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**Abstract**

This chapter is an up-to-date survey of the state-of-the art in consumer demand analysis. We review (and evaluate) advances in a number of related areas, in the spirit of the recent survey paper by Barnett and Serletis (2008). In doing so, we only deal with consumer choice in a static framework, ignoring a number of important issues, such as, for example, the effects of demographic or other variables that affect demand, welfare comparisons across households (equivalence scales), and the many issues concerning aggregation across consumers.

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1

# 1    Introduction

This chapter is an up-to-date survey of static consumer demand analysis.   We review and evaluate recent advances in this attractive research area, in the spirit of a number of earlier surveys of that literature such as those by Barnett and Serletis (2008), LaFrance (2001), Lewbel (1997), Blundell (1988), and Brown and Deaton (1972).

It is not our intention in this chapter to cover all theories of consumer behavior.   We only deal with consumer choice in a static framework.   In doing so, we also ignore a number of important issues. In particular, we do not cover Varian's (1982) nonparametric revealed preference approach to demand analysis [see the recent survey paper by Barnett and Serletis (2008) for a brief review], the effects of demographic or other variables that affect demand, welfare comparisons across households (equivalence scales), and the many issues concerning aggregation across consumers — see Lewbel (1991), Kirman (1992), Stoker (1993), and Hildenbrandt (1994) regarding these issues.

The chapter is organized as follows.  Section 2 briefly presents directly specified demand equations, with no reference to the utility function.    Section 3 reviews the neoclassical theory of consumer choice, and section 4 deals with functional form issues.    Section 5 discusses functional forms in terms of their ability to capture the Engel curve structure of cross-sectional data.   Section 6 discusses estimation issues and the final section concludes.

# 2    Demand Systems without Utility Reference

There is an old tradition in applied demand analysis, which specifies the demand system directly with no reference to the utility function.   Under this approach, the demand for a good $i$, $x_i$, is specified as a function of nominal income, $y$, and prices, $p_1, \cdots, p_n$, where $n$ is the number of goods.

Consider, for example, the log-log demand system,

$$\log x_i = \alpha_i + \eta_{iy} \log y + \sum_{j=1}^{n} \eta_{ij} \log p_j, \quad i = 1, \cdots, n, \tag{1}$$

where $\alpha_i, \eta_{iy}$, and $\eta_{ij}$ are constant coefficients.   The coefficient $\eta_{iy}$ is the income elasticity of demand for good $i$, $\eta_{iy} = d \log x_i / d \log y$.   It measures the percentage change in $x_i$ per 1 percent change in $y$, with prices constant.   If $\eta_{iy} > 0$, the $i$th good is classified as normal and if $\eta_{iy} < 0$, it is classified as inferior (its consumption falls with increasing income). Moreover, if $\eta_{iy} > 1$, the $i$th good is classified as a luxury, and if $\eta_{iy} < 1$, it is classified as a necessity.   The coefficient $\eta_{ij}$ is the uncompensated (Cournot) price elasticity of good $i$, $\eta_{ij} = d \log x_i / d \log p_j$.   It measures the percentage change in $x_i$ per 1 percent change in $p_j$, with nominal income and the other prices constant.   If $\eta_{ij} > 0$, the goods are gross

substitutes, if $\eta_{ij} < 0$, they are gross complements, and if $\eta_{ij} = 0$, they are independent. If $i = j$, we would expect $\eta_{ii} < 0$, ruling out Giffen goods.

Another example of a demand system without reference to the utility function is Working's (1943) model,

$$s_i = \alpha_i + \beta_i \log y, \quad i = 1, \cdots, n, \tag{2}$$

expressing the budget share of good $i$, $s_i = p_i x_i / y$, as a linear function of logged income, $\log y$. Since the budget shares sum to 1, $\sum_{i=1}^{n} s_i = 1$, the parameters in equation (2) satisfy $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{i=1}^{n} \beta_i = 0$. As equation (2) does not involve prices, it is applicable to cross sectional data that offer limited variation in relative prices but substantial variation in income levels.

Under the assumption that prices are constant, multiplying the budget share of good $i$, $s_i = p_i x_i / y$, by its income elasticity, $\eta_{iy}$, yields

$$s_i \eta_{iy} = \frac{p_i x_i}{y} \frac{\partial x_i}{\partial y} \frac{y}{x_i} = p_i \frac{\partial x_i}{\partial y} = \frac{\partial (p_i x_i)}{\partial y} = \theta_i,$$

where $\theta_i = \partial (p_i x_i) / \partial y$ is the marginal share of good $i$. Unlike budget shares, marginal budget shares are not always positive (for example, marginal shares are negative in the case of inferior goods), but like budget shares, marginal shares also sum to 1, $\sum_{i=1}^{n} \theta_i = 1$.[1]

Multiplying both sides of (2) by $y$ and differentiating the resulting equation, $p_i x_i = \alpha_i y + \beta_i y \log y$, with respect to $y$, yields

$$\theta_i = \alpha_i + \beta_i (1 + \log y),$$

which after using equation (2) reduces to

$$\theta_i = \beta_i + s_i. \tag{3}$$

Equation (3) relates the marginal budget share of good $i$, $\theta_i$, to its budget share, $s_i$. It shows that the marginal share and the budget share differ by $\beta_i$, and that the marginal share, like the budget share, is changing over time.

Finally, if we divide both sides of equation (3) by $w_i$, we get the income elasticity of good $i$, $\eta_{iy} = 1 + \beta_i / s_i$, suggesting that good $i$ is a luxury if $\beta_i > 0$ and a necessity if $\beta_i < 0$. The model also predicts that as income increases, all goods become less luxurious. For example,

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[1] To see that marginal shares sum to 1, take the differential of the budget constraint, $y = \sum_{i=1}^{n} p_i x_i$, to get,

$$dy = \sum_{i=1}^{n} p_i dx_i + \sum_{i=1}^{n} x_i dp_i.$$

Since prices are assumed to be constant, the above implies $\sum_{i=1}^{n} \partial (p_i x_i) / \partial y = \sum_{i=1}^{n} \theta_i = 1$.

with constant prices, in the case of luxury goods, $s_i$ increases as $y$ increases, causing $\eta_{iy}$ to fall towards 1; in the case of necessities, $s_i$ declines as $y$ increases, causing $\eta_{iy}$ to also decline.

As already noted, Working's model is applicable to household level data where prices exhibit little variation. To apply this model to time series data that offer substantial variation in relative prices but less variation in income, the model has to be extended by adding a substitution term, as in equation (1).

# 3    Neoclassical Consumer Theory

Consider $n$ consumption goods that can be selected by a consuming household. The household's problem is

$$\max_{\{x_1,\cdots,x_n\}} u(x_1,\cdots,x_n) \quad \text{subject to} \quad \sum_{i=1}^{n} p_i x_i = y,$$

or in matrix notation

$$\max_{\boldsymbol{x}} u(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{p}'\boldsymbol{x} = y, \tag{4}$$

where $\boldsymbol{x}$ is the $n \times 1$ vector of goods; $\boldsymbol{p}$ is the corresponding vector of prices; and $y$ is the household's total income.

The first order conditions for a maximum can be found by forming an auxiliary function known as the Lagrangian

$$\mathcal{L} = u(\boldsymbol{x}) + \lambda \left( y - \sum_{i=1}^{n} p_i x_i \right),$$

where $\lambda$ is the Lagrange multiplier. By differentiating $\mathcal{L}$ with respect to $x_i$ $(i = 1, \cdots, n)$, and using the budget constraint, we obtain the $(n+1)$ first order conditions

$$\frac{\partial u(\boldsymbol{x})}{\partial x_i} - \lambda p_i = 0, \qquad i = 1, \cdots, n;$$

$$y - \sum_{i=1}^{n} p_i x_i = 0,$$

where the partial derivative $\partial u(\boldsymbol{x})/\partial x_i$ is the marginal utility of good $i$.

What do these first order conditions tell us about the solution to the utility maximization problem? Notice that the first $n$ conditions can be written as

$$\frac{\partial u(\boldsymbol{x})/\partial x_1}{p_1} = \frac{\partial u(\boldsymbol{x})/\partial x_2}{p_2} = \cdots = \frac{\partial u(\boldsymbol{x})/\partial x_n}{p_n} = \lambda, \tag{5}$$

which simply says that, in equilibrium, the ratio of marginal utility to price must be the same for all goods. Alternatively, for any two goods $i$ and $j$, the above condition can be rewritten as

$$\frac{\partial u(\boldsymbol{x})/\partial x_i}{\partial u(\boldsymbol{x})/\partial x_j} = \frac{p_i}{p_j},$$

which says that, in equilibrium, the ratio of marginal utilities (also known as the marginal rate of substitution) must equal the respective price ratio.

Notice that according to equation (5), the optimal Lagrange multiplier is utility per unit of good $k$ divided by the number of dollars per unit of good $k$ ($k = 1, \cdots, n$), reducing to utility per dollar. By this interpretation, the optimal Lagrange multiplier is also called the marginal utility of income.

## 3.1  Marshallian Demands

The first-order conditions for utility maximization can be used to solve for the $n$ optimal (i.e., equilibrium) values of $\boldsymbol{x}$,

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p}, y). \tag{6}$$

These utility maximizing quantities demanded are known as the Marshallian ordinary demand functions. In fact, system (6) is the demand system, giving the quantity demanded as a function of the prices of all goods and income. Demand systems are the systems whose parameters we want to estimate and whose properties we want to analyze in empirical demand analysis. Demand systems are also expressed in budget share form, $\boldsymbol{s}$, where $s_j = p_j x_j(\boldsymbol{p}, y)/y$ is the income share of good $j$.

As an example, consider the Cobb-Douglas utility function,

$$u(\boldsymbol{x}) = \prod_{i=1}^{n} x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots \tag{7}$$

with $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Setting up the Lagrangian for this problem, we get the following first order conditions

$$\frac{\alpha_i}{x_i} \prod_{i=1}^{n} x_i^{\alpha_i} - \lambda p_i = 0, \qquad i = 1, \cdots, n;$$

$$y - \sum_{i=1}^{n} p_i x_i = 0,$$

which, when solved for the $n$ optimal values of $\boldsymbol{x}$, yield the Marshallian demand functions

$$x_i = \alpha_i \frac{y}{p_i}, \qquad i = 1, \cdots, n, \tag{8}$$

5

since $\sum_{i=1}^{n} \alpha_i = 1$.

It is to be noted that Marshallian demands satisfy the following properties:

i) positivity;

ii) adding up (or summability), $\boldsymbol{p}'\boldsymbol{x}(\boldsymbol{p}, y) = y$, or, if the demand system is written in share form, $\boldsymbol{\iota}'\boldsymbol{s} = 1$, where $\boldsymbol{\iota}$ is a vector of ones;

iii) homogeneity of degree zero in $(\boldsymbol{p}, y)$. That is, $\boldsymbol{x}(\boldsymbol{p}, y) = \boldsymbol{x}(t\boldsymbol{p}, ty)$ for all $t$, implying the absence of money illusion (meaning that the optimal consumption levels are invariant to proportionate changes in all prices and money income);

iv) the matrix of substitution effects (provided the derivatives exist and are continuous),

$$\boldsymbol{S} \;=\; \left[ \partial \boldsymbol{x}(\boldsymbol{p}, y)/\partial \boldsymbol{p}' + \frac{\partial \boldsymbol{x}(\boldsymbol{p}, y)}{\partial y} \boldsymbol{x}(\boldsymbol{p}, y)' \right],$$

is symmetric and negative semidefinite (implying that the substitution effect of each good with respect to its own price is always nonpositive).

These properties of the demand system are frequently referred to as the 'integrability conditions,' since they permit the reconstruction of the preference ordering from the demand system. See, for example, Hurwicz and Uzawa (1971). If they are tested empirically and cannot be rejected, then we can infer that there exists a utility function that generates the demand system. To put it differently, demand behavior is consistent with the theory of utility maximization, if and only if the integrability conditions are satisfied.

## 3.2 Indirect Utility

The maximum level of utility given prices $\boldsymbol{p}$ and income $y$, $h(\boldsymbol{p}, y) = u\left[\boldsymbol{x}(\boldsymbol{p}, y)\right]$, is the indirect utility function and reflects the fact that utility depends indirectly on prices and income. In the case, for example, of Cobb-Douglas preferences the indirect utility function is obtained by substituting the demand system (8) into the direct utility function (7) to get

$$h(\boldsymbol{p}, y) = \prod_{i=1}^{n} x_i^{\alpha_i}$$

$$= \prod_{i=1}^{n} \left( \frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} \frac{y}{p_i} \right)^{\alpha_i} = y \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}, \tag{9}$$

since $\sum_{i=1}^{n} \alpha_i = 1$.

The direct utility function and the indirect utility function are equivalent representations of the underlying preference preordering. In fact, there is a duality relationship between the direct utility function and the indirect utility function, in the sense that maximization of $u(\boldsymbol{x})$ with respect to $\boldsymbol{x}$, with given $(\boldsymbol{p}, y)$, and minimization of $h(\boldsymbol{p}, y)$ with respect to $(\boldsymbol{p}, y)$, with given $\boldsymbol{x}$, leads to the same demand functions.

While the direct utility function has greater intuitive appeal than the indirect utility function, being able to represent preferences by an indirect utility function has its advantages. This is so, because the indirect utility function has prices exogenous in explaining consumer behavior. Moreover, we can easily derive the demand system by straightforward differentiation, without having to solve a system of simultaneous equations, as is the case with the direct utility function approach. In particular, a result known as Roy's identity

$$\boldsymbol{x}(\boldsymbol{p}, y) = -\frac{\partial h(\boldsymbol{p}, y)/\partial \boldsymbol{p}}{\partial h(\boldsymbol{p}, y)/\partial y}, \tag{10}$$

allows us to derive the demand system, provided that $\boldsymbol{p} > \boldsymbol{0}$ and $y > 0$. Alternatively, the logarithmic form of Roy's identity,

$$\boldsymbol{s}(\boldsymbol{p}, y) = -\frac{\partial \log h(\boldsymbol{p}, y)/\partial \log \boldsymbol{p}}{\partial \log h(\boldsymbol{p}, y)/\partial \log y}, \tag{11}$$

or Diewert's (1974, p. 126) modified version of Roy's identity,

$$s_j(\boldsymbol{v}) = \frac{v_j \nabla h(\boldsymbol{v})}{\boldsymbol{v}' \nabla h(\boldsymbol{v})}, \tag{12}$$

can be used to derive the budget share equations, where $\boldsymbol{v} = [v_1, \cdots, v_n]$ is a vector of income normalized prices, with the $j$th element being $v_j = p_j/y$ and $\nabla h(\boldsymbol{v}) = \partial h(\boldsymbol{v})/\partial \boldsymbol{v}$. Applying, for example, Roy's identity (10) to the Cobb-Douglas indirect utility function (9) yields the Dobb-Douglas demand system (8).

The indirect utility function is continuous in $(\boldsymbol{p}, y)$ and has the following properties:

   i) positivity;

  ii) homogeneity of degree zero in $(\boldsymbol{p}, y)$;

 iii) decreasing in $\boldsymbol{p}$ and increasing in $y$;

  iv) strictly quasi-convex in $\boldsymbol{p}$;

   v) satisfies Roy's identity, (10).

Together, properties (i)-(iv) are called the 'regularity conditions.' In the terminology of Caves and Christensen (1980), an indirect utility function is 'regular' at a given $(\boldsymbol{p}, y)$, if it satisfies the above properties at that $(\boldsymbol{p}, y)$. Similarly, the 'regular region' is the set of prices and incomes at which an indirect utility function satisfies the regularity conditions.

## 3.3  Hicksian Demands

Dual to the utility maximization problem is the problem of minimizing the cost or expenditure necessary to obtain a fixed level of utility, $u$, given market prices, $\boldsymbol{p}$,

$$C(\boldsymbol{p}, u) = \min_{\boldsymbol{x}} \boldsymbol{p}'\boldsymbol{x} \quad \text{subject to} \quad u(\boldsymbol{x}) \geq u.$$

For example, with Cobb-Douglas preferences, the Lagrangian for this problem is

$$\mathcal{L} = \sum_{i=1}^{n} p_i x_i + \lambda \left( u - \prod_{i=1}^{n} x_i^{\alpha_i} \right),$$

with the following first order conditions

$$p_i - \lambda \frac{\alpha_i}{x_i} \prod_{i=1}^{n} x_i^{\alpha_i} = 0, \qquad i = 1, \cdots, n;$$

$$u - \prod_{i=1}^{n} x_i^{\alpha_i} = 0,$$

which, when solved for the optimal values of $\boldsymbol{x}$, yield the expenditure minimizing demands, denoted by $\widetilde{\boldsymbol{x}}$,

$$\widetilde{x}_i(\boldsymbol{p}, u) = \frac{\alpha_i}{p_i} u \prod_{j=1}^{n} \left( \frac{p_j}{\alpha_j} \right)^{\alpha_j}, \qquad i = 1, \cdots, n. \tag{13}$$

The expenditure minimizing demands are also known as Hicksian or compensated demands; they tell us how $\boldsymbol{x}$ is affected by prices with $u$ held constant.[2]  Finally, substituting the Hicksian demands into the cost function yields

$$C(\boldsymbol{p}, u) = \sum_{i=1}^{n} p_i \widetilde{x}_i$$

$$= \sum_{i=1}^{n} p_i \left[ \frac{\alpha_i}{p_i} u \prod_{j=1}^{n} \left( \frac{p_j}{\alpha_j} \right)^{\alpha_j} \right] = u \prod_{j=1}^{n} \left( \frac{p_j}{\alpha_j} \right)^{\alpha_j}, \tag{14}$$

since $\sum_{i=1}^{n} \alpha_i = 1$.

Hicksian demands are positive valued and have the following properties:

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[2] To derive (13), we treat good $x_1$ asymmetrically, by solving for the demand for each of the other goods as a function of $x_1$. We then substitute in the utility function, $u = \prod_{j=1}^{n} x_j^{\alpha_j}$, and solve for $x_1$ as a function of $u$ and the other $x'$s. We follow a similar procedure for each of the other $x$'s.

i) homogeneous of degree zero in $\boldsymbol{p}$;

ii) the Slutsky matrix, $[\partial \widetilde{\boldsymbol{x}}(\boldsymbol{p}, u)/\partial \boldsymbol{p}']$, is symmetric and negative semidefinite.

Finally, the cost or expenditure function, $C(\boldsymbol{p}, u) = \boldsymbol{p}'\widetilde{\boldsymbol{x}}(\boldsymbol{p}, u)$, has the following properties:

i) continuous in $(\boldsymbol{p}, u)$;

ii) homogeneous of degree one in $\boldsymbol{p}$;

iii) increasing in $\boldsymbol{p}$ and $u$;

iv) concave in $\boldsymbol{p}$;

v) satisfies Shephard's (1953) lemma

$$\widetilde{\boldsymbol{x}}(\boldsymbol{p}, u) = \frac{\partial C(\boldsymbol{p}, u)}{\partial \boldsymbol{p}}. \tag{15}$$

For example, applying Shephard's lemma (15) to the cost function (14) yields the Hicksian compensated demand functions (13).

## 3.4 Elasticity Relations

A demand system provides a complete characterization of consumer preferences and can be used to estimate the income elasticities, the own- and cross-price elasticities, as well as the elasticities of substitution. These elasticities are particularly useful in judging the validity of the parameter estimates (which sometimes are difficult to interpret, due to the complexity of the demand system specifications).

The elasticity measures can be calculated from the Marshallian demand functions, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p}, y)$. In particular, the income elasticity of demand, $\eta_{iy}(\boldsymbol{p}, y)$, can be calculated as (for $i = 1, \ldots, n$)

$$\eta_{iy}(\boldsymbol{p}, y) = \frac{\partial x_i(\boldsymbol{p}, y)}{\partial y} \frac{y}{x_i(\boldsymbol{p}, y)}.$$

If $\eta_{iy}(\boldsymbol{p}, y) > 0$, the $i$th good is classified as normal at $(\boldsymbol{p}, y)$; and if $\eta_{iy}(\boldsymbol{p}, y) < 0$, it is classified as inferior. Another interesting dividing line in classifying goods according to their income elasticities is the number one. If $\eta_{iy}(\boldsymbol{p}, y) > 1$, the $i$th good is classified as a luxury; and if $\eta_{iy}(\boldsymbol{p}, y) < 1$, it is classified as a necessity. For example, with Cobb-Douglas preferences (7), the Marshallian demands are given by (8), in which case $\eta_{iy} = 1$ (for all $i$), since the Marshallian demands in this case are linear in income.

The uncompensated (Cournot) price elasticities, $\eta_{ij}(\boldsymbol{p}, y)$, can be calculated as (for $i$, $j = 1, \ldots, n$)

$$\eta_{ij}(\boldsymbol{p}, y) = \frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} \frac{p_j}{x_i(\boldsymbol{p}, y)}.$$

If $\eta_{ij}(\boldsymbol{p}, y) > 0$, the goods are gross substitutes (meaning that when $x_j$ becomes more expensive, the consumer increases consumption of good $x_i$ and decreases consumption of good $x_j$). If $\eta_{ij}(\boldsymbol{p}, y) < 0$, they are gross complements (meaning that when $x_j$ becomes more expensive, the consumer reduces the consumption of $x_j$ and also of $x_i$). If $\eta_{ij}(\boldsymbol{p}, y) = 0$, they are independent. With Cobb-Douglas preferences (7), using the Marshallian demands (8), the own-price elasticities are $\eta_{ii} = -\alpha_i/s_i$ (for all $i$), where $s_i = p_i x_i/y$, and the cross-price elasticities are $\eta_{ij} = 0$, since the demand for the $i$th good depends only on the $i$th price.

The definitions given above are in gross terms, because they ignore the income effect — that is, the change in demand of good $x_i$ due to the change in purchasing power resulting from the change in the price of good $x_j$. The Slutsky equation, however, decomposes the total effect of a price change on demand into a substitution effect and an income effect. In particular, differentiating the second identity in

$$x_i(\boldsymbol{p}, y) = x_i(\boldsymbol{p}, C(\boldsymbol{p}, u)) = \widetilde{x}_i(\boldsymbol{p}, u)$$

with respect to $p_j$ and rearranging, we acquire the Slutsky equation

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} = \frac{\partial \widetilde{x}_i(\boldsymbol{p}, u)}{\partial p_j} - x_j(\boldsymbol{p}, y) \frac{\partial x_i(\boldsymbol{p}, y)}{\partial y},$$

for all $(\boldsymbol{p}, y)$, $u = h(\boldsymbol{p}, y)$, and $i, j = 1, \cdots, n$. The derivative $\partial x_i(\boldsymbol{p}, y)/\partial p_j$ is the total effect of a price change on demand, while $\partial \widetilde{x}_i(\boldsymbol{p}, u)/\partial p_j$ is the substitution effect of a compensated price change on demand, and $-x_j(\boldsymbol{p}, y)\partial x_i(\boldsymbol{p}, y)/\partial y$ is the income effect, resulting from a change in price. Hicks (1936) suggested using the sign of the cross-substitution effect (that is, the change in compensated demand) to classify goods as substitutes, whenever $\partial \widetilde{x}_i(\boldsymbol{p}, u)/\partial p_j$ is positive. In fact, according to Hicks (1936), $\partial \widetilde{x}_i(\boldsymbol{p}, u)/\partial p_j > 0$ indicates substitutability, $\partial \widetilde{x}_i(\boldsymbol{p}, u)/\partial p_j < 0$ indicates complementarity, and $\partial \widetilde{x}_i(\boldsymbol{p}, u)/\partial p_j = 0$ indicates independence.

As already noted, one important property of the Slutsky equation is that the cross-substitution effects are symmetric; that is, $\partial \widetilde{x}_i(\boldsymbol{p}, u)/\partial p_j = \partial \widetilde{x}_j(\boldsymbol{p}, u)/\partial p_i$. This symmetry restriction may also be written in elasticity terms, as follows

$$\eta_{iy}(\boldsymbol{p}, y) + \frac{\eta_{ij}(\boldsymbol{p}, y)}{s_j} = \eta_{jy}(\boldsymbol{p}, y) + \frac{\eta_{ji}(\boldsymbol{p}, y)}{s_i}.$$

The symmetrical terms in the above equation are the Allen elasticities of substitution, so that the equation can be written, in terms of Hicksian demand elasticities, as

$$\sigma_{ij}^a = \frac{\widetilde{\eta}_{ij}(\boldsymbol{p}, u)}{s_j} = \sigma_{ji}^a,$$

where $\widetilde{\eta}_{ij}(\boldsymbol{p}, u) = \partial \log \widetilde{x}_i(\boldsymbol{p}, u)/\partial \log p_j$ denotes the Hicksian elasticity of demand and $\sigma_{ij}^a$ denotes the Allen elasticity of substitution between goods $i$ and $j$ — see Allen (1938) for more details. Hence, the Allen elasticity of substitution is the Hicksian demand elasticity divided by the budget share; for this reason reporting both the Hicksian demand elasticity and the Allen elasticity of substitution is redundant. Alternatively, since the Hicksian demand elasticity is related to the Marshallian demand elasticity through the elasticity form of the Slutsky equation, the Allen elasticities of substitution can be written in terms of Marshallian demand elasticities as follows

$$\sigma_{ij}^a = \eta_{iy}(\boldsymbol{p}, y) + \frac{\eta_{ij}(\boldsymbol{p}, y)}{s_j} = \eta_{jy}(\boldsymbol{p}, y) + \frac{\eta_{ji}(\boldsymbol{p}, y)}{s_i} = \sigma_{ji}^a. \tag{16}$$

If $\sigma_{ij}^a > 0$, goods $i$ and $j$ are said to be Allen substitutes, in the sense that an increase in the price of good $j$ causes an increased consumption of good $i$. If, however, $\sigma_{ij}^a < 0$, then the goods are said to be Allen complements, in the sense that an increase in the price of good $j$ causes a decreased consumption of good $i$.

The Allen elasticity of substitution is the traditional measure and has been employed to measure net substitution behavior (with utility held constant) and structural instability in a variety of contexts. There are, however, other elasticities that can be used to assess the substitutability/complementarity relationship between goods and classify goods as complements or substitutes. See Blackorby and Russell (1989) or Davis and Gauger (1996) for more details. For example, the Morishima (1967) net elasticity of substitution can be used to measure the percentage change in relative demands (quantity ratios) with respect to a percentage change in one price. In particular, under the assumption that a change in $p_j/p_i$ is due solely to a change in $p_j$, the Morishima elasticity of substitution for $x_i/x_j$ is given by

$$\sigma_{ij}^m = \frac{\partial \log\left(\widetilde{x}_i(\boldsymbol{p}, u)/\widetilde{x}_j(\boldsymbol{p}, u)\right)}{\partial \log(p_j/p_i)} = \frac{\partial \log \widetilde{x}_i(\boldsymbol{p}, u)}{\partial \log p_j} - \frac{\partial \log \widetilde{x}_j(\boldsymbol{p}, u)}{\partial \log p_j}$$

$$= \widetilde{\eta}_{ij}(\boldsymbol{p}, u) - \widetilde{\eta}_{jj}(\boldsymbol{p}, u) = s_j\left(\sigma_{ij}^a - \sigma_{jj}^a\right), \tag{17}$$

and measures the net change in the compensated demand for good $i$, when the price of good $j$ changes. As can be seen, a change in $p_j$, holding $p_i$ constant, has two effects on the quantity ratio $x_i/x_j$: one on $x_i$ captured by $\widetilde{\eta}_{ij}(\boldsymbol{p}, u)$ and one on $x_j$ captured by $\widetilde{\eta}_{jj}(\boldsymbol{p}, u)$. Goods will be Morishima substitutes (complements), if an increase in the price of $j$ causes $x_i/x_j$ to decrease (increase).

The Morishima elasticity of substitution is a 'two-good one-price' elasticity of substitution, unlike the Allen elasticity of substitution, which is a 'one-good one-price' elasticity of substitution. Another 'two-good one-price' elasticity of substitution that can be used to assess the substitutability/complementarity relationship between goods is the Mundlak

elasticity of substitution [see Mundlak (1968)],

$$\sigma_{ij}^U = \frac{\partial \log \left( x_i(\boldsymbol{p}, y) / x_j(\boldsymbol{p}, y) \right)}{\partial \log(p_j/p_i)}$$

$$= \eta_{ij}(\boldsymbol{p}, y) - \eta_{jj}(\boldsymbol{p}, y)$$

$$= \sigma_{ij}^m + s_j \left( \eta_{jy}(\boldsymbol{p}, y) - \eta_{iy}(\boldsymbol{p}, y) \right). \tag{18}$$

The Mundlak elasticity of substitution, like the Marshallian demand elasticity, is a measure of gross substitution (with income held constant). Goods will be Mundlak substitutes (complements) if an increase in the price of $j$ causes $x_i/x_j$ to decrease (increase).

While either the Allen, Morishima, or Mundlak elasticity of substitution can be used to stratify assets as substitutes or complements, they will yield different stratification sets. See, for example, Davis and Gauger (1996). Thus, the choice of the appropriate elasticity measure is very important. Comparing the Allen and Morishima elasticities of substitution, for example, we see that if two goods are Allen substitutes, $\sigma_{ij}^a > 0$, they must also be Morishima substitutes, $\sigma_{ij}^m > 0$. However, two goods may be Allen complements, $\sigma_{ij}^a < 0$, but Morishima substitutes if $\left| \sigma_{jj}^a \right| > \left| \sigma_{ij}^a \right|$, suggesting that the Allen elasticity of substitution always overstates the complementarity relationship. Moreover, the Allen elasticity of substitution matrix is symmetric, $\sigma_{ij}^a = \sigma_{ji}^a$, but the Morishima elasticity of substitution matrix is not; Blackorby and Russell (1989) show that the Morishima elasticity of substitution matrix is symmetric only when the aggregator function is a member of the constant elasticity of substitution family.

# 4    Demand System Specification

## 4.1    The Differential Approach and the Rotterdam Model

One model that has been frequently used to test the theory and to estimate income elasticities, own- and cross-price elasticities, as well as elasticities of substitution between goods is the Rotterdam model, introduced by Theil (1965) and Barten (1966). As shown in Barnett and Serletis (2008), if we take the total differential of the logarithmic form of the Marshallian demand function for good $i$, $x_i = x_i(\boldsymbol{p}, y)$, then

$$d \log x_i = \eta_{iy} d \log y + \sum_{j=1}^{n} \eta_{ij} d \log p_j,$$

where $\eta_{iy}$ is the income elasticity and $\eta_{ij}$ is the price elasticity of good $i$ with respect to the price of good $j$. Using the Slutsky decomposition in elasticity terms, $\eta_{ij} = \widetilde{\eta}_{ij} - \eta_{iy}s_j$, the above equation can be written as

$$s_i d \log x_i = b_i \left( d \log y - \sum_{j=1}^{n} s_j d \log p_j \right) + \sum_{j=1}^{n} c_{ij} d \log p_j, \tag{19}$$

where $b_i = s_i \eta_{iy} = p_i \partial x_i / \partial y$ is the marginal budget share of the $i$th good and $c_{ij} = s_i \widetilde{\eta}_{ij}$.

Replacing the differentials in (19) by finite approximations and treating the $b_i$'s and $c_{ij}$'s as constant parameters, we get the absolute price version of the Rotterdam model, which is linear in its parameters; another version is the relative price version, which is nonlinear in its parameters. See Barnett and Serletis (this book) for more details regarding the differential approach to demand analysis and the absolute and relative price versions of the Rotterdam model.

## 4.2 The Parametric Approach to Demand Analysis

The Rotterdam model that we just briefly discussed avoids the necessity of using a particular functional form for the utility function. In addition, it is entirely based on neoclassical consumer demand theory, as discussed by Barnett and Serletis (this book). The proof of the aggregated model's consistency with economic theory does not require the existence of a representative consumer, as shown by Barnett (1978).

However, after the publication of Diewert's (1971) important paper, most of the demand modeling literature has taken the approach of specifying the aggregator function with the utility function of the representative consumer. This approach to empirical demand analysis involves specifying a differentiable form for the indirect utility function, and deriving the resulting demand system. Using the demand system and relevant data, we then could estimate the parameters and compute the income elasticities, the own- and cross-price elasticities, as well as the elasticities of substitution of the aggregate representative consumer.

### 4.2.1 Globally Regular Functional Forms

For many years, the literature concentrated on the use of globally regular functional forms; that is, forms that satisfy the theoretical regularity conditions for rational neoclassical economic behavior globally at all positive prices and income. That approach primarily concentrated on specifications having pairwise elasticities of substitution that are constant, independent of the quantities consumed of the pairs of goods. However, that approach ran into a dead end, when Uzawa (1962) proved that it is not possible to produce a model that simultaneously can have constant elasticities of substitution and also can attain arbitrary elasticities of substitution.

For example, the use of a Cobb-Douglas functional form in equation (7) imposes an elasticity of substitution equal to unity between every pair of goods, and its use implies that each good always accounts for a constant share of the expenditure. If this proposition is at odds with the facts, as it is likely to be, the use of the Cobb-Douglas is inappropriate. Also, a constant elasticity of substitution (CES) functional form,

$$u(\boldsymbol{x}) = \sum_{j=1}^{n} (a_j x_j^r)^{1/r}, \qquad \text{where } 0 < a_j < 1, -\infty < r < 1, \tag{20}$$

relaxes the unitary elasticity of substitution restriction imposed by the Cobb-Douglas, but imposes the restriction that the elasticity of substitution between any pair of goods is always constant, $1/(1-r)$. Again this is contrary to fact in almost all cases, except for the 2-good case, in which there is only one pairwise elasticity of substitution.

The list of specific functional forms is boundless, but the defining property of the more popular of these entities is that they imply limitations on the behavior of the consumer that may be incorrect in practice. While the issue of their usefulness is ultimately an empirical question, we feel that the constant elasticity-of-substitution, globally-regular class of functions should be rejected, when the sample size is adequate to permit estimation of less restrictive models, partly in view of the restrictive nature of their implicit assumptions, and partly because of the existence of attractive alternatives. Among the alternatives are the Rotterdam model and the flexible functional forms, to which we now turn. We shall make a distinction between (i) 'locally flexible' functional forms, (ii) 'effectively globally regular' forms, (iii) 'normalized quadratic flexible' functional forms, and (iv) 'asymptotically globally flexible' forms.

### 4.2.2 Locally Flexible Functional Forms

A locally flexible functional form is a second-order approximation to an arbitrary function. In the demand systems literature there are two different definitions of second-order approximations, one by Diewert (1971) and another by Lau (1974). Barnett (1983a) has identified the relationship of each of those definitions to existing definitions in the mathematics of local approximation orders and has shown that a second-order Taylor series approximation is sufficient but not necessary for both Diewert's and Lau's definitions of second-order approximation.

Consider an $n$-argument, twice continuously differentiable aggregator function, $h(\boldsymbol{v})$. According to Diewert (1971), $h(\boldsymbol{v})$ is a flexible functional form if it contains enough parameters so that it can approximate an arbitrary twice continuously differentiable function $h^*$ to the second order at an arbitrary point $\boldsymbol{v}^*$ in the domain of definition of $h$ and $h^*$. Thus $h$ must

have enough free parameters to satisfy the following $1 + n + n^2$ equations

$$h(\boldsymbol{v}^*) = h^*(\boldsymbol{v}^*); \tag{21}$$

$$\nabla h(\boldsymbol{v}^*) = \nabla h^*(\boldsymbol{v}^*); \tag{22}$$

$$\nabla^2 h(\boldsymbol{v}^*) = \nabla^2 h^*(\boldsymbol{v}^*), \tag{23}$$

where $\nabla h(\boldsymbol{v}) = \partial h(\boldsymbol{v})/\partial \boldsymbol{v}$ and $\nabla^2 h(\boldsymbol{v}) = \partial^2 h(\boldsymbol{v})/\partial v_i v_j$ denotes the $n \times n$ symmetric matrix of second-order partial derivatives of $h(\boldsymbol{v})$ evaluated at $\boldsymbol{v}$. The symmetry property follows from the assumption that $h(\boldsymbol{v})$ is twice continuously differentiable.

Since both $h$ and $h^*$ are assumed to be twice continuously differentiable, we do not have to satisfy all $n^2$ equations in (23) independently, since the symmetry of second derivatives (sometimes known as Young's theorem) implies that $\partial^2 h(\boldsymbol{v}^*)/\partial v_i \partial v_j = \partial^2 h(\boldsymbol{v}^*)/\partial v_j \partial v_i$ and $\partial^2 h^*(\boldsymbol{v}^*)/\partial v_i \partial v_j = \partial^2 h^*(\boldsymbol{v}^*)/\partial v_j \partial v_i$ for all $i$ and $j$. Thus the matrices of second order partial derivatives $\nabla^2 h(\boldsymbol{v}^*)$ and $\nabla^2 h^*(\boldsymbol{v}^*)$ are both symmetric matrices. Hence, there are only $n(n + 1)/2$ independent equations to be satisfied in the restrictions (23), so that a general locally flexible functional form must have at least $1 + n + n(n + 1)/2$ free parameters.

To illustrate Diewert's flexibility concept, let us consider the basic translog indirect utility function, introduced by Christensen *et al.* (1975),

$$\log h(\boldsymbol{v}) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log v_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ji} \log v_i \log v_j, \tag{24}$$

where $v_j = p_j/y$, $\alpha_0$ is a scalar, $\boldsymbol{\alpha}' = [\alpha_1, \cdots, \alpha_n]$ is a vector of parameters, and $\boldsymbol{B} = [\beta_{ij}]$ is an $n \times n$ symmetric matrix of parameters, for a total of $1 + n + n(n + 1)/2$ parameters. To show that (24) is a flexible functional form, we need to show that $\alpha_0$, $\boldsymbol{\alpha}'$, and $\boldsymbol{B}$ in (24) satisfy conditions (21)-(23). With (24), conditions (21)-(23) can be written as (respectively)

$$\alpha_0 + \sum_{i=1}^{n} \alpha_i \log v_i^* + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \log v_i^* \log v_j^* = \log h^* \left(\boldsymbol{v}^*\right);$$

$$\alpha_i + \sum_{j=1}^{n} \beta_{ij} \log v_j^* = \nabla_{\log v_i^*} \log h^* \left(\boldsymbol{v}^*\right), \qquad i = 1, \cdots, n;$$

$$\beta_{ij} = \nabla^2_{\log v_i^* \log v_j^*} \log h^* \left(\boldsymbol{v}^*\right), \qquad 1 \leq i \leq j \leq n.$$

15

To show that we can satisfy these conditions, we can choose $\alpha_0$ and the elements of $\boldsymbol{\alpha}'$ and $\boldsymbol{B}$ as follows

$$\alpha_0 = \log h^* \left(\boldsymbol{v}^*\right) - \sum_{i=1}^{n} \alpha_i \log v_i^* - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \log v_i^* \log v_j^*;$$

$$\alpha_i = \nabla_{\log v_i^*} \log h^* \left(\boldsymbol{v}^*\right) - \sum_{j=1}^{n} \beta_{ij} \log v_j^*, \quad i = 1, \cdots, n;$$

$$\beta_{ij} = \nabla^2_{\log v_i^* \log v_j^*} \log h^* \left(\boldsymbol{v}^*\right), \quad 1 \leq i \leq j \leq n.$$

Another locally flexible functional form in the translog family of functional forms is the generalized translog (GTL), introduced by Pollak and Wales (1980). The GTL reciprocal indirect utility function is written as

$$\log h(\boldsymbol{v}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \log \left[ \frac{p_k}{(y - \sum_{k=1}^{n} p_k \gamma_k)} \right]$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} \beta_{kj} \log \left[ \frac{p_k}{(y - \sum_{k=1}^{n} p_k \gamma_k)} \right] \log \left[ \frac{p_j}{(y - \sum_{k=1}^{n} p_k \gamma_k)} \right], \qquad (25)$$

where $\alpha_0$ is a scalar, $\boldsymbol{\alpha}' = [\alpha_1, \cdots, \alpha_n]$ is a vector of parameters, $\boldsymbol{\gamma}' = [\gamma_1, \cdots, \gamma_n]$ is a vector of 'committed' quantities, and $\boldsymbol{B} = [\beta_{ij}]$ is an $n \times n$ symmetric matrix of parameters, for a total of $\left(n^2 + 3n + 2\right)/2$ parameters. It is assumed that the consumer first purchases the minimum required quantities and thereby expends $\boldsymbol{p}'\boldsymbol{\gamma}$. The consumer is then left with the supernumerary expenditure, $y - \boldsymbol{p}'\boldsymbol{\gamma}$, to allocate in a discretionary manner. The share equations, derived using the logarithmic form of Roy's identity (11) are (for $i = 1, \cdots, n$)

$$s_i = \frac{p_i \gamma_i}{y} + \left[ 1 - \frac{\left(\sum_{k=1}^{n} p_k \gamma_k\right)}{y} \right] \times \frac{\alpha_i + \sum_{j=1}^{n} \beta_{ij} \log \left[ p_j / \left(y - \sum_{k=1}^{n} p_k \gamma_k\right) \right]}{\sum_{j=1}^{n} \alpha_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \log \left[ p_j / \left(y - \sum_{k=1}^{n} p_k \gamma_k\right) \right]}. \qquad (26)$$

With $n$ goods the GTL model's share equations contain $n(n+5)/2$ parameters.

It is to be noted that the basic translog (BTL) in (24) is a special case of the GTL. It can be derived by imposing restrictions on the GTL form, namely,

$$\gamma_i = 0, \qquad \text{for all } i. \qquad (27)$$

16

Applying restriction (27) to the GTL (25), yields the BTL reciprocal indirect utility function (24). The share equations of the BTL, derived using again the logarithmic form of Roy's identity, are (for $i = 1, \cdots, n$)

$$s_i = \frac{\alpha_i + \sum_{j=1}^{n} \beta_{ij} \log v_j}{\sum_{j=1}^{n} \alpha_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \log v_j}. \tag{28}$$

With $n$ assets the BTL model's share equations contain $n(n+3)/2$ parameters.

Also, the linear translog (LTL), a homothetic special case of the GTL (or, equivalently, a quasi-homothetic special case of the BTL) can be derived by imposing the restriction

$$\sum_{i=1}^{n} \beta_{ij} = 0, \quad \text{for all } j \tag{29}$$

on the GTL. The LTL model's share equations are (for $i = 1, \cdots, n$)

$$s_i = \frac{p_i \gamma_i}{y} + \left[1 - \frac{(\sum_{k=1}^{n} p_k \gamma_k)}{y}\right] \times \frac{\alpha_i + \sum_{j=1}^{n} \beta_{ij} \log \left[p_j / \left(y - \sum_{k=1}^{n} p_k \gamma_k\right)\right]}{\sum_{j=1}^{n} \alpha_j}. \tag{30}$$

This model has linear Engel curves (income-consumption paths for fixed prices), but does not require them to pass through the origin. With $n$ assets the LTL model's share equations contain $n(n+5)/2$ parameters.

Finally, the homothetic translog flexible form can be derived either by imposing restrictions (27) and (29) on (25) or by imposing restriction (29) on (24). The HTL model's share equations are (for $i = 1, \cdots, n$)

$$s_i = \frac{\alpha_i + \sum_{j=1}^{n} \beta_{ij} \log v_j}{\sum_{j=1}^{n} \alpha_j}. \tag{31}$$

With $n$ assets the HTL model's share equations contain $n(n+3)/2$ parameters. The homothetic translog is a generalization of the Cobb-Douglas and reduces to it when all of the $\beta_{ij}$ are zero.

Notice that estimation of each of (26), (28), (30), and (31) requires a parameter normalization, as the share equations are homogeneous of degree zero in the $\alpha$'s. Usually the normalization $\sum_{i=1}^{n} \alpha_i = 1$ is used.

Another locally flexible functional form is the generalized Leontief (GL), introduced by Diewert (1973) in the context of cost and profit functions. Diewert (1974) also introduced the GL reciprocal indirect utility function,

$$h(\boldsymbol{v}) = \alpha_0 + \sum_{i=1}^{n} \alpha_i v_i^{1/2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} v_i^{1/2} v_j^{1/2}, \tag{32}$$

where $\boldsymbol{B} = [\beta_{ij}]$ is an $n \times n$ symmetric matrix of parameters and $a_0$ and $a_i$ are other parameters, for a total of $(n^2 + 3n + 2)/2$ parameters.

Applying Diewert's (1974) modified version of Roy's identity, (12), to (32) the following share equations result (for $i = 1, \cdots, n$)

$$s_i = \frac{\alpha_i v_i^{1/2} + \sum_{j=1}^{n} \beta_{ij} v_i^{1/2} v_j^{1/2}}{\sum_{j=1}^{n} \alpha_j v_j^{1/2} + \sum_{k=1}^{n} \sum_{m=1}^{n} \beta_{km} v_k^{1/2} v_m^{1/2}}. \tag{33}$$

Since the share equations are homogeneous of degree zero in the parameters, the model requires a parameter normalization. Barnett and Lee (1985) use the following normalization

$$2 \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} = 1.$$

Deaton and Muellbauer (1980) also introduced another locally flexible demand system, the Almost Ideal Demand System (AIDS). It is given by (for $i = 1, \cdots, n$)

$$s_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j + \beta_i \log\left(\frac{y}{P}\right), \tag{34}$$

where the price deflator of the logarithm of income is

$$\log P = \alpha_0 + \sum_{k=1}^{n} \alpha_k \log p_k + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{kj} \log p_k \log p_j.$$

For more details regarding the AIDS, see Deaton and Muellbauer (1980), Barnett and Serletis (2008), and Barnett and Seck (2008).

### 4.2.3 Effectively Globally Regular Flexible Functional Forms

Locally flexible demand models such as the generalized Leontief, translog, and AIDS have revolutionized microeconometrics, by providing access to all neoclassical microeconomic theory in econometric applications. However, as argued by Caves and Christensen (1980), Guilkey and Lovell (1980), Barnett and Lee (1985), and Barnett *et al.* (1985, 1987), among others, most popular locally flexible functional forms have very small regions of theoretical regularity, thereby giving up global integrability.

The problem of choosing between globally regular models (such as the Cobb-Douglas and the CES), which are not locally flexible, and locally flexible models (such as the generalized Leontief, translog, and AIDS), which are not theoretically globally regular, led to the development of locally-flexible functional forms, which have larger regularity regions.

Cooper and McLaren (1996) classify such models as 'effectively globally regular' flexible functional forms. These functions typically have regular regions that include almost all data points in the sample. In addition, the regularity region increases as real expenditure levels grow, as is often the case with time series data. Furthermore, these functions provide more general Engel curve approximations, especially when income varies considerably.

Examples of these functions include Barnett's (1983a, 1985) minflex Laurent (ML) models [see also Barnett (1983b), Barnett and Lee (1985), and Barnett *et al.* (1985, 1987)], based on the Laurent series expansion, the quadratic AIDS (QUAIDS) model of Banks *et al.* (1997), and the general exponential form (GEF) of Cooper and McLaren (1996). Barnett and Serletis (2008) provide a brief discussion of these models.

### 4.2.4 Normalized Quadratic Flexible Functional Forms

The effectively globally regular flexible functional forms (minflex Laurent, quadratic AIDS, and the general exponential form) appear to violate the theoretically appropriate curvature conditions less often than the generalized Leontief, translogs, and the AIDS. However, effectively globally regular flexible functional forms also exhibit regions whithin which the curvature conditions are violated. Even if none of the data lie in those irregular regions, forecasting and simulation could enter those regions. This problem led Diewert and Wales (1988) to propose two locally flexible systems of functional forms for consumer demand functions for which the theoretical curvature conditions can be imposed globally. The first system is derived from a normalized quadratic (NQ) reciprocal indirect utility function and the second is derived from a NQ expenditure function. See Diewert and Wales (1988) or Barnett and Serletis (2008) for more details. But those models lose their flexibility, if monotonicity also is imposed. Regularity requires both curvature and monotonicity conditions.

## 4.3  Asymptotically Globally Flexible Functional Forms

The functional forms discussed so far are capable of approximating an arbitrary function locally (at a single point). A path-breaking innovation in this area has been provided by Gallant (1981) in his introduction of the semi-nonparametric inference approach, which uses series expansions in infinite dimensional parameter spaces. The idea behind the semi-nonparametric approach, is to expand the order of the series expansion, as the sample size increases, until the semi-nonparametric function converges asymptotically to the true data generating process.

Semi-nonparametric functional forms are globally flexible in the sense that the model asymptotically can reach any continuous function. Inferences with this approach do not maintain a specification containing a finite number of parameters, so that asymptotic inferences are free from any specification error. Two globally flexible functional forms in general use are the Fourier flexible functional form, introduced by Gallant (1981), and the Asymptotically Ideal Model (AIM), introduced by Barnett and Jonas (1983) and employed and explained in Barnett and Yue (1988).

These functional forms are discussed in Serletis and Shahmoradi (2005) and also in Barnett and Serletis (2008). Unlike other approaches, this state-of-the-art approach equates economic theory with econometrics by permitting the model asymptotically to span the relevant theoretical space. Recently, Serletis and Shahmoradi (2008) estimated the AIM($k$) demand systems for $k = 1, 2, 3$, where $k$ is the order of the Müntz-Szatz series expansion, on which the AIM model is based. They found that the AIM(3) model, estimated subject to global curvature, currently provides the best specification for research in semi-nonparametric modeling of consumer demand systems.

# 5  Engel Curves and the Rank of Demand Systems

Applied demand analysis uses two types of data: time series data and cross sectional data. Time series data offer substantial variation in relative prices and less variation in income, whereas cross sectional data offer limited variation in relative prices and substantial variation in income levels. In time series data, prices and income vary simultaneously, whereas in household budget data prices are almost constant. Household budget data give rise to the Engel curves (income expansion paths), which are functions describing how a consumer's purchases of some good vary as the consumer's income varies. That is, Engel curves are Marshallian demand functions, with the prices of all goods held constant. Like Marshallian demand functions, Engel curves may also depend on demographic or other nonincome consumer characteristics (such as, for example, age and household composition), which we have chosen to ignore in this chapter.

Engel curves can be used to calculate the income elasticity of a good and hence whether

a good is an inferior, normal, or luxury good, depending on whether income elasticity is less than zero, between zero and one, or greater than one, respectively. They are also used for equivalence scale calculations (welfare comparisons across households) and for determining properties of demand systems, such as aggregability and rank. For many commodities standard empirical demand systems do not provide an accurate picture of observed behavior across income groups. Hence, in the next section we discuss functional forms in terms of their ability to capture the Engel curve structure of the data.

## 5.1  Exact Aggregation

We begin our discussion of the rank of demand systems with the definition of exactly aggregable demand systems. A demand system is 'exactly aggregable' if demands can be summed across consumers to yield closed form expressions for aggregate demand. Exactly aggregable demand systems are demand systems that are linear in functions of $y$, as follows,

$$s_i(\boldsymbol{p}, y) = \sum_{r=1}^{R} c_{ir}(\boldsymbol{p})\varphi_r(y), \tag{35}$$

where the $c_{ir}(\boldsymbol{p})$'s are the coefficients on $\varphi_r(y)$, which is a scalar valued function independent of $\boldsymbol{p}$, and $R$ is a positive integer. Gorman (1981), extending earlier results by Muellbauer (1975, 1976), proved in the context of exactly aggregable demand systems that integrability (i.e., consistency with utility maximization) forces the matrix of Engel curve coefficients to have rank three or less. The rank of a matrix is defined as the maximum number of linearly independent columns. Other related exact aggregation theorems can be found in Banks *et al.* (1997).

## 5.2  The Rank of Demand Systems

Lewbel (1991) extended Gorman's rank idea to all demand systems (not just exactly aggregable demand systems), by defining the rank of a demand system to be the dimension of the space spanned by its Engel curves, holding demographic or other nonincome consumer characteristics fixed. He showed that demands that are not exactly aggregable can have rank higher than three and still be consistent with utility maximization.

Formally, the rank of any given demand system $\boldsymbol{x}(\boldsymbol{p}, y)$ is the smallest value of $R$ such that each $s_i$ can be written as

$$s_i(\boldsymbol{p}, y) = \sum_{r=1}^{R} \phi_{ir}(\boldsymbol{p})f_r(\boldsymbol{p}, y), \tag{36}$$

for some $R \leq n$, where for each $r = 1, \cdots, R$, $\phi_{ir}$ is a function of prices and $f_r$ is a scalar valued function of prices and income. That is, the rank of the system is the number of

linearly independent vectors of price functions. All demand systems have rank $R \leq n$, where $n$ is the number of goods. Clearly, demands that are not exactly aggregable can have rank greater than three (i.e., $R > 3$). Equation (36) is a generalization of the concept of rank. That generalization, defined by Gorman (1981), only applies to exactly aggregable demands. Notice that $f_r$ in equation (36) depends on $\boldsymbol{p}$ and $y$, whereas $\varphi_r$ in equation (35) is not a function of $\boldsymbol{p}$.

Hence, any demand system has rank $R$, if there exist $R$ goods such that the Engel curve of any good equals a weighted average of the Engel curves of those $R$ goods. The rank of an integrable demand system determines the number of price functions on which the indirect utility function and the cost or expenditure function depend on. See, for example, Lewbel (1991).

### 5.2.1 Demand Systems Proportional to Expenditure

Homothetic demand systems, with Engel curves being rays from the origin, have rank one. Rank one demand systems, such as the Cobb-Douglas, CES, and homothetic translog, exhibit expenditure proportionality (so that the budget share of every good is independent of total expenditure). This contradicts Engel's law, according to which the budget share of food is smaller for rich than for poor households.

Rank one demand systems can be written as

$$x_i(\boldsymbol{p}, y) = b_i(\boldsymbol{p})y,$$

and are homothetic. For example, the demand system of the Cobb-Douglas utility function (7) is given by (8), that of the CES utility function (20) is

$$x_i(\boldsymbol{p}, y) = \frac{p_i^{1/(r-1)}}{\sum_{j=1}^n p_j^{r/(r-1)}} y,$$

and that of the homothetic translog, equation (24) with restriction (29) imposed, is given by (31), in budget share form.

Clearly, expenditure proportionality implies marginal budget shares that are constant and in fact equal to the average budget shares. Because of this, the assumption of expenditure proportionality has little relevance in empirical demand analysis.

### 5.2.2 Demand Systems Linear in Expenditure

A demand system that is linear in expenditure is of the form

$$x_i(\boldsymbol{p}, y) = c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})y.$$

If $c_i(\boldsymbol{p}) = 0$ (for all $i$) then demands are homothetic. Gorman (1961) showed that any demand system that is consistent with utility maximization and linear in expenditure must be of the form

$$x_i(\boldsymbol{p}, y) = f_i(\boldsymbol{p}) - \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} f(\boldsymbol{p}) + \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} y$$

$$= f_i(\boldsymbol{p}) + \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} [y - f(\boldsymbol{p})], \tag{37}$$

where $g(\boldsymbol{p})$ and $f(\boldsymbol{p})$ are functions homogeneous of degree one, and $g_i(\boldsymbol{p})$ and $f_i(\boldsymbol{p})$ denote the partial derivative of $g(\boldsymbol{p})$ and $f(\boldsymbol{p})$ with respect to the $i$th price. Such demand systems are generated by an indirect utility function of the 'Gorman polar form,'

$$h(\boldsymbol{p}, y) = \frac{y - f(\boldsymbol{p})}{g(\boldsymbol{p})}. \tag{38}$$

To see this, apply Roy's identity (10) to (38) to get (37).

An example of a demand system linear in expenditure is the 'linear expenditure system,'

$$x_i(\boldsymbol{p}, y) = b_i - \frac{a_i}{p_i} \sum_{k=1}^{n} p_k b_k + \frac{a_i}{p_i} y,$$

generated by the (Stone-Geary) utility function

$$u(\boldsymbol{x}) = \sum_{i=1}^{n} a_i \log (x_i - b_i), \qquad a_i > 0, \ (x_i - b_i) > 0, \ \sum_{i=1}^{n} a_i = 1,$$

which is homothetic relative to the point $\boldsymbol{b} = (b_1, \cdots, b_n)$ as origin, or, equivalently, by an indirect utility function of the Gorman polar form, (38), with $f(\boldsymbol{p}) = \sum_{k=1}^{n} p_k b_k$ and $g(\boldsymbol{p}) = \prod p_k^{a_k}$, with $\sum a_k = 1$, so that $f_i(\boldsymbol{p}) = b_i$ and $g_i(\boldsymbol{p})/g(\boldsymbol{p}) = a_i/p_i$.

Demand systems linear in expenditure are rank two and have linear Engel curves, but not necessarily through the origin. Linearity in expenditure implies marginal budget shares that are independent of the level of expenditure, suggesting that poor and rich households spend the same fraction of an extra dollar on each good. This hypothesis, as well as the hypothesis of expenditure proportionality, are too restrictive for the analysis of household budget data.

### 5.2.3 Demand Systems Linear in the Logarithm of Expenditure

Muellbauer (1975) has studied 'two-term' demand systems of the general form

$$x_i(\boldsymbol{p}, y) = c_i(\boldsymbol{p}) y + b_i(\boldsymbol{p}) f(y), \tag{39}$$

for any function $f(y)$. Homothetic demand are obtained, if $f(y) = 0$. He shows that if $f(y) \neq 0$, then $f(y)$ must be either equal to $y^k$ with $k \neq 1$ [the 'price independent generalized linearity' (PIGL) class] or equal to $y \log y$ [the 'price independent generalized logarithmic' (PIGLOG) class].

Hence, the PIGLOG class of demand systems is linear in the logarithm of total expenditure and has the form

$$x_i(\boldsymbol{p}, y) = c_i(\boldsymbol{p})y + b_i(\boldsymbol{p})y \log y,$$

with expenditure entering linearly and as a logarithmic function of $y$. Muellbauer (1975) has shown that theoretically plausible demand systems of the PIGLOG form must be written as

$$x_i(\boldsymbol{p}, y) = \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})}y - \frac{G_i(\boldsymbol{p})}{G(\boldsymbol{p})}\left[\log y - \log g(\boldsymbol{p})\right]y, \tag{40}$$

where $G(\boldsymbol{p})$ is homogeneous of degree zero, $G(\boldsymbol{p}) = G(\lambda \boldsymbol{p})$, and $g(\lambda \boldsymbol{p})$ is homogeneous of degree one, $g(\lambda \boldsymbol{p}) = \lambda g(\boldsymbol{p})$. The indirect utility function associated with (40) is

$$h(\boldsymbol{p}, y) = G(\boldsymbol{p})\left[\log y - \log g(\boldsymbol{p})\right]. \tag{41}$$

To see this, apply Roy's identity (10) to (41) to get (40).

Examples of PIGLOG demand systems are the log translog (log TL), a special case of the basic translog (28) with $\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij} = 0$ imposed, so that $y$ drops out of the denominator of (28), and the AIDS (34). For example, the AIDS demand system is a special case of (41) with

$$G(\boldsymbol{p}) = \prod_{k=1}^{n}p_k^{-\beta_k}, \qquad \sum_{k=1}^{n}\beta_k = 0,$$

and

$$\log g(\boldsymbol{p}) = \alpha_0 + \sum_{k=1}^{n}\alpha_k \log p_k + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\gamma_{kj}\log p_k \log p_j,$$

with $\gamma_{ij} = \gamma_{ji}$, $\sum_{k=1}^{n}\gamma_{kj} = 0$, and $\sum_{k=1}^{n}\alpha_k = 1$.

It is to be noted, however, that most of the commonly used PIGLOG specifications are rank two, and thus have limited flexibility in modelling the curvature of Engel curves.

### 5.2.4 Demand Systems Quadratic in Expenditure

Lewbel (1987a) has studied 'three-term' demand systems of the following form

$$x_i(\boldsymbol{p}, y) = c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})y + a_i(\boldsymbol{p})f(y). \tag{42}$$

Equation (42) is a special case of Gorman's (1981) equation (35), with $r$ ranging from 1 to 3 and $\varphi_1(y) = 1$, $\varphi_2(y) = y$, and $\varphi_3(y) = f(y)$. Gorman's (1981) main result, that the

24

matrix of Engel curve coefficients cannot have rank higher than three, is true in this case, since that matrix, $[\boldsymbol{c}(\boldsymbol{p})\ \boldsymbol{b}(\boldsymbol{p})\ \boldsymbol{a}(\boldsymbol{p})]$, only has three columns. Lewbel (1987a) showed that in equation (42), $f(y)$ must be either $0$, $y^k$, $y \log y$, or $\log y$, and that the only $f(y)$ that yields full rank-three demand systems is $y^2$. Hence, one way to relax the assumption that demand systems are linear in expenditure is to specify demand systems that are quadratic in expenditure, as follows

$$x_i(\boldsymbol{p}, y) = c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})y + a_i(\boldsymbol{p})y^2.$$

Ryan and Wales (1999), following Howe *et al.* (1979) and van Daal and Merkies (1989), argue that for a quadratic demand system to be theoretically plausible, the demand functions must be of the form

$$x_i(\boldsymbol{p}, y) = \frac{1}{g(\boldsymbol{p})^2} \left( r_i(\boldsymbol{p}) - \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})r(\boldsymbol{p})} \right) [y - f(\boldsymbol{p})]^2$$

$$+ \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} [y - f(\boldsymbol{p})] + f_i(\boldsymbol{p}) + \chi \left( \frac{r(\boldsymbol{p})}{g(\boldsymbol{p})} \right) \left( r_i(\boldsymbol{p}) - \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} r(\boldsymbol{p}) \right), \qquad (43)$$

where there are no restrictions on the function $\chi(\cdot)$ and the functions $f(\boldsymbol{p})$, $g(\boldsymbol{p})$, and $r(\boldsymbol{p})$ are restricted to be homogeneous of degree one in prices. In equation (43), $f_i(\boldsymbol{p})$, $g_i(\boldsymbol{p})$, and $r_i(\boldsymbol{p})$ are the first partial derivatives of $f(\boldsymbol{p})$, $g(\boldsymbol{p})$, and $r(\boldsymbol{p})$ with respect to $p_i$. The demand function (43) can be simplified by assuming $\chi(.) = 0$ and defining $r(\boldsymbol{p})$ to be the product of $g(\boldsymbol{p})$ and a function $h(\boldsymbol{p})$, that is homogeneous of degree zero in prices, so that the coefficient of the quadratic term in (43) becomes $h_i(\boldsymbol{p})/g(\boldsymbol{p})$. In that case (43) reduces to

$$x_i(\boldsymbol{p}, y) = \frac{h_i(\boldsymbol{p})}{g(\boldsymbol{p})} [y - f(\boldsymbol{p})]^2 + \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} [y - f(\boldsymbol{p})] + f_i(\boldsymbol{p}), \qquad (44)$$

whose corresponding indirect utility function is

$$h(\boldsymbol{p}, y) = -\frac{g(\boldsymbol{p})}{y - f(\boldsymbol{p})} - h(\boldsymbol{p}). \qquad (45)$$

To see this, apply Roy's identity (10) to (45) to get (44).

Equation (45) is the general form of the indirect utility function that can generate quadratic Engel curves (that is, rank-three demand systems). The difference between the Gorman polar form indirect utility function (38) and the more general indirect utility function (45) is that the latter adds a term, $h(\boldsymbol{p})$, that is homogeneous of degree zero in prices, to the Gorman polar form indirect utility function (38).

The first functional form proposed along these lines is the quadratic AIDS (known as QUAIDS), which we mentioned in Section 4. As already noted, the QUAIDS is an extension

of the simple AIDS, having expenditure shares linear in log income and in another smooth function of income. See Banks *et al.* (1997) for more details.

Following Banks *et al.* (1997), Ryan and Wales (1999) modified the translog (24), GL (32), and NQ demand systems and introduced three new rank-three demand systems, having expenditure shares quadratic in expenditure. The three new demand systems are called the 'translog-quadratic expenditure system,' 'GL-quadratic expenditure system,' and 'NQ-quadratic expenditure system.'

To demonstrate, we consider the NQ expenditure function, introduced by Diewert and Wales (1988),

$$C(\boldsymbol{p}, u) = \boldsymbol{a}'\boldsymbol{p} + \left( \boldsymbol{b}'\boldsymbol{p} + \frac{1}{2} \frac{\boldsymbol{p}'\boldsymbol{B}\boldsymbol{p}}{\boldsymbol{\alpha}'\boldsymbol{p}} \right) u, \tag{46}$$

where the parameters of the model consist of $\boldsymbol{a}' = [a_1, \cdots, a_n]$, $\boldsymbol{b}' = [b_1, \cdots, b_n]$, and the elements of the $n \times n$ symmetric $\boldsymbol{B} = [\beta_{ij}]$ matrix. The nonnegative vector of predetermined parameters $\boldsymbol{\alpha}' = (\alpha_1, \cdots, \alpha_n)$ is assumed to satisfy

$$\boldsymbol{\alpha}'\boldsymbol{p}^* = 1, \qquad \alpha_j \geq 0 \text{ for } j = 1, \cdots, n, \tag{47}$$

where $p_j^*$ is the $j$th element of the reference vector. Moreover, the following restrictions are also imposed

$$\sum_{j=1}^{n} a_j p_j^* = 0; \tag{48}$$

$$\sum_{j=1}^{n} \beta_{ij} p_j^* = 0, \qquad i = 1, \cdots, n. \tag{49}$$

Hence, there are $n(n+5)/2$ parameters in (46), but the imposition of the above restrictions reduces the number of parameters to $(n^2 + 3n - 2)/2$. The NQ expenditure function defined by (46)-(49) is a Gorman polar form, and the preferences that are dual to it are quasi-homothetic.

Applying Shephard's lemma (15) to (46) yields the share equations of the NQ expenditure system (for $i = 1, \cdots, n$)

$$s_i = a_i v_i + \frac{(1 - \boldsymbol{\alpha}'\boldsymbol{v}) \left( b_i + (\boldsymbol{\alpha}'\boldsymbol{v})^{-1} \boldsymbol{B}\boldsymbol{v} - \frac{1}{2} (\boldsymbol{\alpha}'\boldsymbol{v})^{-2} \boldsymbol{v}'\boldsymbol{B}\boldsymbol{v}\boldsymbol{\alpha} \right) v_i}{\boldsymbol{b}'\boldsymbol{v} + \frac{1}{2} (\boldsymbol{\alpha}'\boldsymbol{v})^{-1} \boldsymbol{v}'\boldsymbol{B}\boldsymbol{v}}. \tag{50}$$

Since the share equations in (50) are homogeneous of degree zero in the parameters, Diewert and Wales (1988) impose the normalization $\sum_{j=1}^{n} b_j = 1$. Also, regarding the curvature properties of the NQ expenditure function, it is locally flexible in the class of expenditure

functions satisfying local money-metric scaling, and it retains this flexibility when concavity needs to be imposed. See Diewert and Wales (1988) for more details.

In developing the NQ-QES, Ryan and Wales (1999) choose the $f(\boldsymbol{p})$, $g(\boldsymbol{p})$, and $h(\boldsymbol{p})$ functions in (45) as follows

$$f(\boldsymbol{p}) = \sum_{k=1}^{n} p_k d_k; \tag{51}$$

$$g(\boldsymbol{p}) = \sum_{k=1}^{n} p_k b_k + \frac{1}{2} \left( \frac{\sum_{k=1}^{n} \sum_{j=1}^{n} \boldsymbol{B}_{kj} p_k p_j}{\sum_{k=1}^{n} \alpha_k p_k} \right); \tag{52}$$

$$h(\boldsymbol{p}) = \sum_{k=1}^{n} \alpha_k \log p_k, \qquad \sum_{k=1}^{n} \alpha_k = 0. \tag{53}$$

Substituting (51)-(53) in (45) and applying Roy's identity (10) to (45) yields the demand system (for $i = 1, \cdots, n$)

$$x_i(\boldsymbol{p}, y) = \frac{\alpha_i}{p_i g(\boldsymbol{p})} \left( y - \sum_{k=1}^{n} p_k d_k \right)^2$$

$$+ \left[ \frac{\left[ b_i + \sum_{k=1}^{n} \boldsymbol{B}_{ik} p_k \middle/ \left( \sum_{k=1}^{n} \alpha_k p_k \right) - \frac{1}{2} \alpha_i \sum_{k=1}^{n} \sum_{j=1}^{n} \boldsymbol{B}_{kj} p_k p_j \middle/ \left( \sum_{k=1}^{n} \alpha_k p_k \right)^2 \right]}{g(\boldsymbol{p})} \right]$$

$$\times \left( y - \sum_{k=1}^{n} p_k d_k \right) + d_i,$$

where $a_k$, $b_k$, $d_k$, and $\boldsymbol{B}_{kj}$ are unknown parameters, and the $a_k > 0$ are predetermined parameters, $k, j = 1, \cdots, n$. The $\boldsymbol{B} \equiv [\beta_{ij}]$ matrix also satisfies the following two restrictions, as in the common NQ model,

$$\beta_{ij} = \beta_{ji}, \qquad \text{for all } i, j;$$

$$\boldsymbol{B} \boldsymbol{p}^* = \boldsymbol{0}, \quad \text{for some } \boldsymbol{p}^* > \boldsymbol{0}.$$

The development of the GL-QES and TL-QES follows a similar pattern. See Ryan and Wales (1999) for more details.

The QUAIDS, translog-quadratic expenditure system, GL-quadratic expenditure system, and NQ-quadratic expenditure system are locally flexible in the Diewert sense and also are rank-three demand systems, thereby allowing more flexibility in modelling income distribution than the AIDS, translog, GL, and NQ models.

### 5.2.5   Fractional Demand Systems

Lewbel (1987b) has studied demand systems of the 'fractional' form

$$x_i(\boldsymbol{p}, y) = \frac{c_i(\boldsymbol{p})f(y) + b_i(\boldsymbol{p})g(y)}{c(\boldsymbol{p})F(y) + b(\boldsymbol{p})G(y)}, \tag{54}$$

where $f(y)$, $g(y)$, $F(y)$, and $G(y)$ are differentiable functions of income and $c_i(\boldsymbol{p})$, $b_i(\boldsymbol{p})$, $c(\boldsymbol{p})$, and $b(\boldsymbol{p})$ are differentiable functions of prices only. He shows that the budget shares of fractional demand systems can always be written as

$$s_i(\boldsymbol{p}, y) = \frac{c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})f(y)}{1 + b(\boldsymbol{p})f(y)}, \tag{55}$$

where $f(y)$ must be either 0, $\log y$, $y^k$, or $\tan(k \log y)$ for $k \neq 0$. As can be seen, fractional demands are proportional to two-term demands. Moreover, if $f(y) = 0$ in equation (55), homothetic demands obtain and if $b(\boldsymbol{p}) = 0$, Gorman polar form demands obtain, either PIGL demands or PIGLOG demands, corresponding to $f(y) = y^k$ or $f(y) = \log y$, respectively. For $f(y) = y^2$ equation (55) reduces to what Lewbel (1987b) refers to as 'EXP' demands; the minflex Laurent demand system that we mentioned in Section 4 is a member of the EXP class of demand systems.

As Lewbel (1987b) puts it, fractional demand systems provide a parsimonious way of increasing the range of income response patterns. In fact, an advantage of fractional demands (54) over three-term demands (42) is that they require the estimation of only one more function of prices, $b(\boldsymbol{p})$, than two-term demands (39), whereas three-term demands require the estimation of one more function of income, $f(y)$, and $n - 1$ functions of prices, $a_i(\boldsymbol{p})$, than two-term demands.

For analyses involving substantial variation in income levels across individuals, increased flexibility in global Engel curve shapes is required and fractional demand systems in the form of equation (55) are likely to be superior to two-term demand systems (such as homothetic, PIGL, and PIGLOG systems) and three-term demand systems (such as the quadratic AIDS, GL-QES, TL-QES, and NQ-QES). Moreover, as already noted, fractional demand systems, like the minflex Laurent, have larger regularity regions than two- and three-term demand systems.

# 6 Estimation Issues

In order to estimate share equation systems, such as (26), (28), (30), (31), (33), (34), and (50), a stochastic version must be specified. Demand systems are usually estimated in budget share closed form, in order to minimize heteroskedasticity problems, with only exogenous variables appearing on the right-hand side. It often is assumed that the observed share in the $i$th equation deviates from the true share by an additive disturbance term $u_i$. Furthermore, it is usually assumed that $\boldsymbol{u} \sim N\left(\boldsymbol{0}, \boldsymbol{\Omega} \otimes \mathbf{I}_T\right)$, where $\boldsymbol{0}$ is a null vector, $\boldsymbol{\Omega}$ is the $n \times n$ symmetric positive definite error covariance matrix, $\mathbf{I}$ is the identity matrix, and $\otimes$ is the Kronecker product.

With the addition of additive errors, the share equation system can be written in matrix form as

$$\boldsymbol{s}_t = \boldsymbol{g}(\boldsymbol{v}_t, \vartheta) + \boldsymbol{u}_t, \tag{56}$$

where $\boldsymbol{s} = (s_1, \cdots, s_n)'$, $\boldsymbol{g}(\boldsymbol{v}, \vartheta) = \left(\boldsymbol{g}_1\left(\boldsymbol{v}, \vartheta\right), \cdots, \boldsymbol{g}_n\left(\boldsymbol{v}, \vartheta\right)\right)'$, $\vartheta$ is the parameter vector to be estimated, and $\boldsymbol{g}_i\left(\boldsymbol{v}, \vartheta\right)$ is given by the right-hand side of systems, such as (26), (28), (30), (31), (33), (34), and (50).

The assumption made about $\boldsymbol{u}_t$ in (56) permits correlation among the disturbances at time $t$, but rules out the possibility of autocorrelated disturbances. This assumption and the fact that the shares satisfy an adding up condition imply that the errors across all equations are linearly related and that the error covariance matrix is singular. Barten (1969) has shown that this problem can be handled by arbitrarily deleting any equation from the system. When the errors are homoskedastic and non-autocorrelated, the resulting estimates are invariant to the equation deleted, and the parameter estimates of the deleted equation can be recovered from the restrictions imposed.

Equation (56) can be estimated using different methods, including maximum likelihood and Bayesian methodology, as recently discussed by Barnett and Serletis (2008). In the case, for example, of maximum likelihood estimation, if the disturbances in (56) are multivariate normally distributed, then maximum likelihood estimation of (56) is equivalent to maximizing the log likelihood function for a sample of $T$ observations,

$$\log L\left(\boldsymbol{s} \,|\boldsymbol{\theta}\right) = -\frac{MT}{2} \ln\left(2\pi\right) - \frac{T}{2} \ln|\boldsymbol{\Omega}| - \frac{1}{2} \sum_{i=1}^{T} \boldsymbol{u}_t' \boldsymbol{\Omega}^{-1} \boldsymbol{u}_t,$$

where $\boldsymbol{\theta} = (\vartheta, \boldsymbol{\Omega})$. In the relevant class of ("seemingly unrelated regression") models, maximization of $\log L\left(\boldsymbol{s} \,|\boldsymbol{\theta}\right)$ is equivalent to minimization of $|\boldsymbol{\Omega}|$, as shown by Barnett (1976), who provided the relevant asymptotics for the maximum likelihood estimator within the relevant class of nonlinear systems under the customary assumptions. See Barnett and Serletis (2008) for more details regarding estimation issues and a number of under-researched complications in this area.

## 6.1 Theoretical Regularity

The usefulness of flexible functional forms depends on whether they satisfy the theoretical regularity conditions of positivity, monotonicity, and curvature, and in the empirical demand systems literature there often has been a tendency to ignore theoretical regularity or not to report the results of full regularity checks. In fact, as Barnett (2002, p. 199) observed in his *Journal of Econometrics* Fellow's opinion article: "without satisfaction of both curvature and monotonicity, the second-order conditions for optimizing behavior fail, and duality theory fails. The resulting first-order conditions, demand functions, and supply functions become invalid."

Once a demand system is estimated, the regularity conditions can be checked as follows:

- Positivity is checked by direct computation of the estimated indirect utility function $\widehat{h}(\boldsymbol{v})$. It is satisfied if $\widehat{h}(\boldsymbol{v}) > 0$, for all $t$.

- Monotonicity is checked by direct computation of the values of the first gradient vector of the estimated indirect utility function. Monotonicity is satisfied if $\nabla\widehat{h}(\boldsymbol{v}) < 0$, where $\nabla\widehat{h}(\boldsymbol{v}) = \partial\widehat{h}(\boldsymbol{v})/\partial\boldsymbol{v}$.

- Curvature requires that the Slutsky matrix be negative semidefinite and can be checked by performing a Cholesky factorization of that matrix. The Cholesky values must be nonpositive, since a matrix is negative semidefinite, if its Cholesky factors are nonpositive. See Lau (1978, Theorem 3.2). Curvature alternatively can be checked by examining the Allen elasticities of substitution matrix, if the monotonicity condition holds. This matrix must be negative semidefinite.

If regularity is not attained, some models can be estimated by imposing regularity, thereby treating the curvature and monotonicity properties as maintained hypotheses. In the case of the locally flexible functional forms, for example, curvature can be imposed using the procedure suggested by Ryan and Wales (1998). But simultaneous imposition of monotonicity on these models, as required for regularity, can seriously damage flexibility. In the context of the globally flexible functional forms (Fourier and AIM) curvature, and monotonicity can be imposed using the procedures suggested by Gallant and Golub (1984). For a discussion of these methods for imposing theoretical regularity on locally and globally flexible functional forms, see Serletis and Shahmoradi (2005, 2007) and Barnett and Serletis (2008).

## 6.2 Elasticity Calculations

As we noted earlier, a system of budget share equations, such as (56), provides a complete characterization of consumer preferences over goods and can be used to estimate the income elasticities, the own- and cross-price elasticities, as well as the elasticities of substitution.

In particular, the elasticities can be calculated directly from the estimated budget share equations, rearranged in the form

$$x_i = \frac{s_i y}{p_i}, \quad i = 1, \ldots, n.$$

For example, the uncompensated (Cournot) price elasticities, $\eta_{ij}(\boldsymbol{p}, y)$, can be calculated as

$$\eta_{ij}(\boldsymbol{v}) = \frac{\partial s_i}{\partial v_j} \frac{v_j}{s_i} - \delta_{ij}, \quad i, j = 1, \cdots, n,$$

where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise. The income elasticities, $\eta_{iy}(\boldsymbol{p}, y)$, can be calculated as

$$\eta_{iy}(\boldsymbol{v}) = -\sum_{j=1}^{n} \eta_{ij}(\boldsymbol{v}), \quad i = 1, \cdots, n.$$

The Allen, Morishima, and Mundlak elasticities of substitution can then be calculated using equations (16), (17), and (18), respectively.

# 7   Conclusion

We have provided a glimpse of one of the most interesting and rapidly expanding areas of current research — the measurement of consumer preferences and the estimation of demand systems. But as we noted in the introduction, we only dealt with consumer choice in a static framework.

The static neoclassical model of consumer choice can be extended to accommodate taste change, the introduction of new goods, and changes in the characteristics of the available goods.[3] One of these (widely used) extensions is the theory of household production, which integrates consumer choice theory with the theory of the firm. See Becker (1965), Lancaster (1966), and Barnett (1977). Finally, merging household production theory with the theory of intertemporal consumer choice gives rise to dynamic household production theory. These and other important extensions of the static neoclassical theory of consumer choice are well beyond the objectives of this chapter. See LaFrance (2001) for a summary of the current status of household production theory, dynamic household production theory, and the microeconomic theory of consumer choice in an intertemporal framework.

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[3]Changing tastes has become the subject of much research, but usually only through the habit formation mechanism. More general explorations of time varying tastes have been rare, with a notable exception being Basmann, Molina, and Slottje (1983).

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