Comonotonic Processes

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July 12, 2007

Abstract

We consider in this paper two Markovian processes X and Y, solutions of a stochastic differential equation with jumps, that are comonotonic, i.e., that are such that for all t, almost surely, X_t is greater in one state of the world than in another if and only if the same is true for Y_t. This notion of comonotonicity can be of great use for finance, insurance and actuarial issues.
We show here that the assumption of comonotonicity imposes strong constraints on the coefficients of the diffusion part of X and Y.

1 Introduction

We want to show that the assumption of comonotonicity for two processes imposes strong constraints on the coefficients of the diffusion part of the processes. This result is to be used for instance in finance, insurance or actuarial appplications where the notion of comonotonicity appears quite naturally (see Yaari (1987) for decision theory applications, Dybvig (1988) for finance applications, and Dhaene et al. (2002 *a* and *b*) for a review of the actuarial literature).

We start by introducing the notion of comonotonicity. We shall first recall its definition for random variables and we extend it for stochastic processes.

Definition 1 *Two real-valued random variables x_1 and x_2 defined on the same probability space (Ω, F, P) are comonotonic if there exists A in F, with probability one, and such that*

$$[x_1(\omega) - x_1(\omega')]\,[x_2(\omega) - x_2(\omega')] \geq 0 \quad \text{for all } (\omega, \omega') \in A \times A$$

or equivalently if the cumulative distribution function $F_{(x_1,x_2)}$ of the pair (x_1, x_2) is given by

$$F_{x_1,x_2}(\xi_1, \xi_2) = \min\left(F_{x_1}(\xi_1), F_{x_2}(\xi_2)\right).$$

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Other characterizations of comonotonic random variables can be found in Denneberg (1994). In particular, if two random variables x_1 and x_2 are such that there exists a nondecreasing function φ for which x_1 can be written in the form $x_1 = \varphi(x_2)$ (or if x_2 can be written in the form $x_2 = \varphi(x_1)$), then x_1 and x_2 are comonotonic. In fact, x_1 and x_2 are comonotonic if and only if they are nondecreasing functions of the same third random variable x_3, which can be chosen to be equal to $x_1 + x_2$ (Denneberg (1994), Proposition 4.5, p.54). Hence, as underlined by Wang and Dhaene (1998) comonotonic risks can be considered as "common monotonic".

This concept of comonotonicity emerges naturally in insurance issues since most risk sharing schemes between insurer and reinsurer or between insured and insurer lead to partial risks that are comonotonic. Furthermore, as proved by Landsberger and Meilijson (1994), all Pareto optimal risk allocations are comonotonic. It is also particularly useful in actuarial science since, as underlined by Dhaene et al. (2002 *a*), the concept of comonotonicity is closely related to Fréchet bounds for multivariate distribution functions and permits approximations for sums of random variables when the distributions of the terms are known, but the stochastic dependence structure between them is unknown, or too cumbersome to work with. Applications of such approximations to for instance the evaluation of insurance portfolios or cash flows, or to the determination of bounds for the price of an arithmetic asian option can be found in Dhaene et al. (2002 *b*).

Definition 2 *Two real-valued adapted processes X^1 and X^2 defined on the same filtered probability space $\left(\Omega, F, (F_t)_{t\geq 0}, P\right)$ are comonotonic if for all $t \geq 0$, the random variables X_t^1 and X_t^2 are comonotonic.*

Notice that if two processes X^1 and X^2 are such that for all t, $X_t^1 = d\left(t, X_t^2\right)$ where for all t, $d(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is some nondecreasing function, then X^1 and X^2 are comonotonic.
Besides, if d is of class $C^{1,2}$ and $X = \left(X^1, X^2\right)$ is a diffusion process of the form

$$dX_t = b_t dt + \sigma_t dW_t$$

where the $\mathbb{R}^2$-valued process $b \equiv \left(b^{X^1}, b^{X^2}\right)^*$, as well as the matrix-valued process $\sigma \equiv \left(\sigma^{X^1}, \sigma^{X^2}\right)^*$, where $\sigma^{X^1} \equiv (\sigma_1, \sigma_2)$ and $\sigma^{X^2} \equiv (\sigma_3, \sigma_4)$, satisfy the usual regularity conditions, then the use of Itô's lemma enables us to get that

$$\begin{aligned} dX_t^1 &= \left\{ d_t\left(t, X_t^2\right) + d_x\left(t, X_t^2\right) b_t^{X^2} + 1/2 d_{xx}\left(t, X_t^2\right) \left|\sigma_t^{X^2}\right|^2 \right\} dt \\ &\quad + d_x\left(t, X_t^2\right) \sigma_t^{X^2} dW_t. \end{aligned}$$

Identifying the diffusion parts, we immediately obtain that for all t,

$$\sigma_t^{X^1} = \sigma_t^{X^2} d_x\left(t, X_t^2\right) \tag{1}$$

so that for all t,

$$\det \sigma(t) = \sigma_1(t)\,\sigma_4(t) - \sigma_3(t)\,\sigma_2(t) = 0 \qquad P \text{ a.s .}$$

In the general diffusion case[1], remark that if X^1 and X^2 are comonotonic, then the law of $\left(X^1, X^2\right)$ is singular with respect to the Lebesgue measure. The problem can be treated as follows. Let $T_a \equiv \inf\{t, \det \sigma_t \sigma_t^* > a\}$. The pair $\left(X_t^1, X_t^2\right)$ is a non-homogeneous diffusion process with transition kernels $P_{s,t}$ and as soon as $\det \sigma_t \sigma_t^* \neq 0$ and σ is continuous, then $P_{s,t}(x, \cdot)$ admits a density with respect to the Lebesgue measure for all t in an interval $[s, s+\varepsilon]$. Since $E[f(X_t)] \geq E\left[P_{T_a, t-T_a} f(X_{T_a})\, 1_{\{T_a < t\}}\right]$ for any nonnegative f, it follows that the joint law of $\left(X_t^1, X_t^2\right)$ is not singular with respect to the Lebesgue measure for some t, as soon as $P(T_a < \infty) > 0$. Hence, if X^1 and X^2 are comonotonic, then $P(T_a = \infty) = 1$ for all $a > 0$, that is $\det \sigma_t \sigma_t^* = 0$ for all t.

We want to get an analogous result in the general case of two processes which are solutions of a stochastic differential equation with jumps. Notice that such jump processes are particularly relevant for insurance applications. Remark that in the case where one of the considered processes can be written as a regular function of the other, then, as above, Itô's Lemma concludes.

Let (Ω, F, P) be a given probability space and $(F_t)_{t\geq 0}$ denote a right-continuous, complete filtration. Let $W = \left\{\left(W_t^1, ..., W_t^d\right)^*; t \geq 0\right\}$ denote a d-dimensional Brownian motion for $(F_t)_{t\geq 0}$. Let $\mathcal{M}$ denote the set of real valued $(2 \times d)$-matrices.

Let n be a finite measure on $\mathbb{R}^k$. Let $\sigma : \mathbb{R}^2 \to \mathcal{M}$ and $b : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^2$ be Borel measurable, bounded and uniformly continuous functions such that for some positive constants A and K,

$$|\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)|^2 + \int_{B(0;1)} |f(x,u) - f(y,u)|^2\, n(du) \leq K\,|x-y|^2 \tag{2}$$

$$|\sigma(x)|^2 + |b(x)|^2 + |f(x,u)|^2 \leq A^2 \tag{3}$$

for x, y in $\mathbb{R}^2$ and u in $\mathbb{R}^k$ where as usual, for $m \in \mathcal{M}$ given by $m = \begin{pmatrix} m_{11} & ... & m_{1d} \\ m_{21} & ... & m_{2d} \end{pmatrix}$, we let $|m| \equiv \sqrt{\sum_{i,j} (m_{ij})^2}$ and for $x \in \mathbb{R}^N$ given by $x = (x_1, ..., x_N)^*$, $|x| \equiv \sqrt{\sum_{i=1}^N (x_i)^2}$.

Let μ be the Poisson measure on $\mathbb{R}_+ \times \mathbb{R}^k$ with intensity $ds \otimes n(du)$ and $\widetilde{\mu} = \mu - ds \otimes n(du)$ its compensated measure. We suppose that μ is independent of the Brownian motion W. Let p be the (F_t)-stationary Poisson point process associated with the counting measure μ (see e.g. Ikeda-Watanabe (1981), Section II-3). Under our conditions, we know that the following stochastic differential

[1] We are grateful to an anonymous referee for providing this short proof in the diffusion case.

equation

$$
\begin{aligned}
X_t &= X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s + \int_0^t \int_{|u|>1} f(X_{s-},u)\,\mu(ds,du) \\
&\quad + \int_0^t \int_{|u|\le 1} f(X_{s-},u)\,\widetilde{\mu}(ds,du) \qquad (4)
\end{aligned}
$$

with given initial condition $X_0 = \left(X_0^1, X_0^2\right)$, where X_0 is supposed to be a square integrable $\mathbb{R}^2$-valued F_0 -measurable random variable, admits a unique $(F_t)_{t\geq 0}$-adapted, càdlàg 2-dimensional solution process. We shall in the remainder of the paper write indifferently $\sigma(X_t)$ (resp. $b(X_t)$) or σ_t (resp. b_t).

In such a framework, we shall prove the following:

Theorem 1 *If the two-dimensional solution process X of Equation* (4) *has comonotonic components X^1 and X^2, then for all $t \geq 0$, its dispersion matrix σ_t almost surely does not have full rank.*

2 Proof of Theorem 1

To prove Theorem 1, we shall assume that there exists $t_0 \geq 0$ such that the dispersion matrix has full rank with a positive probability and show that the two processes X^1 and X^2 cannot be comonotonic. The rough idea is that if the dispersion matrix has full rank at date $t = t_0$, then according to the fact that $W = \left(W^1, ..., W^d\right)^*$ is a d-dimensional Brownian motion, the processes ΔX^1 and ΔX^2 do not necessarily have a "parallel" evolution[2] and as long as we take $X_{t_0}^1$ and $X_{t_0}^2$ in a small enough interval, we will be able to find $\Delta t \equiv t - t_0 \geq 0$ such that the two random variables $X_{t_0+\Delta t}^1$ and $X_{t_0+\Delta t}^2$ are not comonotonic.

In Section 2.1, we exhibit an event B_{t_0} in F_{t_0} on which the dispersion matrix has full rank and each of the random variables $X_{t_0}^1$, $X_{t_0}^2$ and $\sigma_{ij}(t_0)$ for $i = 1, 2$ and $j = 1, ..., d$ is stuck in an interval of given length. In Section 2.2, we show that on some subevents, the problem can be reduced to the one with constant coefficients and a diffusion process. In Section 2.3, we prove that these events have a positive probability and we conclude.

2.1 A Specific Set at $t = t_0$

Suppose that at $t = t_0$, $\det \sigma_t \sigma_t^* \neq 0$ with a positive probability. Without loss of generality, we can assume that $\sigma_{11}(t_0)\,\sigma_{22}(t_0) - \sigma_{21}(t_0)\,\sigma_{12}(t_0) \neq 0$ with a positive probability. We show that there exists an event B_{t_0} in F_{t_0}, with positive probability, on which each of the random variables $X_{t_0}^1$, $X_{t_0}^2$ and $\sigma_{ij}(t_0)$ for $i = 1, 2$ and $j = 1, ..., d$ is stuck in an interval of given length and on which $\sigma_{11}(t_0)\,\sigma_{22}(t_0) - \sigma_{21}(t_0)\,\sigma_{12}(t_0) \neq 0$.

[2] For any process $Y = \{Y_t; t \geq t_0\}$, let ΔY denote the stochastic process $\{Y_t - Y_{t_0}; t \geq t_0\}$.

To do so, consider first $B \equiv \{\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0) \neq 0\}$. By assumption, we have $P(B) \neq 0$. Then there exists a positive real number denoted by ℓ such that the event B_0 given by

$$B_0 \equiv \{|\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0)| \geq \ell\}$$

is of positive probability. Moreover, we can assume that the sign of the expression $\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0)$ remains constant on B_0.

Let n denote any given integer. Let for all k in $\mathbb{Z}$, for $i = 1, 2$ and $j = 1, ..., d$ and $l = 1, 2$,

$$C_k^{i,j} \equiv \left\{\sigma_{ij}(t_0) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right\}$$
$$D_k^l \equiv \left\{X_{t_0}^l \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right\}$$

As

$$B_0 = \cup_{\substack{k_{i,j}\in\mathbb{Z} \\ k'_l\in\mathbb{Z}}} \left[B_0 \bigcap_{i=1,2;j=1,...,d} C_{k_{i,j}}^{i,j} \bigcap_{l=1,2} D_{k'_l}^l\right]$$

there exist $k_{i,j}$ for $i = 1, 2; j = 1, ..., d$ and k'_1, k'_2 in $\mathbb{Z}$ such that the event B_{t_0} given by $B_{t_0} \equiv B_0 \bigcap_{i=1,2;j=1,...,d} C_{k_{i,j}}^{i,j} \bigcap_{l=1,2} D_{k'_l}^l$ has positive probability. It is immediate that B_{t_0} satisfies the conditions mentioned above, the length of the intervals being equal to $1/2^n$. We consider a decreasing sequence of such nested sets $B_{t_0}(n)$. Since $(\sigma_{ij}(t_0))_{i=1,2;j=1,...,d}$ is stuck in a compact set and $|\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0)| \geq \ell$, there exists some n_0, such that for all n greater than n_0, $a_{11}a_{22} - a_{21}a_{12} \neq 0$ holds true for any a_{ij} in $\left[\frac{k_{ij}}{2^n}, \frac{k_{ij}+1}{2^n}\right[$. For such an n_0, we let $\bar{\sigma}_i \equiv \frac{k_{ij}+1}{2^{n_0}}$ and $\underline{\sigma}_i \equiv \frac{k_{ij}}{2^{n_0}}$.

2.2 An Intermediary Lemma

We shall denote by $\tilde{X}$ the stochastic process $\left\{\tilde{X}_t = \left(\tilde{X}_t^1, \tilde{X}_t^2\right)^*; t \geq t_0\right\}$ given by

$$\widetilde{X}_t = X_{t_0} + \sigma_{t_0}\Delta W_t$$

and by Z the stochastic process $\left\{Z_t = \left(Z_t^1, Z_t^2\right)^*; t \geq t_0\right\}$ given by

$$\begin{aligned} Z_t &= \int_{t_0}^t b_s ds + \int_{t_0}^t (\sigma_s - \sigma_{t_0}) dW_s + \int_{t_0}^t \int_{|u|>1} f(X_{s-}, u)\, \mu(ds, du) \\ &\quad + \int_{t_0}^t \int_{|u|\leq 1} f(X_{s-}, u)\, \widetilde{\mu}(ds, du) \end{aligned}$$

Then for all $t \geq t_0$, $X_t = \tilde{X}_t + Z_t$ and $\Delta X = \Delta\tilde{X} + \Delta Z$.

Finally, for a given $\eta \in R_+^*$, let Z^η be given by

$$\begin{aligned} Z_t^\eta &= \int_{t_0}^t b_s ds + \int_{t_0}^t \varphi^\eta \left(\sigma_s - \sigma_{t_0}\right) dW_s + \int_{t_0}^t \int_{|u|>1} f\left(X_{s-}, u\right) \mu\left(ds, du\right) \\ &\quad + \int_{t_0}^t \int_{|u|\leq 1} f\left(X_{s-}, u\right) \widetilde{\mu}\left(ds, du\right) \end{aligned}$$

where $\varphi^\eta(x)$ stands for $x 1_{|x|\leq\eta} + \frac{x}{|x|} 1_{|x|>\eta}$.

Using the Lipschitz condition on σ, we know that for all given $\eta \in \mathbb{R}_+^*$, there exists a positive real number $\varepsilon(\eta)$ such that for all x and y in $\mathbb{R}^2$ satisfying $|x-y| \leq \varepsilon(\eta)$,

$$|\sigma(x) - \sigma(y)| \leq \eta$$

For all $(\lambda, \Delta t, \eta, n) \in \left(\mathbb{R}_+^*\right)^3 \times \mathbb{N}$, we let $B^1_{\lambda,\Delta t,\eta,n}$ denote the set

$$\left\{\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right\} \cap \left\{\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)\right\} \cap \left\{\begin{array}{c} \Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda \\ \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda \end{array}\right\}$$

and $B^2_{\lambda,\Delta t,\eta,n}$ denote the set

$$\left\{\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right\} \cap \left\{\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)\right\} \cap \left\{\begin{array}{c} \Delta\tilde{X}^1_{t_0+\Delta t} \leq -\lambda \\ \Delta\tilde{X}^2_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda \end{array}\right\}$$

For $l = 1, 2$, we let $A^l_{\lambda,\Delta t,\eta,n} \equiv B^l_{\lambda,\Delta t,\eta,n} \cap B_{t_0}(n)$ and we prove the following

Lemma 1 *If there exist $[(\lambda_1, \eta_1), (\lambda_2, \eta_2)] \in \left(R_+^*\right)^2 \times \left(R_+^*\right)^2$, $\Delta t \in \mathbb{R}_+^*$, and $n \in \mathbb{N}$ for which $P\left[A^l_{\lambda_i,\Delta t,\eta_i,n}\right] > 0$ for $l = 1, 2$, then the two processes X^1 and X^2 cannot be comonotonic.*

Proof Let us see first what happens on $A^1_{\lambda,\Delta t,\eta,n}$: we have $\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)$ hence for all $s \in [t_0, t_0 + \Delta t]$

$$\left|\sigma_s - \sigma_{t_0}\right| \leq \eta$$

so that for all $s \in [t_0, t_0 + \Delta t]$, $Z_s = Z_s^\eta$ and $|\Delta Z_{t_0+\Delta t}| = \left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda$.

As $\Delta X = \Delta\tilde{X} + \Delta Z$, we get on $A^1_{\lambda,\Delta t,\eta,n}$ that $\Delta X^1_{t_0+\Delta t} = \Delta\tilde{X}^1_{t_0+\Delta t} + \Delta Z^1_{t_0+\Delta t} \geq \frac{1}{2^n}$ and $\Delta X^2_{t_0+\Delta t} \leq 0$.

Now, using the same method, we get that for all $(\lambda, \Delta t, \eta, n) \in \left(\mathbb{R}_+^*\right)^3 \times \mathbb{N}$, we have $\Delta X^1_{t_0+\Delta t} \leq 0$ and $\Delta X^2_{t_0+\Delta t} \geq \frac{1}{2^n}$ on $A^2_{\lambda,\Delta t,\eta,n}$.

As $X^1_{t_0}$ and $X^2_{t_0}$ both belong to a (semi-open) interval of given length equal to $\frac{1}{2^n}$ on $A^l_{\lambda_l,\Delta t,\eta_l,n}$, we get that for all $(\omega, \omega') \in A^1_{\lambda_1,\Delta t,\eta_1,n} \times A^2_{\lambda_2,\Delta t,\eta_2,n}$, $X^1_{t_0+\Delta t}(\omega) > X^1_{t_0+\Delta t}(\omega')$ whereas $X^2_{t_0+\Delta t}(\omega) < X^2_{t_0+\Delta t}(\omega')$ so that

$$\left[X^1_{t_0+\Delta t}(\omega) - X^1_{t_0+\Delta t}(\omega')\right] \times \left[X^2_{t_0+\Delta t}(\omega) - X^2_{t_0+\Delta t}(\omega')\right] < 0$$

for all $(\omega, \omega') \in A^1_{\lambda_1,\Delta t,\eta_1,n} \times A^2_{\lambda_2,\Delta t,\eta_2,n}$, and the two random variables $X^1_{t_0+\Delta t}$ and $X^2_{t_0+\Delta t}$ cannot be comonotonic, which completes the proof of the lemma.□

So the lemma reduces the proof of our theorem to the finding of $[(\lambda_l, \eta_l)]_{l=1,2} \in \left(\mathbb{R}^*_+\right)^2 \times \left(\mathbb{R}^*_+\right)^2$, $n \in \mathbb{N}$ and $\Delta t \in \mathbb{R}^*_+$ for which the two events $A^1_{\lambda_1,\Delta t,\eta_1,n}$ and $A^2_{\lambda_2,\Delta t,\eta_2,n}$ have positive probability.

2.3 End of the Proof of Theorem 1

We consider first the set $A^1_{\lambda,\Delta t,\eta,n}$ and we only need to show that there exist $(\lambda, \Delta t, \eta, n) \in \left(\mathbb{R}^*_+\right)^3 \times \mathbb{N}$ for which

$$\begin{aligned}
& P\left\{\left(\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right) \cap B_{t_0}\right\} + P\left\{\left(\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)\right) \cap B_{t_0}\right\} \\
& + P\left\{\begin{pmatrix} \Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda \\ \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda \end{pmatrix} \cap B_{t_0}\right\} \\
> \;& 2P\left(B_{t_0}\right). \qquad (5)
\end{aligned}$$

We first consider the set $\left\{\begin{pmatrix} \Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda \\ \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda \end{pmatrix} \cap B_{t_0}\right\}$. We shall denote by $\hat{X}$ the stochastic process $\left\{\hat{X}_t = \left(\hat{X}^1_t, \hat{X}^2_t\right)^*; t \geq t_0\right\}$ given by

$$\hat{X}_t = X_{t_0} + a^0 \Delta W_t$$

where $a^0 \equiv \begin{pmatrix} a^0_{11} & a^0_{12} & 0 & ... & 0 \\ a^0_{21} & a^0_{21} & 0 & ... & 0 \end{pmatrix}$ for some real numbers $a^0_{ij} \in \left[\frac{k_{ij}}{2^n}, \frac{(k_{ij}+1)}{2^n}\right[$ for $i, j = 1, 2$. Then

$$\Delta\tilde{X}_{t_0+\Delta t} = \Delta\hat{X}_{t_0+\Delta t} + \left[\sigma_{t_0} - a^0\right] \Delta W_{t_0+\Delta t}$$

On $B_{t_0}(n)$, $\sigma_{ij}(t_0) \in \left[\frac{k_{ij}}{2^n}, \frac{k_{ij}+1}{2^n}\right[$, so that $\left|\sigma_{ij}(t_0) - a^0_{ij}\right| < \frac{1}{2^n}$. It is easy to see that for a given positive real number ξ, if $\Delta\hat{X}^1_{t_0+\Delta t} \geq 2\lambda+\xi$, $\Delta\hat{X}^2_{t_0+\Delta t} \leq -2\lambda-\xi$, $\left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\lambda 2^n - 1}{2}$ for $j = 1, 2$, $\left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\xi}{(d-2)A}$ for $j = 3, ..., d$, then $\Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda$ and $\Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda$. So

$$\begin{aligned}
& P\left[B_{t_0} \cap \left\{\Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda; \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda\right\}\right] \\
\geq \;& P\left[B_{t_0} \cap \left\{\begin{array}{c} \Delta\hat{X}^1_{t_0+\Delta t} \geq 2\lambda + \xi \\ \Delta\hat{X}^2_{t_0+\Delta t} \leq -2\lambda - \xi \end{array}; \begin{array}{c} \left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\lambda 2^n - 1}{2}, \ j = 1, 2 \\ \left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\xi}{(d-2)A}, \ j = 3, ..., d \end{array};\right\}\right] \\
\geq \;& \frac{P\left(B_{t_0}\right) P\left(B^\xi\right)}{2\pi\Delta t} \int_{\substack{a^0_{11}x + a^0_{12}y \geq 2\lambda+\xi \\ a^0_{21}x + a^0_{22}y \leq -2\lambda-\xi \\ |x| \leq \frac{\lambda 2^n - 1}{2}, |y| \leq \frac{\lambda 2^n - 1}{2}}} e^{-\frac{x^2+y^2}{2\Delta t}} dxdy
\end{aligned}$$

where $B^{\xi}=\left\{\left|\Delta W^{j}_{t_0+\Delta t}\right|\leq\frac{\xi}{(d-2)A},j=3,...,d\right\}$ because μ and W are independent and independent of F_{t_0}.

Let us now consider the other sets involved in Inequality (5), i.e., the sets $B_{t_0}\cap\left\{\left|\Delta Z^{\eta}_{t_0+\Delta t}\right|\leq\lambda\right\}$ and $B_{t_0}\cap\left\{\sup_{s\in[t_0,t_0+\Delta t]}|\Delta X_s|\leq\varepsilon(\eta)\right\}$.
As for Z^{η}, we have

$$\begin{aligned}P\left[\left\{\left|\Delta Z^{\eta}_{t_0+\Delta t}\right|\leq\lambda\right\}\right] &\geq 1-P\left[\left|\int_{t_0}^{t}b_s ds\right|>\frac{\lambda}{4}\right]-P\left[\left|\int_{t_0}^{t}\varphi^{\eta}\left(\sigma_s-\sigma_{t_0}\right)dW_s\right|>\frac{\lambda}{4}\right]\\ &\quad -P\left[\left|\int_{t_0}^{t}\int_{|u|>1}f\left(X_s,u\right)\mu\left(ds,du\right)\right|>\frac{\lambda}{4}\right]\\ &\quad -P\left[\left|\int_{t_0}^{t}\int_{|u|\leq 1}f\left(X_s,u\right)\widetilde{\mu}\left(ds,du\right)\right|>\frac{\lambda}{4}\right]\end{aligned}$$

By Itô's isometry, we get

$$\begin{aligned}P\left[\left|\int_{t_0}^{t}\varphi^{\eta}\left(\sigma_s-\sigma_{t_0}\right)dW_s\right|>\frac{\lambda}{4}\right] &\leq \frac{16}{\lambda^2}E\left[\left|\int_{t_0}^{t}\varphi^{\eta}\left(\sigma_s-\sigma_{t_0}\right)dW_s\right|^2\right]\\ &\leq \frac{32\eta^2\left(\Delta t\right)}{\lambda^2}.\end{aligned}$$

It is immediate that

$$P\left[\left|\int_{t_0}^{t}b_s ds\right|>\frac{\lambda}{4}\right]\leq\frac{4A\left(\Delta t\right)}{\lambda}.$$

Now,

$$\begin{aligned}P\left[\left|\int_{t_0}^{t}\int_{|u|>1}f\left(X_s,u\right)\mu\left(ds,du\right)\right|>\frac{\lambda}{4}\right] &\leq P\left[A\mu\left(\left[t_0,t\right]\times\left\{|z|>1\right\}\right)>\frac{\lambda}{4}\right]\\ &\leq \frac{4}{\lambda}E\left[A\mu\left(\left[t_0,t\right]\times\left\{|z|>1\right\}\right)\right]\\ &\leq \frac{4A\left(\Delta t\right)n\left\{|z|>1\right\}}{\lambda}\end{aligned}$$

and

$$\begin{aligned}P\left[\left|\int_{t_0}^{t}\int_{|u|\leq 1}f\left(X_s,u\right)\widetilde{\mu}\left(ds,du\right)\right|>\frac{\lambda}{4}\right] &\leq \frac{16}{\lambda^2}E\left[\left|\int_{t_0}^{t}\int_{|u|\leq 1}f\left(X_s,u\right)\widetilde{\mu}\left(ds,du\right)\right|^2\right]\\ &\leq \frac{16}{\lambda^2}\int_{t_0}^{t}ds\int_{|u|\leq 1}E\left[\left|f\left(X_s,u\right)\right|^2\right]n\left(du\right)\\ &\leq \frac{16A^2\left(\Delta t\right)n\left\{|z|\leq 1\right\}}{\lambda^2}\end{aligned}$$

On the other hand,

$$\begin{aligned}
P\left\{\sup_{s\in[t_0,t_0+\Delta t]}|\Delta X_s|\leq\varepsilon(\eta)\right\} \quad\geq\quad & 1-P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}b_u du\right|>\frac{\varepsilon(\eta)}{4}\right\}\\
& -P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\sigma_u dW_u\right|>\frac{\varepsilon(\eta)}{4}\right\}\\
& -P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|u|>1}f(X_{s-},u)\,\mu(ds,du)\right|>\frac{\varepsilon(\eta)}{4}\right\}\\
& -P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|u|\leq 1}f(X_{s-},u)\,\widetilde{\mu}(ds,du)\right|>\frac{\varepsilon(\eta)}{4}\right\}
\end{aligned}$$

We easily get

$$P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\sigma_u dW_u\right|>\frac{\varepsilon(\eta)}{4}\right\}\leq\frac{128\,(\Delta t)\,A^2}{[\varepsilon(\eta)]^2}.$$

$$P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}b_u du\right|>\frac{\varepsilon(\eta)}{4}\right\}\leq\frac{16\,(\Delta t)^2\,A^2}{[\varepsilon(\eta)]^2}$$

$$\begin{aligned}
& P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|u|>1}f(X_{s-},u)\,\mu(ds,du)\right|>\frac{\varepsilon(\eta)}{4}\right\}\\
\leq\quad & P\left\{A\mu([t_0,t_0+\Delta t]\times\{|z|>1\})>\frac{\varepsilon(\eta)}{4}\right\}\\
\leq\quad & \frac{4A}{\varepsilon(\eta)}E\left[\mu([t_0,t_0+\Delta t]\times\{|z|>1\})\right]\\
\leq\quad & \frac{4A\,(\Delta t)\,n(\{|z|>1\})}{\varepsilon(\eta)}
\end{aligned}$$

$$\begin{aligned}
& P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|u|\leq 1}f(X_{s-},u)\,\widetilde{\mu}(ds,du)\right|>\frac{\varepsilon(\eta)}{4}\right\} && (6)\\
\leq\quad & \frac{16}{\varepsilon(\eta)^2}E\left[\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|u|\leq 1}f(X_{s-},u)\,\widetilde{\mu}(ds,du)\right|^2\right] && (7)\\
\leq\quad & \frac{4\times 16}{\varepsilon(\eta)^2}E\left[\left|\int_{t_0}^{t_0+\Delta t}\int_{|u|\leq 1}f(X_{s-},u)\,\widetilde{\mu}(ds,du)\right|^2\right] && (8)\\
\leq\quad & \frac{64}{\varepsilon(\eta)^2}\int_{t_0}^{t_0+\Delta t}ds\int_{|u|\leq 1}E\left[|f(X_{s-},u)|^2\right]n(du)\\
\leq\quad & \frac{64A^2\,(\Delta t)\,n(\{|z|\leq 1\})}{\varepsilon(\eta)^2}
\end{aligned}$$

where (8) is obtained by Doob's inequality and the fact that $\int_{t_0}^{s}\int_{|u|\leq 1} f\left(X_{s-},u\right)\widetilde{\mu}\left(ds,du\right)$ is a martingale (Ikeda Watanabe, p62).

Then, as mentioned at the beginning of the subsection, if there exist $t^* \leq \delta\left(\eta\right)$ and $\left(\lambda,\eta,n\right)\in\left(\mathbb{R}_+\right)^2\times\mathbb{N}$ for which for all $\Delta t\leq t^*$ the condition

$$\begin{aligned}
&2P\left(B_{t_0}\right)-\frac{32\eta^2\left(\Delta t\right)}{\lambda^2}-\frac{4A\left(\Delta t\right)}{\lambda}-\frac{4A\left(\Delta t\right)n\left\{|z|>1\right\}}{\lambda}-\frac{16A^2\left(\Delta t\right)n\left\{|z|\leq 1\right\}}{\lambda^2}\\
&-\frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}-\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}-\frac{4A\left(\Delta t\right)n\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}-\frac{64A^2\left(\Delta t\right)n\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2}\\
&+\frac{P\left(B^{\xi}\right)P\left(B_{t_0}\right)}{2\pi\Delta t}\int_{\substack{a^0_{11}x+a^0_{12}y\geq 2\lambda+\xi\\ a^0_{21}x+a^0_{22}y\leq -2\lambda-\xi\\ |x|\leq\frac{\lambda 2^n-1}{2},|y|\leq\frac{\lambda 2^n-1}{2}}} e^{-\frac{x^2+y^2}{2\Delta t}}dxdy\\
>\;&2P\left(B_{t_0}\right) \qquad (9)
\end{aligned}$$

holds, then our problem is solved. Inequality (9) is equivalent to

$$\begin{aligned}
&\frac{P\left(B^{\xi}\right)}{2\pi\Delta t}\int_{\substack{a^0_{11}x+a^0_{12}y\geq 2\lambda+\xi\\ a^0_{21}x+a^0_{22}y\leq -2\lambda-\xi\\ |x|\leq\frac{\lambda 2^n-1}{2},|y|\leq\frac{\lambda 2^n-1}{2}}} e^{-\frac{x^2+y^2}{2\Delta t}}dxdy\\
>\;&\frac{32\eta^2\left(\Delta t\right)}{\lambda^2}+\frac{4A\left(\Delta t\right)}{\lambda}+\frac{4A\left(\Delta t\right)n\left\{|z|>1\right\}}{\lambda}+\frac{16A^2\left(\Delta t\right)n\left\{|z|\leq 1\right\}}{\lambda^2} \qquad (10)\\
&+\frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{4A\left(\Delta t\right)n\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}+\frac{64A^2\left(\Delta t\right)n\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2} \qquad (11)
\end{aligned}$$

Letting $\xi=\lambda$, $u=\frac{x}{\sqrt{\Delta t}}$, $v=\frac{y}{\sqrt{\Delta t}}$ and $\mu=\frac{\lambda}{\sqrt{\Delta t}}$, the inequality is equivalent to

$$L_1\equiv\frac{P\left(B^{\lambda}\right)}{2\pi}\int_{\substack{M\binom{u}{v}\geq\binom{3\mu}{3\mu}\\ |u|\leq\frac{\lambda 2^n-1}{2\sqrt{\Delta t}},|v|\leq\frac{\lambda 2^n-1}{2\sqrt{\Delta t}}}} e^{-\frac{1}{2}\left(u^2+v^2\right)}dudv>L_2$$

for

$$\begin{aligned}
L_2\quad\equiv\quad&\frac{32\eta^2}{\mu^2}+\frac{4A\sqrt{\left(\Delta t\right)}}{\mu}+\frac{4A\sqrt{\left(\Delta t\right)}n\left\{|z|>1\right\}}{\mu}+\frac{16A^2n\left\{|z|\leq 1\right\}}{\mu^2}\\
&+\frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{4A\left(\Delta t\right)n\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}+\frac{64A^2\left(\Delta t\right)n\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2}
\end{aligned}$$

where $M\equiv\begin{pmatrix} a^0_{11} & a^0_{12}\\ -a^0_{21} & -a^0_{22}\end{pmatrix}$. As we have seen in Section 3.1, for $n\geq n_0$, we know that for $i,j=1,2$, $a^0_{ij}\in\left[\underline{\sigma}_{ij},\overline{\sigma}_{ij}\right]$ on B_{t_0} and for all $a\in\prod_{i,j=1}^2\left[\underline{\sigma}_{ij},\overline{\sigma}_{ij}\right]$, $a_{11}a_{22}-a_{21}a_{12}\neq 0$. Then there exist real numbers $\bar{\gamma}_{ij}$'s for which, letting $\bar{M}\equiv\begin{pmatrix}\bar{\gamma}_{11} & \bar{\gamma}_{12}\\ \bar{\gamma}_{21} & \bar{\gamma}_{22}\end{pmatrix}$, $\bar{M}$ is invertible and

$$\bar{M}\begin{pmatrix}u\\ v\end{pmatrix}\geq\begin{pmatrix}3\mu\\ 3\mu\end{pmatrix}\Rightarrow M\begin{pmatrix}u\\ v\end{pmatrix}\geq\begin{pmatrix}3\mu\\ 3\mu\end{pmatrix}.$$

Then,

$$L_1 \geq \frac{P\left(B^\lambda\right)}{2\pi} \int_{\substack{\binom{u}{v} \in \bar{M}^{-1}\left([3\mu;+\infty[^2\right) \\ |u| \leq \frac{\lambda 2^n - 1}{2\sqrt{\Delta t}}, |v| \leq \frac{\lambda 2^n - 1}{2\sqrt{\Delta t}}}} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv$$

Since $\bar{M}^{-1}\left([3\mu;+\infty[^2\right)$ is independent of n and since we can choose n as large as we want (greater than n_0), we only need to solve

$$\frac{P\left(B^\lambda\right)}{2\pi} \int_{\binom{u}{v} \in \bar{M}^{-1}\left([3\mu;+\infty[^2\right)} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv > L_2.$$

As for $P\left(B^\lambda\right)$, we have

$$\begin{aligned} P\left(B^\lambda\right) &= P\left\{\left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\lambda}{2\,(d-2)\,A}, j=3,...,d\right\} \\ &\geq \prod_{j=3}^{d} \left\{1 - \frac{4\,(d-2)^2 A^2}{\lambda^2} E\left[\left(\Delta W^j_{t_0+\Delta t}\right)^2\right]\right\} \\ &\geq \left[1 - \frac{4\,(d-2)^2 A^2}{\lambda^2}\Delta t\right]^{d-2}. \end{aligned}$$

Let $\varphi(\mu) \equiv \frac{1}{2\pi}\int_{\binom{u}{v} \in \bar{M}^{-1}\left([3\mu;+\infty[^2\right)} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv$. We fix then μ such that

$$\frac{4A}{\mu} + \frac{4An\{|z|>1\}}{\mu} + \frac{16A^2 n\{|z|\leq 1\}}{\mu^2} < \frac{1}{6}\varphi(\mu)$$

and $\left[1 - \frac{4(d-2)^2 A^2}{\lambda^2}\Delta t\right]^{d-2} > \frac{1}{2}$, we find η such that $\frac{32\eta^2}{\mu^2} < \frac{1}{6}\varphi(\mu)$, then $(\Delta t) < 1$ such that $\frac{128(\Delta t)A^2}{[\varepsilon(\eta)]^2} + \frac{16(\Delta t)^2 A^2}{[\varepsilon(\eta)]^2} + \frac{4A(\Delta t)n(\{|z|>1\})}{\varepsilon(\eta)} + \frac{64A^2(\Delta t)n(\{|z|\leq 1\})}{\varepsilon(\eta)^2} < \frac{1}{6}\varphi(\mu)$ and $\lambda \equiv \mu\sqrt{\Delta t}$. This enables us to get

$$\begin{aligned} P\left(B^\lambda\right)\varphi(\mu) &> \frac{1}{2}\varphi(\mu) \\ &> \frac{32\eta^2}{\mu^2} + \frac{4A}{\mu} + \frac{4An\{|z|>1\}}{\mu} + \frac{16A^2 n\{|z|\leq 1\}}{\mu^2} \\ &\quad \frac{128\,(\Delta t)\,A^2}{[\varepsilon(\eta)]^2} + \frac{16\,(\Delta t)^2 A^2}{[\varepsilon(\eta)]^2} + \frac{4A\,(\Delta t)\,n\,(\{|z|>1\})}{\varepsilon(\eta)} + \frac{64A^2\,(\Delta t)\,n\,(\{|z|\leq 1\})}{\varepsilon(\eta)^2}. \\ &> \frac{32\eta^2}{\mu^2} + \frac{4A\sqrt{(\Delta t)}}{\mu} + \frac{4A\sqrt{(\Delta t)}n\{|z|>1\}}{\mu} + \frac{16A^2 n\{|z|\leq 1\}}{\mu^2} \\ &\quad \frac{128\,(\Delta t)\,A^2}{[\varepsilon(\eta)]^2} + \frac{16\,(\Delta t)^2 A^2}{[\varepsilon(\eta)]^2} + \frac{4A\,(\Delta t)\,n\,(\{|z|>1\})}{\varepsilon(\eta)} + \frac{64A^2\,(\Delta t)\,n\,(\{|z|\leq 1\})}{\varepsilon(\eta)^2}. \end{aligned}$$

We have then the existence of $(\lambda_1, (\Delta t)_1, \eta_1, n_1) \in \left(\mathbb{R}_+^*\right)^3 \times \mathbb{N}$ such that Inequation (5) holds.

Proceding in the exact same way for the set $A^2_{\lambda,\Delta t,\eta,n}$, we get the existence of $(\lambda_2, (\Delta t)_2, \eta_2, n_2) \in \left(\mathbb{R}_+^*\right)^3 \times \mathbb{N}$ such that $P\left[A^2_{\lambda_2,(\Delta t)_2,\eta_2,n_2}\right] > 0$; now, taking $n = \sup(n_1, n_2)$ and $\Delta t = \inf[(\Delta t)_1, (\Delta t)_2]$, we obtain that there exist $[(\lambda_i, \eta_i)]_{i=1,2} \in (\mathbb{R}_+^*)^2 \times (\mathbb{R}_+^*)^2$, $n \in \mathbb{N}$ and $\Delta t \in \mathbb{R}_+^*$ for which $P\left[A^i_{\lambda_i,\Delta t,\eta_i,n}\right] > 0$ for $i = 1, 2$, which, using Lemma 1, completes the proof.

2.4 m-dimensional Processes

We now assume that the process X is an m-dimensional Markov process, solution of a stochastic differential equation with jumps, for m possibly greater than 2. As in the preceding subsection, let $W = \left\{\left(W_t^1, ..., W_t^d\right)^*; t \geq 0\right\}$ denote a d -dimensional Brownian motion for $(F_t)_{t\geq 0}$. Let $\mathcal{M}^{m,d}$ denote the set of real valued $(m \times d)$ -matrices. Let $\sigma : \mathbb{R}^m \to \mathcal{M}^{m,d}$ and $b : \mathbb{R}^m \to \mathbb{R}^m$ and $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ be Borel measurable and uniformly continuous functions such that for some positive constants A and K, Inequations(2) and (3) are satisfied. Under these conditions, we know that the stochastic differential equation (4) with given initial condition $X_0 = \left(X_0^1, ..., X_0^m\right)$, where X_0 is supposed to be a square integrable R^m-valued F_0 -measurable random variable, admits a unique continuous, $(F_t)_{t\geq 0}$-adapted m -dimensional solution process $X = \left\{\left(X^1, ..., X^m\right)^*\right\}$. We shall prove the following

Theorem 2 *If the real-valued solution processes X^1 and X^2 of Equation (4) are comonotonic, then for all t, their dispersion coefficients are linked by the following relation*

$$\sigma_{1j}(t)\,\sigma_{2j'}(t) - \sigma_{2j}(t)\,\sigma_{1j'}(t) = 0 \qquad P\ a.s. \quad \text{for all } 1 \leq j, j' \leq d.$$

Proof The proof is similar to the one made in the case $m = 2$. We consider the same specific set B_{t_0} at time t_0 and the same sets $B^1_{\lambda,\Delta t,\eta,n}$ and $B^2_{\lambda,\Delta t,\eta,n}$ for all $(\lambda, \Delta t, \eta, n) \in (\mathbb{R}_+)^3 \times \mathbb{N}$. Lemma 1 remains valid. Then, we show exactly like in the preceding section that there exist $(\lambda, \Delta t, \eta, n) \in \left(\mathbb{R}_+^*\right)^3 \times \mathbb{N}$ for which the condition of Lemma 1 holds.$\square$

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