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Abstract

I use a sequential-auction model to mimic the environment of Internet auction sites, such as eBay. For a sequence of auctions, new buyers may enter the auction site after some of the auctions has completed and only bid for the remaining auctions. Because an incumbent buyer may have revealed their own valuation in earlier auctions while a new entrant do not, their expectations about the future are asymmetric. As a result, a buyer with a lower valuation may win an auction while a buyer with a higher valuation may restrain from bidding higher, resulting an inefficient allocation. On the contrary, selling the multiple items in a single simultaneous auction results in an efficient outcome. The profit from selling all items together in one simultaneous auction is less than that from selling them sequentially.

1 Introduction

Internet auctions have grown rapidly in the past decade. Although there have been many researches focusing on Internet auctions[1], one important feature of Internet auctions remain lack of attention. Contrary to most traditional brick-and-mortar auctions, buyers do not arrive at an Internet auction site simultaneously. For a sequence of auctions, new buyers may enter the auction site only later in the sequence.

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[1]See Bajari and Hortaçsu (2004); Ockenfels, Reiley, and Sadrieh (2006) for surveys on these studies.

In the tradition models of sequential auctions, all buyers arrive before the first auction takes place. (See, for example, Weber (1983); Milgrom and Weber (2000).) An auction begins after the previous one has completed. If a buyer does not win in the previous auction, she can participate in the next one. Because of the future options, buyers would lower their bids in the previous auction. With *independent private* values, Weber (1983) shows that the expected price of each auction is the same for both the sequential first-price and second-price auctions because the effect of winners leaving the auction site and the effect of bidders shading their bids in early auctions to account for future options cancel out. Huang, Chen, Chen, and Chow (2008) propose a model to analyze Internet auctions with buyer entry. They find parallel auctions can be viewed as sequentially auctions because buyers only bid on the auction which is the first one to end among all remaining auctions. The allocation is efficient even when entry is possible. Nonetheless, they only analyze independent private valuations among buyers. In this paper, I find that the efficiency does not hold if valuations are allowed to be affiliated.

When buyers' valuations are *affiliated*, the analysis is much more complicated. There is still no complete answer for a fully general auction model. Milgrom and Weber (2000) find that sequential first-price auctions would have ascending expected prices because buyers are willing to bid more in later auctions after observing the information revealed in earlier auctions. As for sequential second-price auctions, Wang (2006) shows that the expected prices also ascend in a sequential second-price auction with *affiliated private* values. Zeithammer (2006) extends the model of sequential auctions from selling identical items to selling heterogenous items. Contrary to the theoretical findings about affiliated value auctions, empirical studies tend to find descending prices across auctions.[2]

The model in this paper consists of a sequence of ascending auctions. Each of them sells an identical good. All buyers have inelastic unit demand for the good. Buyers enter the auction site sequentially and stay until winning a good. I characterize a perfect Bayesian equilibrium in this game. An important result of this paper is that the allocation of the auction mechanism

[2]For instance, see Ashenfelter (1989).

is not necessarily efficient. It is possible that a buyer with a lower valuation obtains the good while a buyer with a higher valuation does not. Allowing new buyers to enter *per se* does not necessarily result in inefficiency in the allocation of the items. However, when buyers' valuations are affiliated, inefficiency results from the information asymmetry among buyers.

The driving force behind the the inefficiency is the asymmetry of buyers. While some buyers participate in an earlier auction and reveal their own valuations to all other buyers by submitting bids, new buyers do not. Since new buyers also know their own private valuations, they have better information than old buyers. Because of affiliation in valuations, the asymmetry in turn creates different expectations about the valuations of future incoming buyers. New buyers get information rents from their better information and bid *less* aggressively, other things being equal. Hence, the allocation of the sequential auctions is not always efficient.

I also find transaction prices to be identical across auctions in the sequential auction model considered in this paper. This price is higher than the price of selling all the items together in a single multiple-object auction. Consequently, in order to maximize the expected profit, a seller with several identical items to sell should offer them in a sequence of single-object auctions rather than in one multiple-object simultaneous auction.

The inefficiency remains possible if sequential ascending auctions are replaced by sequential sealed-bid second-price auctions. Information asymmetry is still a reason for inefficiency. Nonetheless, in addition to selling the good to the less informed buyer when she has a lower valuation, efficiency may also occur in the opposition case, selling the good to the more informed buyer when she has a lower valuation.

The rest of the paper is organized as the following. In the next section, I introduce a model which mimics sequential eBay auctions with buyer entries. A perfect Baysian equilibrium is presented in Section 3. In Section 4, I compare the eBay type auction with other formats of auctions. To provide a concrete idea of the model equilibrium, I show an numerical example in Section 5. Concluding remarks are in the final section.

2 Model

Consider an affiliated private value model with three auctions and four buyers. Under the *private* value assumption, each buyer has perfect information about her own valuation for the item. Under the *affiliated* value assumption, buyers' valuations are positively correlated. When there are only two auctions, the information obtained in the first auction is useless in the second auction because it is a dominant strategy to bid one's own valuation in the second (final) auction. The simplest model to show the effect of asymmetric information is to have three auctions and four buyers. The model can be extended to have more auctions without changing the result of the information asymmetry among buyers.

Following the empirical findings in Zeithammer and Adams (2006), I model eBay auctions as ascending auctions.[3] Assume that three auctions, each with one identical object to be sold, will be held sequentially in periods $t = 1, 2, 3$, respectively.[4]

One buyer enters in each period $t = 0, 1, 2, 3$. Let the random variable V_i represent the valuation of the buyer who enters in period $i - 1$, $i = 1, 2, 3, 4$. The realized value of V_i is denoted as v_i. While the joint distribution of (V_1, V_2, V_3, V_4) is common knowledge among all buyers, the realized value v_i is not. The valuations V_i are affiliated in the sense that the conditional expectation $E[V_i|V_{-i} = v_{-i}]$ increases in each components of v_{-i}, where $v_{-i} \equiv (v_1, \ldots, v_{-i}, v_{i+1}, \ldots, v_4)$. The distribution function of (V_1, V_2, V_3, V_4) is non-degenerate. The support of each V_i is an interval $[\underline{v}, \overline{v}]$. To simplify the exposition, assume the joint distribution of V_1 and V_2 are symmetric. Every buyer is risk-neutral and demands only one object. It is costless to bid. There is no discount between periods. A buyer's surplus is her valuation v_i less the transaction price if she wins in one of the auctions. The surplus is zero otherwise. A buyer wants to maximize her surplus. Since all buyers has inelastic unit demand, a winning bidder will not participate in future auctions.

Auctions are conducted as a "button auction" in each period. All buyers press on a

[3]They empirically test the bidding behavior on eBay and find the data are better described by ascending auctions rather than second-price sealed-bid auctions.

[4]I also discuss a sequence of sealed-bid second-price auctions tower the end of paper.

button at the beginning of a period. The standing price starts from $\underline{v}$ and keeps rising. A buyer depresses the button once she decides to quit from the current auction. When a buyer quits from one auction, she cannot re-enter the same auction. The standing price rises until only one buyer left. The last remaining buyer wins the auction and pays the final standing price. The final standing price is announced to all buyer, including future buyers.

Since buyers' valuations are random variables, the transaction prices are also random variables. Let P_t^* denote the transaction price of the auction in period t. Its realized value is represented by the lowercase p_t^*.

3 Perfect Bayesian Equilibrium

In this section, I will derive a perfect Baysian equilibrium of this three-period auction game by backward induction. Because the joint distribution of (V_1, V_2) is symmetric, without loss of generality, I only need to consider subgames after the histories with $V_1 \geq V_2$. For the subgames after the histories $V_1 \leq V_2$, I simply need to exchange the scripts 1 and 2 in the analysis. Therefore, I will assume $V_1 \geq V_2$ for the rest of the paper.

3.1 The Final Auction

As is well-known in the literature, a buyer's dominant strategy in a second-price ascending auction is to bid until her own valuation. The bidding strategy of a buyer with valuation v in the final auction is to bid until the price rises to her own valuation, $\beta_3(v) = v$. This strategy is independent of information. As a result, the winner of Auction 3 is the buyer with the higher valuation in this period. The transaction price equals to the lower valuation among the two buyers.

3.2 The Second Auction

Consider the history after the buyer with the higher valuation in period one has won the first auction. The two buyers active in period two have valuations v_2 and v_3, respectively. Since the dominant strategy in the final auction is to bid until a buyer's own valuation, regardless of other buyers' valuations. There is no strategic concern in the second auction to conceal one's own valuation.

Suppose the bidding strategy in the first auction is a strictly increasing function of a buyer's valuation. Then the realized valuation v_2 can be inferred from the transaction price of Auction 1, p_1^*, and the bidding function for the first auction β_1. Denote the inferred value of Buyer 2's valuation by $\hat{v}_2 = \beta_1^{-1}(p_1^*)$, it is public information in this period. In the equilibrium path, $\hat{v}_2 = v_2$. On the contrary, the new buyer's valuation v_3 is her private information. The two buyers process *different information sets*. Consequently, they would have different expectations on the valuation about the future entrant, V_4, which in turn affect their bidding strategies.

Let $S_2^n(v_3; \hat{v}_2)$ be the *new* buyer's expected surplus of participating in the next auction, where v_3 is her own valuation and $\hat{v}_2$ is the inferred value of her opponent's valuation. Note that the current standing price p_2 does not enter the expected payoff on the equilibrium path because there is no information gain from observing the price.

$$
\begin{aligned}
S_2^n(v_3; \hat{v}_2) &= \int_{\underline{v}}^{v_3} (v_3 - v_4) dF_4(v_4 | V_2 = \hat{v}_2, V_3 = v_3) \\
&= \int_{\underline{v}}^{v_3} F_4(v_4 | V_2 = \hat{v}_2, V_3 = v_3) dv_4.
\end{aligned} \tag{1}
$$

Buyer 3's expected surplus depends on Buyer 2's valuation $\hat{v}_2$ because her surplus depends on the distribution of Buyer 4's valuation, which is affiliated with V_2.

The following lemma shows Buyer 3's bidding strategy for the second auction.

Lemma 1. *Buyer 3 keeps bidding on Auction 2 until the payoff from the Auction 2 is just indifferent to the expected future payoff,* $v_3 - p_2 = S_2^n(v_3; \hat{v}_2)$.

Proof. The surplus from winning Auction 2 at price p_2 is $v_3 - p_2$. It decreases in the standing price p_2. Because the expected surplus of entering the next auction $S_2^n(v_3; \hat{v}_2)$, which is independent of the price p_2. Therefore, there exists a price p^* such that $v_3 - p^* \geq S_2^n(v_3; \hat{v}_2)$

if and only if $p_2 \leq p^*$. Buyer 3 prefers Auction 2 over Auction 3 if and only if $p^2 < p^*$. $\square$

The maximal amount Buyer 3 is willing to bid on Auction 2 can be denoted as

$$\beta_2^n(v_3; \hat{v}_2) = v_3 - S_2^n(v_3; \hat{v}_2). \tag{2}$$

Lemma 2. *$\beta_2^n(v_3; \hat{v}_2)$ is strictly increasing in v_3 for any given $\hat{v}_2$.*

Proof. For any $v_3' > v_3$,

$$\begin{aligned}\beta_2^n(v_3'; \hat{v}_2) - \beta_2^n(v_3; \hat{v}_2) = &- \int_{\underline{v}}^{v_3} \left[F_4(v_4|V_2 = \hat{v}_2, V_3 = v_3') - F_4(v_4|V_2 = \hat{v}_2, V_3 = v_3)\right] dv_4 \\ &+ \int_{v_3}^{v_3'} \left[1 - F_4(v_4|V_2 = \hat{v}_2, V_3 = v_3')\right] dv_4.\end{aligned}$$

Because V_3 and V_4 are affiliated, $F_4(v_4|V_2 = \hat{v}_2, V_3 = v_3') \leq F_4(v_4|V_2 = \hat{v}_2, V_3 = v_3)$ for all v_4. In addition, $F_4(v_3|V_2 = v_2, V_3 = v_3') < 1$ for $v_3 < \overline{v}$. Consequently, $\beta_2^n(v_3; v_2)$ is strictly increasing in $v_3 \in (0, \overline{v})$. By continuity, $\beta_2^n(v_3; v_2)$ is strictly increasing in v_3 on the entire support $[0, \overline{v}]$. $\square$

Because of the monotonicity of $\beta_2^n(\,\cdot\,; \hat{v}_2)$, it is invertible for any given $\hat{v}_2$. Denote the inverse function by

$$\hat{v}_3(\,\cdot\,; \hat{v}_2) \equiv [\beta_2^n]^{-1}(\,\cdot\,; \hat{v}_2),$$

which Buyer 2 uses to update her expectation about Buyer 3's valuation.

Next, consider the old buyer (Buyer 2)'s decision in Auction 2. Her expected surplus of entering the future auction is

$$\begin{aligned}S_2^o(v_2; \hat{v}_3) &= \int_{\underline{v}}^{v_2} (v_2 - v_4) dF_4(v_4|V_2 = v_2, V_3 \geq \hat{v}_3) \\ &= \int_{\underline{v}}^{v_2} F_4(v_4|V_2 = v_2, V_3 \geq \hat{v}_3) dv_4,\end{aligned} \tag{3}$$

where $\hat{v}_3 = \hat{v}_3(p_2; \hat{v}_2)$ is the inferred valuation of Buyer 3. While Buyer 3 knows both buyers' valuations, Buyer 2 only knows her own valuation v_2 for sure and infers a lower bound for Buyer 3's valuation $\hat{v}_3$ from the current standing price p_2.

For any given $(v_2, \hat{v}_2)$, define $c(v_2, \hat{v}_2)$ as the solution c to the equation.

$$v_2 - S_2^o(v_2; c) = c - S_2^n(c; \hat{v}_2) \tag{4}$$

I will show that $c(v_2, \hat{v}_2)$ is the critical value to determine whether Buyer 2 or Buyer 3 wins the second auction. To prove that the function is well-defined, I impose the following assumption on the distribution of V_3.

Assumption 1. *The marginal distribution of V_3 conditional on V_2 has increasing hazard rates.*

$$\frac{d}{dv_3}\left[\frac{f_3(v_3|V_2 = v_2)}{1 - F_3(v_3|V_2 = v_2)}\right] > 0.$$

Lemma 3. *In the equilibrium path, $c(v_2, v_2)$ is well-defined. Moreover, $c(v_2, v_2) \geq v_2$. The inequality holds strictly when V_3 and V_4 are strictly affiliated. When V_3 and V_4 are independent, $c(v_2, v_2) = v_2$.*

Proof. In the equilibrium path $\hat{v}_2 = v_2$, equation (4) can be rewritten as

$$\int_{\underline{v}}^{v_2} [1 - F_4(v_4|V_2 = v_2, V_3 \geq c)]\, dv_4 = \int_{\underline{v}}^{c} [1 - F_4(v_4|V_2 = v_2, V_3 = c)]\, dv_4. \tag{5}$$

Since V_3 and V_4 are weakly affiliated, I have $F_4(v_4|V_2 = v_2, V_3 \geq v_3) \leq F_4(v_4|V_2 = v_2, V_3 = v_3)$ for any v_2, v_3, and v_4. The left hand side of equation (5) is greater than or equal to the right hand side when $c = v_2$. On the other hand, the integrands on both sides of (5) are identical when $c = \overline{v}$. Consequently, the left hand side of (5) is less than or equal to the right hand side if $c = \overline{v}$. As a result, by the intermediate value theorem, there exists a solution $c \in [v_2, \overline{v}]$ to equation (5) for any given value of v_2.

When V_3 and V_4 are strictly affiliated, $F_4(v_4|V_2 = v_2, V_3 \geq v_3) < F_4(v_4|V_2 = v_2, V_3 = v_3)$. Therefore, the solution c must be strictly greater than v_2. On the contrary, when V_3 and V_4 are independent, the integrands are the same on both sides. As a result $c(v_2, v_2) = v_2$ if V_3

and V_4 are independent.

Let $H(v_3) \equiv \int_{\underline{v}}^{v_2} F_4(v_4|V_2 = v_2, V_3 = v_3)dv_4$. Applying integration by parts, I have

$$\begin{aligned} H(v_3) &= \int_{\underline{v}}^{v_2} (v_2 - v_4) dF_4(v_4|V_2 = v_2, V_3 = v_3) \\ &= E[\max\{v_2 - V_4, 0\}|V_2 = v_2, V_3 = v_3]. \end{aligned}$$

Since V_3 and V_4 are affiliated, V_3 and $-\max\{v_2 - V_4, 0\}$ are affiliated for any given v_2. Therefore, $H(v_3)$ is a decreasing function.

Next, I will show H is convex. Consider the change of variable $z \equiv F_4(v_4|V_2 = v_2, V_3 = v_3)$. I then have $H(v_3) = \int_0^1 \max\{v_2 - F_4^{-1}(z|V_2 = v_2, V_3 = v_3), 0\}dz$. Furthermore, because $\max\{v_2 - v_4, 0\}$ is a convex function of v_4, and $F_4(v_4|V_2 = v_2, V_3 = v_3)$ is a smooth function in v_3,

$$\begin{aligned} \max\left\{v_2 - F_4^{-1}(z|V_2 = v_2, V_3 = v_3 + \varepsilon), 0\right\} &+ \max\left\{v_2 - F_4^{-1}(z|V_2 = v_2, V_3 = v_3 - \varepsilon), 0\right\} \\ &\geq 2\max\left\{v_2 - F_4^{-1}(z|V_2 = v_2, V_3 = v_3), 0\right\} \end{aligned}$$

for any $z \in (0, 1)$ and for $\varepsilon > 0$ small enough. Integrating the both sides of the above inequality over z, I obtain $H(v_3 + \varepsilon) + H(v_3 - \varepsilon) \geq 2H(v_3)$. This is equivalent to say that H is convex.

To show the uniqueness of the solution c in equation (5), I will compare the derivatives on both sides of (5) with respect to c. Note that $\int_0^{v_2} F_4(v_4|V_2 = v_2, V_3 \geq v_3)dv_4 = E[H(V_3)|V_2 = v_2, V_3 \geq v_3]$. Its derivative is

$$\begin{aligned} \frac{dE[H(V_3)|V_2 = v_2, V_3 \geq v_3]}{dv_3} &= \frac{d}{dv_3}\left[\frac{\int_{v_3}^{\overline{v}} H(x) f_3(x|V_2 = v_2)}{1 - F_3(v_3|V_2 = v_2)}\right] \\ &= -\frac{H(v_3) f_3(v_3|V_2 = v_2)}{1 - F_3(v_3|V_2 = v_2)} + \frac{f_3(v_3|V_2 = v_2)\int_c^{\overline{v}} H(x) f_3(x|V_2 = v_2)dx}{[1 - F_3(v_3|V_2 = v_2)]^2} \\ &= \frac{f_3(v_3|V_2 = v_2)}{1 - F_3(v_3|V_2 = v_2)}\left\{E[H(V_3)|V_2 = v_2, V_3 \geq v_3] - H(v_3)\right\} \end{aligned}$$

This is negative because H is decreasing. Moreover, by the convexity of H,

$$\begin{aligned}0 &\geq E[H(V_3)|V_2 = v_2, V_3 \geq v_3] - H(v_3) = E[H(V_3) - H(v_3)|V_2 = v_2, V_3 \geq v_3] \\ &\geq E[H'(v_3)(V_3 - v_3)|V_2 = v_2, V_3 \geq v_3] = H'(v_3)E[(V_3 - v_3)|V_2 = v_2, V_3 \geq v_3]. \quad (6)\end{aligned}$$

Because V_3 has increasing hazard rates for given $V_2 = v_2$, the conditional distribution of V_3 giving $V_2 = v_2$ and $V_3 \geq v_3$ also has increasing hazard rates. As a result,

$$\begin{aligned}\frac{f_3(v_3|V_2 = v_2)}{1 - F_3(v_3|V_2 = v_2)} &= \frac{f_3(v_3|V_2 = v_2, V_3 \geq v_3)}{1 - F_3(v_3|V_2 = v_2, V_3 \geq v_3)} \\ &\leq \frac{\int_{v_3}^{\overline{v}} f_3(x|V_2 = v_2, V_3 \geq v_3)dx}{\int_{v_3}^{\overline{v}}[1 - F_3(V_3|V_2 = v_2, V_3 \geq v_3)]dx} = \frac{1}{E[(V_3 - v_3)|V_2 = v_2, V_3 \geq v_3]}, \quad (7)\end{aligned}$$

where the last equality uses integration by parts on the denominators. Combine the inequalities in (6) and (7). I obtain

$$0 \geq \frac{dE[H(V_3)|V_2 = v_2, V_3 \geq v_3]}{dv_3} \geq H'(v_3),$$

or, equivalently,

$$0 \geq \frac{d}{dv_3}\int_{\underline{v}}^{v_2} F_4(v_4|V_2 = v_2, V_3 \geq v_3)dv_4 \geq \frac{d}{dv_3}\int_{\underline{v}}^{v_2} F_4(v_4|V_2 = v_2, V_3 = v_3)dv_4. \quad (8)$$

Suppose there exists two solutions c and c' to equation (5) with $c' > c > v_2$. Plugging c and c' into equation (5) respectively and taking their difference, I obtain

$$\begin{aligned}&\int_{\underline{v}}^{v_2} \left[F_4(v_4|V_2 = v_2, V_3 > c) - F_4(v_4|V_2 = v_2, V_3 > c')\right] dv_4 \\ =& \int_{\underline{v}}^{c} \left[F_4(v_4|V_2{=}v_2, V_3{=}c) - F_4(v_4|V_2{=}v_2, V_3{=}c')\right] dv_4 + \int_{c}^{c'} \left[1 - F_4(v_4|V_2{=}v_2, V_3{=}c')\right] dv_4.\end{aligned}$$

However, this is a contradiction because combining this equality with the inequality (8) would imply

$$
\begin{aligned}
&\int_{\underline{v}}^{v_2} \left[F_4(v_4|V_2 = v_2, V_3 > c) - F_4(v_4|V_2 = v_2, V_3 > c')\right] dv_4 \\
=&\int_{\underline{v}}^{v_2} \left[F_4(v_4|V_2 = v_2, V_3 = c) - F_4(v_4|V_2 = v_2, V_3 = c')\right] dv_4 \\
&+\int_{v_2}^{c}\left[F_4(v_4|V_2=v_2, V_3=c) - F_4(v_4|V_2=v_2, V_3=c')\right] dv_4 + \int_{c}^{c'}\left[1 - F_4(v_4|V_2=v_2, V_3=c')\right] dv_4 \\
>&\int_{\underline{v}}^{v_2} \left[F_4(v_4|V_2 = v_2, V_3 > c) - F_4(v_4|V_2 = v_2, V_3 > c')\right] dv_4.
\end{aligned}
$$

□

To show the monotonicity of the critical value function $c(v_2, v_2)$, the affiliations between valuations (V_2, V_3, V_4) have to be restricted. As long as the second derivative, $\partial^2 E[V_4|V_2 = v_2, V_3 = v_3]/\partial v_2 \partial v_3$, is not too negative, the following condition holds.

Assumption 2. *At $v_3 = c(v_2, v_2)$,*

$$
\frac{\partial}{\partial v_2} E\left[\min\{v_2, V_4\}|V_2 = v_2, V_3 \geq v_3\right] \geq \frac{\partial}{\partial v_2} E\left[\min\{v_3, V_4\}|V_2 = v_2, V_3 = v_3\right].
$$

A sufficient condition for Assumption 2 is $\partial E[V_4|V_2 = v_2, V_3 = v_3]/\partial v_2 < 1$ and $\partial^2 E[V_4|V_2 = v_2, V_3 = v_3]/\partial v_2 \partial v_3 \geq 0$. For instance, when (V_2, V_3, V_4) are joint normal distribution with zero mean and unit variance, $E[V_4|V_2 = v_2, V_3 = v_3] = \rho_{24} v_2 + \rho_{34} v_3$. The sufficient condition holds for this example.

Lemma 4. *In the equilibrium path, $c(v_2, v_2)$ strictly increases in v_2.*

Proof. Given Assumption 2, monotonicity follows from applying the implicit function theorem to equation (5). □

Lemma 5. *On the equilibrium path, for any given v_2, Buyer 2 prefers Auction 2 over Auction 3 if and only if the standing price of Auction 2 is less than the maximal price bid by Buyer 3 with valuation $c(v_2, v_2)$.*

Proof. Buyer 2's surplus of winning Auction 2 at price p_2 is $v_2 - p_2$. I need to show that $v_2 - p_2 \geq S_2^o(v_2; \hat{v}_3(p_2; v_2))$ if and only if $p_2 \leq \beta_2^n(c(v_2, v_2); v_2)$. Since $\beta_2^n(\cdot; v_2)$ is monotonically increasing and $\hat{v}_3(\cdot; v_2)$ is its inverse, it is equivalent to show that $v_2 - \beta_2^n(\hat{v}_3; v_2) \geq S_2^o(v_2; \hat{v}_3)$ if and only if $\beta_2^n(\hat{v}_3; v_2) \leq \beta_2^n(c(v_2, v_2); v_2)$. The last expression is in turn equivalent to $\hat{v}_3 \leq c(v_2, v_2)$. Lemma 3 shows that $c(v_2, v_2)$ is the unique solution to $v_2 - S_2^o(v_2; \hat{v}_3) = \hat{v}_3 - S_2^n(\hat{v}_3; v_2)$ on the equilibrium path and $v_2 - \beta_2^n(\hat{v}_3; v_2) \geq S_2^o(v_2; \hat{v}_3)$ if and only if $\hat{v}_3 \leq c(v_2, v_2)$. ☐

Buyer 2's bidding strategy is to keep bidding on Auction 2 until the surplus from the current auction is equal to the expected surplus from the future one.

$$v_2 - p_2 = S_2^o(v_2; \hat{v}_3(p_2; \hat{v}_2)).$$

Hence, her bidding strategy $\beta_2^o(v_2; \hat{v}_2)$ is the solution of p_2 to the above equation. Because Buyer2 can correctly anticipate Buyer 3's bidding strategy β_2^n, her own strategy only depends on her own valuation v_2 and the revealed valuation $\hat{v}_2$, but not on Buyer 3's revealed valuation $\hat{v}_3$. Lemma 5 shows that the maximal amount Buyer 2 is willing to bid on the second auction is $\beta_2^o(v_2; v_2) = v_2 - S_3^o(v_2; c(v_2, v_2)) = \beta_2^n(c(v_2, v_2); v_2)$ on the equilibrium path.

Lemma 6. *On the equilibrium path, the winner of Auction 2 is Buyer 2 if $v_3 \leq c(v_2, v_2)$ and the winner of Auction 2 is Buyer 3 if $v_3 \geq c(v_2, v_2)$*

Proof. The winner of Auction 2 is Buyer 2 if $\beta_2^o(v_2; v_2) \geq \beta_2^n(v_3; v_2)$. On the other hand, the winner is Buyer 3 if $\beta_2^o(v_2; v_2) \leq \beta_2^n(v_3; v_2)$. By Lemma 5, I know $\beta_2^o(v_2; v_2) = \beta_2^n(c(v_2, v_2); v_2)$ on the equilibrium path. Since $\beta_2^n(v_3; v_2)$ is an increasing function in v_3. Buyer 3 wins Auction 2 if and only if $v_3 \geq c(c_2, v_2)$. ☐

Proposition 1. *The lowest valuation for Buyer 3 to win the second auction is greater than Buyer 2's valuation, $c(v_2, v_2) \geq v_2$. The inequality is strict if the valuations of the last two buyers, V_3 and V_4, are strictly affiliated. If V_3 and V_4 are independent, $c(v_2, v_2) = v_2$. Consequently, the allocation of the goods is not always efficient.*

Proof. This is the combined results from Lemma 3 and Lemma 6. □

This proposition shows that Buyer 2 may win the second auction even if her valuation v_2 is smaller than Buyer 3's valuation. The intuition behind this result is the information asymmetry between the two buyers. Buyer 3 gets an *information rent* from knowing more about their valuations. Consequently, Buyer 3 drops from the second auction earlier than Buyer 2 even if they have identical valuations.

The transaction price of the second auction in equilibrium is

$$p_2^* = \begin{cases} \beta_2^n(v_3; v_2) = v_3 - S_3^n(v_3; v_2) = E[\min\{v_3, V_4\}|V_2 = v_2, V_3 = v_3], & \text{if } v_3 \leq c(v_2, v_2); \\ \beta_2^o(v_2; v_2) = v_2 - S_3^o(v_2; c(v_2, v_2)) = E[\min\{v_2, V_4\}|V_2 = v_2, V_3 \geq c(v_2, v_2)], & \text{if } v_3 \geq c(v_2, v_2). \end{cases}$$

3.3 The First Auction

Because the valuations V_1 and V_2 are assumed to be symmetric, I only need to consider the bidding strategy used by one of the buyers. Without loss of generality, let us consider the buyer with valuation v_2 under the condition $V_1 \geq v_2$.

For any monotonic bidding strategy for the first auction, the loser's valuation v_2 is revealed after the first auction completes. Therefore, buyers need to account for the effect of revealing information on their future surplus when bidding in the first auction.

If a buyer with valuation v_2 does not win the first auction and enters future auctions, she may either win or lose in Auction 2. In the first case, she gets a surplus equal to her own valuation minus the highest bid Buyer 3 is willing to submit for Auction2. In the later case, she loses in Auction 2 at the price $\beta_2^o(v_2; \hat{v}_2)$. Note that her future outcome depends on the inferred valuation $\hat{v}_2$, which is determined by her bidding strategy in the first auction. Her expected surplus of entering future auctions is

$$S_1(v_2, \hat{v}_2) = \int_{\underline{v}}^{c(v_2,\hat{v}_2)} [v_2 - \beta_2^n(v_3; \hat{v}_2)] dF_3(v_3|V_2 = v_2) + \int_{c(v_2,\hat{v}_2)}^{\overline{v}} S_2^o(v_2; c(v_2, \hat{v}_2)) dF_3(v_3|V_2 = v_2). \quad (9)$$

Note that the assumption $V_1 \geq V_2$ has been suppressed. In the equilibrium path, the expected surplus can also be expressed as

$$S_1(v_2, \hat{v}_2) = \Pr(V_3 \leq c(v_2, \hat{v}_2)) E[v_2 - \min\{V_4, V_3\} | V_2 = v_2, V_3 \leq c(v_2, \hat{v}_2)] \\ + \Pr(V_3 \geq c(v_2, \hat{v}_2)) S_3^o(v_2; c(v_2, \hat{v}_2)). \quad (10)$$

Denote the bidding function in the first auction by $\beta_1(v_2)$. It is the maximal standing price a buyer with valuation v_2 is willing to bid for Auction 1. If β_1 is a strictly increasing function, choosing the maximal price p_1 to bid in the auction is equivalent to choosing the valuation $\hat{v}_2$ to be inferred.

Proposition 2. *For any given valuation v_2, there exists a unique price $\beta_1(v_2)$ such that a buyer with valuation v_2 prefers winning in Auction 1 over future auctions if and only if the current price is less than $\beta_1(v_2)$. Furthermore, β_1 is strictly increasing.*

Proof. Let

$$\beta_1(v) = v - S_1(v, v). \quad (11)$$

I will show that the function β_1 indeed satisfies the property stated by the lemma.

First, claim that β_1 is a strictly increasing function. Define

$$\gamma(v_2, v_3) \equiv \begin{cases} E[\min\{v_3, V_4\} | V_2 = v_2, V_3 = v_3], & \text{if } v_3 \leq c(v_2, v_2); \\ E[\min\{v_2, V_4\} | V_2 = v_2, V_3 > c(v_2, v_2)], & \text{if } v_3 \geq c(v_2, v_2). \end{cases}$$

From equation (9), I have $\beta_1(v_2) = v_2 - S_1(v_2, v_2) = \int_0^{\overline{v}} \gamma(v_2, v_3) dF_3(v_3 | V_2 = v_2)$. For any

$v_2' > v_2$, Lemma 4 implies $c(v_2', v_2') > c(v_2, v_2)$.

$$\gamma(v_2', v_3) - \gamma(v_2, v_3) =$$
$$\begin{cases} E[\min\{v_3, V_4\}|V_2 = v_2', V_3 = v_3] - E[\min\{v_3, V_4\}|V_2 = v_2, V_3 = v_3], & \text{if } v_3 \leq c(v_2, v_2); \\ E[\min\{v_3, V_4\}|V_2 = v_2', V_3 = v_3] - E[\min\{v_2, V_4\}|V_2 = v_2, V_3 \geq c(v_2, v_2)], & \text{if } c(v_2, v_2) \leq v_3 \leq c(v_2', v_2'); \\ E[\min\{v_2, V_4\}|V_2 = v_2', V_3 \geq c(v_2', v_2')] - E[\min\{v_2, V_4\}|V_2 = v_2, V_3 \geq c(v_2, v_2)], & \text{if } v_3 \geq c(v_2', v_2'). \end{cases}$$

By affiliation and the fact $v_2 \leq c(v_2, v_2)$, for each of the three cases in the above equation, $\gamma(v_2', v_3) - \gamma(v_2, v_3) \geq 0$. In particular, there is a positive measure of v_3 falling in the third case, $v_3 \geq c(v_2', v_2')$, and the inequality is strict in this case, i.e., $\gamma(v_2', v_3) - \gamma(v_2, v_3) > 0$. Moreover, because $\gamma(v_2, v_3)$ increases in v_3, affiliation also implies $\int_{\underline{v}}^{\overline{v}} \gamma(v_2', v_3) dF_3(v_3|V_2 = v_2') \geq \int_{\underline{v}}^{\overline{v}} \gamma(v_2', v_3) dF_3(v_3|V_2 = v_2)$. Consequently,

$$\beta_1(v_2') = \int_{\underline{v}}^{\overline{v}} \gamma(v_2', v_3) dF_3(v_3|V_2 = v_2') \geq \int_{\underline{v}}^{\overline{v}} \gamma(v_2', v_3) dF_3(v_3|V_2 = v_2)$$
$$\geq \int_{\underline{v}}^{\overline{v}} \gamma(v_2, v_3) dF_3(v_3|V_2 = v_2) = \beta_1(v_2).$$

Suppose β_1 is the function used by buyers in the next auction to infer Buyer 2's valuation. A buyer with valuation v_2 prefers winning Auction 1 at price p_1 over waiting for the future auctions if and only if $v_2 - p_1 \geq S_1(v_2, \hat{v}_2)$, where $\hat{v}_2 = \beta^{-1}(p_1)$. It is equivalent to $v_2 - \beta_1(\hat{v}_2) \geq S_1(v_2, \hat{v}_2)$. In equilibrium, v_2 will always be inferred from the monotonic bidding function β_1. Therefore, the maximal price a buyer with valuation v_2 willing to bid in the first auction is $\beta_1(v_2) = v_2 - S_1(v_2, v_2)$. □

The valuation of Buyer 2 will always be correctly inferred from the transaction price of Auction 1, p_1^*, from the monotonic bidding function β_1. To simplify the notation, I will slightly abuse the notation and denote the critical value in the equilibrium path by $c(v_2) \equiv c(v_2, v_2)$.

Lastly, the transaction price of Auction 1 is

$$p_1^* = \beta_1(v_2)$$

3.4 Expected Transaction Prices

The *ex ante* expected transaction prices are identical across auctions. This is contrary to the findings of ascending transaction prices of sequential auctions in Milgrom and Weber (2000).[5]

Proposition 3. *The ex ante expected transaction prices are identical across auctions.*

$$E[p_1^*] = E[p_2^*] = E[p_3^*]$$

Proof. The transaction price of the final auction is determined by the lower valuation among the two buyers participating in this auction.

$$p_3^* = \begin{cases} \min\{v_3, v_4\}, & \text{if } v_3 \leq c(v_2); \\ \min\{v_2, v_4\}, & \text{if } v_3 \geq c(v_2). \end{cases}$$

Therefore, its expected value is

$$\begin{aligned} E[P_3^*] =& \Pr(V_3 \leq c(V_2))E[\min\{V_3, V_4\}|V_3 \leq c(V_2)] + \Pr(V_3 \geq c(V_2))E[\min\{V_2, V_4\}|V_3 \geq c(V_2)] \\ =& \Pr(V_3 \leq c(V_2), V_4 \leq V_3)E[V_4|V_3 \leq c(V_2), V_4 \leq V_3] \\ &+ \Pr(V_3 \leq c(V_2), V_4 \geq V_3)E[V_3|V_3 \leq c(V_2), V_4 \geq V_3] \\ &+ \Pr(V_3 \geq c(V_2), V_4 \leq V_2)E[V_4|V_3 \geq c(V_2), V_4 \leq V_2] \\ &+ \Pr(V_3 \geq c(V_2), V_4 \geq V_2)E[V_2|V_3 \geq c(V_2), V_4 \geq V_2]. \end{aligned}$$

Recall that $S_3^n(v_3; v_2) = v_3 - E[\min\{V_4, v_3\}|V_2 = v_2, V_3 = v_3]$ and $S_3^o(v_2; v_3) = v_2 - E[\min\{V_4, v_2\}|V_2 = v_2, V_3 > c(v_2)]$.

[5] In my model, the effect of information revelation is intentionally abstracted away.

The expected transaction price of the second auction is

$$\begin{aligned}
E[P_2^*] =& \Pr(V_3 \leq c(V_2, V_2)) E[V_3 - S_2^n(V_3; V_2) | V_3 \leq c(V_2)] \\
& + \Pr(V_3 \geq c(V_2, V_2)) E[v_2 - S_2^o(V_2; c(V_2)) | V_3 \geq c(V_2)] \\
=& \Pr(V_3 \leq c(V_2, V_2)) E\left\{E_{V_4}[\min\{V_4, V_3\} | V_2, V_3] | V_3 \leq c(V_2)\right\} \\
& + \Pr(V_3 \geq c(V_2, V_2)) E\left\{E_{V_4}[\min\{V_4, V_2\} | V_2, V_3 > c(V_2)] | V_3 \geq c(V_2)\right\} \\
=& \Pr(V_3 \leq c(V_2), V_4 \leq V_3) E[V_4 | V_3 \leq c(V_2), V_4 \leq V_3] \\
& + \Pr(V_3 \leq c(V_2), V_4 \geq V_3) E[V_3 | V_3 \leq c(V_2), V_4 \geq V_3] \\
& + \Pr(V_3 \geq c(V_2), V_4 \leq V_2) E[V_4 | V_3 \geq c(V_2), V_4 \leq V_2] \\
& + \Pr(V_3 \geq c(V_2), V_4 \geq V_2) E[V_2 | V_3 \geq c(V_2), V_4 \geq V_2].
\end{aligned}$$

Finally, the transaction price of the first auction can be written as

$$\begin{aligned}
p_1^* =& v_2 - S_1(v_2, v_2) \\
=& \int_{\underline{v}}^{c(v_2, v_2)} \beta_2^n(v_3; v_2) dF_3(v_3 | V_2 = v_2) + \int_{c(v_2)}^{\overline{v}} \left[v_2 - S_3^o(v_2; c(v_2))\right] dF_3(v_3 | V_2 = v_2) \\
=& \Pr(V_3 \leq c(v_2), V_4 \leq V_3) E[V_4 | V_3 \leq c(v_2), V_4 \leq V_3] \\
& + \Pr(V_3 \leq c(v_2), V_4 \geq V_3) E[V_3 | V_3 \leq c(v_2), V_4 \geq V_3] \\
& + \Pr(V_3 \geq c(v_2)) \left[v_2 - S_3^o(v_2; c(v_2))\right],
\end{aligned}$$

where the second equality comes from $\beta^n(v_3; v_2) = E[\min\{V_4, v_3\} | V_2 = v_2, V_3 = v_3]$. Taking expectation on the valuation V_2, I have

$$\begin{aligned}
E[P_1^*] =& \Pr(V_3 \leq c(V_2), V_4 \leq V_3) E[V_4 | V_3 \leq c(V_2), V_4 \leq V_3] \\
& + \Pr(V_3 \leq c(V_2), V_4 \geq V_3) E[V_3 | V_3 \leq c(V_2), V_4 \geq V_3] \\
& + \Pr(V_3 \geq c(V_2), V_4 \leq V_2) E[V_4 | V_3 \geq c(V_2), V_4 \leq V_2] \\
& + \Pr(V_3 \geq c(V_2), V_4 \geq V_2) E[V_2 | V_3 \geq c(V_2), V_4 \geq V_2].
\end{aligned}$$

Consequently, $E[P_1^*] = E[P_2^*] = E[P_3^*]$. □

4 Comparisons

In this section, I consider two alternative ways to sell the items. The first one is to sell all item togethers in one single auction. The second one uses a sequence of sealed-bid second price auctions.

4.1 Single Multiple-Object Auction

Suppose the sellers can cooperate and sell all the items in a single multiple-object English auction. Instead of auctioning them sequentially, the auction takes place until all buyers have arrived. Then, the three items are sold by a simultaneous multiple-object auction. Specifically, there is a single standing price which starts at zero and keeps rising. At any standing price, a buyer may either stay in the auction or drop out. Once dropping out, it is not allowed to re-enter the auction. The standing price rises until only three buyers left. Each of these three remaining buyers can get one item and pay the final standing price.

Under the assumption of affiliated private valuation, a buyer knows her personal valuation for sure. There is no information updating during the course of the auction. A buyer's dominant strategy is to stay in the auction if and only if the current standing price is less than her valuation.

In equilibrium, the standing price stops at the fourth-highest valuation among all four buyers. Consequently, the equilibrium transaction price, denoted by p^{sim}, equals to the lowest valuation among the four buyers. That is, $p^{sim} = \min\{v_i\}$.

Proposition 4. *The expected transaction price under sequential auctions is less than or equal to the expected transactio price under simultaneous auctions. The inequality is strict if and only if the valuations of the last two buyers are strictly affiliated.*

Proof. The expected transaction price under simultaneous auctions is

$$\begin{aligned}
E[p^{sim}] =& E[\min\{v_i\}] \\
=& \Pr(V_3 \le V_2, V_4 \le V_3)E[V_4|V_3 \le V_2, V_4 \le V_3] \\
&+ \Pr(V_3 \le V_2, V_4 \ge V_3)E[V_3|V_3 \le V_2, V_4 \ge V_3] \\
&+ \Pr(V_3 \ge V_2, V_4 \le V_2)E[V_4|V_3 \ge V_2, V_4 \le V_2] \\
&+ \Pr(V_3 \ge V_2, V_4 \ge V_2)E[V_2|V_3 \ge V_2, V_4 \ge V_2]. \qquad (12)
\end{aligned}$$

On the other hand, the expected transaction price under simultaneous auctions is

$$\begin{aligned}
E[P_1^*] = E[P_2^*] = E[P_3^*] =& \Pr(V_3 \le c(V_2), V_4 \le V_3)E[V_4|V_3 \le c(V_2), V_4 \le V_3] \\
&+ \Pr(V_3 \le c(V_2), V_4 \ge V_3)E[V_3|V_3 \le c(V_2), V_4 \ge V_3] \\
&+ \Pr(V_3 \ge c(V_2), V_4 \le V_2)E[V_4|V_3 \ge c(V_2), V_4 \le V_2] \\
&+ \Pr(V_3 \ge c(V_2), V_4 \ge V_2)E[V_2|V_3 \ge c(V_2), V_4 \ge V_2]. \qquad (13)
\end{aligned}$$

By Proposition 1, I know $c(v_2) \ge v_2$. Since $v_2 < v_3$ for $v_3 \in (v_2, c(v_2))$, equations (12) and (13) imply $E[p^{sim}] \le E[P_1^*] = E[P_2^*] = E[P_3^*]$.

Furthermore, when V_3 and V_4 are strictly affiliated, $c(v_2) > v_2$. As a result, $E[p^{sim}] < E[P_1^*] = E[P_2^*] = E[P_3^*]$. □

This proposition implies a profit-maximizing seller should not to bundle all items together into a single multiple-object auction. Instead, if items are sold sequentially, the aggregate expected profit would be higher. Even though the allocation of sequential auctions is socially inefficiently, sellers can exploit information asymmetry to gain their profit.

4.2 Sequential Sealed-Bid Auctions

Instead of treating Internet auctions as ascending auctions, the model can to apply to a sequence of sealed-bid second price auction. The main difference from ascending auctions is the information available to buyers during the bidding process. For a sealed-bid auction, a buyer cannot revised her belief from observing the current standing price.

The analysis of the final auction is unchanged since it is still a dominant strategy to bid one's own valuation, $\beta_3^s(v) = v$. (I use the superscript, s, to denote *sealed-bid* auctions.) For the second auction, the bidding strategy of the new buyer is also unchanged.

$$\beta_2^{ns}(v_3; \hat{v}_2) = v_3 - \int_{\underline{v}}^{v_3} F_4(v_4 | V_2 = \hat{v}_2, V_3 = v_3) dv_4.$$

Nevertheless, the biding function of the old buyer changes to

$$\beta_2^{os}(v_2) = v_2 - \int_{\underline{v}}^{v_2} F_4(v_4 | V_2 = v_2) dv_4.$$

The critical value $c^s(v_2, \hat{v}_2)$ to determine whether Buyer 3 is the winner of the second auction or not is the solution c to the equation,

$$v_2 - \int_{\underline{v}}^{v_2} F_4(v_4 | V_2 = v_2) dv_4 = c - \int_{\underline{v}}^{c} F_4(v_4 | V_2 = \hat{v}_2, V_3 = c) dv_4.$$

It is easy to see that $c^s(v_2, \hat{v}_2)$ is well-defined since the right hand side of the above equation is strictly increasing. Because $F_4(v_4 | V_2 = v_2) \leq F_4(v_4 | V_2 = v_2, V_3 = v_3)$ when v_2 and v_3 are both closed to $\underline{v}$, but $F_4(v_4 | V_2 = v_2) \geq F_4(v_4 | V_2 = v_2, V_3 = v_3)$ when v_2 and v_3 are both closed to $\overline{v}$. Consequently, on the equilibrium path, both $c^s(v_2, v_2) \leq v_2$ and $c^s(v_2, v_2) \geq v_2$ are possible. The allocation of the auctions are still not always efficient, but two inefficient allocations are possible: Either selling the good to Buyer 2 when $v_3 > v_2$ or selling it to Buyer 3 when $v_3 < v_2$.

Proposition 5. *The lowest valuation for Buyer 3 to win the second auction is greater than Buyer 2's valuation $c^s(v_2, v_2) \geq v_2$ when v_2 and v_3 are both closed to $\underline{v}$. On the contrary, the lowest valuation for Buyer 3 to win the second auction is less than Buyer 2's valuation $c^s(v_2, v_2) \leq v_2$ when v_2 and v_3 are both closed to $\overline{v}$. Consequently, the allocation of the goods could be inefficient.*

The intuition behind this result is clear. When both buyers have high valuations, Buyer 3 has better information than Buyer 2. As a result, she has a stronger incentive to bid more aggressively to avoid facing Buyer 4, who is likely to have high valuation due to affiliation. On the other hand, when both buyers have low valuations, Buyer 3's better information would help her to bid less since the price of the final auction is likely to be low.

5 Numerical Example

In this section, I present a simple example which has a closed-form formula for the expected surplus functions.

Suppose only V_3 and V_4 are affiliated, but V_1 and V_2 are independent of any other valuation. In addition, assume that $V_3 = \rho X_0 + (1-\rho)X_3$ and $V_4 = \rho X_0 + (1-\rho)X_4$ for $0 \leq \rho \leq 1/2$, where X_0, X_3, and X_4 are drawn independently from the uniform distribution $UNIF(0,1)$. The parameter ρ is a measure of the affiliation between V_3 and V_4. In fact, the correlation between the two random variables is ρ^2. The valuations of the first two buyers, V_1 and V_2, are also drawn independently from $UNIF(0,1)$. Although the distribution functions of V_1 and V_2 are smooth on the entire support $[0,1]$, the distribution functions of V_3 and V_4 both have kinks at ρ and $1-\rho$.

Because of the independence of V_1 and V_2, the conditional distributions can be written as $F_4(v_4|V_2 = v_2, V_3 = c(v_2)) = \Pr(V_4 \leq v_4|V_3 = c(v_2))$ and $F_4(v_4|V_2 = v_2, V_3 \geq c(v_2)) = \Pr(V_4 \leq v_4|V_3 \geq c(v_2))$ in this example. For the purpose of solving for the critical value $c(v_2)$ for any $v_2 \in (0,1)$, I need to know $\Pr(V_4 \leq v_4|V_3 = v_3)$ and $\Pr(V_4 \leq v_4|V_3 \geq v_3)$ for any $v_4 \leq v_3$.

To compute Buyer 3's expectation on Buyer 4's valuation, I need $\Pr(V_4 \leq v_4|V_3 = v_3)$. There are three cases. When $v_3 \leq \rho$,

$$\Pr(V_4 \leq v_4|V_3 = v_3) = \frac{v_4^2}{2(1-\rho)v_3}$$

for $v_4 \leq v_3$. When $\rho \leq v_3 \leq 1-\rho$,

$$\Pr(V_4 \leq v_4 | V_3 = v_3) = \begin{cases} \frac{v_4^2}{2(1-\rho)\rho}, & \text{if } v_4 \leq \rho; \\ \frac{2v_4-\rho}{2(1-\rho)}, & \text{if } \rho \leq v_4 \leq v_3. \end{cases}$$

When $v_3 \geq 1-\rho$,

$$\Pr(V_4 \leq v_4 | V_3 = v_3) = \begin{cases} 0, & \text{if } v_4 \leq v_3 - 1 + \rho; \\ \frac{(1-\rho-v_3+v_4)^2}{2(1-\rho)(1-v_3)}, & \text{if } v_3 - 1 + \rho \leq v_4 \leq \rho; \\ \frac{2v_4-v_3+1-2\rho}{2(1-\rho)}, & \text{if } \rho \leq v_4 \leq v_3. \end{cases}$$

To compute Buyer 2's expectation on Buyer 4's valuation, I need $\Pr(V_4 < v_4 | V_3 > v_3)$. When $v_3 \leq \rho$,

$$\Pr(V_4 \leq v_4 | V_3 \geq v_3) = \frac{v_4^2\left[1 - \frac{v_3}{1-\rho} + \frac{v_4}{3(1-\rho)}\right]}{2\rho(1-\rho) - v_3^2}$$

for $v_4 \leq v_3$. When $\rho \leq v_3 \leq 1-\rho$,

$$\Pr(V_4 \leq v_4 | V_3 \geq v_3) = \begin{cases} \frac{\frac{v_4^2}{2\rho(1-\rho)}\left[1-\frac{v_3}{1-\rho}+\frac{v_4}{3(1-\rho)}\right]}{1-\frac{v_3}{1-\rho}+\frac{\rho}{2(1-\rho)}}, & \text{if } v_4 \leq \rho; \\ \frac{\left(v_4-\frac{\rho}{2}\right)(1-\rho-v_3)+\frac{\rho}{2}\left(v_4-\frac{2}{3}\rho\right)}{(1-\rho)^2-v_3(1-\rho)+\frac{1}{2}\rho(1-\rho)}, & \text{if } \rho \leq v_4 \leq v_3. \end{cases}$$

When $v_3 \geq 1-\rho$,

$$\Pr(V_4 \leq v_4 | V_3 \geq v_3) = \begin{cases} 0, & \text{if } v_4 \leq v_3 - 1 + \rho; \\ \frac{(1-\rho-a+v_4)^3}{3(1-\rho)(1-v_3)^2}, & \text{if } v_3 - 1 + \rho \leq v_4 \leq \rho; \\ \frac{v_4-\rho}{1-\rho} + \frac{1-a}{3(1-\rho)}, & \text{if } \rho \leq v_4 \leq v_3. \end{cases}$$

Buyer 3's expected surplus of entering the third auction is

$$S_2^n(v_3, \hat{v}_2) = \begin{cases} \frac{v_3^2}{6(1-\rho)}, & \text{if } v_3 \leq \rho; \\ \frac{\rho^2+3v_3^2-3\rho v_3}{6(1-\rho)}, & \text{if } \rho \leq v_3 \leq 1-\rho; \\ \frac{(1-v_3)^2}{6(1-\rho)} - \frac{\rho}{2} + \frac{v_3}{2}, & \text{if } v_3 \geq 1-\rho. \end{cases}$$

Buyer 2's inferred valuation, $\hat{v}_2$, has no effect on the expected surplus because V_2 and V_4 are independent.

Buyer 2's expected surplus depends on Buyer 3's valuation, which can be inferred from the current standing price by the inverse of Buyer 3's bidding function $\hat{v}_3(p_2)$.

$$S_2^o(v_2;\hat{v}_3) = \begin{cases} \frac{[1-\frac{\hat{v}_3}{1-\rho}]\frac{v_2^3}{3}+\frac{v_2^4}{12(1-\rho)}}{2\rho(1-\rho)-\hat{v}_3^2}, & \text{if } \hat{v}_3 \leq \rho; \\ \frac{[1-\frac{\hat{v}_3}{1-\rho}]\frac{v_2^3}{3}+\frac{v_2^4}{12(1-\rho)}}{2\rho(1-\rho)-2\rho\hat{v}_3+\rho^2}, & \text{if } \rho \leq \hat{v}_3 \leq 1-\rho, v_2 \leq \rho; \\ \frac{\frac{\rho^2}{3}-\frac{\rho^2\hat{v}_3}{3(1-\rho)}+\frac{\rho^3}{12(1-\rho)}}{2-2\hat{v}_3-\rho} + \frac{(v_2-\rho)\left[v_2(1-\rho-\hat{v}_3)+\rho\left(\frac{v_2}{2}-\frac{\rho}{6}\right)\right]}{2(1-\rho)\left(1-\frac{\rho}{2}-\hat{v}_3\right)}, & \text{if } \rho \leq \hat{v}_3 \leq 1-\rho, \rho \leq v_2 \leq \hat{v}_3; \\ 0, & \text{if } a \geq 1-\rho, v_2 \leq \hat{v}_3 - 1 + \rho; \\ \frac{(1-\rho-\hat{v}_3+v_2)^4}{12(1-\rho)(1-\hat{v}_3)^2}, & \text{if } \hat{v}_3 \geq 1-\rho, \hat{v}_3-1+\rho \leq v_2 \leq \rho; \\ \frac{(1-\hat{v}_3)^2}{12(1-\rho)} + \frac{v_2^2}{2(1-\rho)} + \frac{1-\hat{v}_3-3\rho}{3(1-\rho)}v_2 + \frac{\frac{\rho^2}{2}-\frac{(1-\hat{v}_3)\rho}{3}}{1-\rho}, & \text{if } \hat{v}_3 \geq 1-\rho, \rho \leq v_2 \leq \hat{v}_3. \end{cases}$$

The function $c(v_2)$ is implicitly defined in equation (4). Consequently, for a given value of v_2, the function $c(v_2)$ is equal to the solution c in the following equations.

$$v_2 - S_3^o(v_2;c) = c - S_3^n(c;v_2).$$

The solid line in Figure 1 shows the graph of $c(v_2)$ for the cases with $\rho = 0$, $\rho = 0.25$ and $\rho = 0.5$. When $\rho = 0$, the graph of $c(v_2)$ coincides with the 45° line. For positive correlation between V_3 and V_4 ($\rho > 0$), the graph of $c(v_2)$ is above the 45° line for all $v_2 \in (0,1)$. As a result, there is a positive probability of inefficiency in the sense of selling the good to Buyer 2 in Auction 2 when Buyer 3 has a higher valuation than Buyer 2. The allocation is inefficient in these cases.

The expected transaction prices for sequential auctions are slightly higher than those for simultaneous auctions. Figure 2 shows the transaction prices for these two types of auctions at different correlation levels between the two valuations V_3 and V_4. When the correlation coefficient between these two random variables is 0.25 (i.e. $\rho = 0.5$), the expected transaction price of simulation auctions is 0.22% lower than that of sequential auctions.

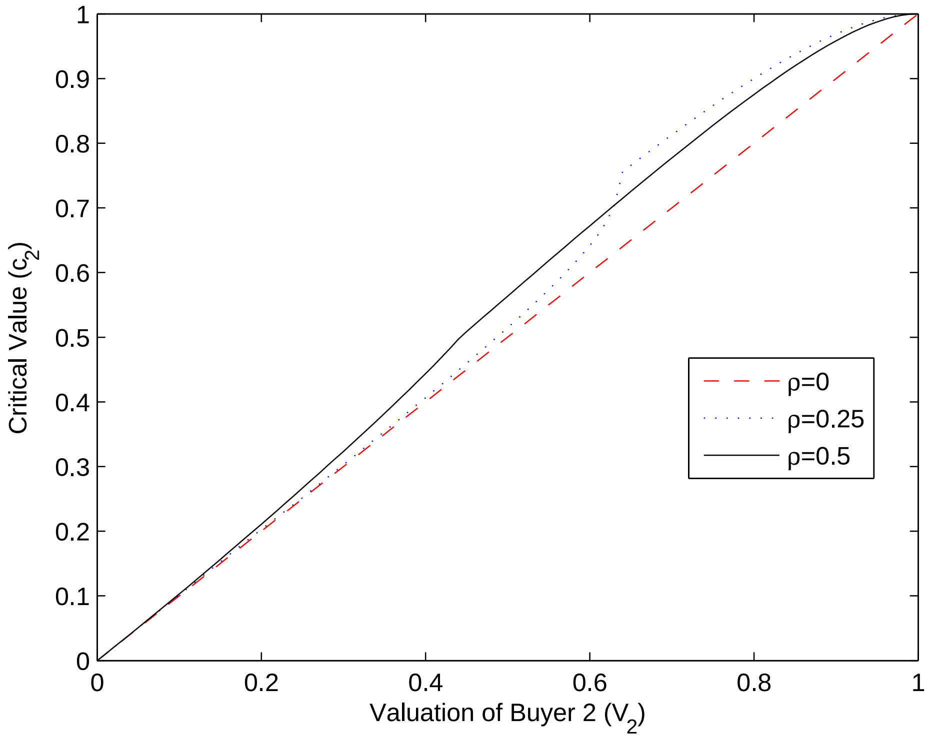

Figure 1: The minimal valuation for Buyer 3 to win in Auction 2

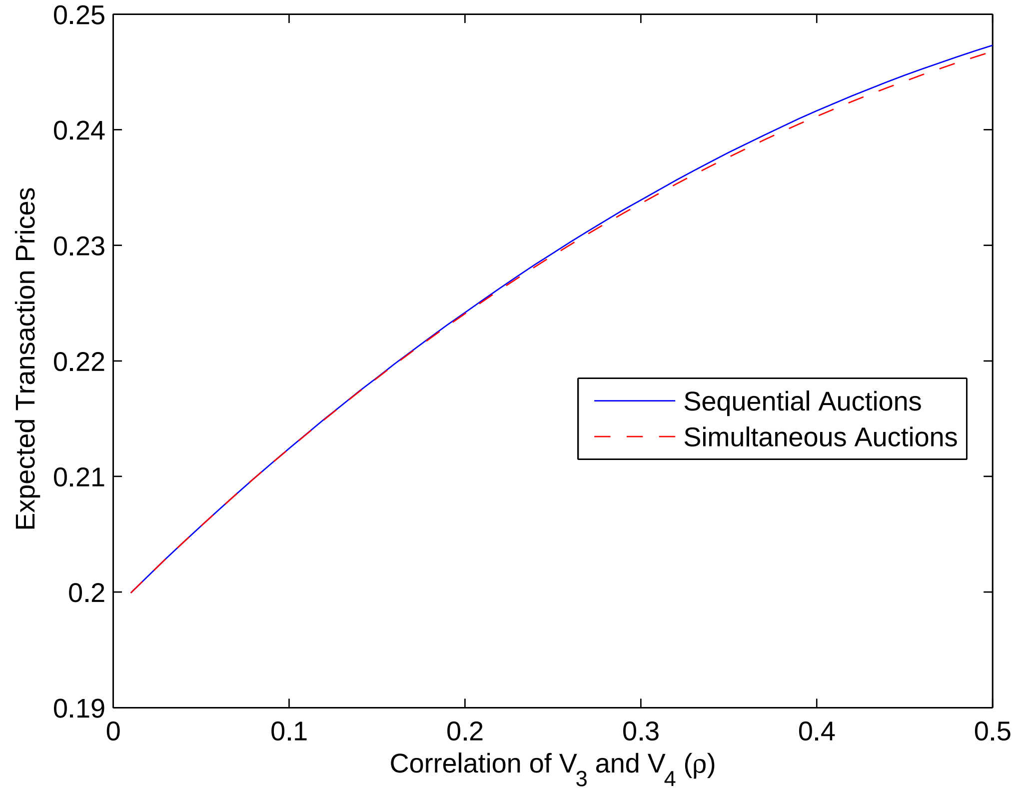

Figure 2: The expected transaction prices

6 Conclusion

One important feature of Internet auctions which is ignored in the traditional auction literature is the entry of new buyers. In this paper, I show that, when multiple items of an identical good are sold in sequential auctions and buyers have affiliated private valuation over the good, entry would result in inefficient allocation of the goods. A less informed buyer may win an auction even if her valuation is lower than other buyers. The inefficiency is a consequence of information asymmetry between buyers. On the contrary, if all items are sold in one multiple-object auction. The allocation is efficient, but sellers on average receive less joint profit. The inefficiency remains possible if the mechanism is changed from a sequence of ascending auctions to a sequence of sealed-bid second-price auctions.

This paper consider a simple situation to demonstrate the inefficiency due to entry to sequential auctions. I conjecture the result to hold in a more general model, but rigid analysis remains to be done in the future extension.

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