Learning from Private and Public Observation of Other's Actions

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Abstract

We study how a continuum of agents learn about disseminated information by observing others' actions. Every period each agent observes a public and private noisy signal centered around the aggregate action taken by the population. The public signal represents an endogenous aggregate variable such as a price or a quantity. The private signal represents the information gathered through private communication and local interactions. We identify conditions such that the average learning curve is S-shaped: learning is slow initially, intensifies rapidly, and finally converges slowly to the truth. We show that increasing public information always slows down learning in the long run and, under some conditions, reduces welfare. Lastly, optimal diffusion of information requires that agents "strive to be different": agents need to be rewarded for choosing actions away from the population average.

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1 Introduction

Households and firms learn about the state of the economy in both a public and a private fashion. They gather public information by observing noisy endogenous aggregates such as asset prices or the macroeconomic figures published by agencies. At the same time, some information remains dispersed because households and firms know more about their local markets than about the economy as a whole. Hence, they also gather some information in a private fashion, by interacting among each other. In this paper, we study the interaction of these public and private learning channels in a large population. We provide conditions such that information diffuses along an S-shaped learning curve. We show that, in the long run, more public information always slows down the diffusion of private information and sometimes reduces welfare. Lastly, we provide an analysis of optimal information diffusion.

We consider an economy populated by a continuum of agents who, at time zero, receive some public and private signals about the state of the world. This is the only source of exogenous information arrival in the model. After that, each agent takes an action at every moment. The state is revealed at a random time and the payoff to each agent is decreasing in the distance between her current action and the revealed state. However, before the state is revealed, the initial private information of agents slowly diffuses in the population through the following mechanism. At all times, each agent observes two signals. The first signal is public, shared by everyone, and is designed to represent an endogenous aggregate variable such as a price or some macroeconomic aggregate. The second one is private, only observed by the agent: it represents the information gathered through private communication and local interactions. The two signals are endogenous: they are centered around the current average action in the population. Because the average action reflects agents' current forecasts about the state of the world, the observation of these signals allow our continuum of agents to progressively learn about the state of the world.

We show that there exists an equilibrium in which agents eventually learn the truth. The average belief in the population converges to the state of the world, but it does so along an S-shaped curve as long as the initial information is *sufficiently dispersed*. This is because agents become more and more confident about the information they gathered privately, and make forecasts that are increasingly sensitive to this private information. Since the private information of others is the only new thing there is to learn and because of the informational noise, a larger sensitivity to private information increases the informativeness of the average forecast, accelerates learning, and creates a convex learning curve. The learning curve is concave at the end because convergence to the truth implies that learning must eventually slow down. In addition, when agents learn privately, they learn independently from one another: this means that agents learning histories are increasingly heterogeneous, and implies that the dispersion of beliefs increases at the beginning. Of course, the dispersion of beliefs eventually converges to zero as agents learn the truth.

Public information ends up crowding out private information, and a more precise initial public signal will eventually reduce the precision of agents beliefs. The culprit is an information externality. Indeed, with an increase in the precision of public information, each agent's action becomes more sensitive to the public information which is already known by everyone and hence useless for learning. At the same time the average action becomes less sensitive to the private information of others which is the only new thing there is to learn. Together with the presence of an observational noise, this makes it harder for an agent to learn from the average forecast, and eventually slows down information diffusion. Note that, because of our continuum-of-players assumption, an agent has no incentive to take this effect into account when choosing his action.

Whether public information is socially beneficial depends on the trade-off between a short-run gain and a long-run loss. The short-run gain is to make agents better informed, while the long-run loss is to slow down the diffusion of private information in the population. We prove that, as long as agents are sufficiently patient, a given marginal increase in the precision of the public signal is always welfare reducing. Hence, differently from Morris and Shin [2002], even in the absence of a payoff externality better public information can reduce welfare.

In the last part of the paper we study the problem of a planner who maximizes utilitarian welfare by telling agents what forecast to make as a deterministically time-varying affine function of their public and private forecasts. In equilibrium, agents do not take into account the impact on information diffusion of their actions and hence a learning externality appears. The planner internalizes this externality by requiring that agents strive to be different: they should make forecasts that are more sensitive to their private information than in equilibrium. We show in addition that, as long as the initial information is sufficiently dispersed, the planner's sensitivity deviates from the equilibrium sensitivity non-monotonically over time: the optimal sensitivity is close to the equilibrium sensitivity at the beginning, far away in the middle, and close again at the end. This is because the cost of increasing the sensitivity is very large at the beginning, while the benefit is very small at the end. Namely, at the beginning, each agent has a tiny amount of private information which varies widely around the true state of the world. Increasing the sensitivity to private information would create a large increase in the dispersion of agents forecasts and result in a huge welfare loss. At the end, on the other hand, agents know almost all the private information, and the benefit of increasing the sensitivity in order to speed up learning is very small. Finally, we show that the planner's solution can be decentralized in a dynamic version of Morris and Shin [2002] beauty contest, by rewarding agents for making forecasts away from the population play. In other words, agents should be rewarded for being different.

Literature Review

Our work is related to the recent literature on the social value of public information (see Morris and Shin [2002], Hellwig [2005], and Angeletos and Pavan [2005]). In these models, public information may reduce welfare because a static payoff externality gives individual agents a coordination motive, but at the same time washes out in the social welfare function. Our contribution is to study an alternative mechanism based on a dynamic information externality: in our model public information crowds out the diffusion of information in the population. Morris and Shin [2005] set up an overlapping-generations beauty-contest game in which a similar learning externality arises: they show that, when it discloses its information, a central bank ends up learning less from private agents' actions. While Morris and Shin's model is designed to study the learning process of the central bank, our work focuses on the learning process and welfare of private agents.[1]

Information externalities have been studied in the social learning literature (see, among many others, Vives [1993], Chamley and Gale [1994], and Vives [1997]). The maintained assumption of these models is that agents learn from public signals. The present paper adds to this literature the possibility of learning from each other privately. The assumptions that agents learn either from public or private signals end up having strikingly different implications. Indeed, when agents learn from public signals, the learning speed is decreasing over time. This key implication is reversed in our model because agents learn from the learning

[1]Because private agents in Morris and Shin's model do not inherit the information set of previous generations, they do not suffer from the dynamic learning externality emphasized in our paper. Hence, in the absence of a payoff externality, a release of public information always improves their welfare (although it reduces the precision of the information gathered by the central bank). This result stands in contrast with the welfare results of our paper.

of others, which creates an information snowballing effect: initially, learning speed increases over time. This implies that information diffuses along a S-shape, a pattern documented by a number of empirical studies of social learning (see Chapter 9 of Chamley [2004] and see also Jovanovic and Nyarko [1995]). Recent work on social learning focused on learning in networks: Bala and Goyal [1998], Gale and Kariv [2003], Smith and Sorensen [2005] study deterministic networks with finite number of agents, Banerjee and Fudenberg [2004] provide a continuum-of-agents setup (see also DeMarzo et al. [2003] for a network of boundedly rational agents). Because they lack tractability, these models end up focusing almost exclusively on the question of convergence to the truth. Our continuous-time model with a continuum of agents can be solved in closed form, which allows us to take the learning-in-network literature a step further, with an analysis of transitional dynamics, welfare, and the impact of public information.

Another related literature uses search-and-matching model in order to study how agents learn from local interactions with others. In Wolinsky [1990] seminal work, information diffuses at the individual level but stays constant at the aggregate level: indeed, agents leave the economy after trading and uninformed agents continuously enter the economy (see also the recent work of Kircher and Postlewaite [2006]). The issue of convergence when information diffuses on the aggregate has been subsequently addressed in Green [1991], Blouin and Serrano [2001], and in the independent work of Duffie and Manso [2006]. Wallace [1997], Katzman et al. [2003], and Araujo and Shevchenko [2006] address learning about a money supply shock in Trejos and Wright [1995] random-matching model. For tractability, they assume that the money supply becomes public after either one or two periods. Araujo and Camargo [2006] relax this assumption in a Kiyotaki and Wright [1989] model, and study the government incentives to expand the money supply. Our setup is somewhat simpler than these models because agents do not learn from trading but from observing the action of others. The benefit of this simplification is that we can explicitly characterize the transitional dynamics of beliefs and study the welfare impact of public information.

The rest of the paper is organized as follows. Section 2 introduces the setup. Section 3 solves for an equilibrium. Section 4 studies the dynamics of diffusion and the role of the private and public signals. Section 5 studies the impact on welfare of changes in the quality of public information, and how the social planner's problem can be decentralized in a dynamic beauty context. Section 6 concludes, and appendix collects all the proofs not in the main text.

2 Set up

Time is continuous and runs forever. We fix a probability space $\{\Omega, \mathcal{G}, Q\}$ together with an information filtration $\{\mathcal{G}_t, t \geq 0\}$ satisfying the usual conditions (Protter [1990]).

Our economy is populated by a continuum of agents, who seek to forecast the same random variable $x \in \mathbb{R}$. At time zero, agents receive a public signal about x, as well as some independent private signals which make them asymmetrically informed. Given the continuum assumption, if any particular agent could gather the signals of all others, she will know the realized value of x. The objective of this paper is to study situations where such pooling is not feasible and we model explicitly how the initial information diffuse slowly in the population through endogenously generated public and private signals.

Preferences.

We assume that, at each time, each agent $i \in [0, 1]$ prepares a forecast a_{it} of the random variable x, which we take to be normally distributed with mean zero and precision $\bar{P}$. The realization of x is to be announced at some random "day of reckoning" $\tau > 0$, which is Poisson distributed with intensity $r > 0$, and independent from everything else. At the day τ of reckoning, an agent receives the payoff $-(a_{i\tau} - x)^2$. This quadratic specification ensures that, at each time, an agent's optimal forecast is the expectation

$$a_{it} = E\left[x \,|\, \mathcal{G}_{it}\right] \tag{1}$$

of the random variable x, conditional on her information filtration $\{\mathcal{G}_{it}\,,\, t \geq 0\}$ that we define below.

Information.

The information available to an agent evolves over time in the following fashion. First at time zero, agents become asymmetrically informed about x: each of them receives both a public signal Z_0 and a private signal z_{i0}

$$Z_0 = x + \tfrac{W_0}{\sqrt{\Pi_0}} \tag{2}$$

$$z_{i0} = x + \tfrac{\omega_{i0}}{\sqrt{\pi_0}}, \tag{3}$$

where W_0 and $(\omega_{i0})_{i\in[0,1]}$ are normally distributed with mean zero and variance 1, pairwise independent, and independent from everything else. The precision of the public and private signals are measured by Π_0 and π_0, respectively.

Note that if agents could pool their time-zero private signals, they would be able to infer the exact realization of the state of the world x. A standard way of obtaining such pooling of information is to assume that agents observe an endogenous aggregate variable. For example, if agents were able to observe the cross-sectional average action

$$A_t \equiv \int_0^1 a_{it}\, di, \tag{4}$$

in the first period, $t = 0$, they would be able to infer the exact value of x.

In this paper, we prevent agents from instantly pooling all their private information by introducing noise in the observation of the average action. We assume that agents observe a noisy public signal of the average action, Z_t, solving the following stochastic differential equation

$$dZ_t = A_t dt + \frac{dW_t}{\sqrt{P_\varepsilon}} \tag{5}$$

with initial condition given by equation (2). This public signal represents the information conveyed by some endogenous aggregate variable. This could be, for instance, an asset price in a noisy-rational model of financial markets.

On the other hand, each agent i also observes a private signal z_{it} of the average action satisfying

$$dz_{it} = A_t dt + \frac{d\omega_{it}}{\sqrt{p_\varepsilon}}, \tag{6}$$

with initial condition given by equation (3). Such a signal captures the decentralized gathering of information. One could think for instance, of local interaction and private communication such as gossips.[2]

We let W and $(\omega_i)_{i\in[0,1]}$, in equations (5) and (6), be pairwise independent Wiener pro-

[2]The work of Amador and Weill [2006] suggests that specification (6) may arise when each agent continuously observes, with idiosyncratic noise, the forecast of other randomly chosen agents. Intuitively, observing the forecast of a randomly chosen agent amounts to sample from a distribution centered around the average forecast A_t. When the time between period and the precision of the noise go to zero at the same rate, Amador and Weill [2006] informally arrive at specification (6).

cesses with initial conditions W_{i0}, ω_{i0}, independent from x and from τ.

Note that one crucial assumption in specification (5)-(6) is that agents' signals are distributed about the cross-sectional average forecast A_t. This assumption, which is standard in the social learning literature (see Chamley [2004] and references therein), means that agents do not learn by directly observing the information of others, but instead by observing a noisy average of their actions.

Equilibrium.

Given a process A for the average action, we can define the information sets of generated in the economy as

Definition 1. *Let A be a process for the average action. The public information is represented by the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ generated by $\{Z_s,\ 0 \leq s \leq t\}$. Similarly, an agent's private information is represented by the filtration $\{\mathcal{F}_{it},\ t \geq 0\}$ generated by $\{z_{is},\ 0 \leq s \leq t\}$. The filtration $\mathcal{G}_{it}$ generated by $\mathcal{F}_t \cup \mathcal{F}_{it}$ represents all the information available to agent $i \in [0,1]$ at any time $t > 0$.*

We can now state our notion of equilibrium.

Definition 2. *An* equilibrium *is pair of processes $\{a_i, A\}$ where a_i is a $\mathcal{G}_{it}$-adapted individual announcement process and A is a $\mathcal{G}_t$-adapted average announcement process, such that:*

(i) at each time, a_{it} solves equation (1),

(ii) and A_t solves equation (4).

The notion of equilibrium is standard. The average announcement process A determines the information sets of the agents, both the public and the private filtrations. Given their information, agents maximize their expected utility. And finally, the cross-sectional average forecast of the population A is consistent with the agents individually optimal forecasts.

3 An Equilibrium

In this section we explicitly construct an equilibrium. We start by showing that an agent's forecast of x at any time is the convex combination of two forecasts: a public forecast that summarizes the public information in the economy and a private one that summarizes the particular agent's private information. To do this, we guess

Hypothesis (H). There exists two continuous precision functions π_t and Π_t such that observing Z_t is equivalent to observing $\tilde{Z}_t$, and observing (Z_t, z_{it}) is equivalent to observing $(\tilde{Z}_t, \tilde{z}_{it})$, where

$$d\tilde{Z}_t = x\,dt + \frac{dW_t}{\sqrt{\Pi_t}} \tag{7}$$

$$d\tilde{z}_{it} = xdt + \frac{d\omega_{it}}{\sqrt{\pi_t}}. \tag{8}$$

Formally, let $\tilde{\mathcal{F}}_t$ be the filtration generated by $\{Z_0, \tilde{Z}_u,\ 0 < u \leq t\}$, and let $\tilde{\mathcal{F}}_{it}$ be the filtration generated by $\{z_{i0}, \tilde{z}_{iu},\ 0 < u \leq t\}$. Then $\mathcal{F}_t = \tilde{\mathcal{F}}_t$ and $\mathcal{G}_{it} = \mathcal{F}_t \cup \mathcal{F}_{it} = \tilde{\mathcal{F}}_t \cup \tilde{\mathcal{F}}_{it}$.

Hypothesis (H) states that observing the original public and private signals Z_t and z_{it}, which are centered around the average forecast in the population, is equivalent to observing two transformed signals centered instead around the true value of the parameter x. These transformed signals are generated with the same noises (W, ω_i), but with time-varying precisions (Π_t, π_t) which are determined in equilibrium.

We first introduce some notations. Conditional only on the history of the transformed private signal $\tilde{z}$, the initial private signal z_0, together with a totally diffuse initial prior, we define an agent's best private forecast of x by $\hat{x}_{it} = E\left[x \mid \tilde{\mathcal{F}}_{it}\right]$. Similarly, conditional on the history of the transformed public signal $\tilde{Z}$, the initial public signal Z_0, together with the initial common prior that x is normally distributed with mean zero and precision $\bar{P}$, we denote an agent's best private forecast of the state by $\hat{X}_t \equiv E\left[x \mid \tilde{\mathcal{F}}_t\right] = E\left[x \mid \mathcal{F}_t\right]$. We refer to $\hat{X}_t$ as the **public forecast** at time t and $\hat{x}_{it}$ as agent i's **private forecast**. Lastly, we let the precision of these private and public forecasts be denoted by

$$p_t = \left\{E\left[(x - \hat{x}_{it})^2 \mid \tilde{\mathcal{F}}_{it}\right]\right\}^{-1} \tag{9}$$

$$P_t = \left\{E\left[(x - \hat{X}_t)^2 \mid \mathcal{F}_t\right]\right\}^{-1}, \tag{10}$$

where we anticipate that the precision p_t of the private forecast is the same for all agents at any given time t.

Our objective is to derive the dynamics of the action of any given agent i as a combination of both his private and the public forecasts, $\hat{x}_{it}$ and $\hat{X}_t$. Bearing this in mind, we first determine the dynamics of the public forecast, $\hat{X}_t$, and of its precision, P_t; and of the private forecast $\hat{x}_{it}$ and its precision p_t.

In order to see how we obtain the law of motion of $\hat{X}_t$, it is convenient to consider the following discrete time approximation of the filtering problem (the formal proof in continuous time can be found in the appendix). Suppose that at the beginning of time t, the public forecast is that x is normally distributed with mean $\hat{X}_t$ and precision P_t. During the small time interval $[t, t+\Delta]$, equation (5) means that agents receive a public signal, $\Delta\tilde{Z}_t$, which is approximately equal to[3]

$$\begin{aligned} \Delta\tilde{Z}_t &= x\Delta + \sqrt{\Delta}\frac{\varepsilon_t}{\sqrt{\Pi_t}} \\ \Leftrightarrow \quad \frac{\Delta\tilde{Z}_t}{\Delta} &= x + \frac{\varepsilon_t}{\sqrt{\Pi_t \Delta}}, \end{aligned} \tag{11}$$

for some standard normal random variable ε_t. Equation (11) means that agents receive a signal centered about x, with precision $\Pi_t\Delta$. Given normality, the public forecast about x after having observed $\Delta\tilde{Z}_t$ is then a weighted average of the prior forecast and the newly received signal,

$$\hat{X}_{t+\Delta} = \left(\frac{P_t}{P_t + \Pi_t\Delta}\right)\hat{X}_t + \left(\frac{\Pi_t\Delta}{P_t + \Pi_t\Delta}\right)\frac{\Delta\tilde{Z}_t}{\Delta}, \tag{12}$$

where the weights on the prior and the signal reflect their relative precisions. Subtracting $\hat{X}_t$ on both sides, and plugging equation (11), the change in the public forecast is

$$\hat{X}_{t+\Delta} - \hat{X}_t = \frac{\Pi_t}{P_t + \Pi_t\Delta}\left[\Delta\left(x - \hat{X}_t\right) + \sqrt{\Delta}\frac{\varepsilon_t}{\sqrt{\Pi_t}}\right], \tag{13}$$

suggesting that, in the continuous-time limit as Δ goes to zero, the public forecast $\hat{X}_t$ solves the stochastic differential equation,

$$d\hat{X}_t = -\frac{\Pi_t}{P_t}\left[\left(x - \hat{X}_t\right)dt + \frac{dW_t}{\sqrt{\Pi_t}}\right]. \tag{14}$$

To complete the characterization of $\hat{X}_t$ it is also necessary to obtain the law of motion for P_t. Note that at each time interval $[t, t+\Delta]$, the public signal has precision $\Pi_t\Delta$. Since, the precision of the posterior is the sum of the precision of the prior, and of the precision of the

[3]The variance of the increment of Wienner process is proportional to the time interval, which implies the presence of $\sqrt{\Delta}$ in the discrete time approximation.

signal, it follows that $P_{t+\Delta} = P_t + \Pi_t \Delta$, and hence letting Δ go to zero,

$$dP_t = \Pi_t \, dt$$

which determines the law of motion for P_t.

A similar analysis determines the law of motion of the private belief $\hat{x}_{it}$ and its precision p_t under hypothesis (H). The following proposition, whose proof can be found in the appendix, formalizes these results.

Proposition 1 (Dynamics of Private and Public Forecasts)**.** *Suppose that hypothesis (H) holds. The public and private forecasts $(\hat{X}_t, \hat{x}_{it})$ and the precisions (P_t, p_t) solve the system of stochastic differential equations*

$$d\hat{X}_t = \frac{\Pi_t}{P_t} \left[\left(x - \hat{X}_t \right) \, dt + \frac{dW_t}{\sqrt{\Pi_t}} \right] \tag{15}$$

$$d\hat{x}_{it} = \frac{\pi_t}{p_t} \left[\left(x - \hat{x}_{it} \right) \, dt + \frac{d\omega_{it}}{\sqrt{\pi_t}} \right] \tag{16}$$

$$dP_t = \Pi_t \, dt \tag{17}$$

$$dp_t = \pi_t \, dt, \tag{18}$$

with the initial conditions $P_0 = \bar{P} + \Pi_0$ and $p_0 = \pi_0$. In addition, the above system can be integrated into

$$\hat{X}_t = \left(1 - \frac{\bar{P}}{P_t} \right) x + \frac{1}{P_t} \left[\sqrt{\Pi_0} W_0 + \int_0^t \sqrt{\Pi_u} dW_u \right] \tag{19}$$

$$\hat{x}_{it} = x + \frac{1}{p_t} \left[\sqrt{\pi_0} \omega_{i0} + \int_0^t \sqrt{\pi_u} d\omega_{iu} \right] \tag{20}$$

Proof. In the appendix. □

We next show that an agent's optimal action is a weighted average of his private forecast and the public forecast, where the weights are given by the relative precision of those forecasts,

Corollary 1 (Optimal Announcement)**.** *Suppose that hypothesis (H) holds. Then, the expectation of x conditional on the information available to agent's $i \in [0, 1]$ at time $t > 0$ is*

$$a_{it} = E \left[x \mid \mathcal{G}_{it} \right] = \frac{P_t}{p_t + P_t} \hat{X}_t + \frac{p_t}{p_t + P_t} \hat{x}_{it}, \tag{21}$$

and the posterior precision of her beliefs is

$$\left\{E\left[(x-a_{it})^2 \;\mid \mathcal{G}_{it}\right]\right\}^{-1} = P_t + p_t. \tag{22}$$

Proof. In the appendix. □

Once the belief of an agent has been characterized by equation (21), we can aggregate the actions in the population, and obtain the average

$$A_t = \frac{P_t}{p_t + P_t}\,\hat{X}_t + \frac{p_t}{p_t + P_t}\,x. \tag{23}$$

So, under hypothesis (H) we have characterized the dynamics of the beliefs of an agent after observing signals according to equation (7) and equation (8), and computed the implied aggregate behavior. To complete the characterization of equilibrium, we will now provide an intuitive verification of our Hypothesis (H), and compute the law of motions for the precisions of the public and the private forecasts, P_t and p_t – the formal proof is left to the Appendix.

Consider for instance the public signal, $dZ_t = A_t\,dt + dW_t/\sqrt{P_\varepsilon}$ and suppose that A_t evolves according to equation (23). Note that $\hat{X}_t$ is public under hypothesis (H) and known at time t. Hence, an agent can subtract from dZ_t, the part $P_t/(p_t + P_t)\hat{X}_t$ of A_t that she already knows. This implies that the public signal is observationally equivalent to $p_t/(p_t + P_t)x\,dt + dW_t/\sqrt{P_\varepsilon}$. Dividing this through by $p_t/(p_t + P_t)$, we find that observing the public signal is equivalent to

$$d\tilde{Z}_t = x\,dt + \frac{dW_t}{\sqrt{P_\varepsilon}\frac{p_t}{p_t+P_t}},$$

as conjectured, implying that $\Pi_t = P_\varepsilon\left[p_t/(p_t + P_t)\right]^2$.

Similarly, together with the public signal which determines A_t, the private signal is observationally equivalent to

$$d\tilde{z}_{it} = x\,dt + \frac{d\omega_{it}}{\sqrt{p_\varepsilon}\frac{p_t}{p_t+P_t}},$$

implying that $\pi_t = p_\varepsilon\left[p_t/(p_t + P_t)\right]^2$. This confirms that hypothesis (H) holds. Hence we can state the following Theorem:

Theorem 1 (Equilibrium). *Let $\Pi_t = P_\varepsilon(p_t/(p_t + P_t))^2$ and $\pi_t = p_\varepsilon(p_t/(p_t + P_t))^2$ and where p_t and P_t evolve according to Proposition 1. The pair of processes $\{a_i, A\}$ where a_i solves equation (21) and A solves equation (23) is an equilibrium.*

Proof. In the Appendix. □

Note that Π_t and π_t represent the precisions of the newly received public and private signals. The informativeness of these newly received signals at any time is a function of the precisions of the private and public forecasts, p_t and P_t respectively: it increases with p_t and decreases with P_t. Hence, the more agents know privately, the more informative their new signals become and the faster they learn. Improvements in the public forecast has the opposite effect: they slow down subsequent learning. This suggests that public and private learning affect differently the diffusion of information in the economy. The next section sheds some light into these different effects while studying the dynamics of the system.

4 Information Dynamics

This section analyzes the dynamics of the equilibrium described above. In the first two subsections, we study the dynamics of the precisions of public and private forecasts. We show that , in the limit as time goes to infinity, each agent learns the realization of the random variable x. We also provide comparative statics and relate our results to that of Vives [1997]. In the third subsection we study cross-sectional beliefs: we show that, if the initial private information is sufficiently dispersed, the mean belief in the population converges to the truth along a S-shaped curve, and that the dispersion of beliefs in the population converges to zero along a hump-shaped curve.

4.1 Dynamics of Precision

A Closed-form Solution

From (17) and (18) and using the equilibrum values of Π_t and π_t, the precisions of the public and the private forecasts evolve according to the Ordinary Differential Equations (ODE)

$$\dot{P}_t = \left(\frac{p_t}{P_t + p_t}\right)^2 P_\varepsilon \tag{24}$$

$$\dot{p}_t = \left(\frac{p_t}{P_t + p_t}\right)^2 p_\varepsilon, \tag{25}$$

Note also that, by (22), the precision of agent i's beliefs is the sum of the precisions of the public and his private forecasts: $P_t + p_t$.

Note that ODE (24) is equal to ODE (25) multiplied by $P_\varepsilon/p_\varepsilon$. So, as long as $p_\varepsilon > 0$, this implies that $P_t - (P_\varepsilon/p_\varepsilon)p_t$ stays constant over time, meaning that $P_t - P_0 = P_\varepsilon/p_\varepsilon(p_t - p_0)$. Plugging the previous equation into equation (25), the ODE (25) of p_t becomes

$$\dot{p}_t = \left(\frac{p_t}{p_t + \alpha/\beta}\right)^2 \frac{p_\varepsilon}{\beta^2}, \tag{26}$$

where $\alpha = P_0 - P_\varepsilon/p_\varepsilon p_0$ and $\beta = 1 + P_\varepsilon/p_\varepsilon$. Hence given an initial condition for the precision p_0 of the private forecast, and using equation (26), it is possible to characterize the dynamics of p_t and hence the dynamics of the entire system,

Proposition 2 (Dynamics of Precisions)**.** *If $p_\varepsilon > 0$ then*

(i) The precision of the public forecasts P_t is an affine function of the private forecasts,

$$P_t = \alpha + (\beta - 1)p_t \ ,$$

where $\alpha = P_0 - \frac{P_\varepsilon}{p_\varepsilon}p_0$ and $\beta = 1 + \frac{P_\varepsilon}{p_\varepsilon}$.

(ii) The precision of the private forecast monotonically converges to infinity, $\lim_{t\to\infty} p_t = \infty$.

(iii) The ratio $p_t/(p_t + P_t)$ monotonically converges to $p_\varepsilon/(p_\varepsilon + P_\varepsilon)$.

(iv) The precision of the private forecasts p_t, is such that

$$t = \frac{H(p_t) - H(p_0)}{p_\varepsilon}$$

where $H(p_t) = 2\alpha\beta \log p + \beta^2 p - \alpha^2/p$*,*

(v) As $t \to \infty$ the precision of the total beliefs, $p_t + P_t$, is such that

$$p_t + P_t = \left(\frac{p_\varepsilon}{p_\varepsilon + P_\varepsilon}\right)^2 (p_\varepsilon + P_\varepsilon)\, t - 2\alpha \log(t) + O\left(\log\left(\frac{\log(t)}{t}\right)\right). \tag{27}$$

Proof. In the Appendix. □

There is an special case not covered in the proposition. Suppose that there is no private learning, $p_\varepsilon = 0$; agents only learn through public signals, $P_\varepsilon > 0$. This case was studied by Vives [1997] in a discrete-time model, and below we replicate his results in our continuous-time setup.

The Vives case: $p_\varepsilon = 0$

Vives studies a discrete time version of our set up with the difference that there is no private learning. In our set up this occurs for $p_\varepsilon = 0$. For this case however, the results in Proposition 2 do not apply. The following proposition solves for the precision of total beliefs in Vives' case.

Proposition 3 (Vives)**.** *When $p_\varepsilon = 0$, the precision of the total beliefs, $p_t + P_t$, is*

$$p_t + P_t = p_0 + P_t = 3^{1/3}\left(p_0^2 P_\varepsilon t + \frac{(p_0 + P_0)^3}{3}\right)^{1/3} \tag{28}$$

Proof. Let $y_t = P_t + p_0$. Given that $p_t = p_0$ we have from (24) that

$$\dot{y}_t = \frac{1}{y_t^2} p_0^2 P_\varepsilon$$

It is now easy to verify that (28) is the solution to this differential equation given initial condition $y_0 = p_0 + P_0$. □

This proposition replicates Vives' asymptotic result. When agents observe only public signals of the average action in the population, the speed of learning is dramatically reduced in the limit. In this case, the precision of total beliefs goes to infinity at rate $t^{1/3}$. This is slower than the rate at which the beliefs would converge if agents were to observe noisy signals of x every period with a constant noise – in that case the precision of belief would go to infinity at rate t. Note that this is also the rate at which the precision converges to infinity for the case where $p_\varepsilon > 0$ (as seen by equation (27)). Hence, when learning is based only on public signals, the speed of learning is dramatically reduced. Later sections will discuss more about this particular case.

Some Intuitions

We have decomposed what a given agent knows into a public forecast (shared with everyone else in the economy) and a private forecast (containing all the information observed by that agent and no-one else). Note that from equation (21), an agent's belief at any point in time about x is a weighted average of his private forecast and the public forecast. Agents learn in this economy by observing the average action, given by equation (23), with noise. To obtain equation (23), we just need to note that the average private belief in the population, $\int \hat{x}_{it} di$, is x: the idiosyncratic errors in the private forecasts wash out.

In the case of the public signal, an agent observes the aggregate action plus some noise with precision P_ε. Given the functional form of the average action, we have shown in the proof of Theorem 1 that observing the average action plus that noise is equivalent to a signal with mean x and precision

$$\Pi_t = \left(\frac{p_t}{p_t + P_t} \right)^2 P_\varepsilon. \tag{29}$$

A similar analysis works for the case of the private signal.

Hence, the precisions of the signals observed by an agent at any time are endogenous and are a function of the precision of all agents beliefs at that time. Note in particular that when the private forecasts are more precise (the higher p_t for given P_t), the more informative the endogenous generated signals are! When the private forecast is more precise, agents take actions that are more sensitive to them; and because agents learn only from the private part of the beliefs of others this increases the amount of "new" information aggregated when observing the average action plus noise.

From equation (29) it is possible to informally obtain the law of motions for the precision of both the public and the private information. Note that at time t, after observing the public signal, the precision of the public forecasts increases exactly by the precision of the newly observed signal. This is exactly what equation (24) states. A similar argument explains equation (25).

To understand the asymptotic result stated in the proposition 2, note that part (ii) of the proposition shows that in the limit, as time goes to infinity, the precision of the signals generated by the average action converges. Hence, the sum of the precisions of the endogenously generated private and the public signals converges in the limit to

$$\lim_{t\to\infty}\left(\frac{p_t}{p_t+P_t}\right)^2(p_\varepsilon+P_\varepsilon)=\left(\frac{p_\varepsilon}{p_\varepsilon+P_\varepsilon}\right)^2(p_\varepsilon+P_\varepsilon) \tag{30}$$

As can be seen from (27), this is the coefficient on t in the asymptotic expansion of the precision of total beliefs. When t goes to infinity, it is as if agents repeatedly observe signals about x with precision given by (30). Differently from Vives [1997], as long as p_ε is not 0, learning does not slows down dramatically asymptotically.

4.2 The Role of Public and Private Signals

Having solved the model, we now proceed to study the role of public and private information in the dynamics of information diffusion. To simplify notation, we refer to p_t and P_t as the equilibrium value of the precisions of the private and public forecasts at time t, while keeping in mind that these precisions are functions of the initial precisions p_0, P_0, and the exogenous noise, p_ε, and P_ε.

Let's start from the Vives case. For this case we have an explicit equation for the precision of total beliefs at time t (equation 28). We can think of the quality of public information as being determined from two different sources: the initial prior's precision P_0 and the public noise's precision P_ε. Increasing P_0 and increasing P_ε, increases the quality of public information. As can be seen from equation (28) such increases always increase the precision of total beliefs: they increase the amount learned at all times. The next proposition states this.

Proposition 4 (Vives)**.** *When $p_\epsilon = 0$, then both, an increase in the precision of the prior, P_0, or an increase in the precision of the public noise, P_ε, increases the precision of total* *beliefs at all times:*

$$\frac{\partial(p_0 + P_t)}{\partial P_0} > 0, \text{ for all } t \geq 0$$
$$\frac{\partial(p_0 + P_t)}{\partial P_\varepsilon} > 0, \text{ for all } t > 0$$

Proof. Follows by inspection of equation (28). □

Public information is benign in the Vives model, because it accelerates learning: better public information increases the precision of agents beliefs at all times.[4] However, as we will now show, things change dramatically when there is also private learning in the economy: better public information crowds out the private learning and eventually reduces the total amount learned by all agents. Before showing this last result, let us first proof that public information crowds out all types of learning.

Proposition 5 (Public Info Crowds Out of Learning)**.** *When $p_\varepsilon > 0$ and $P_\varepsilon > 0$, improving the quality of the public signals (increasing P_ε or P_0) decreases the precision of private forecasts at all times:*

$$\frac{\partial p_t}{\partial P_0} < 0, \text{ for all } t > 0 \quad \text{and} \quad \frac{\partial p_t}{\partial P_\varepsilon} < 0, \text{for all } t > 0$$

and eventually decreases the precision of the public forecast: there exists finite t_1 and t_2 such that

$$\frac{\partial P_t}{\partial P_0} < 0, \text{ for all } t > t_1 \quad \text{and} \quad \frac{\partial P_t}{\partial P_\varepsilon} < 0, \text{ for all } t > t_2$$

Proof. Note first that none of these changes affects p_0 (the initial condition of the public forecasts which is given by the precision of the initial private signals). By equation (26), which describes the law of motion for p_t, we see that an increase in P_0 increases α and reduces strictly $\dot{p}_t$ for all p_t. Hence p_t falls for all t. Note that the same is true for an increase in P_ε, which increases β and strictly reduces $\dot{p}_t$ for all p_t.

For the second part, note that equation (27) and using that $P_t = (P_t + p_t)(\beta - 1)/\beta + \alpha/\beta$,

[4]In the discrete time version of the Vives the result of Proposition 4 is not always correct. However, a similar result holds if we perform the comparative statics from the second period onwards.

implies that

$$P_t = \frac{p_\varepsilon^2 P\varepsilon}{(p_\varepsilon + P_\varepsilon)^2} t - 2\frac{(P_0 p_\varepsilon - p_\varepsilon P_\varepsilon)P_\varepsilon}{p_\varepsilon(p_\varepsilon + P_\varepsilon)} \log(t) + O(\log(\log(t))) \tag{31}$$

As can be seen from this equation, P_ε has a negative effect on the highest order term, and hence marginally increasing it will eventually reduce P_t. Note that P_0 has no effect in the highest order term, but has a negative effect in the $\log(t)$ term, hence marginally increasing P_0 will eventually reduce P_t as well. □

The previous proposition presents and important result: more precise public signals will eventually reduce the amount known! The intuition for such strong result relies on the endogeneity of learning. As the public forecast becomes more precise (because of more precise signals), agents will put more weight on the public belief when choosing their actions, and less on their private beliefs. This endogenous change generated by a more precise public signal, has the effect of reducing the amount of private information that can be inferred from their actions. Hence the endogenous signals become less informative about the true parameter and learning is impaired. This implies that the precision of the private forecasts at all times is reduced. However, the proposition is stronger: it shows that eventually the public forecasts are also impaired. In the one hand, better public signals slow down learning as explained above, but on the other hand, better public signals directly contribute to a more precise public forecast. The proposition shows that the first effect always dominates for large enough t. We can now state the following corollary.

Corollary 2. *When $P_\varepsilon > 0$ and $p_\varepsilon > 0$, increasing the precision of the initial prior, P_0, or increasing the precision of the public noise, P_ε, eventually reduces the precision of total beliefs.*

That is, there exists finite values t_1 and t_2 such that,

$$\frac{\partial(p_t + P_t)}{\partial P_0} < 0, \textit{ for all } t > t_1$$
$$\frac{\partial(p_t + P_t)}{\partial P_\epsilon} < 0, \textit{ for all } t > t_2$$

When agents learn privately, $p_\varepsilon > 0$, better public information eventually reduces the **total** amount learned.

Note that the dynamics of the private information are governed by

$$\dot{p}_t = \left(\frac{p_t}{P_0 + p_t + \frac{P_\varepsilon}{p_\varepsilon}(p_t - p_0)} \right)^2 p_\varepsilon \tag{32}$$

And either an increase in p_0 or an increase in p_ε, increases the $\dot{p}_t$ for any given $p_t > p_0$, and hence it will increase the precision of the private forecast at all times. The dynamics of the public forecast are similarly given by

$$\dot{P}_t = \left(\frac{p_t}{P_0 + p_t + \frac{P_\varepsilon}{p_\varepsilon}(p_t - p_0)} \right)^2 P_\varepsilon \tag{33}$$

and hence, an increase in p_0 or p_ε increases $\dot{P}_t$ for given P_t, and implies a higher public forecast at all times. The following proposition now follows,

Proposition 6. *An increase in p_0 or p_ε, increases the precision of both the private forecast and the public forecast at all times.*

4.3 Dynamics of Cross-sectional Beliefs

We close this section on equilibrium with the dynamics of cross-sectional beliefs. In the present normal-quadratic framework, the distribution of beliefs in the population is also normal and can be characterized in closed form.

We focus our attention on the average distribution of beliefs across realizations of public signal. That is, at each time $t > 0$, we look at the distribution a_{it} conditional on the realization of x but unconditional on the public information. Combining equations (19), (20) and (21),

$$\begin{aligned} a_{it} \quad = \quad & \frac{p_t}{p_t + P_t} \left\{ x + \frac{1}{p_t} \left[\sqrt{\pi_0} \omega_{i0} + \int_0^t \sqrt{\pi_u}\, d\omega_{iu} \right] \right\} \\ & + \frac{P_t}{p_t + P_t} \left\{ x \left(1 - \frac{\bar{P}}{P_t} \right) + \frac{1}{P_t} \left[\sqrt{\Pi_0} W_0 + \int_0^t \sqrt{\Pi_u}\, dW_{iu} \right] \right\}, \end{aligned} \tag{34}$$

which implies that, conditional on x, the belief a_{it} is normally distributed. Taking expectations on both sides conditional on x shows that the mean belief is

$$E\left[a_{it} \,|\, x\right] = x \left(1 - \frac{\bar{P}}{p_t + P_t} \right), \tag{35}$$

and that the cross-sectional dispersion of the beliefs is

$$V\left[a_{it}\,|\,x\right] = \frac{p_t + P_t - \bar{P}}{(p_t + P_t)^2}. \tag{36}$$

We can also analyze the dynamics of the cross-sectional belief distribution. Let $\rho_t = E[a_{it}|x]$, then

$$\dot{\rho}_t = x\bar{P}\frac{\dot{p}_t + \dot{P}_t}{(p_t + P_t)^2} = x\bar{P}(p_\varepsilon + P_\varepsilon)\left(\frac{p_t}{(\beta p_t + \alpha)^2}\right)^2 \equiv x f(p_t)$$

Hence $\dot{\rho}$ is negative or positive depending on whether x is negative or positive: the path of $E[a_{it}|x]$ monotonically approaches x.

Now, note that

$$f'(p_t) = \left(\frac{2}{p_t} - \frac{4\beta}{\beta p_t + \alpha}\right) f(p_t) = 2\frac{\alpha - \beta p_t}{(\beta p_t + \alpha)p_t} f(p_t)$$

Hence $f'(p_t) < 0$ when $p_t > \alpha/\beta$ and $f'(p_t) > 0$ when $p_t < \alpha/\beta$. Given that p_t is increasing through time, this implies that $|\dot{\rho}|$ starts decreasing from the time where $p_t > \alpha/\beta$, and is increasing before that. Hence a necessary and sufficient condition for the path of ρ to be S-shaped is that $p_0 < \alpha/\beta$. By noticing that this last inequality is equivalent to $p_0 < P_0/(2\beta - 1) = (\bar{P} + \Pi_0)/(1 + 2P_\varepsilon/p_\varepsilon)$, we have thus shown the following result,

Proposition 7 (S-shaped means if dispersed private information)**.** *The path of $E[a_{it}|x]$ monotonically approaches x as times tends to infinity, and*

1. *if $p_0 < (\bar{P} + \Pi_0)/(1 + 2P_\varepsilon/p_\varepsilon)$, then there exists a $t_0 > 0$ such that $|dE[a_{it}|x]/dt|$ is increasing for all $t < t_0$ and decreasing for all $t > t_0$,*

2. *if $p_0 > (\bar{P} + \Pi_0)/(1 + 2P_\varepsilon/p_\varepsilon)$, then $|dE[a_{it}|x]/dt|$ is decreasing for all t.*

Proof. In the text. □

A similar analysis can be done with the cross-sectional dispersion of beliefs. Let $\sigma_t = V[a_{it}|x]$. First note that $\lim_{t\to\infty} \sigma_t = 0$. Also,

$$\dot{\sigma}_t = \sigma_t(\dot{p}_t + \dot{P}_t)\left(\frac{1}{p_t + P_t - \bar{P}} - \frac{2}{p_t + P_t}\right) = \frac{\sigma_t(\dot{p}_t + \dot{P}_t)}{(p_t + P_t)(p_t + P_t - \bar{P})}\left(2\bar{P} - p_t - P_t\right)$$

So, when $p_t + P_t < 2\bar{P}$ then σ_t is increasing through time, and when $p_t + P_t > 2\bar{P}$, then σ_t is decreasing. Given that $p_0 + P_0 < 2\bar{P}$ is equivalent to $p_0 < \bar{P} - \Pi_0$, the following result follows now directly

Proposition 8 (Hump-shaped dispersion)**.** *The* $\lim_{t\to\infty} V[a_{it}|x] = 0$*, and*

1. *if* $p_0 < \bar{P} - \Pi_0$*, then there exists a* $t_1 > 0$ *such that* $V[a_{it}|x]$ *is increasing for* $t < t_1$ *and decreasing towards zero for* $t > t_1$*.*

2. *if* $p_0 > \bar{P} - \Pi_0$*, then* $V[a_{it}|x]$ *is monotonically decreasing towards zero for all* t*.*

Proof. In the text. □

The next figure presents a particular example of the dynamics of cross-sectional beliefs for the case where the path of the mean is S-shaped and the dispersion of belief is hump-shaped.

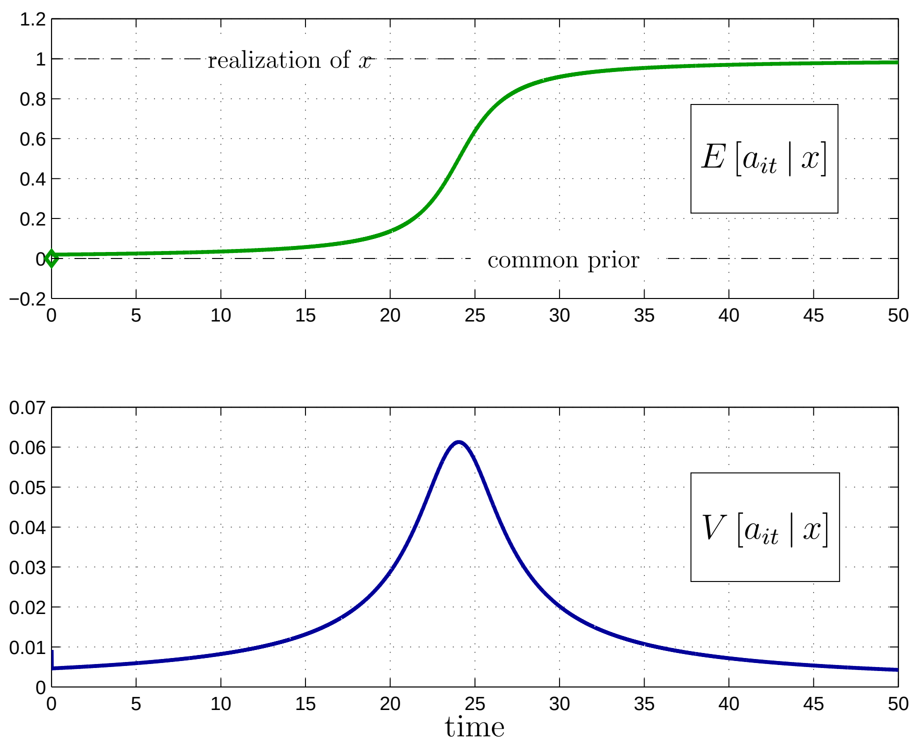

Figure 1: The figure shows the time path of the mean and dispersion of cross-sectional beliefs a_{it}, conditional on the realization of x. We choose $\bar{P} = 2.04$, $\pi_0 = \Pi_0 = 0.02$, $p_\varepsilon = P_\varepsilon = 10$. In order to calculate the time path of (p_t, P_t), we use the Euler method (see Judd [1999]) with step size $h = 0.01$ to discretize the ODEs (25) and (24).

It is also possible to say whether the time in the inflexion of the path of $E[a_{it}|x]$ occurs before or after the time of the highest dispersion,

Proposition 9. *Let t_0 and t_1 be defined as in propositions 7 and 8,*

1. *if $\Pi_0/P_\varepsilon < \pi_0/p_\varepsilon$, then $t_0 < t_1$,*

2. *if $\Pi_0/P_\varepsilon > \pi_0/p_\varepsilon$, then $t_0 > t_1$,*

3. *if $\Pi_0/P_\varepsilon = \pi_0/p_\varepsilon$, then $t_0 = t_1$.*

Proof. Note that t_0 is defined by $p_{t_0} = \alpha/\beta$ and t_1 is defined by $p_{t_1} + P_{t_1} = 2\bar{P}$, or equivalently $p_{t_1} = (2\bar{P} - \alpha)/\beta$. Hence, if $(2\bar{P} - \alpha)/\beta > \alpha/\beta$, then $p_{t_1} > p_{t_0}$. So, if $\bar{P} > \alpha$, then $p_{t_1} > p_{t_0}$ or equivalently $t_1 > t_0$. The result follows by noticing that $\bar{P} > \alpha$ is equivalently to $\Pi_0/P_\varepsilon < \pi_0/p_\varepsilon$. □

5 Welfare

Whether public information is socially beneficial depends on the tradeoff between a short-term gain and a long-term loss. Public information initially improves the precision of agents, generating a short term gain. The long-term loss, as shown in Proposition 5 comes from a learning externality: public information eventually slows down the diffusion of private information in the population. In the first subsection that follows we provide conditions ensuring that the long-term loss dominates: namely, we show that if agents are sufficiently patient, then a marginal increase in public information always reduces utilitarian welfare. Hence, differently from Morris and Shin [2002], even in the absence of a payoff externality more public information can be welfare reducing.

In the second subsection we analyze the problem of optimal information diffusion, subject to the learning technology. We show that the planner improves information diffusion by requiring that agents strive to be different: they should make forecasts that are more sensitive to their private forecast than in the equilibrium. In addition, the socially optimal sensitivity of agents' forecasts to private forecasts deviates from the equilibrium sensitivity non-monotonically over time. The optimal sensitivity is close the equilibrium sensitivity at the beginning, far away in the middle, and close again at the end.

5.1 The Equilibrium Welfare Cost of Public Information

We take our welfare criterion to be the equally weighted sum of agents' expected utility. By the Law of Large Number, this criterion coincides with the ex-ante utility of a representative agent,

$$W = -E\left[\int_0^\infty (a_{it} - x)^2 re^{-rt}\, dt\right], \tag{37}$$

where re^{-rt} is the probability density that the game ends at time $t > 0$. Because $a_{it} = E[x \mid \mathcal{G}_{it}]$ and $E\left[(a_{it} - x)^2 \mid \mathcal{G}_{it}\right] = 1/(p_t + P_t)$, an application of Fubini's Theorem implies that (37) can be written

$$W = -\int_0^\infty \frac{re^{-rt}}{p_t + P_t}\, dt. \tag{38}$$

Public information increases the total precision $p_t + P_t$ of agents beliefs in the short run but, as shown in Proposition 5, it decreases $p_t + P_t$ in the long run. Hence, because (38) is the present value of $1/(p_t + P_t)$ it is natural to conjecture that, as long as r is close enough to 0, public information reduces welfare. Although intuitive, this result is not obvious because, even when r goes to zero, the trade-off between the short-term gain and the long-term loss remains non trivial. Indeed, since $1/(p_t + P_t)$ converges to zero, the welfare flows are vanishingly small.

Note that from equation (25),

$$\dot{p}_t = \left(\frac{p_t}{\alpha + \beta p_t}\right)^2 p_\varepsilon \tag{39}$$

Hence the welfare can be rewritten as,

$$W = -\int_0^\infty \left(\frac{\alpha + \beta p_t}{p_t^2}\right) \frac{r}{p_\varepsilon} e^{-rt}\, (\dot{p}_t\, dt) \tag{40}$$

Now, from Proposition 2, we have

$$t = \frac{H(p_t) - H(p_0)}{p_\varepsilon} \tag{41}$$

for $H(p) \equiv 2\alpha\beta \log p + \beta^2 p - \alpha^2/p$. Given that p_t monotonically approaches infinity through time, changing the integrating variable from t to p in equation (40), yields

$$W = -\frac{r}{p_\varepsilon} \int_{p_0}^\infty \left(\frac{\alpha + \beta p}{p^2}\right) e^{-r\left(\frac{H(p) - H(p_0)}{p_\varepsilon}\right)}\, dp. \tag{42}$$

Note that the equilibrium welfare function W depends on the initial precision P_0 of public information only through $\alpha = P_0 - P_\varepsilon/p_\varepsilon p_0$. This means that a marginal increase in the precision of the initial public signal decreases social welfare if and only if $\partial W/\partial\alpha < 0$. Based on this remark we show

Theorem 2 (Welfare Cost of Public Information)**.** *For all p_0, there exists an $\eta > 0$ such that $0 < r < \eta$ implies $\partial W/\partial P_0 < 0$.*

Proof. In the appendix □

Theorem 2 means that, for any initial level p_0 of private information, a marginal increase in public information reduces welfare, as long as the intensity r of finishing the game is low enough. Hence, in contrast with Morris and Shin [2002], an increase in public information can reduce welfare even when payoffs induce no coordination motives.

The Theorem does not imply, however, that welfare is a monotonically decreasing function of P_0. Indeed, an infinite increase in precision would reveal the state of the world and would clearly improve welfare. By continuity, one might expect that a sufficiently large release of public information would also improve welfare. This intuition is confirmed by the numerical calculation of Figure 2: it shows that welfare is a non-monotonic function of P_0. It first decreases but eventually increases if P_0 is large enough.

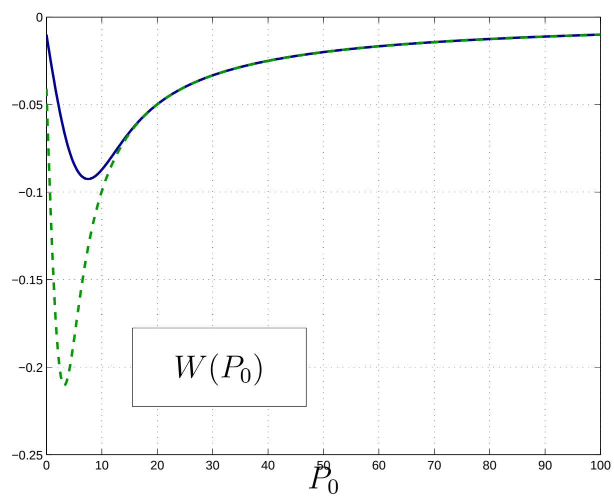

Figure 2: The figure shows welfare as a function of the precision P_0 of public information. We choose $p_0 = 0.05$, $p_\varepsilon = 1$, and $P_\varepsilon = 0.01$. The blue plain line is for an intensity $r = 0.001$ of finishing the game, and the green dotted line is for an intensity $r = 0.005$.

5.2 Optimal Information Diffusion

In this subsection we study the socially optimal diffusion of information. We let a planner choose the sensitivity of agents' forecasts to their private and public forecasts, in order to maximize utilitarian welfare. Choosing a sensitivity that is greater than the equilibrium sensitivity generates a dynamic welfare gain. Indeed, because of the information externality, it speeds up the dissemination of private information in case the game continues. On the other hand, this also generates a static welfare loss in case the game ends, because it increases (on average) the dispersion of agents' forecast around the realized value of x. Note, however, that the equilibrium sensitivity minimizes the dispersion of forecasts at each time, implying that the static welfare loss is of second order, while the dynamic welfare gain is of first order. Therefore, the socially-optimal sensitivity to private forecast greater than the equilibrium sensitivity. In addition, we show that the optimal sensitivity deviates from the equilibrium sensitivity non-monotonically over time. It is close to the equilibrium sensitivity at the beginning, far way in the middle, and close again at the end.

5.2.1 The planner's problem

A planner chooses a $\mathcal{G}_{it}$-adapted forecast process a_i in order to maximize the ex-ante utility (37) of a randomly chosen agent, subject to the learning technology. In setting up our planning problem, we follow Vives [1997] and restrict attention to the class of deterministic affine forecasts, whereby an agent's forecast is restricted to be

$$a_{it} = \gamma_t \hat{x}_{it} + \Gamma_t \hat{X}_t \tag{43}$$

for some deterministic functions (γ_t, Γ_t) of time and where, as before, $\hat{x}_{it}$ is an agent's private forecast and $\hat{X}_t$ is the public forecast.[5]

Learning Dynamics. Given our restriction (43), we can solve for the learning dynamics exactly as before. Starting with hypothesis (H), we use equations (19) and (20) of Proposition

[5] Although the existence of a linear equilibrium makes it natural to study affine forecast, we could not prove that an unrestricted optimum is indeed affine. In section 5.2.2, we illustrate one virtue of an optimal affine rule (43): it can be implemented by letting agents play a beauty contest game à la Morris and Shin [2002].

1 to find that

$$A_t = \gamma_t \int \hat{x}_{it}\, di + \Gamma_t \hat{X}_t = \gamma_t x + \Gamma_t \hat{X}_t. \tag{44}$$

We then intuitively verify hypothesis (H), as before. Recall that the public signal is $dZ_t = A_t\, dt + dW_t/\sqrt{P_\varepsilon}$. Since an agent can subtract from A_t, the the part $\Gamma_t \hat{X}_t$ that she already knows, the public signal is observationally equivalent to $\gamma_t x\, dt + dW_t/\sqrt{P_\varepsilon}$. Dividing this through by γ_t, we find that the public signal is observationally equivalent to $d\tilde{Z}_t = x\, dt + dW_t/(\sqrt{P_\varepsilon}\gamma_t)$, meaning that the precision Π_t of the public signal is $\Pi_t = P_\varepsilon \gamma_t^2$. A similar reasoning shows that the precision π_t of the private signal is $\pi_t = p_\varepsilon \gamma_t^2$. The following Lemma provides the formal verification result:

Lemma 1. *When $a_{it} = \gamma_t \hat{x}_{it} + \Gamma_t \hat{X}_t$, hypothesis (H) is verified with $\Pi_t = P_\varepsilon \gamma_t^2$ and $\pi_t = p_\varepsilon \gamma_t^2$.*

Simplification of the objective. We use the results of Proposition 1 to simplify the planner's objective. We first substitute equations (19) and (20) into (43) to find that

$$\begin{aligned} a_{it} - x &= x\left[\gamma_t + \Gamma_t\left(1 - \frac{\bar{P}}{P_t}\right) - 1\right] \\ &\quad + \frac{\gamma_t}{p_t}\left[\sqrt{\pi_0}\omega_{i0} + \int_0^t \sqrt{\pi_u} d\omega_{iu}\right] + \frac{\Gamma_t}{P_t}\left[\sqrt{\Pi_0} W_0 + \int_0^t \sqrt{\Pi_u} dW_u\right] \end{aligned}$$

After squaring and taking expectations on both sides, we find that

$$\begin{aligned} E\left[(a_{it} - x)^2\right] &= \frac{1}{\bar{P}}\left[\gamma_t + \Gamma_t\left(1 - \frac{\bar{P}}{P_t}\right) - 1\right]^2 + \frac{\gamma_t^2}{p_t^2}\left[\pi_0 + \int_0^t \pi_u\, du\right] + \frac{\Gamma_t^2}{P_t^2}\left[\Pi_0 + \int_0^t \Pi_u\, du\right] \\ &= \frac{1}{\bar{P}}\left[\gamma_t + \Gamma_t\left(1 - \frac{\bar{P}}{P_t}\right) - 1\right]^2 + \frac{\gamma_t^2}{p_t} + \frac{\Gamma_t^2}{P_t}\left[1 - \frac{\bar{P}}{P_t}\right], \end{aligned} \tag{45}$$

where the first line follows because $E(x) = 0$ and $E(x^2) = V(x) = 1/\bar{P}$, and because x, ω_i and W are mutually independent. The second line follows from the fact that $p_t - p_0 = p_t - \pi_0 = \int_0^t \pi_u\, du$ and $P_t - P_0 = P_t - \Pi_0 - \bar{P} = \int_0^t \Pi_u\, du$, meaning simply that the precision of the posterior is the sum of the precisions of the signals. Given the initial condition (p_0, P_0), the planner's problem is to choose a time path for (γ_t, Γ_t) in order to maximize

$$-\int_0^\infty E\left[(a_{it} - x)^2\right] re^{-rt}\, dt, \tag{46}$$

subject to the ODEs

$$\dot{p}_t = p_\varepsilon \gamma_t^2 \tag{47}$$

$$\dot{P}_t = P_\varepsilon \gamma_t^2. \tag{48}$$

Note that these laws of motion imply, as before, that P_t is an affine transformation of p_t. In particular, $P_t = \alpha + (\beta - 1)p_t$ where α and β are as in part (i) of Proposition 2.

While the sensitivity Γ_t to the public forecast enters the objective (45), it does not enter ODEs (47) and (48), meaning that it has no impact on the time path (p_t, P_t) of precisions. This implies that the optimal optimal Γ_t maximizes the flow welfare (45) given γ_t. Taking first-order conditions with respect to Γ_t on the right-hand side of (45) immediately shows:

Lemma 2. *Given any time path for γ_t, the planner's objective (46) is maximized by $\Gamma_t = 1 - \gamma_t$.*

Plugging $\Gamma_t = 1 - \gamma_t$ into (45) we find that

$$rE\left[(a_{it} - x)^2\right] = r\left(\frac{\gamma_t^2}{p_t} + \frac{(1-\gamma_t)^2}{P_t}\right) \equiv u(p_t, P_t, \gamma_t).$$

We let an *admissible control* be some measurable function $\gamma : \mathbb{R}_+ \to [0, 1]$ (Corollary 3 will show that the constraint $\gamma \in [0, 1]$ is not binding). Given an admissible control, the state (p_t, P_t) evolves according to the ODE (47) and (48), where the initial condition (p_0, P_0) is given. The planner's intertemporal utility is

$$v(p_0, P_0, \gamma) = -\int_0^\infty u(p_t, P_t, \gamma_t)e^{-rt}\, dt.$$

and the *planner's problem* is to find the supremum of $v(p_0, P_0, \gamma)$, subject to the constraint that γ is an admissible control. An *optimal control* is a solution of the planner's problem. We let $V(p_0, P_0)$ be the planner's maximum attainable utility. We first show some

Proposition 10 (elementary properties)**.** *The value function $V(p, P)$ is i) strictly negative, ii) bounded below by $-1/(p + P)$, iii) increasing in both its arguments, and iv) continuous. Moreover, v) for every $p \le p'$ and every $P \le P'$,*

$$(1 - e^{-rT}) \min\left\{\frac{p' - p}{(p' + p_\varepsilon T)^2}, \frac{P' - P}{(P' + P_\varepsilon T)^2}\right\} \le V(p', P') - V(p, P) \le \frac{p' - p}{p^2} + \frac{P' - P}{P^2} \tag{49}$$

for any T.

Proof. In the appendix. □

The negativity follows immediately from the fact that the flow welfare is negative. The lower bound is obtained in the worse case when agents learn nothing and keep the same precision $p + P$ of beliefs forever.

Proposition 10 shows that public information always increases the value of the planner. Note the sharp contrast with what can happen in the equilibrium characterized in previous sections. Indeed, as we know from Theorem 2, an increase in public information can decrease equilibrium welfare. However, if the planner could control the sensitivity of agents actions to the public and their private forecasts, more precise public information always increases welfare.

To establish properties of an optimal control, we rely on the theory of viscosity solutions of Hamilton-Jacobi-Bellman (HJB) equations (see Bardi and Capuzzo-Dolcetta [1997]). This allows us to use continuous-time dynamic programming techniques without assuming smoothness of the value function. Let us first define a generalized notion of directional derivative[6]

$$\partial^+ V\left(p, P; p_\varepsilon, P_\varepsilon\right) \equiv \limsup_{t \to 0^+} \frac{V(p + p_\varepsilon t, P + P_\varepsilon t) - V(p, P)}{t}. \tag{50}$$

Our main dynamic programming result is that this generalized directional derivative can be used for stating a Hamilton-Jacobi-Bellman (HJB): indeed, it represents the impact on the rate of change of the value function, of increasing the control γ^2. The following proposition, whose proof is found in the appendix, is a direct application of the results in Bardi and Capuzzo-Dolcetta [1997]:

Proposition 11 (HJB)**.** *The value function solves the HJB equation*

$$rV(p, P) = \sup_{\gamma \in [0,1]} \left\{ u(p, P, \gamma) + \gamma^2 \partial^+ V(p, P; p_\varepsilon, P_\varepsilon) \right\}, \tag{51}$$

for all (p, P). Let $\gamma^*(p, P)$ solve (51):

$$\gamma^*(p, P) \equiv \min \left\{ 1, \frac{p}{p + P - pP\partial^+ V(p, P; p_\varepsilon, P_\varepsilon)/r} \right\}. \tag{52}$$

[6] As a remark, note that part (v) of Proposition 10 implies that $V(p, P)$ is Lipschitz continuous in any open set of R^2_+. By Rademacher's Theorem, this implies that the classical derivative of V exists almost everywhere. Hence the generalized directional derivative $\partial^+ V$ coincides with the classical derivative for almost all states.

Then $\gamma_t^* = \gamma^*(p_t, P_t)$ where $p_t = p_0 + \int_0^t (\gamma_t^*)^2 p_\varepsilon dt$, and $P_t = P_0 + \int_0^t (\gamma_t^*)^2 P_\varepsilon dt$ is an optimal control for the planner's problem with initial conditions $p_0 > 0$, $P_0 > 0$.

Proof. In the appendix. □

The proposition establishes several results at once. First it shows that the value function is a solution of a HJB equation when using the generalized derivative. It also tell us that there exists an optimal control, which is generated by the the feedback rule (52).

This proposition is also useful in proving properties of the optimal control $\gamma^*(p, P)$. First, evaluating the right-hand-side of (51) at $\gamma = 1$, one finds

$$0 > rV(p, P) \geq -\frac{r}{p} + \partial^+ V(p, P; p_\varepsilon, P_\varepsilon),$$

which implies that $\partial^+ V(p, P; p_\varepsilon, P_\varepsilon)/r < 1/p$. Plugging this inequality into (52) immediately implies that

Corollary 3 (the constraint $\gamma \leq 1$ is never binding)**.** *An optimal feedback rule γ^* is such that $\gamma^*(p, P) < 1$ for all $(p, P) \in R_+^2$.*

Proof. In the appendix. □

In addition, because $V(p, P)$ is increasing in both its argument, it follows that $\partial^+ V(p, P, p_\varepsilon; P_\varepsilon) \geq 0$. Note that part (v) of Proposition 10 implies as well that

$$\frac{V(p + p_\varepsilon t, P + P_\varepsilon t) - V(p, P)}{t} \geq (1 - e^{-rT/2}) \min \left\{ \frac{p_\varepsilon}{(p + p_\varepsilon T)^2}, \frac{P_\varepsilon}{(P + P_\varepsilon T)^2} \right\}$$

for any $T > 0$ and $t < T/2$. Given that the right hand side of the above inequality is strictly positive for any finite T, taking limsup on the left-hand side implies that $\partial^+ V(p, P, p_\varepsilon; P_\varepsilon) > 0$. Now, using that $\partial^+ V(p, P, p_\varepsilon; P_\varepsilon) > 0$ in (52) implies that

Proposition 12. *An optimal feedback rule is such that*

$$\gamma^*(p, P) > \frac{p}{p + P}.$$

for any strictly positive pair (p, P).

Proposition 12 means that the planner mitigates the learning externality by requiring the sensitivity $\gamma^*(p, P)$ to private forecasts to be strictly greater than the equilibrium sensitivity of $p/(p+P)$. Since the planner finds it optimal to diffuse information faster than in an equilibrium where all information ends up being revealed, this intuitively suggests that

Corollary 4 (full revelation is socially optimal)**.** *Consider any optimal control and the associated time path* (p_t^*, P_t^*) *of precisions with initial state* $(p_0, P_0) > 0$*. Then both* p_t^* *and* P_t^* *go to infinity as* t *goes to infinity*

Proof. In the appendix. □

The following last theorem characterizes features of an optimal control at the boundaries, for either small and large precisions:

Theorem 3 (Equilibrium behavior is socially optimal at infinity and at zero)**.**

1. *For any* $\alpha \in \mathbb{R}$*, let* $L(p) = \alpha + P_\varepsilon/p_\varepsilon p$*. Then:*

$$\lim_{p\to\infty} \left| \gamma^*(p, L(p)) - \frac{p}{p + L(p)} \right| = 0.$$

2. *In addition, for all* $P_0 > 0$

$$\lim_{(p,P)\to(0,P0)} \gamma(p, P) = 0.$$

Proof. In the appendix. □

To explain the first part of Theorem 3, consider an optimal path (p_t^*, P_t^*). We then know that there exists some $\alpha \in \mathbb{R}$ such that, at each time, $P_t^* = \alpha + P_\varepsilon/p_\varepsilon p_t^* = L(p_t^*)$. Moreover, both the private and the public precision go to infinity. Taken together with the Theorem, these imply that equilibrium behavior is optimal as time goes to infinity.

The second part of the Theorem states that the socially optimal sensitivity goes to zero as the private precision goes to zero. This means that the equilibrium behavior is socially optimal at the beginning of time, as long as the precision of the initial private information is close enough to zero.

5.2.2 Implementation in a beauty contest game

In the previous subsection, we solved for the solution of planner's problem when the planner could choose the sensitivity of agents actions to the public and their private forecasts.

We showed that a solution to that problem existed. Efficiency implied that public information always increased welfare and agents where to weight more their private forecasts when choosing their actions than in the equilibrium play. In this subsection we propose a way of decentralizing such an allocation. Our main result in this subsection is to show that a sequence of appropriately chosen zero-sum beauty contest games a la Morris and Shin [2002] implements the planners allocation as an equilibrium. The beauty contest implementation is characterized by the "strive to be different" property: agents have to be rewarded when choosing actions away from the current average population play.

Our beauty contest representation is as follows. If the game ends at time $t > 0$, we follow Morris and Shin [2002] and let the payoff of agent $i \in [0, 1]$ be given by,

$$-(a_{it} - x)^2 - \frac{b_t}{1 - b_t}(L_{it} - \bar{L}_t), \tag{53}$$

where $b_t \in (-\infty, 1)$ and

$$L_{it} = \int_0^1 (a_{jt} - a_{it})^2 \, dj \tag{54}$$

$$\bar{L}_t = \int_0^1 L_{it} \, di. \tag{55}$$

The first term of (53) is the payoff of the forecasting game we studied so far. The second term is the payoff of the beauty contest game. Note that the cross-sectional sum $\int_0^1 (L_{it} - \bar{L}) \, di$ of agents' beauty-contest losses is equal to zero. This means that the beauty contest is a zero sum game that can be run with a balance budget.

According to equation (53), an agent trades off the distance of his action to the random variable x against the distance from the average action in the population. The parameter b_t captures the strength of the beauty contest: larger b_t means that an agent worries more about staying close to average forecast.

A strategy is some $\mathcal{G}_{it}$-adapted forecast process, and the solution is taken to be the Bayesian equilibrium of the dynamic game. Note that, at any point in time, given his beliefs, and taking as given the strategies of all other agents, an agent's strategy maximizes his expected payoff of the *current period*. This follows because a particular agent's action is negligible by the continuum-of-agents assumption. In particular, it has to be the case that

$$\begin{aligned} a_{it} &= (1-b_t)E\left[x \,|\, \mathcal{G}_{it}\right] + b_t E\left[A_t \,|\, \mathcal{G}_{it}\right] \\ &= (1-b_t)\left(\frac{p_t}{p_t+P_t}\hat{x}_{it} + \frac{P_t}{p_t+P_t}\hat{X}_t\right) + b_t E\left[A_t \,|\, \mathcal{G}_{it}\right], \end{aligned} \tag{56}$$

where A_t denotes the average forecast in the population, $\hat{x}_{it}$ and $\hat{X}_{it}$ denote the private and public beliefs, and p_t and P_t denote their respective precisions. We guess and verify that there is an equilibrium in which an agent forecast takes the linear form

$$a_{it} = \gamma_t \hat{x}_{it} + (1-\gamma_t)\hat{X}_t, \tag{57}$$

for some sensitivity γ_t. Proposition 1 implies that $A_t = \int a_{it}\, di = \gamma_t x + (1-\gamma_t)\hat{X}_t$, and therefore that

$$\begin{aligned} E\left[A_t \,|\, \mathcal{G}_{it}\right] &= \gamma_t E\left[x \,|\, \mathcal{G}_{it}\right] + (1-\gamma_t)\hat{X}_t \\ &= \gamma_t\left(\frac{p_t}{p_t+P_t}\hat{x}_{it} + \frac{P_t}{p_t+P_t}\hat{X}_t\right) + (1-\gamma_t)\hat{X}_t. \end{aligned} \tag{58}$$

Plugging (58) into the best reply (56) and identifying unknown coefficients with equations (57) shows that there exists a

Proposition 13 (Morris and Shin [2002] linear equilibrium.)**.** *There exists a linear equilibrium of the beauty contest game in which agents forecasts take the linear form (57), with*

$$\gamma_t = \frac{(1-b_t)p_t}{(1-b_t)p_t + P_t}. \tag{59}$$

We now show that a solution of the planning problem can be implemented in a sequence of beauty contest games, by choosing an appropriate time path b_t^* of the beauty-contest parameter. Indeed, the equilibrium γ_t lies in $(0,1)$ and we know from Corollaries 3 and 12 that a solution to the planning problem satisfies $\gamma_t^* \in (0,1)$. Now because the beauty contest equilibrium sensitivity γ_t is strictly monotonic in parameter b_t, it follows that there exists a unique beauty-contest parameter $b_t^* \in (-\infty, 1)$ solving $\gamma_t^* = (1-b_t^*)p_t^* / \left((1-b_t^*)p_t^* + P_t^*\right)$. Written in terms of the optimal feedback rule we define $b^*(p,P)$ to be,

$$b^*(p,P) \quad = \quad 1 - \frac{\gamma^*(p,P)}{1-\gamma^*(p,P)}\frac{P}{p} \tag{60}$$

Using the same methods as in the proof of Theorem 3, one can also characterize the behavior of the optimal beauty contest parameter b_t^* at infinity. We summarize our results in the following proposition.

Proposition 14 (Implementation and Striving to be Different)**.** *Let $p_0, P_0 \in R_+$ be the initial conditions. There exists a process $b_t^* \in (-\infty, 1]$ indexing a sequence of beauty contests, such that the equilibrium of the associated dynamic beauty contest game coincides with the solution of the planner's problem and where $b_t^* = b^*(p_t^*, P_t^*)$ with $p_t^* = p_0 + \int_0^t \gamma^*(p_\tau^*, P_\tau^*)^2 p_\varepsilon d\tau$ and $P_t^* = P_0 + \int_0^t \gamma^*(p_\tau^*, P_\tau^*)^2 P_\varepsilon d\tau$. The following holds as well,*

1. *$b_t^* < 0$ for all $t > 0$,*
2. *$\lim_{t\to\infty} b_t^* = 0$,*

The solution of the planner problem can be implemented by letting agents play a continuous-time beauty contest game with time varying parameter b_t^*. In such a game, agents are strictly rewarded for choosing actions away from population play. However such a reward vanishes as time goes to infinity and agents learn the truth. We will show numerically in the following section, that such a reward also vanishes in states where private information is sufficiently dispersed (p_0 small enough).

5.2.3 Numerical example

We conclude our study with a numerical illustration. We solve a discrete-time approximation of the planner's problem using a value function iteration algorithm (see, e.g., Chapter 12 of Judd [1999]). The approximation is obtained by setting some small step $h > 0$, letting the flow welfare be $u(p_t, P_t, \gamma_t) \times h$, and taking the laws of motion of p_t and P_t to be $p_{t+h} = p_t + p_\varepsilon \gamma_t^2 h$, and $P_{t+h} = P_t + P_\varepsilon \gamma_t^2 h$. Lastly, it is convenient to reduce the dimension of the state space by noting that $P_t = \alpha + (\beta - 1)p_t$, where $\beta = 1 + P_\varepsilon/p_\varepsilon$ and $\alpha = P_0 - (\beta - 1)p_0$.

We take the intensity of finishing the game to be $r = 1$, the initial conditions to be $(p_0, P_0) = (0.05, 1)$, and the precision of the private and public signals be $(p_\varepsilon, P_\varepsilon) = (1, 0.1)$. The upper panel of Figure 3 shows the time path of the planner's sensitivity γ_t^*, and the lower panel shows the time path of the optimal beauty contest parameter b_t^*. The time path of b_t^* is not monotonic. Figure 4 shows that the difference $\gamma_t^* - p_t^*/(p_t^* + P_t^*)$ between the socially optimal and the equilibrium sensitivity is not monotonic.

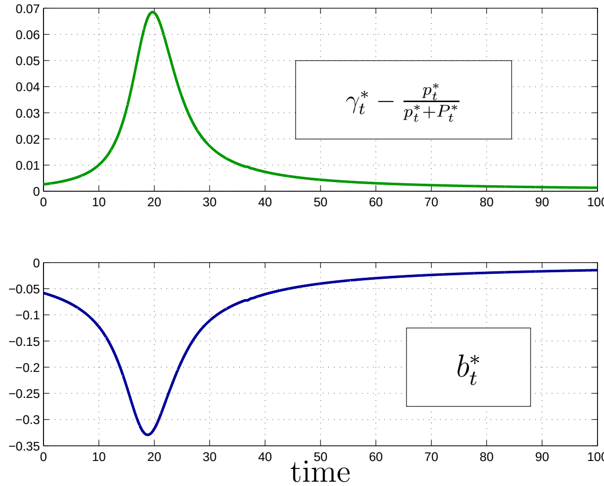

Figure 3: Optimal sensitivity and beauty contest parameters

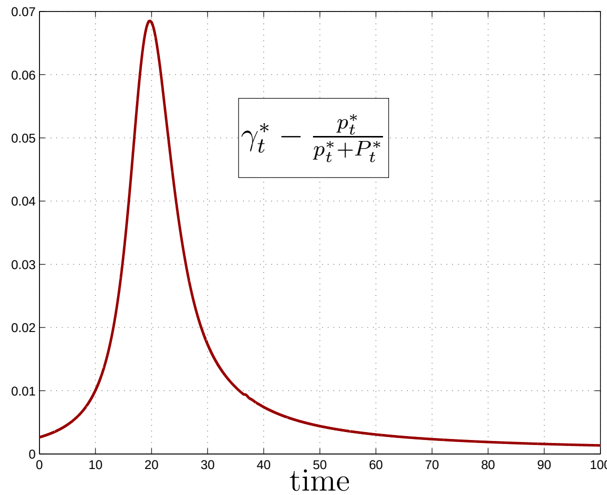

Figure 4: The difference between the planner's and the equilibrium sensitivity.

6 Conclusion

This papers studies how private information diffuses among a continuum of agents who learn from both public and private observations of others' actions. We provide conditions such that agents learn the truth along a S-shape curve, reflecting an information snowballing effect. We show that greater public information at the beginning always slows down the the diffusion of private information in the economy, and sometimes reduces welfare. We solve an optimal information diffusion problem and show that the planner speeds up diffusion by requiring that agents strive to be different, in the sense of taking action that far away from the population average. In addition, we show that the planner deviates from the equilibrium solution non-monotonically over time. Socially-optimal and equilibrium actions are close to each other at the beginning when information is very dispersed, far away in the middle, and close again at the end when information is almost fully revealed. Further work may address the optimal timing of public information release in this economy.

A Proofs

A.1 Proof of Proposition 1

Equations (15)-(18) follows from a direct application of one-dimensional Kalman filtering formula (see, for instance, Oksendal [1985], pages 85-105). In order to derive equation (19), we multiplying both sides of (15) by P_t. We find

$$\begin{aligned}
& P_t d\hat{X}_t = \Pi_t \left[\left(x - \hat{X}_t \right) dt + \frac{dW_t}{\sqrt{\Pi_t}} \right] \\
\Rightarrow \quad & P_t d\hat{X}_t + dP_t \left(\hat{X}_t - x \right) = \sqrt{\Pi_t} dW_t \\
\Rightarrow \quad & d\left[P_t \left(\hat{X}_t - x \right) \right] = \sqrt{\Pi_t} dW_t \\
\Rightarrow \quad & P_t \left(\hat{X}_t - x \right) - P_0 \left(\hat{X}_0 - x \right) = \int_0^t \sqrt{\Pi_u} dW_u \\
\Rightarrow \quad & \hat{X}_t = \frac{P_0}{P_t} \hat{X}_0 + \left(1 - \frac{P_0}{P_t} \right) x + \frac{1}{P_t} \int_0^t \sqrt{\Pi_u} dW_u \qquad (61)
\end{aligned}$$

where the second line follows from the fact that $dP_t = \Pi_t$. Because P_t is a deterministic function of time it follows that $d[(\hat{X}_t - x)P_t] = d\hat{X}_t P_t + (\hat{X}_t - x)dP_t$, which implies the third line. The fourth line follows from integrating the third line from $u = 0$ to $u - t$, and the fifth line follows from rearranging. Now note that $\hat{X}_0$ and P_0 are the posterior mean and precision at time zero, after observing the public signal $Z_0 = x + W_0/\sqrt{\Pi_0}$ and starting from the common prior that x is normally distributed with a mean of zero and a precision $\bar{P}$. Therefore, $P_0 = \bar{P} + \Pi_0$ and

$$\hat{X}_0 = \left(1 - \frac{\bar{P}}{\bar{P} + \Pi_0} \right) x. \qquad (62)$$

Equation (19) then follows from plugging (62) back into (61). Equation (20) follows from exactly the same algebraic manipulation with one difference. In the case of the private forecast, $\hat{x}_{i0}$ and p_0 are the posterior mean and precision at time zero, after observing the private signal $z_{i0} = x + \omega_{i0}/\sqrt{\pi_0}$, and a totally diffuse prior. (That is the precision of the prior is equal to zero.) Therefore, we have that $p_0 = \pi_0$ and $\hat{x}_{i0} = z_{i0}$.

A.2 Proof of Corollary 1

The result follows easily by noticing that, under assumption H, the signals that generated the private and the public forecast are independent conditional on x. Hence, an agent's total forecast will be a linear combination of the public and his private forecasts, with weights given by their respective precisions.

A.3 Proof of Theorem 1

The only thing left to check is hypothesis (H), which is done in appendix A.6, letting $\gamma_t = p_t/(p_t + P_t)$ and $\Gamma_t = P_t/(p_t + P_t)$.

A.4 Proof of Proposition 2

Part (i): Proved in the text.

Part (ii): The dynamics of p_t are given by

$$\dot{p}_t = \left(\frac{1}{\beta + \alpha/p_t}\right)^2 p_\varepsilon$$

This implies that p_t is increasing through time. Letting $\kappa = 1/\beta^2$ if $\alpha < 0$ and $\kappa = 1/(\beta + \alpha/p_0)^2$ otherwise, it follows that

$$\dot{p}_t \geq \kappa p_\varepsilon$$

where $\kappa > 0$, and,

$$p_t = p_0 + \int_0^t \dot{p}_\tau d\tau \geq p_0 + \kappa p_\varepsilon t,$$

and hence p_t converges to infinity as t goes to infinity.

Part (iii): Let's compute the dynamics of the ratio p_t/P_t. Given that $P_t = (\beta - 1)p_t + \alpha$, and given that p_t tends to infinity, it follows,

$$\lim_{t\to\infty} \frac{p_t}{P_t} = \lim_{t\to\infty} \frac{1}{\beta - 1 + \alpha/p_t} = \frac{1}{\beta - 1} = \frac{p_\varepsilon}{P_\varepsilon}$$

The short run dynamics of p_t/P_t are given by

$$\frac{d}{dt} \log \frac{p_t}{P_t} = d\frac{\log p_t - \log P_t}{dt} = \frac{\dot{p}_t}{p_t} - \frac{\dot{P}_t}{P_t}$$

Using $p_t = \left(\frac{p_t}{p_t + P_t}\right)^2 p_\varepsilon$, and $\dot{P}_t = \left(\frac{p_t}{p_t + P_t}\right)^2 P_\varepsilon$, we get that

$$d\frac{\log p_t - \log P_t}{dt} = \left(\frac{p_t}{p_t + P_t}\right)^2 \left(\frac{p_\varepsilon}{p_t} - \frac{P_\varepsilon}{P_t}\right)$$

The ratio p_t/P_t is strictly increasing whenever $p_t/P_t < p_\varepsilon/P_\varepsilon$ and strictly decreasing when $p_t/P_t > p_\varepsilon/P_\varepsilon$. So p_t/P_t monotonically approaches $p_\varepsilon/P_\varepsilon$.

Part (iv): Let $y_t \equiv p_t + P_t$. Note that $y_t = \alpha + \beta p_t$ and

$$\dot{y}_t = \left(\frac{y_t - \alpha}{y_t}\right)^2 \frac{p_\varepsilon + P_\varepsilon}{\beta^2}$$

The solution of this differential equation is given by,

$$G(y_t, t) \equiv 2\alpha \log(y_t - \alpha) + y_t - \frac{\alpha^2}{y_t - \alpha} - p_\varepsilon^2/(P_\varepsilon + p_\varepsilon)t = C_0 \tag{63}$$

which after some re-arranging implies part (iv).

Part (v): First note that $\lim_{t\to\infty} y_t = \infty$ which implies by equation (63) that

$$\lim_{t\to\infty} \frac{y_t}{t} = p_\varepsilon^2/(P_\varepsilon + p_\varepsilon). \tag{64}$$

Note that

$$\left(y_t - \frac{p_\varepsilon^2}{p_\varepsilon + P_\varepsilon} t + 2\alpha \log(t) \right) = -2\alpha \log\left(\frac{y_t}{t} - \frac{\alpha}{t} \right) + \frac{\alpha^2}{y_t - \alpha} - C_0 \tag{65}$$

which dividing by $\log(t)$, taking limits and using (64), implies

$$\lim_{t\to\infty} \frac{\left(y_t - \frac{p_\varepsilon^2}{p_\varepsilon + P_\varepsilon} t \right)}{\log(t)} = -2\alpha$$

So, y_t/t is of order $O(\log(t)/t)$. Hence the right-hand side of equation (65) is of the order $O\left(\log\left(\log(t)/t\right)\right)$. The asymptotic expansion of y_t is then

$$y_t = \frac{p_\varepsilon^2}{p_\varepsilon + P_\varepsilon} t - 2\alpha \log(t) + O\left(\log\left(\frac{\log(t)}{t} \right) \right) \tag{66}$$

A.5 Proof of Theorem 2

We take the derivative of W with respect to α. We find

$$\begin{aligned} \frac{\partial W}{\partial \alpha} &= -\frac{r}{p_\varepsilon} \int_{p_0}^{\infty} \left\{ \frac{1}{p^2} - \frac{\alpha + \beta p}{p^2} \frac{r}{p_\varepsilon} \frac{\partial}{\partial \alpha} \left(H(p) - H(p_0) \right) \right\} e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp \\ &= -\frac{r}{p_\varepsilon} \int_{p_0}^{\infty} \left\{ \frac{1}{p^2} - \frac{\alpha + \beta p}{p^2} \frac{r}{p_\varepsilon} \left[-2\alpha \left(\frac{1}{p} - \frac{1}{p_0} \right) + 2\beta \log\left(\frac{p}{p_0} \right) \right] \right\} e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp. \end{aligned} \tag{67}$$

We now integrate the first term $\int_{p_0}^{\infty} (-1/p^2) e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp$ of integral (67) by part, noting that $-1/p^2 = d/dp\,(1/p - 1/p_0)$. This gives

$$\begin{aligned} & -\int_{p_0}^{\infty} \frac{1}{p^2} e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp \\ = & \left[\left(\frac{1}{p} - \frac{1}{p_0} \right) e^{-r/p_\varepsilon (H(p) - H(p_0))} \right]_0^{\infty} + \int_{p_0}^{\infty} \left(\frac{1}{p} - \frac{1}{p_0} \right) H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp \\ = & \int_{p_0}^{\infty} \left(\frac{1}{p} - \frac{1}{p_0} \right) H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp, \end{aligned} \tag{68}$$

because $H(p) \to \infty$ as $p \to \infty$. We manipulate the second term of the integral as follows:

$$\begin{aligned} & \int_{p_0}^{\infty} \frac{\alpha + \beta p}{p^2} \frac{r}{p_\varepsilon} \left[-2\alpha \left(\frac{1}{p} - \frac{1}{p_0} \right) + 2\beta \log\left(\frac{p}{p_0} \right) \right] e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp \\ = & \int_{p_0}^{\infty} \frac{H'(p)}{H'(p)} \frac{\alpha + \beta p}{p^2} \frac{r}{p_\varepsilon} \left[\frac{2\alpha (p - p_0)}{p p_0} + 2\beta \log\left(\frac{p}{p_0} \right) \right] e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp \\ = & \int_{p_0}^{\infty} \left[\frac{p}{\alpha + \beta p} \right]^2 \frac{\alpha + \beta p}{p^2} \left[\frac{2\alpha (p - p_0)}{p p_0} + 2\beta \log\left(\frac{p}{p_0} \right) \right] \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp \\ = & \int_{p_0}^{\infty} \frac{1}{\alpha + \beta p} \left[\frac{2\alpha (p - p_0)}{p p_0} + 2\beta \log\left(\frac{p}{p_0} \right) \right] \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))}\, dp, \end{aligned} \tag{69}$$

where the third line follows from the fact that $H'(p) = [(\alpha + \beta p)/p]^2$. Plugging (68) and (69) into the above equation (67) gives:

$$\frac{\partial W}{\partial \alpha} = \frac{r}{p_\varepsilon} \int_{p_0}^{\infty} \Phi(p, p_0) \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))} \, dp, \tag{70}$$

where

$$\Phi(p, p_0) = \left[\frac{1}{p} - \frac{1}{p_0}\right] + \frac{1}{\alpha + \beta p} \left[\frac{2\alpha(p - p_0)}{p p_0} + 2\beta \log\left(\frac{p}{p_0}\right)\right].$$

Now since $\Phi(p, p_0) \to -1/p_0$ as $p \to \infty$, there exists some p^* such that $\Phi(p, p_0) < -1/(2p_0)$ for all $p > p^*$. Letting $M^* = \sup_{p \in [p_0, p^*]} \Phi(p, p_0)$, equation (70) implies that

$$\begin{aligned}
\frac{\partial W}{\partial \alpha} &= \frac{r}{p_\varepsilon} \int_{p_0}^{p^*} \Phi(p, p_0) \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))} \, dp + \frac{r}{p_\varepsilon} \int_{p^*}^{\infty} \Phi(p, p_0) \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))} \, dp \\
&\leq \frac{r}{p_\varepsilon} \left\{ M^* \int_{p_0}^{p^*} \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))} \, dp - \frac{1}{2p_0} \int_{p^*}^{\infty} \frac{r}{p_\varepsilon} H'(p) e^{-r/p_\varepsilon (H(p) - H(p_0))} \, dp \right\} \\
&= \frac{r}{p_\varepsilon} \left\{ M^* \left(1 - e^{-r/p_\varepsilon (H(p) - H(p_0))}\right) - \frac{1}{2p_0} e^{-r/p_\varepsilon (H(p) - H(p_0))} \right\}
\end{aligned}$$

The term inside the curly brackets is negative as long as r is small enough, and we are done.

A.6 Proof of Lemma 1

We show that the filtration generated by Z and $\tilde{Z}$ are the same, and also that the filtration generated by $(\tilde{Z}, \tilde{z}_i)$ and (Z, z_i) are the same. First, after plugging Equation (23) into Equation (5) and (6), and after using the formula for Π_t and π_t, we find that

$$\begin{aligned}
dZ_t &= \Gamma_t \hat{X}_t \, dt + \gamma_t d\tilde{Z}_t \\
dz_{it} &= \Gamma_t \hat{X}_t \, dt + \gamma_t d\tilde{z}_{it}
\end{aligned}$$

Since, by construction, $\hat{X}$ is adapted to the filtration generated by $\tilde{Z}$, this implies that the filtration $\mathcal{F}_t$ generated by Z_t is included in the filtration $\tilde{\mathcal{F}}_t$ generated by $\tilde{Z}$. It also shows that the filtration $\mathcal{G}_{it}$ generated by (Z_t, z_{it}) is included in the filtration $\tilde{\mathcal{F}}_t \cup \tilde{\mathcal{F}}_{it}$ generated by $(\tilde{Z}_t, \tilde{z}_{it})$. To show the reverse inclusions, first rearrange the above equation into

$$d\tilde{Z}_t = \frac{1}{\gamma_t} \left(dZ_t - \Gamma_t \hat{X}_t \, dt\right) \tag{71}$$

$$d\tilde{z}_{it} = \frac{1}{\gamma_t} \left(dz_{it} - \Gamma_t \hat{X}_t \, dt\right) \tag{72}$$

Now, we also know from Proposition 1 that

$$d\hat{X}_t = \frac{\Pi_t}{P_t} \left[\left(x - \hat{X}_t\right) dt + \frac{dW_t}{\sqrt{\Pi_t}}\right] = \frac{\Pi_t}{P_t} \left(-\hat{X}_t \, dt + d\tilde{Z}_t\right)$$

After plugging equations (71) in the equation above and rearranging, we find:

$$d\hat{X}_t = \frac{\Pi_t}{P_t}\left\{-\left(1+\frac{\Gamma_t}{\gamma_t}\right)\hat{X}_t\,dt + \frac{1}{\gamma_t}\,dZ_t\right\}$$

Therefore, $\hat{X}$ is adapted to the filtration generated by Z. Together with (71), this means that $\tilde{Z}$ is adapted to the filtration $\mathcal{F}_t$ generated by Z. Together with (71) and (72) this implies that $(\tilde{Z}, \tilde{z}_i)$ is adapted to the filtration $\mathcal{G}_{it}$ generated by (Z, z_i).

A.7 Proof of Proposition 10

i) Follows from the fact that the flow welfare $u(p, P, \gamma) < 0$.

ii) Consider applying the control $\gamma_t = p_t/(p_t + P_t)$ for all t. The flow welfare is

$$u\left(p_t, P_t, \frac{p_t}{p_t + P_t}\right) = -\frac{1}{p_t + P_t} \geq -\frac{1}{p_0 + P_0} \tag{73}$$

where the inequality follows from the fact that, with $\gamma_t \geq 0$, both p_t and P_t are increasing functions of time. Integrating (73) against re^{-rt} from $t = 0$ to $t = \infty$ gives $v(p_0, P_0, \gamma) \geq -1/(p_0 + P_0)$, and taking the suppremum over all admissible control implies $V(p_0, P_0) \geq -\frac{1}{p_0+P_0}$.

iii) Consider some admissible control γ and the initial conditions (p_0, P_0) and (p_0', P_0), where $p_0' > p_0$. Let (p_t, P_t) and (p_t', P_t') be the corresponding time paths of precisions. The the ODEs (47) and (48) imply that $p_t' \geq p_t$ and $P_t' = P_t$, and therefore that

$$\begin{aligned} & u(p_t, P_t, \gamma_t) \geq u(p_t', P_t', \gamma_t) \\ \Rightarrow\quad & v(p_0, P_0, \gamma) \geq v(p_0', P_0, \gamma) \\ \Rightarrow\quad & V(p_0, P_0) \geq v(p_0', P_0, \gamma), \end{aligned} \tag{74}$$

where the third inequality follows from the definition of the value function $V(p_0, P_0)$. Since inequality (74) holds for any admissible control, we can take the suppremum over all admissible control to obtain $V(p_0', P_0) \geq V(p_0, P_0)$, implying that the value function is increasing in p_0. A similar proof shows that the value function is increasing in P_0.

iv) It is a direct consequence of point v).

v) **For the upper bound**. Consider two initial conditions (p_0, P_0) and (p_0', P_0'), with $p_0' > p_0$ and $P_0' > P_0$. By definition of a supremum, for any $\varepsilon > 0$, there exists some admissible control γ' such that

$$V(p_0', P_0') \leq \int_0^\infty re^{-rt} u(p_t', P_t', \gamma_t')\,dt + \varepsilon, \tag{75}$$

where (p_t', P_t') denotes the associated time path of precisions. In addition, by definition of the value function

$$V(p_0, P_0) \geq \int_0^\infty re^{-rt} u(p_t, P_t, \gamma_t)\,dt, \tag{76}$$

where (p_t, P_t) is the time path of precision obtained by also applying γ' but starting at (p_0, P_0). Note that, because we apply the same control starting at (p_0', P_0') and (p_0, P_0), ODEs (47) and (48) imply that

$$p_t' - p_t = p_0' + \int_0^t p_\varepsilon \gamma_t'^2 \, dt - p_0 - \int_0^t p_\varepsilon \gamma_t'^2 \, dt = p_0' - p_0 \tag{77}$$

$$P_t' - P_t = P_0' + \int_0^t P_\varepsilon \gamma_t'^2 \, dt - P_0 - \int_0^t P_\varepsilon \gamma_t'^2 \, dt = P_0' - P_0 \tag{78}$$

Now subtracting inequality (76) to (75) gives

$$\begin{aligned}
0 \le V(p_0', P_0') - V(p_0, P_0) &\le \int_0^\infty re^{-rt} \left(u(p_t, P_t, \gamma_t') - u(p_t, P_t, \gamma_t')\right) dt + \varepsilon \\
&= \int_0^\infty re^{-rt} \left[\gamma_t'^2 \left(\frac{1}{p_t} - \frac{1}{p_t'}\right) + (1-\gamma_t')^2 \left(\frac{1}{P_t} - \frac{1}{P_t'}\right)\right] dt + \varepsilon \\
&\le \int_0^\infty re^{-rt} \left(\frac{p_t' - p_t}{p_t p_t'} + \frac{P_t' - P_t}{P_t P_t'}\right) dt + \varepsilon \qquad (79) \\
&\le \int_0^\infty re^{-rt} \left(\frac{p_0' - p_0}{p_0^2} + \frac{P_0' - P_0}{P_0^2}\right) dt + \varepsilon \qquad (80) \\
&= \frac{p_0' - p_0}{p_0^2} + \frac{P_0' - P_0}{P_0^2} + \varepsilon.
\end{aligned}$$

where inequality (79) follows from the fact that $p_t' \ge p_t$, $P_t' \ge P_t$ and $\gamma_t \in [0,1]$, and inequality (80) follows from (77), (78) together with the fact that $p_t' \ge p_t \ge p_0$ and $P_t' \ge P_t \ge P_0$. Taking ε to zero, the upper bound obtains.

For the lower bound. For any $\varepsilon > 0$, there is some control γ be such that $v(p_0, P_0, \gamma) + \varepsilon > V(p_0, P_0)$. Let (p_t, P_t) denotes the time path of precisions starting from (p_0, P_0) and applying control γ. Let (p_t', P_t') be the time path of precisions applying the same control γ but starting instead from (p_0', P_0'). Note that as before equations (77) and (78) hold. Given that $\gamma \le 1$, we have that $p_t \le p_0 + p_\varepsilon t$ and $P_t \le P_0 + P_\varepsilon t$. Then it follows that

$$\begin{aligned}
V(p_0', P_0') - V(p_0, P_0) &\ge v(p_0', P_0', \gamma) - v(p_0, P_0, \gamma) - \epsilon \\
&= \int_0^\infty re^{-rt} \left[\gamma_t^2 \left(\frac{1}{p_t} - \frac{1}{p_t'}\right) + (1-\gamma_t)^2 \left(\frac{1}{P_t} - \frac{1}{P_t'}\right)\right] dt - \epsilon \\
&= \int_0^\infty re^{-rt} \left[\gamma_t^2 \frac{p_0' - p_0}{p_t p_t'} + (1-\gamma_t)^2 \frac{P_0' - P_0}{P_t P_t'}\right] dt - \varepsilon \\
&\ge \int_0^T re^{-rt} \left[\gamma_t^2 \frac{p_0' - p_0}{(p_t')^2} + (1-\gamma_t)^2 \frac{P_0' - P_0}{(P_t')^2}\right] dt - \varepsilon \\
&\ge \int_0^T re^{-rt} \left[\frac{1}{\frac{(p_t')^2}{p_0' - p_0} + \frac{(P_t')^2}{P_0' - P_0}}\right] dt - \varepsilon \qquad (81) \\
&\ge \int_0^T re^{-rt} \left[\frac{1}{\frac{(p_0' + p_\varepsilon T)^2}{p_0' - p_0} + \frac{(P_0' + P_\varepsilon T)^2}{P_0' - P_0}}\right] dt - \varepsilon \qquad (82)
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^T re^{-rt}\left[\frac{1}{\max\left\{\frac{(p_0'+p_\varepsilon T)^2}{p_0'-p_0},\frac{(P_0'+P_\varepsilon T)^2}{P_0'-P_0}\right\}}\right]dt-\varepsilon\\
&\geq \int_0^T re^{-rt}\left[\min\left\{\frac{p_0'-p_0}{(p_0'+p_\varepsilon T)^2},\frac{P_0'-P_0}{(P_0'+P_\varepsilon T)^2}\right\}\right]dt-\varepsilon\\
&\geq (1-e^{-rT})\min\left\{\frac{p_0'-p_0}{(p_0'+p_\varepsilon T)^2},\frac{P_0'-P_0}{(P_0'+P_\varepsilon T)^2}\right\}-\varepsilon
\end{aligned}
$$

where equation (81) follows from using calculating the minimum of the term in brackets, and equation (82) follows because $p_t' \leq p_0' + p_\varepsilon T$ and $P_t' \leq P_0 + P_\varepsilon T$ for $t \in [0, T]$. Taking ε to zero the lowerbound obtains.

A.8 Proofs of Proposition 11, Corollary 3, Proposition 12, and Corollary 4

Proof of Proposition 11. For every (p, P), we let

$$
\Phi(p,P) \quad \equiv \quad \left\{\lambda\left[u(p,P,\gamma) \quad p_\varepsilon\gamma^2 \quad P_\varepsilon\gamma^2\right]+(1-\lambda)\left[u(p,P,\gamma') \quad p_\varepsilon\gamma'^2 \quad P_\varepsilon\gamma'^2\right],(\gamma,\lambda)\in[0,1]^2\right\},
$$

and we denote by (u, q, Q) a generic element of $\Phi(p, P)$. Lastly, we let the upper Dini derivative at (p, P) with direction (q, Q) be given by,

$$
\partial V^+(p,P;q,Q) \equiv \limsup_{t\to 0^+,(q',Q')\to(q,Q)} \frac{V(p+q't,P+Q't)-V(p,P)}{t}. \tag{83}
$$

In our case, given that V is locally Lipschitz continuous, equation (83) collapses to[7]

$$
\partial V^+(p,P;q,Q) \equiv \limsup_{t\to 0^+} \frac{V(p+qt,P+Qt)-V(p,P)}{t}. \tag{84}
$$

Note that the directional derivative $\partial V^+(p, P; q, Q)$ is positively homogeneous. That is, for any $\mu \geq 0$, $\partial V^+(p, P; \mu q, \mu Q) = \mu \partial V^+(p, P; q, Q)$.

Proposition 2.8, page 104 in Bardi and Capuzzo-Dolcetta [1997] states that the value function is a

[7] The proof is as follows. Take a sequence (q_n, Q_n) converging to (q, Q) and a sequence (t_n) converging to zero. Suppose that $(V(p + q_n t_n, P + Q_n t_n) - V(p, q))/t_n$ has a limit ℓ. We can write

$$
\begin{aligned}
&\left|\frac{V(p+q_nt_n,P+Q_nt_n)-V(p,P)}{t_n}-\ell\right|\\
\leq\quad &\left|\frac{V(p+q_nt_n,P+Q_nt_n)-V(p+qt_n,P+Qt_n)}{t_n}\right|+\left|\frac{V(p+q_nt_n,P+Q_nt_n)-V(p,P)}{t_n}-\ell\right|\\
\leq\quad & K\left|\max\{q_n-q,Q_n-Q\}\right|+\left|\frac{V(p+qt_n,P+Qt_n)-V(p,P)}{t_n}-\ell\right|.
\end{aligned}
$$

where the last equality follows from the local Lipschitz condition for n large enough. Note that the first term on the right-hand side of the last inequality converges to zero. This means that the sequences $(V(p + qt_n, q + Qt_n) - V(p, q))/t_n$ has the same limit ℓ as the sequence $(V(p + q_n t_n, q + Q_n t_n) - V(p, q))/t_n$. Taking the supremum of all such limits shows that the lim sup in (83) coincides with the lim sup in (84).

viscosity solution of an appropriate HJB equation.[8]

Then we use Theorem 2.40, in page 128, together with Remark 2.43, page 131, to show that value function solves the following HJB equation,

$$rV(p,P) = \sup_{(u,q,Q)\in\Phi(p,P)} \left\{u + \partial V^{+}(p,P;q,Q)\right\}. \tag{85}$$

Note that, if $(u,q,Q) \in \Phi(p,P)$, then

$$\begin{bmatrix} q & Q \end{bmatrix} = (\lambda\gamma^2 + (1-\lambda)\gamma'^2) \begin{bmatrix} p_\varepsilon & P_\varepsilon \end{bmatrix},$$

for some $(\gamma,\gamma',\lambda) \in [0,1]^3$. Together with the fact that the directional derivative is positively homogeneous, this implies that

$$\partial V^{+}(p,P;q,Q) = (\lambda\gamma^2 + (1-\lambda)\gamma'^2)\partial V^{+}(p,P;p_\varepsilon,P_\varepsilon).$$

Therefore, for all $(u,q,Q) \in \Phi(p,P)$, there exists $(\gamma,\gamma',\lambda) \in [0,1]^3$ such that

$$\begin{aligned} & u + \partial V^{+}(p,P;q,Q) \\ = \; & \lambda u(p,P,\gamma) + (1-\lambda)u(p,P,\gamma') + (\lambda\gamma^2 + (1-\lambda)\gamma'^2)\partial V^{+}(p,P;p_\varepsilon,P_\varepsilon) \\ \leq \; & u\left(p,P,\sqrt{\lambda\gamma^2 + (1-\lambda)\gamma'^2}\right) + (\lambda\gamma^2 + (1-\lambda)\gamma'^2)\partial V^{+}(p,P;p_\varepsilon,P_\varepsilon) \qquad (86) \\ = \; & u\left(p,P,\gamma''\right) + \gamma''^2\partial V^{+}(p,P;p_\varepsilon,P_\varepsilon). \end{aligned}$$

where inequality (86) follows from the concavity of $g \mapsto u(p,P,\sqrt{g})$ and $\gamma'' \equiv \sqrt{\lambda\gamma^2 + (1-\lambda)\gamma'^2} \in [0,1]$. Plugging this inequality back into the HJB equation (85) implies that

$$rV(p,P) = \sup_{\gamma\in[0,1]} \left\{u(p,P,\gamma) + \gamma^2\partial V^{+}(p,P;p_\varepsilon,P_\varepsilon)\right\}. \tag{87}$$

For the existence of the optimal control we use Theorem 2.61 part (ii) in page 142 of Bardi and Capuzzo-Dolcetta [1997] together with Remark 2.62 in page 142, which imply that

$$\gamma_t^* = \gamma^*(p_t^*, P_t^*) \equiv \arg\max_{\gamma\in[0,1]} \left\{u(p_t^*, P_t^*, \gamma) + \gamma^2\partial V^{+}(p_t^*, P_t^*; p_\varepsilon, P_\varepsilon)\right\} \tag{88}$$

where $p_t^* = p_0 + \int_0^t (\gamma_t^*)^2 dt$ and $P_t^* = P_0 + \int_0^t (\gamma_t^*)^2 dt$ is an optimal control.

Proof of Corollary 3 and Proposition 12. We first note that $V(p,P) < 0$. Therefore, evaluating the right-hand side of (87) at $\gamma = 1$, we find that $0 > rV(p,P) \geq -r/p + \partial^{+}V(p,P;p_\varepsilon,P_\varepsilon)$, so that $\partial V^{+}(p,P;p_\varepsilon,P_\varepsilon) < r/p$. This implies that the right-hand side of (88) is a strictly concave function of γ.

[8] The infinite-horizon optimal control problem of Bardi and Capuzzo-Dolcetta [1997] is formulated with the state space $\mathbb{R}^N$ and requires that the utility flow function to be bounded. This is different from our problem in which the state space is $\mathbb{R}^2_+$ and the utility flow unbounded as either p or P go to zero. In appendix A.11 we show that we can nevertheless apply their results by fixing some strictly positive $(p_{\min}, P_{\min})$ and using the change of variable $p = \max\{x, p_{\min}\}$ and $P = \max\{X, P_{\min}\}$, where $(x,X) \in \mathbb{R}^2$.

Therefore, taking first-order conditions shows that the unconstrained optimum of (87) is

$$\gamma^*(p,P) = \frac{p}{p+P-pP\partial V^+(p,P;p_\varepsilon,P_\varepsilon)/r}.$$

That $\partial V(p,P;p_\varepsilon,P_\varepsilon) \geq 0$ implies that $\gamma^*(p,P) \geq p/(p+P)$. That $\partial V^+(p,P;p_\varepsilon,P_\varepsilon) < r/p$ implies that $\gamma < 1$.

Proof of Corollary 4. In order to show Corollary 4 first note that, because ODEs (47) and (48) are proportional to each others, it follows that $P_t^* = \alpha + (\beta-1)p_t^*$, where $\beta = 1 + P_\varepsilon/p_\varepsilon$ and $\alpha = P_0 - P_\varepsilon/p_\varepsilon p_0$. plugging this back into the ODE (47) and applying Proposition 12, we obtain

$$\begin{aligned}
& \dot{p}_t^* \geq p_\varepsilon \left(\frac{p_t^*}{\alpha+\beta p_t^*}\right)^2 \\
\Rightarrow \quad & \dot{p}_t^* \left(\frac{\alpha^2}{(p_t^*)^2} + \frac{2\alpha\beta}{p_t^*} + \beta^2\right) \geq p_\varepsilon \\
\Rightarrow \quad & \frac{d}{dt} H(p_t^*) \geq p_\varepsilon, \qquad (89)
\end{aligned}$$

almost everywhere $t \geq 0$ and where $H(p) = -\alpha^2/p + 2\alpha\beta\log(p) + \beta^2 p$. We can integrate both sides of (89) to find that $H(p_t^*) \geq H(p_0) + p_\varepsilon t$. Since $H(p)$ is increasing and goes to infinity as t goes to infinity, we find that p_t^* and hence P_t^* both go to infinity as t goes to infinity.

A.9 Proof of Theorem 3

1. **Proof of the first part**. First note that (49) implies that

   $$0 \leq \frac{V(p+p_\varepsilon t, P+P_\varepsilon t) - V(p,P)}{t} \leq \frac{p_\varepsilon}{p^2} + \frac{P_\varepsilon}{P^2}$$

   and therefore that

   $$\partial^+ V(p,P;p_\varepsilon,P_\varepsilon) \leq \frac{p_\varepsilon}{p^2} + \frac{P_\varepsilon}{P^2} \qquad (90)$$

   Consider the optimal control γ_t^* and the associated time path (p_t^*, P_t^*) of precisions. Plugging (90) into (52) we find that

   $$0 \leq \gamma_t^* \leq \frac{p_t^*}{p_t^* + P_t^* - p_\varepsilon P_t^*/p_t^* - P_\varepsilon p_t^*/P_t^*}. \qquad (91)$$

   Corollary 4 shows that both p_t^* and P_t^* go to infinity as t goes to infinity. In addition, the ODEs (47) and (48) are proportional to one another, implying that $P_t^* = \alpha + (\beta-1)p_t^*$, where, as usual, $\alpha = P_0 - P_\varepsilon/p_\varepsilon p_0$ and $\beta = 1 + P_\varepsilon/p_\varepsilon$. Therefore, both p_t^*/P_t^* and P_t^*/p_t^* have finite limits as t goes to infinity. Hence, (91) implies that $|\gamma_t^* - p_t^*/(p_t^* + P_t^*)|$ goes to zero as t goes to infinity.

2. **Proof of the second part**. We first show a preliminary result.

   Lemma 3. *For every $P_0 > 0$, $V(p,P)$ can be extended by continuity at $(0,P_0)$.*

   Proof. In order to prove the Lemma, we first note that $\lim_{p\to 0^+} V(p,P_0)$ exists because $p \mapsto V(p,P_0)$

is increasing and bounded below. Denoting this limit by $L(0, P_0)$, we then write, for any (p, P),

$$\begin{aligned}|L(0,P_0) - V(p,P)| &\leq |L(0,P_0) - V(p,P_0)| + |V(p,P_0) - V(p,P)| \\ &\leq |L(0,P_0) - V(p,P_0)| + \frac{P_\varepsilon}{(\min\{P,P_0\})^2}|P - P_0|\end{aligned}$$

where the second inequality follows from (49). The first term on the right-hand side goes to zero by definition of $L(0, P_0)$. Since the second term on the right-hand side also goes to zero as $P \to P_0$, we have shown that $V(p, P)$ goes to $L(0, P_0)$ as (p, P) goes to $(0, P_0)$, and we can extend $V(p, P)$ by continuity at $(0, P_0)$ with $V(0, P_0) = L(0, P_0)$. □

Now solving $\gamma^*(p, P) = p/(p + P - pP\partial^+ V(p, P; p_\varepsilon, P_\varepsilon)/r)$ for $\partial^+ V(p, P)$ shows that

$$\gamma^*(p,P)^2 \partial V^+(p,P;p_\varepsilon,P_\varepsilon) = r\left[\frac{\gamma^*(p,P)^2}{p} + \frac{\gamma^*(p,P)^2 - \gamma^*(p,P)}{P}\right].$$

Plugging this back into the HJB equation and rearranging shows that

$$V(p,P) = -\frac{1}{P}\left(1 - \gamma^*(p,P)\right). \tag{92}$$

We now show

Lemma 4. *For any $P_0 > 0$,*

$$\lim_{(p,P)\to(0,P_0)} V(p,P) = -\frac{1}{P_0}. \tag{93}$$

Proof. Lemma 3 shows that the limit of equation (93). Suppose that it is strictly greater than $-1/P_0$. Equations (92) then implies that the control $\gamma^*(p, P)$ also has a limit as (p, P) goes to $(0, P_0)$, and that this limit is strictly positive. Hence, there exists some $\delta > 0$ and $k > 0$ such that, if $\max\{|p|, |P - P_0|\} < \delta$, then $\gamma^*(p, P) > k$.

Now consider any initial condition $(\hat{p}_0, \hat{P}_0)$ such that $\max\{|\hat{p}_0|, |\hat{P}_0 - P_0|\} < \delta/2$. Let $\hat{\gamma}_t^*$ be the associated optimal control and $(\hat{p}_t^*, \hat{P}_t^*)$ the associated time path of precision. Because the optimal control is less than 1, the rate of change of precisions is bounded above, implying that there exists some time T such that, for *any* initial condition $(\hat{p}_0, \hat{P}_0)$ such that $\max\{|\hat{p}_0|, |\hat{P}_0 - P_0|\} < \delta/2$, we have $\max\{|\hat{p}_t^*|, |\hat{P}_t^* - P_0|\} < \delta$ for all $t \in [0, T]$, and therefore $\hat{\gamma}_t^* = \gamma^*(\hat{p}_t^*, \hat{P}_t^*) > k$. So now we can write

$$\begin{aligned}V(\hat{p}_0,\hat{P}_0) &= -\int_0^\infty r\left(\frac{(\hat{\gamma}_t^*)^2}{\hat{p}_t^*} + \frac{(1-\hat{\gamma}_t^*)^2}{\hat{P}_t^*}\right)e^{-rt}\,dt \\ &\leq -\int_0^T \frac{rk^2}{\hat{p}_t^*}e^{-rt}\,dt \\ &\leq -\int_0^T \frac{rk^2}{\hat{p}_0 + p_\varepsilon t}e^{-rt}\,dt \\ &= e^{-rT} rk^2/p_e \log\left(\frac{\hat{p}_0 + p_\varepsilon T}{\hat{p}_0}\right).\end{aligned}$$

We obtain the first inequality because: i) the integrand is negative so the integral over $[0, \infty)$ is bounded above by the integral over the smaller interval $[0, T]$, ii) $-\gamma_t^2/p_t - (1-\gamma_t)^2/P_t \leq -\gamma_t^2/p_t$, and

iii) $\hat{\gamma}^*_t > k$. The second inequality follows from the fact that $\gamma^*_t \leq 1$ which implies that $\hat{p}^*_t \leq \hat{p}_0 + p^*_\varepsilon t$. The third inequality follows by direct integration. Letting $(\hat{p}_0, \hat{P}_0)$ go to $(0, P_0)$ then shows that $V(\hat{p}_0, \hat{P}_0)$ goes to minus infinity, which is a contradiction. □

The second part of the Theorem then follows from the above lemma together with equation (92).

A.10 Proof of Proposition 14

That $b^* < 0$ follows directly from $\gamma^*(p, P) > p/(p + P)$. The only thing left to show is the limit of b^*_t at infinity, which follows from equation (90).

A.11 An Auxiliary Planner's Problem

In order to apply the results of Bardi and Capuzzo-Dolcetta [1997], we need to formulate the optimal control problem in $\mathbb{R}^N$ and make sure that the utility flow is bounded. To that end, we fix some strictly positive $(p_{\min}, P_{\min})$ and we consider the auxiliary utility flow

$$u(p_t, P_t, \gamma; p_{\min}, P_{\min}) = -r\left(\frac{\gamma_t^2}{\max\{p_t, p_{\min}\}} + \frac{(1-\gamma_t)^2}{\max\{P_t, P_{\min}\}}\right).$$

As before, we let an *admissible control* be some measurable function $\gamma : \mathbb{R}_+ \to [0, 1]$. Given an admissible control, the state (p_t, P_t) evolves according exactly as before according to the ODEs (47) and (48), where the initial condition $(p_0, P_0) \in \mathbb{R}^2$ is given. In particular, in this auxiliary problem, we do not restrict the state to be positive. The planner's intertemporal utility is

$$v(p_0, P_0, \gamma) = -\int_0^\infty u(p_t, P_t, \gamma_t)e^{-rt}\, dt.$$

and the *planner's problem* is to find the supremum of $v(p_0, P_0, \gamma)$, subject to the constraint that γ is an admissible control. An *optimal control* is a solution of the planner's problem. We let $V(p_0, P_0; p_{\min}, P_{\min})$ be the planner's maximum attainable utility.

Because the utility flow of this auxiliary planner's problem is bounded, it is easy to check that the regularity conditions (A0)-(A4) page 97-88 in Bardi and Capuzzo-Dolcetta [1997] hold. This allows us to apply Proposition 2.8 page 104 showing that the value function solves an HJB equation. In order to state the HJB equation with the generalized directional derivative, as in Proposition 11 of the present paper, we need to show that the value function is locally Lipschitz continuous, as follows: going through exactly the same steps as in the proof of Proposition 10, it is easy to show that the following modified Lipchitz condition holds. For all $p' \geq p$ and $P' \geq P$,

$$V(p', P'; p_{\min}, P_{\min}) - V(p, P; p_{\min}, P_{\min}) \leq \frac{p' - p}{(\max\{p, p_{\min}\})^2} + \frac{P' - P}{(\max\{P, P_{\min}\})^2}.$$

Now note that the planner's value in the auxiliary and the original problems coincide for all $(p, P) \geq (p_{\min}, P_{\min})$. Indeed, consider such an initial condition and any admissible control in the auxiliary problem. Then, because precision is increasing, we have $p_t \geq p_{\min}$ and $P_t \geq P_{\min}$ at all times. Therefore, we can drop the max in the utility flow meaning that $v(p, P, \gamma) = v(p, P, \gamma; p_{\min}, P_{\min})$. Taking the supremum over all admissible control shows that $V(p,P) = V(p,P;p_{\min},P_{\min})$ for all $p \geq p_{\min}$ and $P \geq P_{\min}$. Clearly, the two functions also share the same generalized directional derivative. Hence, they must solve the same HJB for all $p \geq p_{\min}$ and $P \geq P_{\min}$.

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