteed gain of Player 1 is equal to the guaranteed loss of Player 2). If the variance $\mathbf{D}[C_{\mathbf{p}}]$ is infinite, then, as n tends to ∞, the sequence $V_n(\mathbf{p})$ diverges.

On the contrary, if the variance $\mathbf{D}[C_{\mathbf{p}}]$ is finite, then, as n tends to ∞, the sequence of values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$ is bounded from above and converges. The limit $H(\mathbf{p})$ is a continuous, concave, piecewise linear function with countable number of domains of linearity. The sets $\Theta(k)$, $k = 1, 2, \ldots$ of distributions $\mathbf{p}$ with integer mathematical expectation $\mathbf{E}[C(\mathbf{p})] = k$ form its domains of non-smoothness. If $\mathbf{E}[C_{\bar{p}}]$ is an integer, then $H(\mathbf{p}) = \mathbf{D}[C_{\mathbf{p}}]/2$. If $\mathbf{E}[C_{\mathbf{p}}] = k + \alpha$, where k is an integer, $\alpha \in [0, 1]$, then $H(\mathbf{p}) = (\mathbf{D}[C_{\mathbf{p}}] - \alpha(1 - \alpha))/2$.

As the sequence $V_n(\mathbf{p})$ is bounded from above, it is reasonable to consider the games $G_\infty(\mathbf{p})$ with infinite number of steps. We show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$. We construct explicitly the optimal strategies for these games.

Let $\mathbf{p} \in \Theta(k)$. If the random variable $C_{\mathbf{p}}$ takes the value k, then the "approximate" information of Player 2 turns to be the exact one and in fact the information advantage of Player 1 disappears. The gain of Player 1 is equal to zero and he can stop the game without any loss for himself. Otherwise, the first optimal move of Player 1 makes use of actions $k - 1$ and k with equal total probabilities and with posteriors $\mathbf{p}(\cdot|k - 1) \in \Theta(k - 1)$, $\mathbf{p}(\cdot|k) \in \Theta(k + 1)$. For these posteriors the equalities $\mathbf{p}(k|k - 1) = \mathbf{p}(k|k) = 0$ hold.

Thus the insider optimal strategy generates a symmetric random walk of posterior mathematical expectations over $\mathbb{Z}_+$ with absorption. For $\mathbf{p} \in \Theta(k)$ the expected duration of this random walk is equal to the variance of the liquidation price of a share. The value of infinite game is equal to the expected duration of this random walk multiplied by the constant one-step gain $1/2$ of informed Player 1.

2. Repeated games with one-sided information modeling multistage bidding

We consider the repeated games $G_n(\mathbf{p})$ with incomplete information on one side (Aumann and Maschler, 1995) modeling the bidding described in introduction.

Two players with opposite interests have money and single-type shares. The liquidation price of a share may take any nonnegative integer values $s \in S = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

At stage 0 a chance move determines the liquidation value of a share for the whole period of bidding n according to the probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$ over S known to both Players. Player 1 is informed about the result of chance move s, Player 2 is not. Player 2 knows that Player 1 is an insider.

At each subsequent stage $t = 1, \ldots, n$ both Players simultaneously propose their prices for one share, $i_t \in I = \mathbb{Z}_+$ for Player 1 and $j_t \in J = \mathbb{Z}_+$ for Player 2. The pair (i_t, j_t) is announced to both Players before proceeding to the next stage. The maximal bid wins and one share is transacted at this price. Therefore, if $i_t > j_t$, Player 1 gets one share from Player 2 and Player 2 receives the sum of money i_t from Player 1. If $i_t < j_t$, Player 2 gets one share from Player 1 and Player 1 receives the sum j_t from Player 2. If $i_t = j_t$, then no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 and with countable state space $S = \mathbb{Z}_+$ and with countable action spaces $I = \mathbb{Z}_+$ and $J = \mathbb{Z}_+$. One-step gains of Player 1 are given with the matrices $A^s = [a^s(i,j)]_{i\in I, j\in J}$, $s \in S$,

$$a^s(i,j) = \begin{cases} j - s, & \text{for } i < j\ ; \\ 0, & \text{for } i = j\ ; \\ -i + s, & \text{for } i > j\ . \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a^s(i_t, j_t).$$

This description is common knowledge to both Players.

At step t it is enough for both Players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of Player 1's previous actions only. Thus a strategy σ for Player 1 informed on the state is a sequence of moves

$$\sigma = (\sigma_1, \ldots, \sigma_t, \ldots),$$

where the move $\sigma_t = (\sigma_t(s))_{s\in S}$ and $\sigma_t(s) : I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage t, given the state k and previous observations. Here $\Delta(\cdot)$ is the set of probability distributions over $(\cdot)$.

A strategy τ for uninformed Player 2 is a sequence of moves

$$\tau = (\tau_1, \ldots, \tau_t, \ldots),$$

where $\tau_t : I^{t-1} \to \Delta(J)$.

Observe that here we define infinite strategies fitting for games of arbitrary duration. A pair of strategies (σ, τ) induces a probability distribution $\Pi_{(\sigma,\tau)}$ over $(I \times J)^\infty$. The payoff function of the game $G_n(\mathbf{p})$ is

$$K_n(\mathbf{p}, \sigma, \tau) = \sum_{s\in S} p_s h_n^s(\sigma, \tau),$$

where

$$h_n^s(\sigma, \tau) = \mathbf{E}_{(\sigma,\tau)}[\sum_{t=1}^{n} a^s(i_t, j_t)]$$

is the s-component of the n-step vector payoff $h_n(\sigma, \tau)$ for the pair of strategies (σ, τ). Here the expectation is taken with respect to the probability distribution $\Pi_{(\sigma,\tau)}$.

For the initial probability $\mathbf{p}$, the strategy σ ensures the n-step payoff

$$w_n(p, \sigma) = \inf_{\tau} K_n(p, \sigma, \tau).$$

The strategy τ ensures the n-step vector payoff $h_n(\tau)$ with the components

$$h_n^s(\tau) = \sup_{\sigma(s)} h_n^s(\sigma(s), \tau).$$

Now we describe the recursive structure of $G_{n+1}(\mathbf{p})$. A strategy σ may be regarded as a pair $(\sigma_1, (\sigma(i))_{i\in I})$, where $\sigma_1(i|s)$ is a probability on I depending on s, and $\sigma(i)$ is a strategy depending on the first action $i_1 = i$.

Analogously, a strategy τ may be regarded as a pair $(\tau_1, (\tau(i))_{i\in I})$, where τ_1 is a probability on J.

A pair $(\mathbf{p}, \sigma_1)$ induces the probability distribution π over $S \times I$, $\pi(s, i) = p(s)\sigma_1(i|s)$. Let

$$\mathbf{q} \in \Delta(I), \quad q_i = \sum_S p_s \sigma_1(i|s),$$

be the marginal distribution of π on I (total probabilities of actions), and let

$$\mathbf{p}(i) \in \Delta(S), \quad p_s(i) = p_s\sigma_1(i|s)/q_i,$$

be the conditional probability on S given $i_1 = i$ (a posterior probability).

Conversely, any set of total probabilities of actions $\mathbf{q} \in \Delta(I)$ and posterior probabilities $(\mathbf{p}(i) \in \Delta(S))_{i\in I}$, satisfying the equality

$$\sum_{i\in I} q_i \mathbf{p}(i) = \mathbf{p},$$

define a certain random move of Player 1 for the current probability $\mathbf{p}$. The posterior probabilities contain the whole of essential for Player 1 information about the previous history of the game. Thus to define a strategy of Player 1 it is sufficient to define the random move of Player 1 for any current posterior probability.

The following recursive representation for the payoff function corresponds to the recursive representation of strategies:

$$K_{n+1}(\mathbf{p}, \sigma, \tau) = K_1(\mathbf{p}, \sigma_1, \tau_1) + \sum_{i \in I} q_i K_n(\mathbf{p}(i), \sigma(i), \tau(i)).$$

Let, for all $i \in I$, the strategy $\sigma(i)$ ensure the payoff $w_n(\mathbf{p}(i), \sigma(i))$ in the game $G_n(\mathbf{p}(i))$. Then the strategy $\sigma = (\sigma_1, (\sigma(i))_{i \in I})$ ensures the payoff

$$w_{n+1}(\mathbf{p}, \sigma) = \min_{j \in J} \sum_{i \in I} [\sum_{s \in S} p_s \sigma_1(i|s) a(s, i, j) + q_i w_n(\mathbf{p}(i), \sigma(i))]. \tag{1}$$

Let, for all $i \in I$, the strategy $\tau(i)$ ensure the vector payoff $\mathbf{h}_n(\tau(i))$. Then the strategy $\tau = (\tau_1, (\tau^n(i))_{i \in I})$ ensures the vector payoff $\mathbf{h}_{n+1}(\tau)$ with the components

$$h^s_{n+1}(\tau) = \max_{i \in I} \sum_{j \in J} \tau_1(j)(a(s, i, j) + h^s_n(\tau(i)) \quad \forall s \in S. \tag{2}$$

The game $G_n(\mathbf{p})$ has a value $V_n(\mathbf{p})$ if

$$\inf_{\tau} \sup_{\sigma} K_n(\mathbf{p}, \sigma, \tau) = \sup_{\sigma} \inf_{\tau} K_n(\mathbf{p}, \sigma, \tau) = V_n(\mathbf{p}).$$

Players have optimal strategies σ^* and τ^* if

$$V_n(\mathbf{p}) = \inf_{\tau} K_n(\mathbf{p}, \sigma^*, \tau) = \sup_{\sigma} K_n(\mathbf{p}, \sigma, \tau^*),$$

or, in above introduced notation,

$$V_n(\mathbf{p}) = w_n(\mathbf{p}, \sigma^*) = \sum_{s \in S} p_s h^s_n(\tau^*).$$

For probability distributions $\mathbf{p}$ with finite supports, the games $G_n(\mathbf{p})$, as games with finite state and action spaces, have values $V_n(\mathbf{p})$. The functions V_n are continuous and concave in $\mathbf{p}$. Both players have optimal strategies σ^* and τ^*.

Consider the set M^1 of probability distributions $\mathbf{p}$ with finite first moment $m^1[\mathbf{p}] = \sum_{s=0}^{\infty} p_s \cdot s < \infty$. For $\mathbf{p} \in M^1$, the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share, has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}] = m^1[\mathbf{p}]$. The set M^1 is a convex subset of Banach space $L^1(\{s\})$ of sequences $\mathbf{l} = (l_s)$ with a norm

$$||\mathbf{l}||^1_s = \sum_{s=0}^{\infty} |l_s| \cdot s.$$

Let $\mathbf{p}_1, \mathbf{p}_2 \in M^1$. Then, for "reasonable" strategies σ and τ,

$$|K_n(p_1, \sigma, \tau) - K_n(p_2, \sigma, \tau)| < n||\mathbf{p}_1 - \mathbf{p}_2||^1_s.$$

Therefore, the payoff of game $G_n(\mathbf{p})$ with $\mathbf{p} \in M^1$ can be approximated by the payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support. Next theorem follows immediately from this fact.

Theorem 1. *If $\mathbf{p} \in M^1$, then games $G_n(\mathbf{p})$ have values $V_n(\mathbf{p})$. The values $V_n(\mathbf{p})$ are positive and do not decrease, as the number of steps n increases.*

Remark 1. If the random variable $C_{\mathbf{p}}$ does not belong to L^2, then, as n tends to ∞, the sequence $V_n(\mathbf{p})$ diverges.

3. Upper bound for values $V_n(\mathbf{p})$

Here we consider the set M^2 of probability distributions $\mathbf{p}$ with finite second moment

$$m^2[\mathbf{p}] = \sum_{s=0}^{\infty} p_s \cdot s^2 < \infty.$$

For $\mathbf{p} \in M^2$, the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share, belongs to L^2 and has a finite variance $\mathbf{D}[C_{\mathbf{p}}] = m^2[\mathbf{p}] - (m^1[\mathbf{p}])^2$.

The set M^2 is a closed convex subset of Banach space $L^1(\{s^2\})$ of mappings $\mathbf{l} : Z_+ \to R$ with a norm

$$||\mathbf{l}||_s^1 = \sum_{s=0}^{\infty} |l_s| \cdot s^2.$$

The main result of this section is that, for $\mathbf{p} \in M^2$, as $n \to \infty$, the sequence $V_n(\mathbf{p})$ of values remains bounded.

To prove this we define recursively the set of infinite "reasonable" strategies $\tau^m, m = 0, 1, \ldots$ of Player 2, suitable for the games $G_n(\mathbf{p})$ with arbitrary n.

Definition 1. The first move τ_1^m is the action $m \in J$. The moves τ_t^m for $t > 1$ depend on the last observed pair of actions (i_{t-1}, j_{t-1}) only:

$$\tau_t^m(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - 1, & \text{for } i_{t-1} < j_{t-1} \ ; \\ j_{t-1}, & \text{for } i_{t-1} = j_{t-1} \ ; \\ j_{t-1} + 1, & \text{for } i_{t-1} > j_{t-1} \ . \end{cases}$$

Remark 2. The definition of strategies τ^m includes the previous actions of both players. In fact, these strategies can be implemented on the basis of Player 1's previous actions only.

Proposition 1. *The strategies τ^m ensure the vector payoffs $\mathbf{h_n}(\tau^\mathbf{m}) \in \mathbf{R}_+^\mathbf{S}$ with components given by*

$$h_n^s(\tau^m) = \sum_{l=0}^{n-1} (m - s - l)^+, \tag{3}$$

for $s \leq m$,

$$h_n^s(\tau^m) = \sum_{l=0}^{n-1} (s - m - 1 - l)^+, \tag{4}$$

for $s > m$, where $(a)^+ := \max\{0, a\}$.

Proof. The proof is by induction on the number of steps n.
$\mathbf{n = 1}$. For $s < m$, Player 1's best reply is any action $k < m$ and

$$h_1^s(\tau^m) = \max_i a_{i,m}^s = a_{k,m}^s = m - s.$$

For $s = m$, Player 1's best reply is any action $k \leq m$ and

$$h_1^m(\tau^m) = \max_i a_{i,m}^m = a_{k,m}^m = 0.$$

For $s = m + 1$, Player 1's best replies are actions m and $m + 1$ and

$$h_1^{m+1}(\tau^m) = \max_i a_{i,m}^{m+1} = a_{m,m}^{m+1} = a_{m+1,m}^{m+1} = 0.$$

For $s > m + 1$, Player 1's best reply is action $m + 1$ and

$$h_1^s(\tau^m) = \max_i a_{i,m}^s = a_{m+1,m}^s = (s - m - 1).$$

Therefore,

$$\mathbf{h}_1(\tau^k) = (k, k - 1, \ldots, 1, 0, 0, 1, \ldots).$$

This proves Proposition 1 for $n = 1$.
$\mathbf{n} \to \mathbf{n + 1}$. Assume that the vector payoffs $h_n(\tau^k)$ are given with (3) and (4). We have according to (2)

$$h_{n+1}^s(\tau^m) = \max_i \begin{cases} a_{i,m}^s + h_n^s(\tau^{m-1}), & \text{for } i < m \ ; \\ a_{i,m}^s + h_n^s(\tau^m), & \text{for } i = m \ ; \\ a_{i,m}^s + h_n^s(\tau^{m+1}), & \text{for } i > m \ . \end{cases}$$

For $s < m$, the first move of Player 1's best reply is any action $i < m$. It results in

$$h^s_{n+1}(\tau^m) = a^s_{i,m} + h^s_n(\tau^{m-1}) = (m-s) + \sum_{l=0}^{n-1}(m-s-1-l)^+ = \sum_{l=0}^{n}(m-s-l)^+.$$

For $s = m$, the first move of Player 1's best reply is any action $i < m$ and $i = m$. It results in

$$h^m_{n+1}(\tau^m) = a^m_{i,m} + h^m_n(\tau^{m-1}) = a^m_{m,m} + h^m_n(\tau^m) = 0.$$

For $s = m+1$, the first moves of Player 1's best replies are actions m and $m+1$. It results in

$$h^{m+1}_{n+1}(\tau^m) = a^{m+1}_{m,m} + h^{m+1}_n(\tau^m) = a^{m+1}_{m+1,m} + h^{m+1}_n(\tau^{m+1}) = 0.$$

For $s > m+1$, the first move of Player 1's best reply is action $m+1$. It results in

$$h^s_{n+1}(\tau^m) = a^s_{m+1,m} + h^s_n(\tau^{m+1}) = (s-m-1) + \sum_{l=0}^{n-1}(s-m-2-l)^+ = \sum_{l=0}^{n}(s-m-1-l)^+.$$

This proves Proposition 1 for $n+1$.

□

Theorem 2. *For $\mathbf{p} \in M^2$, the values $V_n(\mathbf{p})$ are bounded from above by a continuous, concave, and piecewise linear function $H(\mathbf{p})$ over M^2. Its domains of linearity are*

$$L(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [k, k+1]\}, \quad k = 0, 1, \ldots.$$

Its domains of non-smoothness are

$$\Theta(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] = k\}.$$

The equality holds

$$H(\mathbf{p}) = (\mathbf{D}[\mathbf{p}] - \alpha(\mathbf{p})(1-\alpha(\mathbf{p})))/2, \tag{5}$$

where $\alpha(\mathbf{p}) = \mathbf{E}[\mathbf{p}] - ent[\mathbf{E}[\mathbf{p}]]$ and $ent[x]$, $x \in R^1$ is the integer part of x.

Proof. It is easy to see that

$$\lim_{n\to\infty} h^s_n(\tau^m) = h_\infty(\tau^m) = (s-m)(s-m-1)/2.$$

Thus there is the following not depending on n upper bound for $V_n(\mathbf{p})$:

$$V_n(\mathbf{p}) \le \min_m \sum_{s=0}^{\infty} p_s(s-m)(s-m-1)/2, \quad m = 0, 1, \ldots. \tag{6}$$

Observe that, for $\mathbf{E}[\mathbf{p}] = m + \alpha$,

$$\sum_{s=0}^{\infty} p_s(s-m)(s-m-1)/2 = [(m^2+m) - (2m+1)\sum_{s=0}^{\infty} p_s s + \sum_{s=0}^{\infty} p_s s^2]/2$$

$$= [\sum_{s=0}^{\infty} p_s s^2 - (m+\alpha)^2 - \alpha + \alpha^2]/2 = [\mathbf{D}[\mathbf{p}] - \alpha(1-\alpha)]/2.$$

Consequently, for $\mathbf{E}[\mathbf{p}] \in [k, k+1]$ the minimum in formula (6) is attained on the k-th vector payoff, and the equality (5) holds. In particular, for $\mathbf{p} \in \Theta(k)$ ($\mathbf{E}[\mathbf{p}] = k$),

$$H(\mathbf{p}) = \sum_{s=0}^{\infty} p_s(s-k)(s-k-1)/2 = \sum_{s=0}^{\infty} p_s(s-k)(s-k+1)/2 = \mathbf{D}[\mathbf{p}]/2.$$

□

Corollary 1. *The strategies $\tau^m, m = 0, 1, \ldots$ guarantee the same upper bound $H(\mathbf{p})$ for the upper value of the infinite game $G_\infty(\mathbf{p})$.*

4. Structure of convex sets $\Theta(r)$ and linear functions over them

The sets $\Theta(r)$, $r = 1, 2, \ldots$ are closed convex subsets of Banach space $L^1(\{s^2\})$. In this section we give a representation of the set $\Theta(r)$ as a convex hull of its extreme points and a decomposition of linear functions over this set corresponding to this representation.

The extreme points of the set $\Theta(r)$ are distributions $\mathbf{p}^r(k,l) \in \Theta(r)$ with two-point supports $\{r-l, r+k\}$

$$p^r_{r-l}(k,l) = \frac{k}{k+l}, \quad p^r_{r+k}(k,l) = \frac{l}{k+l}, \tag{7}$$

$k = 0, 1, 2, \ldots$, $l = 0, 1, \ldots, r$, $k + l > 0$. Note that $\mathbf{p}^r(0,l) = \mathbf{p}^r(k,0) = \mathbf{e}^r$, where $\mathbf{e}^r$ is the degenerate distribution with one-point support $e^r_r = 1$.

Proposition 2. *Any $\mathbf{p} \in \Theta(r)$ has the following representation as a convex combination of extreme points (7):*

$$\mathbf{p} = p_r \cdot \mathbf{e}^r + \sum_{k=1}^{\infty}\sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \cdot \mathbf{p}^r(k,l), \tag{8}$$

with the coefficients

$$\alpha_{kl}(\mathbf{p}) = \frac{k+l}{\sum_{t=1}^{r} t p_{r-t}} p_{r-l} p_{r+k}. \tag{9}$$

Corollary 2. *Any linear function f over $\Theta(r)$ has the following representation as a convex combination of its values at extreme points.*

$$f(\mathbf{p}) = p_r \cdot f(\mathbf{e}^r) + \sum_{k=1}^{\infty}\sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \cdot f(\mathbf{p}^r(k,l)), \tag{10}$$

with the coefficients $\alpha_{kl}(\mathbf{p})$ given by (9).

In particular, the continuous linear function $\mathbf{D}$ over $\Theta(r)$, equal to zero at $\mathbf{e}^r$, has the following representation as a convex combination of values at extreme points $\mathbf{D}[\mathbf{p}^r(k,l)] = k \cdot l$ corresponding to decomposition (8):

$$\mathbf{D}[\mathbf{p}] = \sum_{k=1}^{\infty}\sum_{l=1}^{r} \frac{k+l}{\sum_{t=1}^{r} t p_{r-t}} p_{r-l} p_{r+k} \cdot k \cdot l.$$

Consequently, we obtain another representation for the function $H(\mathbf{p})$ over $\Theta(r)$.

$$H(\mathbf{p}) = \sum_{k=1}^{\infty}\sum_{l=1}^{r} \frac{k+l}{\sum_{t=1}^{r} t p_{r-t}} p_{r-l} p_{r+k} \cdot k \cdot l/2. \tag{11}$$

Observe that there exist the following "canonical" decompositions of extreme points $\mathbf{p}^r(k,l)$ with $k, l > 0$, that generate "quasi-optimal" strategies of Player 1 for games $G_n(\mathbf{p}^r(k,l))$ (see [3]):

$$\mathbf{p}^r(k,l) = (\mathbf{p}^{r+1}(k-1, l+1) + \mathbf{p}^{r-1}(k+1, l-1))/2.$$

These "canonical" decompositions can be expanded to the whole set $\Theta(r)$ by formulas

$$\mathbf{p} = p_r \cdot \mathbf{e}^r + (1-p_r)/2 \cdot \mathbf{p}^- + (1-p_r)/2 \cdot \mathbf{p}^+,$$

where

$$\mathbf{p}^- = \frac{1}{1-p_r}\sum_{k=1}^{\infty}\sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \cdot \mathbf{p}^{r-1}(k+1, l-1) \in \Theta(r-1),$$

$$\mathbf{p}^+ = \frac{1}{1-p_r}\sum_{k=1}^{\infty}\sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \cdot \mathbf{p}^{r+1}(k-1, l+1) \in \Theta(r+1),$$

or component-wise

$$p^-_s = \begin{cases} p_s \dfrac{\sum_{j=0}^{r-1}(r-1-j)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s > r\ ; \\ 0, & \text{for } s = r\ ; \\ p_s \dfrac{\sum_{j=r+1}^{\infty}(j-r+1)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s < r\ , \end{cases} \tag{12}$$

$$p_s^+ = \begin{cases} p_s \dfrac{\sum_{j=0}^{r-1}(r+1-j)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s > r\ ; \\ 0, & \text{for } s = r\ ; \\ p_s \dfrac{\sum_{j=r+1}^{\infty}(j-r-1)p_j}{\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s < r\ . \end{cases} \tag{13}$$

In the next section we show that these "canonical" decompositions also generate "quasi-optimal" strategies of Player 1 for general games $G_n(\mathbf{p})$.

Further, making use of representation (10) we construct a sequence of continuous, piecewise linear functions B_n over M^2 of the same form as the function H. Their domains of linearity are $L(r)$ and domains of non-smoothness are $\Theta(r)$. Such functions are completely defined by their values over the set $\cup_{r=1}^{\infty}\Theta(r)$. Observe that the functions $B_n(\mathbf{p})$ are continuous linear functions over each set $\Theta(r)$, $r = 1, 2, \ldots$.

Definition 2. *a) For the distributions $\mathbf{p}^r(k,l)$ given by (7) the values $B_n(\mathbf{p}^r(k,l))$ are given with the recurrent equalities*

$$B_n(\mathbf{p}^r(k,l)) = [1 + B_{n-1}(\mathbf{p}^{r+1}(k-1,l+1)) + B_{n-1}(\mathbf{p}^{r-1}(k+1,l-1))]/2, \tag{14}$$

with the boundary conditions

$$B_{n-1}(\mathbf{p}^{r+k}(0,l+k)) = B_{n-1}(\mathbf{p}^{r-l}(k+l,0)) = 0,$$

and with the initial condition $B_0(\mathbf{p}^r(k,l)) = 0$.

b) For the interior points $\mathbf{p} \in \Theta(r)$, the values $B_n(\mathbf{p})$ are convex combinations of its values at extreme points

$$B_n(\mathbf{p}) = \sum_{k=1}^{\infty}\sum_{l=1}^{r} \frac{k+l}{\sum_{t=1}^{r} t p_{r-t}} p_{r-l}p_{r+k} \cdot B_n(\mathbf{p}^r(k,l)).$$

In the next section we show that functions B_n are lower bounds for the values $V_n(\mathbf{p})$ over M^2, being the gains of Player 1 corresponding to his "quasi-optimal" strategies.

5. Asymptotics of values $V_n(\mathbf{p})$

In this section we show that, for $\mathbf{p} \in M^2$, as n tends to ∞, the sequence of values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$ converges to $H(\mathbf{p})$. To prove this, we construct Player 1's strategy σ^p that ensure lower bounds $B_n(\mathbf{p})$ converging to $H(\mathbf{p})$, for any $\mathbf{p} \in M^2$.

The strategy σ^p is a stationary strategy (does not depend on the step number). Such strategy is given by its first move for any $\mathbf{p} \in M^2$.

Definition 3. *Let $\mathbf{p} \in \Theta(r)$.*

If the state $s = r$, then the strategy σ^p stops the game.

Otherwise, the first move of the strategy σ^p makes use of two actions $r-1$ and r with probabilities

$$\sigma_1^p(r-1|s) = \begin{cases} \dfrac{\sum_{j=0}^{r-1}(r-j-1)p_j}{2\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s > r\ ; \\ \dfrac{\sum_{j=r+1}^{\infty}(j-r+1)p_j}{2\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s < r\ ; \end{cases}$$

$$\sigma_1^p(r|s) = \begin{cases} \dfrac{\sum_{j=0}^{r-1}(r-j+1)p_j}{2\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s > r\ ; \\ \dfrac{\sum_{j=r+1}^{\infty}(j-r-1)p_j}{2\sum_{j=0}^{r-1}(r-j)p_j}, & \text{for } s < r\ . \end{cases}$$

Thus these actions occur with total probabilities $q_{r-1} = q_r = (1-p_r)/2$. The posterior probability distributions are

$$\mathbf{p}(\cdot|r-1) = \mathbf{p}^- \in \Theta(r-1), \quad \mathbf{p}(\cdot|r) = \mathbf{p}^+ \in \Theta(r+1),$$

where $\mathbf{p}^-$ and $\mathbf{p}^+$ are given by (12) and (13).

For interior points $\mathbf{p} \in L(r)$ with $\mathbf{E}[\mathbf{p}] = r + \alpha$, first moves of strategies σ^p are convex combinations of the first moves at boundary points $\mathbf{p}^r \in \Theta(r)$ and $\mathbf{p}^{r+1} \in \Theta(r+1)$ such that $\mathbf{p} = \alpha\mathbf{p}^{r+1} + (1-\alpha)\mathbf{p}^r$.

Remark 3. It follows from Theorem 2. that, for $\mathbf{p} \in \Theta(r)$, if the random variable $C_{\mathbf{p}}$ takes the value r, then the gain of Player 1 is equal to zero and he can stop the game without any loss for himself.

Proposition 3. *For $\mathbf{p} \in \cup_{r=1}^{\infty}\Theta(r)$, the strategy σ^p ensures the payoff*

$$w_n(\mathbf{p}, \sigma^p) = B_n(\mathbf{p}).$$

in the game $G_n(\mathbf{p})$.

Proof. It is sufficient to prove Proposition for the games $G_n(\mathbf{p}^r(k,l))$ corresponding to extreme points $\mathbf{p}^r(k,l)$ of the sets $\Theta(r)$, $r = 1, 2, \ldots$. The proof is by induction on n.

$n = 1$. The best answer of Player 2 to the first move of the strategy σ^p with $\mathbf{p} = \mathbf{p}^r(k,l)$ is any action l with $l \leq r$. The resulting immediate gain of Player 1 is equal to 1/2. Thus, for $\mathbf{p} = \mathbf{p}^r(k,l)$, the strategy σ^p ensures the payoff $B_1(\mathbf{p}^r(k,l)) = 1/2$ in the one-step game $G_1(\mathbf{p}^r(k,l))$.

$n \to n+1$. Assume that the strategies σ^p with $\mathbf{p} = \mathbf{p}^r(k,l))$ ensure the payoffs $B_n(\mathbf{p}^r(k,l))$ in the games $G_n(\mathbf{p}^r(k,l))$.

For $\mathbf{p} = \mathbf{p}^r(k,l)$, the first move of the strategy σ^p has the immediate gain equal to 1/2. Its posterior probability distributions are $\mathbf{p}^{r-1}(k+1, l-1))$ and $\mathbf{p}^{r+1}(k-1, l+1))$, and both of them occur with probabilities 1/2.

According to the induction assumption and formulas (1), (6), the resulting total gain of Player 1 is equal to

$$[1 + B_n(\mathbf{p}^{r-1}(k+1, l-1)) + B_n(\mathbf{p}^{r+1}(k-1, l+1))]/2 = B_{n+1}(\mathbf{p}).$$

Thus, the strategy σ^p ensures the payoff $B_{n+1}(\mathbf{p})$ in the games $G_{n+1}(\mathbf{p})$ with $\mathbf{p} = \mathbf{p}^r(k,l)$. It is easy to extend this result to all $\mathbf{p} \in \cup_{r=1}^{\infty}\Theta(r)$. □

Theorem 3. *For $\mathbf{p} \in M^2$, the following equalities hold:*

$$\lim_{n\to\infty} V_n(\mathbf{p}) = H(\mathbf{p}).$$

Proof. According to Theorem 2 and Proposition 2 the following inequalities hold:

$$B_n(\mathbf{p}) \leq V_n(\mathbf{p}) \leq H(\mathbf{p}), \quad \forall \mathbf{p} \in M^2.$$

The functions B_n and H are continuous, concave, and piecewise linear with the same domains of linearity $L(r)$, $r = 0, 1, \ldots$. Such functions are completely determined with its values at the domains of non-smoothness $\Theta(r), r = 1, 2, \ldots$.

Because of continuity and concavity of the functions B_n and H, to prove that the sequence B_n converges to H as n tends to ∞, it is enough to show this for $\mathbf{p} \in \Theta(r)$, $r = 1, 2, \ldots$.

The increasing sequence of continuous linear functions B_n over $\Theta(r)$ is bounded from above with the continuous linear function H. Consequently, it has a continuous linear limit function B_∞. To prove Theorem 3 for $\mathbf{p} \in \Theta(r)$ it is enough to show that

$$\lim_{n\to\infty} B_n(\mathbf{p}^r(k,l)) = B_\infty(\mathbf{p}^r(k,l)) = H(\mathbf{p}^r(k,l)) = k \cdot l/2, \quad \forall k, l.$$

It follows from (14) that the limits $B_\infty(\mathbf{p}^r(k,l))$ should satisfy the equalities

$$B_\infty(\mathbf{p}^r(k,l)) = [1 + B_\infty(\mathbf{p}^{r+1}(k-1, l+1)) + B_\infty(\mathbf{p}^{r-1}(k+1, l-1))]/2.$$

with the boundary conditions $B_\infty(\mathbf{p}^{r+k}(0, l+k)) = B_\infty(\mathbf{p}^{r-l}(k+l, 0)) = 0$.

Solving the system of these $k+l-1$ linear equations connecting $k+l-1$ values $B_\infty(\mathbf{p}^{r+m}(k-m, l+m))$, $m = -l+1, -l+2, \ldots, k-1$, for distributions with the same two-point support $\{r-l, r+k\}$, we obtain that

$$B_\infty(\mathbf{p}^r(k,l)) = k \cdot l/2 = H(\mathbf{p}^r(k,l)).$$

According to (10) this proves Theorem 3 for $\mathbf{p} \in \Theta(r), r = 0, 1, \ldots$. Because of the continuity and concavity of the functions V_n it is true for all $\mathbf{p} \in M^2$. □

Corollary 2. *It follows from the proof that the strategy $\sigma(\mathbf{p})$ ensures the payoff $H(\mathbf{p})$ in the infinite game $G_\infty(\mathbf{p})$.*

The strategy $\sigma(\mathbf{p})$ is not optimal in any finite game $G_n(\mathbf{p})$ with $n < \infty$.

5. Solutions for the games $G_\infty(\mathbf{p})$ and random walks

For $\mathbf{p} \in M^2$, as the values $V_n(\mathbf{p})$ are bounded from above, the consideration of games $G_\infty(\mathbf{p})$ with infinite number of steps becomes reasonable.

We restrict the set of Player 1's admissible strategies in these games to the set Σ^+ of strategies employing only the moves ensuring him a nonnegative one-step gain against any action of Player 2. Consequently, the payoff functions $K_\infty(\mathbf{p}, \sigma, \tau)$ of the games $G_\infty(\mathbf{p})$ become definite (may be infinite) at all cases.

We show that the infinite game $G_\infty(\mathbf{p})$ has a value and this value is equal to $H(\mathbf{p})$.

The existence of values for these games does not follow from common considerations and has to be proved. We prove it by providing the optimal strategies explicitly.

Theorem 4. *For $\mathbf{p} \in M^2$, the game $G_\infty(\mathbf{p})$ has a value $V_\infty(\mathbf{p}) = H(\mathbf{p})$. Both Players have optimal strategies.*

The optimal strategy of Player 1 is the strategy σ^p, given by Definition 3.

For $\mathbf{p} \in L(r)$, $r = 0, 1\ldots$, the optimal strategy of Player 2 is the strategy τ^r, given by Definition 1. For $\mathbf{p} \in \Theta(r), r = 1, 2, \ldots$, any convex combination of the strategies τ^{r-1} and τ^r is optimal.

Proof. According to Corollary 2, the strategy $\sigma^p \in \Sigma^+$ ensures the payoff $H(\mathbf{p})$ in the game $G_\infty(\mathbf{p})$. Thus, for any $\mathbf{p} \in M^2$,

$$\sup_{\Sigma^+} \inf_{T} K_\infty(\mathbf{p}, \sigma, \tau) \geq H(\mathbf{p}), \tag{15}$$

and the function H is the lower bound for the lower value of the game G_∞.

On the other hand, according to Corollary 1, the strategies τ^r, $r = 0, 1, \ldots$, ensure the payoff $H(\mathbf{p})$ in the infinite game $G_\infty(\mathbf{p})$. Thus, for any $\mathbf{p} \in M^2$,

$$\inf_{T} \sup_{\Sigma^+} K_\infty(\mathbf{p}, \sigma, \tau) \leq H(\mathbf{p}), \tag{16}$$

and the function H is the upper bound for the upper value of the game G_∞.

As the lower value is always less or equal to the upper value, it follows from (15) and (16) that

$$\sup_{\Sigma^+} \inf_{T} K_\infty(\mathbf{p}, \sigma, \tau) = \inf_{T} \sup_{\Sigma^+} K_\infty(\mathbf{p}, \sigma, \tau) = H(\mathbf{p}) = V_\infty(\mathbf{p}).$$

The strategies $\sigma^p \in \Sigma^+$ and τ^r, $r = 0, 1, \ldots$ ensure the value $H(\mathbf{p}) = V_\infty(\mathbf{p})$ in the infinite game $G_\infty(\mathbf{p})$.

□

For the initial probability distribution $\mathbf{p} \in \Theta(r)$, $r = 1, 2, \ldots$, the random sequence of posterior probability distributions, generated with the optimal strategy σ^p of Player 1, is the symmetric random walk $(\mathbf{p}_t)_{t=1}^\infty$ over domains $\Theta(l)$. Probabilities of jumps to each of adjacent domains $\Theta(l-1)$ and $\Theta(l+1)$ are equal to $(1-p_l)/2$ and probability of absorption is equal to p_l. This is the Markov chain with the state space $\cup_{l=0}^\infty \Theta(l)$, and with the transition probabilities, for $\mathbf{p} \in \Theta(l)$,

$$\Pr(\mathbf{p}, \mathbf{e}^l) = p_l; \quad \Pr(\mathbf{p}, \mathbf{p}^-) = \Pr(\mathbf{p}, \mathbf{p}^+) = (1-p_l)/2,$$

where $\mathbf{p}^-$ and $\mathbf{p}^+$ are given with (12) and (13).

Next arising posterior probability distributions $\mathbf{p}^-$ and $\mathbf{p}^+$ have $p_l = 0$ and thus, for any subsequent visit to the domain $\Theta(l)$, the probability of absorption becomes equal to zero.

For the random walk $(\mathbf{p}_t)_{t=1}^\infty$ with the initial probability distribution $\mathbf{p} \in \Theta(r)$, let $\theta(\mathbf{p})$ be the random Markov time of absorption, i.e.

$$\theta(\mathbf{p}) = \min\{t : \mathbf{p}_t = \mathbf{e}^l\} - 1.$$

The Markov time $\theta(\mathbf{p})$ of absorption of posterior probabilities represents the time of revelation the "true" value of share by Player 2 and, generally speaking, the time of bidding termination.

Proposition 4. *For the random walk $(\mathbf{p}_t)_{t=1}^\infty$ with the initial probability distribution $\mathbf{p} \in \Theta(r)$, the expected duration $\mathbf{E}[\theta(\mathbf{p})]$ of this random walk is equal to the variance $\mathbf{D}[\mathbf{p}]$ of the liquidation price of a share.*

Proof. For the random walk $(\mathbf{p}_t)_{t=1}^\infty$ with the initial probability distribution $\mathbf{p} \in \Theta(r)$, the transition probabilities are linear functions over $\Theta(r)$. Consequently, the expected duration $\mathbf{E}[\theta(\mathbf{p})]$ of this random walk is a linear function over $\Theta(r)$ as well.

The continuous linear function $\mathbf{E}[\theta(\mathbf{p})]$ over $\Theta(r)$, equal to zero at $\mathbf{e}^r$, has the following "canonical" representation as a convex combination of values at extreme points $\mathbf{E}[\theta(\mathbf{p}^r(k,l))]$:

$$\mathbf{E}[\theta(\mathbf{p})] = \sum_{k=1}^{\infty} \sum_{l=1}^{r} \alpha_{kl}(\mathbf{p}) \cdot \mathbf{E}[\theta(\mathbf{p}^r(k,l))],$$

with the coefficients $\alpha_{kl}(\mathbf{p})$ given by (9).

It is well known that

$$\mathbf{E}[\theta(\mathbf{p}^r(k,l))] = k \cdot l = \mathbf{D}[\mathbf{p}^r(k,l)].$$

As the variance $\mathbf{D}[\mathbf{p}]$ is a linear function over $\Theta(r)$, we obtain the assertion of Proposition 4.

□

Remark 4. The result of Theorem 4 turns to be rather intuitive. The value of infinite game is equal to the expected duration of random walk of posterior probability distributions, multiplied by the constant one-step gain 1/2 of informed Player 1.

7. Conclusion

The obtained results on the biddings with countable sets of possible prices and admissible bids demonstrate that the Brownian component in the evolution of prices on the stock market may originate from asymmetric information of stockbrokers on events determining market prices.

References

De Meyer, B. and H. Saley (2002). *On the Strategic Origin of Brownian Motion in Finance.* Int. J. of Game Theory, **31**, 285–319.

Aumann, R. and M. Maschler (1995). *Repeated Games with Incomplete Information.* The MIT Press: Cambridge, Massachusetts - London, England.

Domansky, V. (2007) *Repeated games with asymmetric information and random price fluctuations at finance markets.* Int. J. of Game Theory, **36(2)**, 241–257.

Domansky, V. and V. Kreps (2007). *Moment of revealing of insider information at biddings with asymmetric information of agents.* Proc. of Appl. and Indust. Math., **14(3)**, 399–416 (in Russian).