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Abstract

We show that the seller's optimal reserve price in an all-pay auction with complete information is higher than in a standard auction. We use our results to re-consider some findings of the literature that models lobbying games as all-pay auctions. In particular, we show that the so-called Exclusion Principle appears to rely crucially on the implicit assumption of a "weak" (in terms of bargaining power) seller, and does not hold if she regards bidders' valuations as iid according to a monotonic hazard rate. Our preliminary results for the case of independent but asymmetric bidders make it even more suspicious.

Keywords: all-pay auctions, reserve price, economic theory of lobbying.

JEL *Classification*: D44, D72.

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1. Introduction

Auction models are prototypes of competitive settings, and they are used in several branches of economic literature. In particular, the so-called (first-price) all-pay auction is used (among others) by Hillman and Riley (1989), Baye *et alii* (1993) and Che and Gale (1998) to model the lobbying process. This type of auction fits the lobbying game well, since a lobbyist's contribution is not typically returned if his efforts are unsuccessful,[1] and indeed this literature has elaborated a number of interesting results. In particular, Hillman and Riley (1989) prove that, if there is some asymmetry among bidders/lobbyists, the politically contestable rent is not totally dissipated even in the case of a large number of potential contenders. In addition, Baye *et alii* (1993) show that a seller/politician wishing to maximize political rents may find it in her best interest to exclude certain lobbyists from her "finalist" short list (so-called "Exclusion Principle"), particularly those lobbyists valuing the political prize most (in order to rise incentives to spend for the likely losers). Che and Gale (1998) also show that the imposition of an exogenous cap on individual lobbying contributions may have the adverse effect of increasing total expenditure (by increasing competition among lobbyists).

The previous results contribute to the economic literature on the lobbying process, and in addition also to the pure theory of auction. From the former perspective, it is intriguing that some of them (namely the Exclusion Principle and the possibly adverse effect of a cap on expenditure) appear not to hold in the alternative class of models so-called *à l*a Tullock (1980): see Fang (2002). From the latter perspective, it has to be stressed that the quoted literature refers to the case of complete information,[2] which is a somewhat unusual assumption in auction theory. In fact, it is somewhat unclear which informational assumptions are made: in particular, the role and the information available to the designer (if any) of the auction are left unexplained.

In this paper we start with placing the all-pay auction model with complete information in the context of the auction literature, and derive from this comparison some implications for the economic theory of lobbying (see Boylan, 2000 for a similar approach). In particular, we wish to argue that the previous results are implicitly but crucially based on the assumption that what would be called the “reserve price” in auction theory is null. In turn, this raises the questions of why the "seller" (say a politician) cannot set a positive reserve price, and of what information is ex ante available to her. The only consistent explanations seem to be that the politician who receives the lobbies' contributions does not know their preferences, or has very little bargaining power. After characterizing the equilibrium of the all-pay auction with a reservation price exogenously given, we

[1] This feature is also shared by other economic and social games, such as patent races and sports.

[2] Hillman and Riley (1989: pp. 29-30) also deal with the case of incomplete information among contenders, and Che and Gale (1998: p. 648) claim that their result would hold even under incomplete information if there were asymmetry among bidders.

show that the seller would prefer a strictly positive reserve price, which also increases the overall efficiency of the auction outcome (even if it might decrease the efficacy of the lobbying process through higher rent dissipation). This casts some doubts on the "adverse" results concerning the effects of caps on individual spending and of lobbyist exclusion, because they might disappear once the possibility that the reserve price is optimally set is taken into account. We then argue that more robust results (if any) should be based on the explicit assumption that the seller faces incomplete information while setting her reserve price, and investigate such a case by comparing the all-pay auction with other trading mechanisms. Our preliminary findings in this setting make even more suspicious the case for the Exclusion Principle.

The paper is organized as follows: section 2 compares the (first-price) all-pay auction with complete information with the so-called "standard auctions" (see e.g. Klemperer, 2004: section 1.1.2). Section 3 characterizes the equilibrium of the former under a positive reserve price exogenously given. Section 4 considers the way an optimal reserve price should be set by a seller facing incomplete information when using an all-pay auction rather than a "standard" auction or a take-it-or-leave-it offer (a "trading mechanism" sub-optimal for the seller if there is more than a single buyer). Section 5 discusses the value for the seller of additional competition on the bidders' side and why our approach casts some doubt on the Exclusion Principle and on the result obtained by Che and Gale (1998) concerning the effect of caps on the lobbies' expenditure. Section 6 briefly concludes. Technical proofs are presented in the Appendixes.

2. Reserve price, auction theory and lobbying games

Consider the following setting: n risk-neutral[3] agents (the "buyers") bid for a prize (there is no resale possibility). Bidder i's (private) valuation of the prize is v_i ($i = 1, \ldots, n$), and we order bidders in such a way that $v_1 > v_2 > \ldots > v_{n-1} > v_n > 0$.[4] The rules of the auction can include a reserve (minimum) price $p_r \geq 0$, i.e., a price below which the prize is not assigned. In an important contribution, Milgrom (1987) puts auction theory in the more general context of bargaining theory, and argues that (winner-pay) standard auctions (namely, the oral ascending or descending-bid and the first and second-price sealed-bid auctions) under complete information (i.e., assuming that every

[3] Agents' risk neutrality is assumed throughout the paper: however, this assumption makes no difference for the case of the second-price sealed-bid auction, as it is well known.

[4] The possibility of ties in the valuations is ignored here. This can be justified by assuming that the v_i are ex-ante continuously independently distributed, and so that case has a priori a zero probability. Moreover, for standard auctions the only relevant implication of ties is that $v_1 = v_2$ implies full rent dissipation, independently from the level of the reserve price (if not larger than v_1) and from the number of contenders (on the contrary, in a all-pay auction ties may imply the existence of multiple Nash equilibria which are not necessarily revenue equivalent: see Baye *et alii*, 1996 and footnote 14 below).

details of the setting is common knowledge to all the participants, including the seller) lead to efficient outcomes. In particular, Milgrom (1987: Proposition 1, p. 7) argues that, under complete information, the set of perfect[5] equilibrium outcomes of the standard auctions consists of the Core outcomes of the corresponding exchange game (together with the no-trade outcome if $p_r = v_1$), as the minimum price ranges from the "seller" evaluation of the good to be sold to the highest "buyer" evaluation.[6] Indeed, while there are many Nash equilibria in those auction games, the only "sensible" ones (i.e., such that no agent ever uses weakly dominated strategies) seem to be those such that the prize is allocated to agent 1 for her bid of $r = \mathrm{Max}\{p_r, v_2\}$ (notice that a reserve price $v_0 < p_r < v_2$, where $v_0 \geq 0$ is the seller's valuation of the prize, would have no effect on the auction outcome).

In particular, let us indicate with b_i the bid of agent i. As it is well known, in the case of a second-price sealed-bid auction $b_i = v_i$, $i = 1, \ldots, n$, does constitute a (weakly) dominant strategy equilibrium in which the prize goes to bidder 1 at a price r (as soon as $p_r \leq v_1$). In the case of a first-price sealed-bid auction, the typical equilibrium has $b_1 = r$, and the strategies of the agents $j = 3, \ldots, n$ can be specified arbitrarily, provided that $b_j \in [p_r, v_j)$ with probability 1 (if $p_r > v_j$, agent j bids arbitrary on $[0, p_r)$). The cumulative distribution function of b_2, $F_2(b_2)$, is continuous[7] on a support (weakly) included in $[0, v_2]$ and first-order stochastically dominates $\underline{F}(b) = (v_1 - v_2)/(v_1 - b)$. Notice that in both these auctions (and in the correspondent oral ones)[8] the equilibrium payoffs are $U_1 = v_1 - r$ and $U_j = 0$, $j = 2, \ldots, n$, while the total payment to the seller is r. Milgrom (1987: p. 8) then observes that, from the perspective of (cooperative) game theory, the seller's ability to set any particular reserve price and stick to it measures her bargaining power.

What about an (first-price) all-pay auction version of the previous setting? In such a case, bidder i receives the prize if $b_i > \mathrm{Max}\{b_{j \neq i}\}$ and in that case his payoff is $v_i - b_i$, whereas his payoff is $- b_i$ if he loses (ties are broken randomly). Assuming $p_r = 0$, Hillman and Riley (1989), and Baye *et alii* (1993) and (1996) show that in the unique Nash equilibrium agent 1 uses the uniform distribution $F_1(b_1) = b_1/v_2$ on the support $[0, v_2]$, while agent 2 uses $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$ on the same support (note that this amounts to say that agent 2 randomises between $b_2 = 0$ and the uniform distribution on $[0, v_2]$ with probabilities respectively $1 - v_2/v_1$ and v_2/v_1). Agents $j = 3, \ldots, n$ bid $b_j = 0$

[5] To avoid technical problems, Milgrom (1987: pp. 5-8) actually works with discrete bid spaces.
[6] Milgrom (1987: p. 3) also argues that, if the prize can be later resold, the outcome is stable and especially favourable to sellers in a relatively poor bargaining position with respect to the potential buyers.
[7] For $p_r < v_2$, agent 2 cannot play $b_2 \notin [p_r, v_2)$ with a positive probability, since those strategies are (weakly) dominated for her: thus, she should bid between p_r and v_2 with probability 1. If $p_r > v_2$, then also $F_2(\cdot)$ is arbitrary on a support $\subseteq [0, p_r)$.
[8] As it is well known, the descending ("Dutch") auction is strategically equivalent to the first-price sealed-bid auction, while in an ascending ("English") auction, similarly to a second-price sealed-bid auction, is clearly a dominant strategy for each bidder i to stay in the bidding until the price reaches her value v_i (see e.g. Klemperer, 2004: section 1.1.4).

with probability 1. The prize is then given to agent 1 with probability $1 - v_2/(2v_1) > ½$ and to agent 2 with probability $v_2/(2v_1) < ½$ (note that in the latter event the result is not ex-post efficient, and thus it would not be stable in the case of a resale opportunity). Agent 1 receives a (expected) payoff of $U_1(v_1, v_2) = v_1 - v_2$, while the (expected) payoffs of the other agents are zero; i.e., $U_j(v_1, v_2) = 0$, $j = 2, \dots, n$. The expected total payment to the seller is $p(v_1, v_2) = p_1(v_1, v_2) + p_2(v_1, v_2) = v_2/2 + (v_2/v_1)(v_2/2) = v_2 (1 + v_2/v_1)/2 < v_2$, where p_i is the expected payment of agent i =1,2.

The previous results show that the outcome of an all-pay auction with a null reserve price does *not* belong to the Core of the corresponding exchange game (note that the expected social welfare is $v_2 < W(v_1, v_2) = v_1 - v_2 + p(v_1, v_2) < v_1$).[9] From the perspective of the economic theory of lobbying, they illustrate the possibility that, even if the number of potential contenders is large, asymmetries among players might imply that the political rent is not fully dissipated (see Hillman and Riley, 1989: pp. 18-19). In addition, note that $\partial p/\partial v_1 < 0$ and $\partial p/\partial v_2 > 0$ ($p(\cdot)$ is convex): indeed, Baye *et alii* (1993) show that a politician (the seller in the auction) wishing to maximize her political rents should be willing to select the two active lobbyists (the bidders) i^* and i^*+1 in order to Max $p(v_i, v_{i+1})$. This implies that she might find it in her best interest to exclude lobbyists from 1 to i^*-1 from her "finalists short list", if she is allowed to (there is no point in excluding bidders from i^*+2 to n). This can be worthy to her because while the expected payment from any $i \neq 1$ in the finalist list is necessarily less than the payment expected from 1 in the original auction, the expected payment from $i+1$ may rise with respect to that of 2 and more than compensate the decrease of the other component of total payment. This is the Exclusion Principle, which is intuitively based on the idea to raise (overall) incentives to spend for the active participants by putting them on more equal foots. More formally, the Exclusion Principle works by raising the equilibrium probability of winning of the less favourite contender (between the two who are active in equilibrium). From the perspective of the economic theory of lobbying, Baye *et alii* (1993: p. 290) argues that the politician (the seller), under plausible circumstances, has an adverse incentive to preclude lobbyists most valuing the prize from participating in the lobbying game.

The idea of handicapping the favourite is simple, interesting and it has some counterpart both in the auction literature with incomplete information (if agents' valuations are not identically and independently distributed: see e.g. Myerson, 1981 and Klemperer, 2004: pp. 21-8) and in the sport practice (e.g., in golf competitions).[10] However, note that bidder 1's exclusion decreases (weakly)

[9] For the sake of simplicity, in computing social welfare we ignore v_0 and use the gross seller utility given by p (net utility would be given by $p - v_0$).

[10] Sport event organizers are typically interested in some "competitive balance" among players: a famous example comes from the history of the Giro d'Italia ("Tour of Italy"), the Italian most important cycling stage-race. It is reported that at the beginning of the twentieth century cyclist Alfredo Binda was so much stronger than his possible competitors (he had already won the Giro five times) that the organisers paid him not to participate.

ex-post efficiency (it gives a positive probability to the allocation of the prize to agent $j > 2$ and no chance to 1), and tends[11] to decrease the expected social welfare (of course the outcome is not in the Core). Moreover, there might be other, possibly more efficient, ways (for a seller with some power to affect the auction rules) to motivate the less favourite contenders (for example offering, whether possible, multiple (divided) prizes: see Moldovanu and Sela, 2001). One such a way is investigated in Che and Gale (1998). They introduce an (symmetric) exogenous cap m on bids and show that, in a setting with two contenders, the Nash equilibrium essentially[12] remains unique. However, while the expected utilities, payments and probability of winning of the agents if $m \in (v_2/2, v_2)$ are the same than without any cap (the cap has of course no effect if $m > v_2$),[13] for $m < v_2/2$ the unique equilibrium is $b_1 = b_2 = m$, so that the expected total payment ($2m$) increases for $m \in (v_2(1 + v_2/v_1)/4, v_2/2)$. The intuitive reason is that if the cap is small enough it reduces the ability of agent 1 to pre-empt his weaker competitor. In this case the prize is allocated randomly either to bidders 1 or 2 with probability 1/2: note that this increases the probability of ex-post inefficiency just because it raises the probability that 2 wins (which motivates him to pay more). Moreover, it always decreases expected welfare with respect to the case of no cap. Che and Gale (1998: p. 648) also show that similar circumstances can arise even if there are more than 2 potential contenders. They argue that limits on individual expenditure (such as the ones imposed by the USA Congress to the lobbies in the case of election campaigns, or due to limited financial endowments) may increase total expenditure and lower social surplus.

To conclude this section, we remember the careful reader that under incomplete information (i.e., if the valuation of each bidder is private information to himself), the picture of auction theory looks much more complex and even the properties of the standard auctions heavily depend on the assumptions made on the informational aspects: see e.g. Klemperer (2004). However, in the benchmark case of (private) valuations ex ante identically and independently distributed (iid) according to a common, strictly increasing and atomless distribution $H(v)$ with risk neutral agents, all (winner-pay) standard auctions (and many non-standard ones as the first-price all-pay auction) yield the same expected revenue to the seller and result in each bidder making the same expected payment as a function of his information. This is the famous Revenue Equivalence Theorem: see e.g. Klemperer (2004: p. 17). A corollary of this result is that, under the same assumptions, all the previous auctions are optimal for the seller if she imposes the optimal reserve price. Under the technical condition (see e.g. Krishna, 2002: Appendix A) that $H(v)$ is continuous and has a

[11] A sufficient but not necessary condition is $v_2 > v_1/2$.
[12] There are multiple equilibria for the non-generic cases of $m = v_2/2$ and $m = v_1/2$.
[13] Note that, with a standard auction (and a null reserve price), an increasingly tight cap would decrease monotonically the expected revenue of the seller whenever $m \le v_2$ (in the equilibrium the prize is given randomly to some agent v_i, such that $v_i > m$, for a price m).

monotonic hazard rate (i.e., if $\lambda(v) = h(v)/(1 - H(v))$ is an increasing function of v, where $h(\cdot)$ is the density function which corresponds to $H(\cdot)$), in a standard auction the optimal reserve price $\hat{p}_r*$ for a seller is defined by the condition $\hat{p}_r* = v_0 + 1/\lambda(\hat{p}_r*) > v_0$, and is thus strictly positive: see e.g. Krishna (2002: pp. 25-6). The optimal reserve price $\breve{p}*(n)$ in an all-pay auction with incomplete information is then given by $\breve{p}*(n) = \hat{p}_r*(H(\hat{p}_r*))^{n-1} > 0$: see e.g. Klemperer (2004: p. 43). Note that $\breve{p}*(n)$ is monotonically decreasing with respect to n, and $\lim \breve{p}*(n) \rightarrow 0$ *for* $n \rightarrow \infty$, while $\hat{p}_r*$ does not depend on n.

3. The all-pay auction with complete information and a reserve price

The results concerning the all-pay auction with complete information we quoted in section 2 have been derived assuming a null reserve price. However, together Milgrom (1987) analysis and the results derived under incomplete information at the very least suggest that it would be interesting to know what happens to the equilibrium outcomes under a positive reserve price exogenously given. This is the goal of this section. In fact, it turns out that the Nash equilibrium is unique. In particular, if $v_1 \geq p_r \geq v_2$, very much as in the case of a standard auction, the prize is allocated to agent 1 for here bid of $r = p_r$, while the other agents bid zero with probability 1 (if $v_1 = p_r$ agent 1 is indifferent to receive the prize and there is another Nash equilibrium in which with probability 1 he bids $b_1 = 0$ too).

Things are more interesting if $p_r < v_2$. The relevant results are summarized in Proposition 1.

Proposition 1. *Consider an (first-price) all-pay auction with complete information (no resale possibility). Suppose* $v_2 \geq p_r \geq 0$. *Then, in the unique Nash bidding equilibrium: i)* $F_1(b_1) = b_1/v_2$ *on the support* $[p_r,v_2]$*; ii)* $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$ *on the support* $\{0 \cup [p_r,v_2]\}$*; iii)* $F_j(0) = 1, j = 3, \ldots, n$.

Proof: see Appendix 1. Note that Proposition 1 says that agent 2 randomises between $b_2 = 0$ and the uniform distribution on $[p_r,v_2]$ with probabilities respectively $1 – (v_2 - p_r)/v_1$ and $(v_2 - p_r)/v_1$), and that agent 1 randomises between $b_2 = p_r$ and the uniform distribution on $[p_r,v_2]$ with probabilities respectively p_r/v_2 and $1 - p_r/v_2$. The equilibrium cumulative distribution function of agents 1 and 2 are illustrated in Figure 1.

Also note that, again very much as in the case of a standard auction, and exactly as in the case without a positive reserve price, agent 1 receives an (expected) payoff of $U_1(v_1, v_2, p_r) = v_1 - v_2$, while the (expected) payoffs of the other agents are zero. However, the prize is now won by agent 1 with probability $1 - (v_2^2 - p_r^2)/(2v_1v_2) > 1 - v_2/(2v_1)$; i.e., the introduction of a positive reserve price raises the probability of an ex-post efficient outcome by raising the probability that the prize is allocated to agent 1. Moreover, the expected total payment to the seller is given by ($v_2 \geq p_r$):

$$\tilde{p}(v_1, v_2, p_r) = \tilde{p}_1(v_1, v_2, p_r) + \tilde{p}_2(v_1, v_2, p_r) = \frac{v_2^2 + p_r^2}{2v_2} + \frac{v_2^2 - p_r^2}{2v_1} \quad (1)$$

(note that $\tilde{p}(v_1, v_2, p_r)$ is continuous (and differentiable) for any $v_2 \geq p_r \geq 0$).[14] Equation (1) shows that, as it should be expected, the payment by agent 1 increases (on expectation), while the payment of agent 2 decreases, with respect to the case of a null reserve price. In particular, the increase is given by $(p_r^2/2)(v_2^{-1} - v_1^{-1})$: note that $\partial\tilde{p}/\partial p_r > 0$ and that $\tilde{p}(v_1, v_2, p_r)$ goes continuously from $p(\cdot)$ to v_1 as p_r goes from 0 to v_1 ($\tilde{p} = 0$ if $v_1 < p_r$ and $\tilde{p} = p_r$ if $v_1 > p_r \geq v_2$). This is described in Figure 2. Note, finally, that since $W(v_1, v_2, p_r) = v_1 - v_2 + \tilde{p}(v_1, v_2, p_r)$, also the expected social welfare increases with respect to the case of a null reserve price.

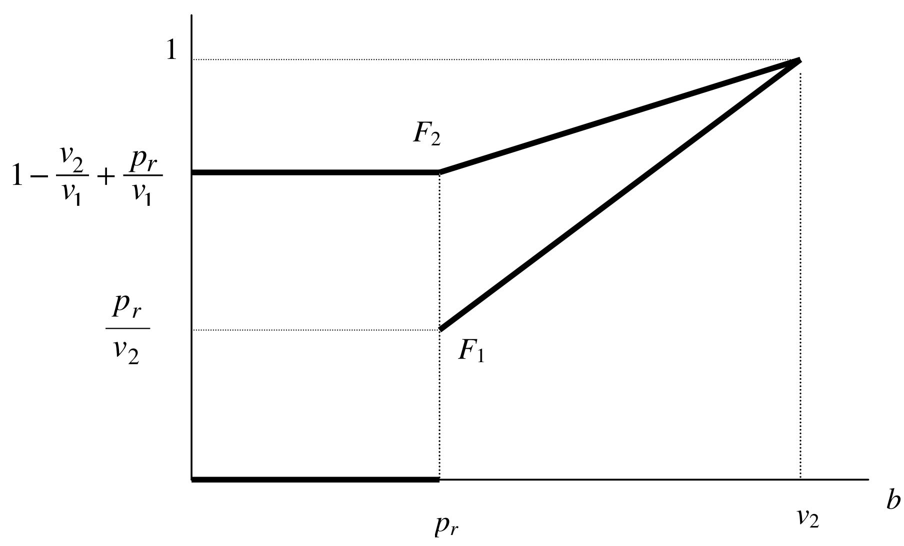

FIGURE 1: The equilibrium distribution functions for $p_r < v_2$

[14] For $v_1 = v_2 \geq p_r > 0$ there is more than 1 Nash equilibrium, and equation (1) does not apply to all of them (see footnote 4).

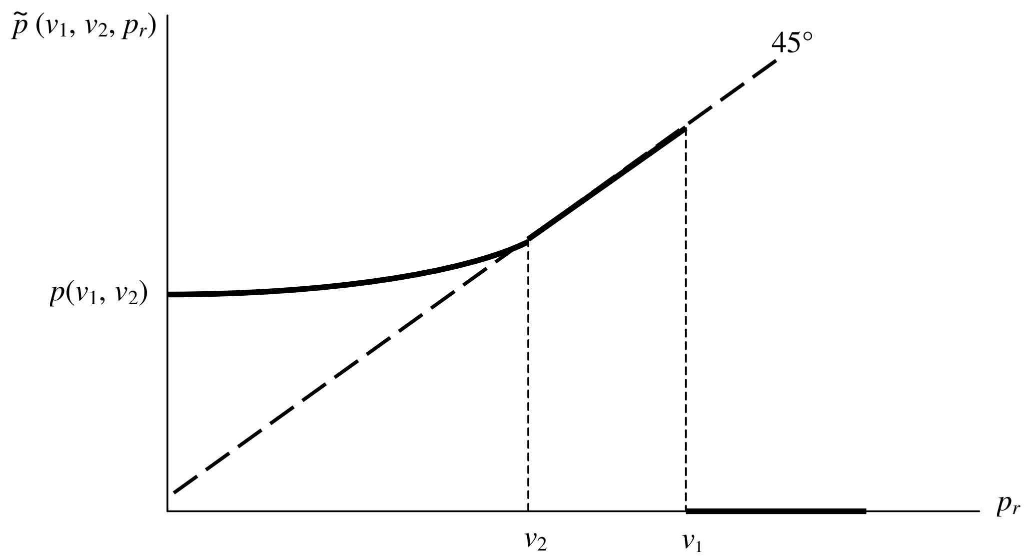

FIGURE 2: The expected revenue as a function of p_r

4. The optimal reserve price for a seller facing incomplete information

Section 3 shows that the outcomes of the (first-price) all-pay auction with complete information do not coincide with the outcomes of the standard auctions if $p_r < v_2$, and are not efficient (they do not belong to the Core). Following Milgrom (1987), we might say that this appears to describe the case of a "weak" seller in terms of bargaining power (indeed, so weak that she seems ready to accept less than v_0, unless the latter value is "sufficiently" small; i.e., unless $v_0 < \tilde{p}\,(v_1, v_2, p_r)$). On the contrary, it is clear that a fully-informed "strong" seller would use $p_r > v_2$, and in fact $p_r = v_1$ (i.e., she would make a take-it-or-leave-it offer to agent 1). Accordingly, we would like to argue that the only assumptions consistent with the use made in the quoted literature of a complete information setting are that the seller, in contrast with the bidders, either is very weak (and cannot commit to a positive reserve price) or she does not know their valuations.

The second case has been recently investigated by Menicucci (2005) and Bertoletti (2005), the latter being a companion paper of the present one. Menicucci (2005) strikingly show that, for some information structures and no reserve price, the Exclusion Principle even applies to the case in which the seller regards the bidders' private valuations as iid. Namely, for the distributional structures he considers, excluding from the all-pay auction with complete information among the ex ante symmetric bidders all but two of them (randomly selected) increases the seller's revenue. Menicucci's example uses a discrete distribution with "small" (the seller is almost certain about the bidders' valuations) uncertainty. However, Bertoletti (2005) (still assuming a null reserve price) show that his result cannot apply to the class of iid continuous distributions with a monotonic hazard rate.

In the following we similarly investigate the case of the optimal reserve price for a seller facing incomplete information. In such a setting, $\tilde{p}\,(v_1, v_2, p_r)$ is the revenue the seller expects ad interim (before bidding takes place but after the definition of a possible "short list" of auction participants), where from her point of view v_1 and v_2 are respectively the first (highest) and the second (second-highest) order statistics of n stochastic variables (see e.g. Krishna, 2002: Appendix C). We generalize the assumptions of Bertoletti (2005) by assuming that v_1 and v_2 are jointly continuously distributed on the support $[\underline{v}, \bar{v}\,]^2$, $\bar{v} > \underline{v} \geq 0$, according to a density function $g(v_1, v_2)$ which is strictly positive for $v_1 > v_2 > \underline{v}$.[15] It follows that the seller should set the optimal reserve price by maximizing with respect to p_r:[16]

$$
\begin{aligned}
P^E(p_r) &= \int_{\underline{v}}^{\bar{v}}\int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) g(v_1, v_2) dv_1 dv_2 \\
&= p_r \int_{\underline{v}}^{p_r}\int_{p_r}^{\bar{v}} g(v_1, v_2) dv_1 dv_2 + \int_{p_r}^{\bar{v}}\int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) g(v_1, v_2) dv_1 dv_2 \\
&= p_r (G_2(p_r) - G_1(p_r)) + \int_{p_r}^{\bar{v}}\int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) g(v_1, v_2) dv_1 dv_2 \\
&= \frac{p_r g_1(p_r)}{\gamma(p_r)} + \int_{p_r}^{\bar{v}}\int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) g(v_1, v_2) dv_1 dv_2,
\end{aligned}
\tag{2}
$$

where $P^E(p_r) = E\{\tilde{p}\,(v_1, v_2, p_r)\}$ (the ex ante expected value of $\tilde{p}$) is a continuous and differentiable function, $G_1(\cdot)$ and $G_2(\cdot)$ are the (marginal) distributions functions of respectively v_1 and v_2, $g_1(\cdot)$ is the density function of v_1 and $\gamma(\cdot) = g_1(\cdot)/(G_2(\cdot) - G_1(\cdot))$. Of course, $G_2(p_r) - G_1(p_r) = \text{Prob}\{v_2 < p_r < v_1\}$. We call γ the *generalized* hazard rate, since it is equal to the hazard rate λ (see section 2) of H if the bidders' valuations are iid according to H. The role played by it in auction theory comes from the fact that $E\{v_1 - v_2\} = E\{1/\gamma(v_1)\}$, as it is easily seen (thus the expected value of the inverse of the generalized hazard rate measures the agents' components of the ex ante expected social welfare). Note that $G_2 \geq G_1$ and thus $\gamma \geq 0$, and $\lim \gamma(v) \to \infty$ for $v \to \bar{v}$. Also note

[15] Note that, obviously, $g(\cdot)$ depends on the joint distribution of the bidders' valuations: see Appendix 4 for the case of independent bidders' valuations.

[16] For the sake of simplicity, in the following we assume $v_0 = 0$.

that $P^E(\bar{v}) = 0$, and $P^E(\underline{v}) > 0$.

A bit of computation also shows that:

$$\frac{dP^E(p_r)}{dp_r} = -g_1(p_r)\varphi(p_r) + p_r \int_{p_r}^{\bar{v}} \int_{v_2}^{\bar{v}} (\frac{1}{v_2} - \frac{1}{v_1}) g(v_1, v_2) dv_1 dv_2 \ , \tag{3}$$

where $\varphi(\cdot) = (\cdot) - 1/\gamma(\cdot)$. We call φ the *standard* virtual value, because obviously $E\{\varphi(v_1)\} = E\{v_2\}$, so that the expectation of the standard virtual value measures the seller's (ex-ante) expected total payment $\hat{P}^E$ in a standard auction with complete information and no reserve price (see A.5 in Appendix 2 and Krishna, 2002: section 5.3). Note that lim $\varphi(v) \to \bar{v}$ for $v \to \bar{v}$. Since $g_1(\underline{v}) = 0$ and $g_1(\bar{v}) > 0$, (3) implies that $dP^E(\bar{v})/dp_r < 0$, and $dP^E(\underline{v})/dp_r > 0$ if $\underline{v} > 0$ (otherwise $dP^E(\underline{v})/dp_r = 0$). Thus the optimal reserve price p_r* is larger[17] than $\underline{v}$, smaller than $\bar{v}$, and must satisfy $dP^E(p_r^*)/dp_r = 0$. To give an example, suppose that each v_i, $i = 1, \ldots, n$, is uniformly iid on the support [0,1]. In this case $g(v_1, v_2) = (n^2 - n)(v_2)^{n-2}$ (see Appendix 4 and e.g. Krishna, 2002: p. 267), and then (3) becomes:

$$\frac{dP^E(p_r)}{dp_r} = np_r^{\,n-1}(1 - 2p_r) + (n^2 - n)p_r \int_{p_r}^{1} \int_{v_2}^{1} (v_2^{\,n-3} - \frac{v_2^{\,n-2}}{v_1}) dv_1 dv_2 \tag{4}$$

which is somehow difficult to solve explicitly. However, it is easy to see that, if $n = 2$, p_r* must satisfy $\ln p_r^* = -(1 + p_r^*)^{-1}$, and thus it is larger than $E\{v_i\} = ½$. This expected value would also be, under incomplete information, the optimal reserve price $\hat{p}_r^*$ in a standard auction, which in turn is larger than the optimal reserve price $\breve{p}^* = 1/4$ of the correspondent all-pay auction (see section 2). It is also clear that p_r* is on the contrary lower than the price, say $\bar{p}_r^*$, a seller in a strong bargaining position should ask (simultaneously) to two agents by a take-it-or-leave-it offer, if their valuations are uniformly iid on [0,1] (the latter price maximizes $p_r(1 - p_r^2)$, and more generally $p_r(1 - G_1(p_r))$). The obvious reason is that, in contrast with the case of standard auctions, with complete information in an all-pay auction the expected revenue is an increasing function of p_r even if $p_r < v_2$. Similarly, in the case of a take-it-or-leave-it offer, the revenue is exactly p_r even if $p_r < v_2$ (see Figure 2), while $\partial\tilde{p}/\partial p_r < 1$.

In fact, the previous results hold for any joint density function $g(\cdot)$ (and any $n \geq 2$) such that the standard virtual value is *monotonic* (obviously, a *sufficient* condition for this is the monotonicity of the generalized hazard rate). The economic content of this technical assumption is

[17] See Appendix 2 and footnote 25.

to guarantee that in a standard auction (with no reserve price) the seller indeed ex ante values more the larger value of v_1 (remember that v_2 matters to her). We assume that such a regularity condition holds. Since $v_2 > \tilde{p}(v_1, v_2, p_r) > p_r$ for any $v_2 > p_r$, the seller also strictly gains (in expected terms) by being able to organize an auction, and with complete information among the bidders more by a standard auction than by an all-pay auction. These results are summarized in Proposition 2.

Proposition 2. *Consider an auction setting with $n \geq 2$ bidders who have complete information. Suppose that the seller regards bidders' valuations as such that v_1 and v_2 are continuously distributed, and call P^E, $\hat{P}^E$ and $\overline{P}^E$ the revenues she expects ex ante by running either an all-pay auction, or a "standard auction", or by using a take-it-or-leave-it offer, with respectively optimal reserve prices given by p_r*, $\hat{p}_r*$ and $\overline{p}_r*$. Then $\hat{P}^E(\hat{p}_r*) > P^E(p_r*) > \overline{P}^E(\overline{p}_r*)$, and $\hat{P}^E(\hat{p}_r*)$ is also the revenue expected by the seller in an optimal second-price sealed-bid auction with incomplete information. Suppose that the standard virtual value $\varphi(x) = x - 1/\gamma(x)$ is monotonic: then $\overline{p}_r* > p_r* > \hat{p}_r*$.*

Proof: see Appendix 2. Note that a special case arises if, as in Bertoletti (2005), the seller regards bidders' valuations (here denoted by v^j, where $j = 1, \ldots, n$ indicates the bidders whose valuations are not ex ante ordered) as iid according to a continuous cumulative distribution function $H(\cdot)$ on the support $[\underline{v}, \overline{v}]$ (the assumption of the Revenue Equivalence Theorem). Then $\hat{p}_r*$ does not depend on n, as we have seen in section 2.[18] The intuition for this result is that, roughly speaking, an increase in p_r causes a marginal expected revenue for the seller which only depends on $[1 - G_{1|2}(p_r)]$ (where $G_{1|2}(\cdot) = (H(\cdot) - H(v_2))/(1 - H(v_2))$ is the distribution function of v_1 conditional on v_2, and does not depend on n) in the case of a standard auction, while there is an additional marginal expected benefit in the case of the two other exchange mechanisms which depends on $G_2(\cdot)$ (that in turn does depend on n). Finally, note that, in such a special case, by the Revenue Equivalence Theorem, $\hat{P}^E(\hat{p}_r*)$ is also equal to the revenue expected by a seller in any optimal standard auction under incomplete information.

5. Some implications for the economic theory of lobbying

The literature which uses the all-pay auction with complete information to model the lobbying process (Hillman and Riley, 1989, Baye *et alii*, 1993 and Che and Gale, 1998) appears to have

[18] It is also easy to see that $\overline{p}_r*$ increases with respect to n.

assumed (somehow implicitly) that the reserve price is null. The previous sections show that this amounts to assume that the seller (the politician), if fully informed, is in a very weak bargaining position, so that the resulting outcomes are not efficient. While this may occasionally be the case, it seems clear that in the general case the politician will be able to set a positive reserve price, since this monotonically increases her expected payoff. Indeed, in the opposite polar case of a politician with a strong bargaining power, we should expect a reserve price at least equal to v_2 (if not v_1), and thus an efficient result. In such a case, there will not generally be full rent dissipation (unless in the extreme cases of either $v_1 = v_2$ or $p_r = v_1$), independently from the number of competitors. But, clearly, the Exclusion Principle does not apply, since it will always be better for the politician to use a reserve price large enough ($p_r \geq v_2$) rather than to exclude from the "finalists short list" some of the lobbyists valuing most the political prize.

However, the situation is more complex if the politician is not strong enough to set a reserve price $p_r \geq v_2$. Since $\partial\tilde{p} / \partial v_1 < 0$ and $\partial\tilde{p} / \partial v_2 > 0$, it is still possible that the exclusion of some agents is in the interest of the politician. In particular, she should choose i, j $(> i)$ *and* p_r in order to maximize $\tilde{p}(v_i, v_j, p_r)$ under the "bargaining" constraints she faces. Notice that, if the reserve price that the politician can adopt does not depend on the agents she selects, she will always choose the largest possible reserve price, say p_r^+, and also agent i+1 when he chooses agent i (things are very much as in Baye *et alii*, 1993). But, in such a case, it cannot be optimal to exclude agent from 1 to i-1 if $p_r^+ \geq v_{i+1}$ (since $\tilde{p}(v_i, v_{i+1}, p_r) > p_r$), which implies that there will be no exclusion at all if $p_r^+ \geq v_3$. Moreover, the assumption that a fully informed politician can credibly exclude some lobbyist from her "short list" while she is unable to ask him a price not larger than his valuation does not seem particularly palatable as a bargaining feature.

Similarly, one might conjecture that a symmetric cap on the individual bids would always be matched by the reserve price set by an informed politician with a bargaining power large enough (i.e., $p_r = m$). Thus, a cap would always decrease the total spending with respect to the case of no cap, decreasing the efficiency of the auction (but possibly raising overall efficiency if campaign spending is *per se* socially harmful: see Che and Gale, 1998). However, while this is certainly so if $m \in (v_2, v_1)$, for smaller (binding) values of m the effect of a cap on the reserve price optimal for the seller still need to be investigated.[19] In particular, Che and Gale (1998: p. 648) show that, if the reserve price is null, the perverse effect they identified arises when $v_k/k > m > v_{k+1}/(k + 1)$ for some $k < n$. In such a case, the revenue expected by the politician is km, which can be (arbitrary) close to v_k, and then possibly larger than $p(v_1, v_2)$. But, again, a politician with some bargaining power might

[19] In an unpublished note D. Menicucci shows that the adverse effect of introducing a cap might arise even if there is a positive reserve price *exogenously given*.

be able to set $p_r \geq v_k$, raising her revenue *at least* (if $v_2 > p_r$) to $\tilde{p}\,(v_1,\, v_2,\, v_k) > v_k$. Thus, the imposition of a cap might de facto decrease overall spending, once the optimal reserve price set by the politician is kept into account.

In fact, we have argued in section 4 that a more interesting assumption is that the seller (the politician) does not know the contenders valuations when she sets her reserve price. In such a setting, results from the case of the standard auctions under incomplete information appear to suggest that, as in Bertoletti (2005), the seller should not find generally useful to exclude participants with independent valuations. For example, Bulow and Klemperer (1996) proved that an English auction with $n + 1$ bidders and *no* reserve price is always more profitable than *any negotiation* with just n participants, if valuations are independent.[20] To grasp the idea in our setting, consider the case of a seller that can either make a take-it-or-leave-it offer to a buyer whose valuation for the prize is uniformly distributed on [0,1], or to run an all-pay auction with complete information for two participants whose valuations are iid uniformly on the support [0,1]. The first opportunity yields her an expected payoff of 0,25 (as we know, she should optimally ask a "reserve price" equal to ½). In the second case she should expect the larger payoff of:

$$P^E(0) = 2\int_0^1 \int_{v_2}^1 \tilde{p}(v_1, v_2, 0)\, dv_1 dv_2 = \frac{5}{18} \tag{6}$$

(the seller would get $\hat{P}^E = 1/3$ if she could use a standard auction), and even more by using optimally a positive reserve price. Can this result be generalized? The answer is certainly yes if bidders' valuations are ex-ante iid with a monotonic hazard rate. In particular, the following proposition holds.

Proposition 3. *Consider an all-pay auction with complete information among bidders and any given reserve price. Suppose that the bidders' valuations are ex-ante iid according to a strictly increasing continuous distribution H(·) with a monotonic hazard rate. In this case the seller maximizes her expected revenue by getting the largest possible number of actual participants.*

Proof: see Appendix 3 (it follows Bertoletti, 2005).

Proposition 3 shows that the Exclusion Principle cannot apply if the seller regards the bidders' valuations as iid according to a monotonic hazard rate (a condition satisfied by many distributions). This implies that $E\{v_1 - v_2\}$ decreases with respect to the number of bidders and this, in turn,

[20] Bulow and Klemperer (1996) also report results for the more general case of ex ante so-called "affiliated" valuations.

implies that a larger n cannot harm the seller (for a discussion see Bertoletti, 2005),[21] whatever is the reserve price adopted. However, we believe that a fair assessment of the Exclusion Principle should rather refer to the case of valuations which are not identically distributed. Even in those cases, the results by Bulow and Klemperer (1996) about the value for the seller of additional (independent) competition on the bidders' side and the following examples motivate the conjecture (whose proof is left for future work) that the seller will generally not find profitable to exclude the contender "more eager" to buy, once she can set optimally her reserve price.

Suppose that there are three possible participants, and everybody knows that valuations are $v^3 = 40$, $v^2 = 38$,[22] and that v^1 is distributed on $[47, \bar{v}]$ according to a continuous, independent cumulative distribution function $H^1(\cdot)$ with density function $h^1(\cdot)$. It is also common knowledge that the agents know the realized value of v^1. If $p_r = 0$, the seller should expect an equilibrium revenue of less than $p(47, 40) \approx 37{,}02$ when all participate. If agent 1 is excluded, this gives the seller an expected payoff of $p(40,38) = 37{,}05$ but, clearly, she can do better by excluding none, setting $p_r = 47$ and receiving that amount for sure. In fact, if $1/h^1(47) > 47$, she should set the higher reserve price such that $p_r = 1/\lambda^1(p_r)$, and expects an even larger revenue equal to $(1 - H^1(p_r))/\lambda^1(p_r)$. Indeed, clearly no exclusion which shifts downwards the entire support of v_1 can be optimal for the seller if she can freely set a reserve price. Similarly, suppose that, while it is common knowledge that $v^3 = 38$, and that v^1 and v^2 are known to the agents, the seller just knows that the valuations of the latter participants are ex ante independently distributed respectively according to $H^1(v^1)$ and $H^2(v^2)$ with the same support $[40, \bar{v}]$ and $H^1(v) \leq H^2(v)$. Again, by excluding agent 1 but setting no reserve price the seller cannot achieve more than 37,05. However, she should expect no less than 40 by optimally setting her reserve price $p_r \geq 40$ and excluding no agent. Moreover, whatever reserve price larger than 40 she might set, she would get a larger expected payoff by excluding agent 2 rather than agent 1.

Let us now go back to the case in which both v^2 and v^3 are iid uniformly on [0,1], while v^1 is independently distributed on the same support with $H^1(v) < v$ for $0 < v < 1$. By excluding agent 1 and optimally setting her reserve price for the remaining two agents the seller cannot achieve more than:

$$\hat{P}^E(\frac{1}{2}) = \frac{1}{4} + 2\int_{1/2}^{1}(y - y^2)dy = \frac{5}{12}, \tag{7}$$

[21] Note that, with iid bidders' valuations, the conditional expectation $E_{1|2}\{v_1 - v_2\}$ is nothing but the so-called *mean residual life* of v_i at v_2 (see e.g. Shaked and Shanthikumar, 1994: section 1.D), and this is decreasing if λ is monotonic.
[22] These numbers are taken by Baye *et alii* (1993: p. 293).

where $\hat{P}^E$ (1/2) is the payoff she could get by running a standard auction. However, by excluding no agent and optimally setting her reserve price she can get more than:

$$\overline{P}^E(\frac{1}{2}) = \frac{1}{2}(1 - \frac{1}{4}H_1(\frac{1}{2})) > \frac{7}{16} > \frac{5}{12}, \tag{8}$$

where $\overline{P}^E$ (1/2) is the payoff she could get by a take-it-or-leave-it offer of 1/2. Indeed, it is intuitive and easy to see that, if potential bidders' valuations are independent, associated to a larger bidders' group there are distributions for the first and the second order statistics such that they first-order stochastically dominate the corresponding distributions associated to a smaller bidder group (see Appendix 4 and Shaked and Shanthikumar, 1994: section 1.B.4). This immediately implies (see A.5 and A.7) larger values for $\hat{P}^E$ and $\overline{P}^E$ when the set of bidders is larger.[23]. The inequalities in (8) even show that it might well be that $\overline{P}^E(\overline{p}_r*; n+1) > \hat{P}^E(\hat{p}_r*; n)$ and then, a fortiori, $P^E(p_r*; n + 1) > P^E(p_r*; n)$, where we have used a notation which stresses the dependence of the seller's expected revenue on the set of bidders (however, the former inequality does not hold for any *n* even if bidders' valuations are uniformly iid).

Finally, we speculate that, as in the case of iid bidders' valuations, a key question is the effect of enlarging the set of bidders on $E\{v_1 - v_2\}$,[24] but this appears hard to characterized in the general case, since the enlargement also affects the generalized hazard rate γ. In particular, with a common support for the independent bidders' valuations, $1/\gamma$ is an average of the different $1/\lambda^j = (1 - H^j)/h^j$ whose (variable) weights are the normalized values of the so-called *reverse* hazard rates $\sigma^j(\cdot) = h^j(\cdot)/H^j(\cdot)$: see Appendix 4. Thus, even a monotonic generalized hazard rate would not be enough to guarantee that $E\{v_1 - v_2\}$ decreases when the set of (independent but not identical) bidders enlarges. However, an especially simple case arises if there are only two types of bidders, say *s* and *w*, with $H^s(\cdot) = (H^w(\cdot))^\theta$, $\theta > 1$: following Krishna (2002: section 4.3), we call them the strong (*s*) and the weak (*w*) bidders, since H^s likelihood-ratio stochastically dominates H^w (see e.g. Krishna, 2002: Appendix B and Shaked and Shanthikumar, 1994: section 1.C). In such a case it is easy to see that weights in the average that defines the generalized hazard rate are constant and equal respectively to $\theta n^s/(\theta n^s + n^w)$ and $n^w/(\theta n^s + n^w)$ (where n^s and n^w are the numbers of the strong and the weak bidders, with $n = n^s + n^w$), and that the addition of any type of bidder generates a distribution for v_1

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[23] Note that, if the standard value is monotonic, $\hat{P}^E(\hat{p}_r*) = E\{\text{Max}\{\varphi(v_1),0\}\}$ and $\overline{P}^E(\overline{p}_r*) = E\{\text{Max}\{\psi_1(v_1),0\}\}$ (see Appendix 2), where the set of bidders affects both functions φ and ψ_1 and the distribution of v_1 (however, with bidders' valuations iid, neither γ nor then φ depend on *n*, while a raise in *n* decreases both λ_1 and ψ_1: see Appendix 4 and Shaked and Shanthikumar, 1994: section 1.B.4).

[24] Note that, up to the second term of its Taylor expansion with respect to v_1 at the right of v_2, $\tilde{p} \approx v_2 - (v_1 - v_2)[1 - (p_r/v_2)^2]/2$.

such that $g_1(v_1; n + 1)$ likelihood-ratio stochastically dominates $g_1(v_1; n)$: see (A.14). Moreover, the monotonicity of λ^w is then sufficient to guarantee the monotonicity of γ.

Under these assumptions, a *sufficient* condition for getting a decrease $E\{v_1 - v_2\}$ by the addition of one strong bidder is:

$$H^s(v) < [\frac{d}{dv}\{1/\gamma(v;n+1)\}]/[\frac{d}{dv}\{1/\gamma(v;n)\}], \tag{9}$$

as it can be seen by taking the difference of the expected values of the generalized hazard rate before and after the addition, and integrating by parts. Since computation shows that $d(1/\lambda^w)/dv > d(1/\lambda^s)/dv$, it is clear that (9) is satisfied and accordingly that adding a "strong" (as defined) independent bidder to the set of bidders does decrease the expected difference of $\{v_1 - v_2\}$. Note that, by the properties of likelihood-ratio stochastic dominance, such an addition decreases both λ_1 and γ and thus raises both $\hat{p}_r*$ and $\overline{p}_r*$. Simulations using the uniform distribution on [0,1] for H^w indeed suggest that it also always increases P^E.

6. Conclusions.

In this paper we have characterized the equilibrium of (first-price) all-pay auctions under complete information and a positive reserve price, and compared it with that of standard auctions. As it is intuitive, a fully informed seller with some bargaining power should set a positive reserve price, since this is profitable for him (and increases overall efficiency). However, once the possibility that the reserve price is optimally set is taken into account, it is unclear if some interesting recent findings in the economic theory of lobbying which uses the all-pay auction framework still apply. Namely, in order to increase her total revenue, the fully-informed seller (of a political rent) might find better to adapt her reserve price rather than to exclude a lobbyist especially eager "to buy" (the so-called Exclusion principle). Similarly, the effects of imposing a cap on the individual campaign contributions become dubious, since a tightening of the cap, that might increase the overall spending, should also change the optimal reserve price. In other words, the previous results appear to apply to the case of a "weak" (in terms of bargaining power) seller, who is unable to stick to a reserve price (in spite of the fact that she is assumed to be fully informed and also able to select the participants to her "short list").

We have also argued that an appealing model should assume that the seller faces incomplete information when setting her optimal reserve price, and characterized such an optimal reserve price under the assumption of a monotonic standard virtual value. Future work will have to assess if some (if any) of the results derived by assuming a fully informed seller can be confirmed in such a setting. However, if the seller regards the bidders' valuations as iid according to a monotonic hazard rate, the Exclusion principle cannot, and our preliminary results for the case of independent valuations, as well as suggestions from the auction literature with incomplete information about the value for the seller of additional competition on the bidders' side cast further doubts on it.

Appendix 1

Proof of Proposition 1. Clearly, for each agent i the set of (weakly) undominated strategies is given by $\{0 \cup [p_r,v_i)\}$. Moreover, it can be shown that in equilibrium no bidder plays $b_i \in (p_r,v_i)$, and no more than one agent bid p_r, with a positive probability. This is so because if at least two of them do the latter, both would have an incentive to move the probability mass slightly higher, so increasing their payoffs (the conditional probability of winning would jump, and so the payoff). If exactly one agent i has a mass point at some $b_i \in (p_r,v_i)$, then no other agent would place density immediately below that bid (it would be better to move that density above the mass point). But then agent i would do strictly better by moving that mass down (see Che and Gale, 1998: p. 645, Lemma 1, and Hillman and Riley, 1989: pp. 22-23, Proposition 1, for a formal proof). Thus, all equilibrium cumulative distribution function $F_i(b_i)$ must be continuous on (p_r,v_i).

Now note that agent 1 can secure himself a payoff equal to $v_1 - v_2 > 0$ by bidding $b_1 = v_2$ with probability 1. It follows that his equilibrium strategy support cannot include $b_1 \in \{[0,p_r) \cup (v_2,v_1)\}$. Suppose that there is an agent $j \neq 1$ who gets in equilibrium a positive expected payoff. Then it must be the case that he bids $b_j > p_r$ with probability 1 (he cannot neither bid zero nor bid p_r with positive probability, because otherwise he would get respectively a null and a negative payoff, while he should be indifferent among all bids that belong to the support of his own equilibrium cumulative distribution function $F_j(b_j)$). And it must also be the case that his infimum bid does coincides with the infimum bid of agent 1, say b^-, because otherwise at least one of them would get a negative payoff by bidding his own infimum bid. In fact, we have found a contradiction, because even by bidding b^- at least one of them must get a negative payoff (the conditional probability of winning is zero). Thus no agent other than 1 can get a positive payoff in the equilibrium, or bid p_r with a positive probability.

In addition, any agent different from 1 bidding more than p_r with a positive probability must have an "infimum" bid ($\geq p_r$) not smaller than b^- (otherwise he would get less than zero from that bid), and at least one must bid b^- (otherwise it would pay to someone to move down some density). Similarly, at least two agents must share the maximum bid, say b^+, larger than p_r. Let us now suppose that two agents different from 1, say j and h, bid more than p_r with a positive probability. It must then be the case that:

$$v_j Prob(j\ wins \mid b_j = b) - b = v_j \Pi_{i\neq j}^{n} F_i(b) - b = v_h \Pi_{i\neq h}^{n} F_i(b) - b\,, \qquad \text{(A.1)}$$

for any $b > p_r$ belonging to the support of both $F_j(\cdot)$ and $F_h(\cdot)$. This implies that, for *all* such a b:

$$\frac{F_h(b)}{F_j(b)} = \frac{v_h}{v_j}\,, \qquad \text{(A.2)}$$

which implies that $F_h(\cdot)$ *strictly* first-order stochastically dominates $F_j(\cdot)$ if $j < h$. Let k the largest agent number among those bidding in equilibrium more than p_r with positive probability. This implies that the maximum bid larger than p_r belongs to the support of both $F_1(\cdot)$ and $F_k(\cdot)$. In turn, this implies that:

$$v_k Prob(k\ wins \mid b_k = b^+) - b^+ = v_k F_1(b^+) - b^+ = v_k - b^+ = 0\,, \qquad \text{(A.3)}$$

but then by bidding b^+ with probability 1 agent 2 would get a positive payoff, unless both $k = 2$ and $b^+ = v_2$.

It follows that in equilibrium only agent 1 and 2 are active, with agent 1 using $F_1(b_1)$ on a support $[b^-,v_2]$, while $F_2(b_2)$ has possibly support $\{0 \cup [b^-,v_2]\}$. Since it must be the case that for any $b \in [b^-,v_2]$:

$$v_2 F_1(b) - b = 0, \quad v_1 F_2(b) - b \geq v_1 - v_2\,, \qquad \text{(A.4)}$$

we can conclude that $b^- = p_r$, that $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$ has in fact the support $\{0 \cup [p_r,v_2]\}$, and that agent 1 uses $F_1(b_1) = b_1/v_2$ on the support $[p_r,v_2]$. *Q.E.D.*

Appendix 2

Proof of Proposition 2. In the case of a standard auction the seller would set the reserve price by maximizing:

$$\begin{aligned}
\hat{P}^E(p_r) &= \frac{p_r g_1(p_r)}{\gamma(p_r)} + \int_{p_r}^{\bar{v}}\int_{v_2}^{\bar{v}} v_2 g(v_1,v_2)dv_1 dv_2 \\
&= p_r(1 - G_1(p_r)) + \int_{p_r}^{\bar{v}}[1 - G_2(v_2)]dv_2 \qquad \text{(A.5)} \\
&= \int_{p_r}^{\bar{v}} \varphi(v_1)dG_1(v_1)
\end{aligned}$$

Note that $\hat{P}^E(p_r) > P^E(p_r)$ for any $p_r < \bar{v}$ (as it is clear from Figure 2). Again, $\hat{P}^E(\bar{v}) = 0$, and $\hat{P}^E(\underline{v}) > 0$. Indeed, (A.5) implies that:

$$\frac{d\hat{P}^E(p_r)}{dp_r} = -g_1(p_r)\varphi(p_r)\,, \tag{A.6}$$

which in turn says that $d\hat{P}^E(\bar{v})/dp_r < 0$ and $d\hat{P}^E(\underline{v})/dp_r = 0$. Thus, also in the case of standard auctions, the optimal reserve price $\hat{p}_r{}^*$ is generally[25] in the interior of the support $[\underline{v}, \bar{v}]$ and, under the assumption of a monotonic standard virtual value, is uniquely identified by the condition $\varphi(\hat{p}_r{}^*) = 0$. Also note that, clearly, the revenue in (A.5) does coincide with that expected by a seller running a (winner-pay) second-price sealed-bid auction under incomplete information and a reserve price. Thus, if the reserve price were optimally set, through the Revenue Equivalence Theorem it would also be equivalent to the revenue expected by the seller in any optimal standard auction with incomplete information. Note that the right-hand side in (A.6) does coincide with the first term in the right-hand side in (3), while the second term in the latter expression is always positive for $p_r \in (\underline{v}, \bar{v})$. It follows that necessarily $p_r{}^* > \hat{p}_r{}^*$.

Similarly, by using a take-it-or-leave-it offer the seller would maximize:

$$\begin{aligned}\overline{P}^E(p_r) &= \frac{p_r g_1(p_r)}{\gamma(p_r)} + p_r \int_{p_r}^{\bar{v}} \int_{v_2}^{\bar{v}} g(v_1, v_2)\, dv_1 dv_2 \\ &= p_r [G_2(p_r) - G_1(p_r) + \int_{p_r}^{\bar{v}} g_2(v_2)\, dv_2] \\ &= p_r (1 - G_1(p_r))\end{aligned} \tag{A.7}$$

with respect to p_r. Note that $\overline{P}^E(p_r) < P^E(p_r)$ for any $p_r < \bar{v}$, $\overline{P}^E(\bar{v}) = 0$ and $\overline{P}^E(\underline{v}) = \underline{v}$. Computation shows that:

$$\begin{aligned}\frac{d\overline{P}^E(p_r)}{dp_r} &= G_2(p_r) - G_1(p_r) - p_r g_1(p_r) + \int_{p_r}^{\bar{v}} \int_{v_2}^{\bar{v}} g(v_1, v_2)\, dv_1 dv_2 \\ &= -\, g_1(p_r)\psi_1(p_r),\end{aligned} \tag{A.8}$$

where $\psi_1(\cdot) = (\cdot) - 1/\lambda_1(\cdot)$ is the virtual value of v_1 (it is the standard virtual of a single agent with ex ante valuation distributed according to G_1: see e.g. Krishna, 2002: section 5.2.3) and $\lambda_1(\cdot) = g_1(\cdot)/(1 - G_1(\cdot))$ his hazard rate (note that $\varphi(v_1) > \psi_1(v_1)$ for $v_1 < \underline{v}$). (A.8) implies $d\overline{P}^E(\bar{v})/dp_r <$

[25] With a monotonic generalized virtual value the optimal reserve price $\hat{p}_r{}^*$ is equal to $\underline{v}$ only if $\varphi(\underline{v}) \geq 0$.

0, and $d\overline{P}^E(\underline{v})/dp_r > 0$. Thus, once again the optimal "reserve" price $\overline{p}_r{}^*$ is in the interior of the support $[\underline{v}, \overline{v}]$. Note that the first term in first expression of the the right-hand side in (A.8) does coincide with the first term in the right-hand side in (3), while the second term in the latter expression is always smaller than the correspondent term in (A.8) for $p_r \in (\underline{v}, \overline{v})$. It follows that necessarily $\overline{p}_r{}^* > p_r{}^*$ if the standard virtual value is monotonic, where the former reserve price is uniquely determined by the condition $\psi_1(\overline{p}_r{}^*) = 0$. Obviously, $\overline{P}^E(\overline{p}_r{}^*) < P^E(p_r{}^*) < \hat{P}^E(\hat{p}_r{}^*)$. *Q.E.D.*

Appendix 3

Proof of Proposition 3. Since the density function of the joint distribution of the first and second order statistics (see Appendix 4 and e.g. Krishna, 2002: p. 267) of n independent draws from H is given by:

$$g(v_1, v_2) = (n^2 - n)(H(v_2))^{n-2} h(v_1) h(v_2) I_{(v_2,\infty)}(v_1) \tag{A.9}$$

(where $I_{(\cdot)}(\cdot)$ is the appropriate indicator function), the density function of v_1 conditional on v_2 is given by:

$$g_{1|2}(v_1|v_2) = \frac{g(v_1, v_2)}{n(n-1)(1 - H(v_2))(H(v_2))^{n-2} h(v_2)} = \frac{h(v_1)}{1 - H(v_2)} \tag{A.10}$$

on the support $[v_2, \overline{v}]$ (note that it does not depend on n). Clearly, $E\{\tilde{p}(v_1, v_2, p_r)\} = E_{v_2}\{E_{v_1|v_2}\{\tilde{p}(v_1, v_2, p_r)\}\}$, with obvious notation for the previous expectations. Now consider, for any given p_r, the function $t(\cdot)$:

$$t(v_2, p_r) = E_{v_1|v_2}\{\tilde{p}(v_1, v_2, p_r)\} = \begin{cases} \int_{v_2}^{\overline{v}} \tilde{p}(v_1, v_2, p_r) \dfrac{h(v_1)}{1 - H(v_2)} dv_1 & \text{for } v_2 \geq p_r, \\ p_r \dfrac{1 - H(p_r)}{1 - H(v_2)} & \text{for } v_2 \leq p_r, \end{cases} \tag{A.11)}$$

and note that it is continuous, and differentiable for any $p_r \neq v_2$. Clearly, $t(\cdot)$ increases with respect to v_2 if $p_r \geq v_2$.

Now consider the case $p_r \leq v_2$ and compute the derivative

$$\frac{\partial t(v_2,p_r)}{\partial v_2} = \int_{v_2}^{\bar{v}} [\frac{\partial \tilde{p}(v_1,v_2,p_r)}{\partial v_2} \frac{1}{1-H(v_2)} + \frac{h(v_2)\tilde{p}(v_1,v_2,p_r)}{[1-H(v_2)]^2}] h(v_1)dv_1 - \frac{h(v_2)\tilde{p}(v_2,v_2,p_r)}{1-H(v_2)}. \quad \text{(A.12)}$$

Then, by using the convexity of $\tilde{p}\,(\cdot)$ with respect to v_1:

$$\frac{\partial t(v_2,p_r)}{\partial v_2} \geq \frac{1}{1-H(v_2)}\{\int_{v_2}^{\bar{v}} \frac{\partial \tilde{p}(v_1,v_2,p_r)}{\partial v_2} h(v_1)dv_1 + \frac{h(v_2)}{1-H(v_2)}\int_{v_2}^{\bar{v}}[v_2 + (v_1-v_2)(\frac{p_r^2}{2v_1^2}-\frac{1}{2})]h(v_1)dv_1 \; - h(v_2)v_2\}$$
$$= \frac{1}{1-H(v_2)}\{\int_{v_2}^{\bar{v}}(\frac{1}{2}-\frac{p_r^2}{2v_1^2}+\frac{v_2}{v_1})h(v_1)dv_1 + \frac{h(v_2)}{2(1-H(v_2))}(\frac{p_r^2}{v_1^2}-1)\int_{v_2}^{\bar{v}}(v_1-v_2)h(v_1)dv_1\} \quad \text{(A.13)}$$
$$= \int_{v_2}^{\bar{v}} \frac{v_2}{v_1}\frac{h(v_1)}{1-H(v_2)}dv_1 + (\frac{1}{2}-\frac{p_r^2}{2v_1^2})[1 - \int_{v_2}^{\bar{v}}\frac{\lambda(v_2)}{\lambda(v_1)}\frac{h(v_1)}{1-H(v_2)}dv_1] = E_{v_1|v_2}\{\frac{v_2}{v_1}+(\frac{1}{2}-\frac{p_r^2}{2v_1^2})(1-\frac{\lambda(v_2)}{\lambda(v_1)})\}.$$

Thus $E_{v_1|v_2}\{p(v_1, v_2)\}$ is an everywhere increasing function of v_2 *if* the hazard rate is monotonic. Finally, since $G_2(v_2; n + 1)$) first-order stochastically dominates $G_2(v_2; n)$ for any number n of bidders with independent valuations (where $G_i(v_i; n)$ is the distribution function of v_i, i = 1,2, for n draws from independent random variables: see Appendix 4), any reduction in n decreases the expected revenue of the seller *if* the hazard rate of $H(\cdot)$ is monotonic. *Q.E.D.*

Appendix 4

Consider the joint distribution of the first and second order statistics of n independent continuous random variables v^j, j = 1, …, n, whose distributions are indicated with $H^j(\cdot)$ ($h^j(\cdot)$ is the corresponding density function) on the common support $[\underline{v}, \bar{v}]$. Clearly, $G_1(v_1) = \prod_{j=1}^{n} H^j(v_1)$, and computation shows that:

$$G_2(v_2) = G_1(v_2)[\sum_{i=1}^{n}\frac{1}{H^i(v_2)} - (n-1)]$$
$$G(v_1,v_2) = [\sum_{i=1}^{n}\frac{H^i(v_1)}{H^i(v_2)} - (n-1)]$$
$$g_1(v_1) = G_1(v_1)\sum_{i=1}^{n}\frac{h^i(v_1)}{H^i(v_1)} \quad \text{(A.14)}$$

$$g_2(v_2) = G_1(v_2)[\sum_{i=1}^{n}\sum_{j=1}^{n}{}_{j\neq i}\frac{h^i(v_2)(1-H^j(v_2))}{H^i(v_2)H^j(v_2)}]$$

$$g(v_1, v_2) = G_1(v_2)[\sum_{i=1}^{n}\sum_{j=1}^{n}{}_{j\neq i}\frac{h^i(v_2)h^j(v_1)}{H^i(v_2)H^j(v_2)}]$$

$$G_2(x) - G_1(x) = G_1(x)[\sum_{i=1}^{n}\frac{1}{H^i(x)} - n]$$

$$\frac{g_1(v_1; n+1)}{g_1(v_1; n)} = H^{n+1}(v_1) + \frac{h^{n+1}(v_1)}{\sum_{i=1}^{n}\frac{h^i(v_1)}{H^i(v_1)}}.$$

Note that $G_i(v_i; n)$ first-order stochastically dominates $G_i(v_i; m)$ for any $n > m$, $i = 1,2$, and that $1/\gamma$ is an average of the different $1/\lambda^j = (1 - H^j)/h^j$ whose weights are the values of the so-called *reverse* hazard rates weights $\sigma^j = h^j/H^j$ divided by $\Sigma\sigma^j$. Moreover, if the random variables are iid according to H and h, then $1/\gamma = (1 - H)/h$ and $g_1(v_i; n + 1)$ even likelihood-ratio stochastically dominates $g_1(v_i; n)$: see e.g. Krishna (2002: Appendix B) and Shaked and Shanthikumar (1994: section 1.C).

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