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Asymmetric English Auctions Revisited

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Abstract

I introduce a property of player's valuations that ensures the existence of an ex post efficient equilibrium in asymmetric English auctions. The use of this property has the advantage of yielding an ex post efficient equilibrium without assuming differentiability of valuations or that signals are drawn from a density. These technical, non economic, assumptions have been ubiquitous in the study of (potentially) asymmetric English auctions. Therefore, my work highlights the economic content of what it takes to obtain efficient ex post equilibria.

I generalize prior work by Echenique and Manelli (2006) and by Birulin and Izmalkov (2003). Relative to Krishna (2003), I weaken his single crossing properties, drop his differentiability and densities assumptions, but I assume that one player's valuation is weakly increasing in other players' signals, while he uses a different assumption (neither stronger nor weaker).

Journal of Economic Literature Classification Numbers: D44, D82

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1 Introduction

This paper gives a minimal set of assumptions that ensures existence of an efficient ex-post equilibrium (once signals are known, players don't want to change their behavior), in asymmetric English auctions. I introduce a new assumption that I call Own Effect Property, and a new method to find equilibria, that does not require the ubiquitous assumptions that value functions are differentiable

*I thank Federico Echenique and Alejandro Manelli for their comments. This paper started as an attempt to weaken some of the assumptions in their paper Echenique and Manelli (2006) on comparative statics. This paper owes them a lot: the main property of this paper is a weak version of their Dominant Effect Property, and the method of proof that I use was first used in an earlier version of their paper.

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and that signals are drawn from a density. Moreover, the Own Effect Property is weaker than all the versions of the single crossing property that have been used in this branch of the literature.

The contribution of this paper is important because the English auction has had a prominent role in allocating objects among a potential set of buyers, both in real life, and in economic theory. The English was the first auction format, having been used since the times of the Roman empire, and it is the most commonly used form of auction to sell goods nowadays (see McAfee and McMillan (1987) and Cassady (1967) who claims that 75% or more of all auctions are English). Its popularity stems from a variety of reasons: it yields more revenue than the other common auction format, the sealed bid, in a variety of contexts (see Milgrom (1989) and Milgrom and Weber, 1982); it allocates the object efficiently in a wider range of environments; and it economizes on information gathering and bid preparation costs (see Milgrom, 1989). Also, relative to other theoretical constructions of reduced use in the real world, like the second-price auction, it does not require the winner to reveal his true valuation, thus avoiding renegotiation between the seller and the highest bidder and also avoiding any conflict that could happen if (for example) in a second-price auction the price paid is significantly less than what the winning bidder stated he was willing to pay. Finally, in a variety of contexts the English Auction has an ex post equilibrium, one which remains an equilibrium even if bidders know each others' valuations or signals. If this is the equilibrium actually played, then players have no ex-post regrets, or reasons to renegotiate, making the auction format attractive. Also, Perry and Reny (2005) have argued that

> "Simultaneous (sealed-bid) auction formats, we contend, often require participants to collect and provide significant amounts of redundant information. For example, the remarkable efficient auction due to Dasgupta and Maskin (2000) requires a bidder to submit to the auctioneer his entire preference profile (over all possible bundles of goods) for all possible vectors of signals of the other bidders. Consequently, the vast majority of information a bidder is required to express is redundant since the only relevant preference profile is that corresponding to the actual realized vector of signals of the others. Perry and Reny's (2002) auction also requires bidders to submit large amounts of redundant information (...) However, given the actual vector of bidder signals, most of these "as if" comparisons are not required to determine the efficient outcome."

They go on to argue that this excessive amount of information gathering may make bidders collect less relevant information about the actual state of the world, leading to an inefficient allocation of the object.

To summarize, I present a simple assumption of payoff functions that is weaker than all the assumptions of its kind, that ensures the existence of an efficient ex-post equilibrium in the Asymmetric English Auction, when one assumes that valuations are non decreasing. This new assumption allows us to work with valuations that are not differentiable, and with types which are not drawn from densities. The method of proof, first presented in Echenique and Manelli (2006), is new and highlights the economic content of the assumptions that ensure efficiency in this kind of auctions. This contribution is relevant, because English auctions have played a prominent role in real life, and in economic theory.

I will now present the model, the assumptions and the results. I postpone the discussion of the literature until Section 3, because it needs a series of definitions. It suffices here to say that the paper most related to this is Krishna (2003). Although he uses differentiability, densities, and two single crossing conditions that are stronger than the one in this paper, his set of sufficient conditions for the existence of efficient ex-post equilibria in Asymmetric English Auctions does not imply my assumptions, since I assume that when some player's valuations are tied, valuations are weakly increasing in other players' signals. This assumption is neither weaker nor stronger than his assumption that one signal's increase causes the sum of valuations to increase.

2 The Model and Main Results

Let $N = \{1, 2..., n\}$ be the set of players. Each player i observes a signal $s_i \in [0, b]$. This signal is only known to player i. Signals are drawn according to some probability measure μ over $[0, b]^n$ that need not posses a density. The signals affect the values that players have for the objects. Player i's valuation is a continuous function $v_i : [0, b]^n \to \mathbf{R}$ that maps profiles of signals (one for each player) into real numbers, and that is strictly increasing in its own signal, so that for all i and all $\mathbf{s}_{-i}$ (vectors are written in bold) in $[0, b]^{n-1}$, $s'_i > s_i$ implies $v_i(s'_i, \mathbf{s}_{-i}) > v_i(s_i, \mathbf{s}_{-i})$. For any $\mathbf{s}$, let $W(\mathbf{s})$ be the set of players i such that $v_i(\mathbf{s}) \geq v_k(\mathbf{s})$ for all k (the set of "winners" at $\mathbf{s}$). Let $|W(\mathbf{s})|$ be the cardinality of $W(\mathbf{s})$. I now introduce the two conditions that are sufficient for the existence of an efficient equilibrium.

Definition. The set of functions v is *Increasing at Ties* if for every $\mathbf{s}$ such that $|W(\mathbf{s})| > 1$ and all $i \in W(\mathbf{s})$

$$\left.\begin{array}{c} \mathbf{s}' \geq \mathbf{s} \\ s'_j > s_j \text{ if and only if } j \in P(\mathbf{s}', \mathbf{s}) \subseteq W(\mathbf{s}) \end{array}\right\} \Rightarrow v_i\left(s'_i, \mathbf{s}'_{-i}\right) \geq v_i\left(s'_i, \mathbf{s}_{-i}\right).$$

The interpretation of the above property is as follows. Suppose $\mathbf{s}$ is a profile of signals for which at least two players have equal and highest valuations. Then, the property requires that if one of the winner's signal increases to s'_i, then the effect of the other player's signals, when they increase from $\mathbf{s}_{-i}$ to $\mathbf{s}'_{-i}$, does not hurt player i. Example 6 of Maskin (2001) shows that even if some sort of single crossing property is satisfied, one still needs that player j's signal does not affect player i's valuation "very" negatively, if an efficient equilibrium is to exist (an equilibrium is efficient if it always allocates the object to one of the players with the highest valuation). Valuations in

Maskin's example are not Increasing at Ties, and that is why he finds that no efficient equilibrium exists. This property is neither weaker nor stronger than Krishna's assumption that when i's signal increases, the sum of all player's valuations increases.

I now turn to the more substantial assumption on valuations.

Definition. The set of functions v satisfies the *Own Effect Property (OEP)* if for every $\mathbf{s}$ such that $|W(\mathbf{s})| > 1$ it happens that

$$\left.\begin{array}{c} \mathbf{s}' \geq \mathbf{s} \\ s_j' > s_j \text{ if and only if } j \in P(\mathbf{s}', \mathbf{s}) \subseteq W(\mathbf{s}) \end{array}\right\} \Rightarrow \max_{j \in P(\mathbf{s}', \mathbf{s})} v_j(\mathbf{s}') \geq \max_{k \notin P(\mathbf{s}', \mathbf{s})} v_k(\mathbf{s}')$$

It states that the effect of an increase in some signals is larger for one of the players whose signal increased than for all the rest of the players. Notice that it is a form of single crossing: if there are only two players, j's valuation is equal to k's and j's signal increases, j's valuation is larger than k's. As will be shown later, it is the weakest form of "single crossing" that has been used in this branch of the literature.

This paper is concerned with the irrevocable exit English Auction introduced by Milgrom and Weber (1982). In this game the auctioneer continuously raises the asking price, starting from zero. A bidder has the option of quitting the auction publicly at any time, and once he quits, he cannot reenter. The winner is the last bidder to remain in the auction, and he pays the price called when the last player (other than him) left the auction. In this game, a **strategy** for a player is a function that determines a price at which to quit, for each realization of the private information, and each history of who left the auction at what price. Formally, a strategy for bidder i is a collection of functions, one for each set of (active) players A and each profile $\mathbf{p}^{N\backslash A}$ of prices at which bidders in $N\backslash A$ quit the auction, $\beta_i^A : [0, b] \times \mathbf{R}_+^{N\backslash A} \rightarrow \mathbf{R}_+$ where $i \in A$, $|A| > 1$ and $\beta_i^A\left(s_i, \mathbf{p}^{N\backslash A}\right) > \max\{p_j : j \in N\backslash A\}$. The value $\beta_i^A\left(s_i, \mathbf{p}^{N\backslash A}\right)$ is the price at which bidder i will drop out if players in $N\backslash A$ dropped at prices $\mathbf{p}^{N\backslash A}$ and nobody quits before. As long as $p < \beta_i^A\left(s_i, \mathbf{p}^{N\backslash A}\right)$ he stays in the auction; he drops out when $p = \beta_i^A\left(s_i, \mathbf{p}^{N\backslash A}\right)$; in any history in which $p > \beta_i^A\left(s_i, \mathbf{p}^{N\backslash A}\right)$ he drops out (this part of the strategy will never be used).

The following is the main result of this paper.

Theorem 1. *Suppose that v is increasing at ties and satisfies the OEP. Then, there is an efficient ex post equilibrium.*

Versions of the previous Theorem have appeared elsewhere. The contribution of this paper lies in the relaxation of several of the assumptions generally made to obtain the results. I now discuss the relation of my assumptions with those that have appeared before in the literature.

2.1 Necessity

I will show that, if one assumes a certain regularity condition on the valuations, then the OEP is necessary for existence of efficient equilibria in undominated strategies.

So far we have said that an equilibrium is efficient if it allocates the object to one of the players with the highest valuation for all profiles of signals. This definition of efficiency is the most demanding if one is concerned with finding sufficient conditions for the existence of an efficient equilibrium. But one could also use another definition of efficiency which is more demanding for necessity, and less so for sufficiency. Let us say that if an equilibrium of the English auction (with valuations v and distribution of signals μ) assigns the object to the highest bidder with $\mu-$probability 1 it is $\mu-$*efficient*.

Suppose now that we want to prove a theorem like "If property P of the profile of valuations v is violated, then there is no $\mu-$efficient equilibrium for any μ." There is no hope for such a Theorem, because if we assume $v_i(\mathbf{0}) = 0$ for all i and set $\mu(\mathbf{0}) = 1$, then any strategy profile that has $\beta_i^N(0, \emptyset) = 0$ (all players quit at $p = 0$, when all players are active, if they have a signal of 0) is a $\mu-$efficient equilibrium. Hence, the statement of the would-be theorem should be "If property P of the profile of valuations v is violated, then there is a μ such that no $\mu-$efficient equilibrium exists."

I say that the set of functions v is *regular* if each v_i is twice differentiable and for all $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ the Jacobian matrix of partial derivatives of subsets of the winners is invertible, or more formally, for all $P \subset W(\mathbf{s})$,

$$D_P v(\mathbf{s}) = \left(\frac{\partial v_i(\mathbf{s})}{\partial s_j}\right)_{i,j \in P}$$

is invertible.

I now show that in the presence of the regularity assumption above, the OEP is necessary. For simplicity, and because the focus of this paper is precisely to get rid of the differentiability assumptions, I present the result for only three players.[1] With more players the arguments are more involved, but the result is still true. I also assume that if two or more players quit at the same price, the tie is broken assigning the object to each player with positive probability.[2]

Theorem 2. Necessity of the OEP. *Suppose there are three players and that the valuations v are regular. If v does not satisfy the OEP at some interior $\mathbf{s}$ and $\mathbf{s}'$ with $\mathbf{s}' > \mathbf{s}$, there is a μ such that no $\mu-$efficient equilibrium in undominated strategies exists.*

[1] I also assume that the equilibrium is undominated, as has been previously done for this kind of auction in Perry and Reny (2005) and Maskin (1992). I believe that the simple argument in Example 1 of Krishna (2003) also requires undominated strategies. I don't know whether the result is true without assuming undominated strategies.

[2] Papers that deal with necessity have usually assumed this either explicitly or implicitly.

I have shown that the two properties introduced in this paper are sufficient for the existence of an efficient ex-post equilibrium. I have also shown that if one is concerned only with almost sure efficiency and requires undominated strategies, then the main property of this paper, the OEP is also necessary. Similar results have appeared elsewhere, so now I discuss the relationship between my assumptions and those of the rest of the literature.

3 The Literature

The OEP is inspired in, and closely related to, the following property used in Echenique and Manelli (2006): v satisfies the *Dominant Effect Property* if for any $\mathbf{s}'$ and $\mathbf{s}$ with some $s'_j > s_j$,

$$\max_{j:s'_j>s_j} v_j\left(\mathbf{s}'\right) - v_j\left(\mathbf{s}\right) > \max_{k:s'_j\leq s_j} v_k\left(\mathbf{s}'\right) - v_k\left(\mathbf{s}\right).$$

The OEP is weaker than the DEP in five dimensions: on the domain of application, OEP applies more generally (i) OEP applies only when $\mathbf{s}' \geq \mathbf{s}$, (ii) it applies only when $|W(\mathbf{s})| > 1$, (iii) it applies only when $s'_j > s_j$ for players j in $W(\mathbf{s})$; the conclusion (what is demanded of the functions) is weaker because (iv) the inequality in the conclusion is weak and (v) OEP does not require that the increments be larger (for players whose signals increase), but only that the final values be larger. Of these weakenings of the condition in Echenique and Manelli, the only relevant one in terms of applicability of the property, is the weak inequality in the conclusion. Of course, sets of functions v that satisfy the OEP and not the DEP (for reasons other than the weak inequality) are easy to construct, but are not very relevant.[3] The strict inequality is different however, since it excludes, for example, simple variations of the common value auction in which one player's signal has the same effect on his valuation than on some other players'. It is worth emphasizing that Echenique and Manelli (2006) is not a paper about auctions, but about comparative statics, so they have defined their DEP in order to yield comparative statics results in a wide variety of contexts, and not just auctions. Hence, it is not surprising that one can weaken their property when using it in a particular setting.

I now turn to the discussion of the most relevant papers and their assumptions.

3.1 The OEP is weaker than the Single Crossing property

As an illustration of the importance of the OEP assumption for auctions, we now show that it is weaker than the Single Crossing condition that I now introduce. Suppose there are two players; the functions $v = (v_1, v_2)$ satisfy the *Single Crossing Condition* if at any $\mathbf{s}$ such that $v_1(\mathbf{s}) = v_2(\mathbf{s})$

$$\frac{\partial v_i(\mathbf{s})}{\partial s_j} < \frac{\partial v_j(\mathbf{s})}{\partial s_j}.$$

[3] Consider for example $b = 1$, $v_1 = 10 + s_1/2$ and $v_2 = s_1 + s_2$. The 10 precludes any equality in valuations, so OEP is trivially satisfied, while the DEP is violated for increases in s_1.

This is the version in Dasgupta and Maskin (2000). In Maskin (1992) the inequality is weak, but applies to all $\mathbf{s}$. Since the Single Crossing is necessary for the existence of efficient equilibria in two player auctions, the OEP is also necessary.[4] Moreover, in addition to being implied by the SC, the OEP does not require that the value functions v be differentiable, as does the Single Crossing Condition.

Theorem 3. OEP is weaker than Single Crossing. *Suppose that there are only two players, and that v_1 and v_2 are differentiable. If v satisfies the SC condition or the (Maskin) Single Crossing, it satisfies the OEP.*

Although this theorem is a consequence of the result in the next section (that the Average Crossing Condition implies the OEP) I present a simple proof of the result.

Proof of Theorem 3. We will assume that the OEP does not hold, and show that this implies that the SC fails. Take any $\mathbf{s}''$ with $|W(\mathbf{s}'')| > 1$ (i.e. $v_1(\mathbf{s}'') = v_2(\mathbf{s}'')$) and an $\mathbf{s}' \geq \mathbf{s}''$ such that $s_1' > s_1''$, $s_2' = s_2''$. We will now show that $v_1(s_1', s_2) \geq v_2(s_1', s_2)$. Let

$$\varepsilon^* = \max\left\{\varepsilon \in [0,1] : v_1\left(\varepsilon \mathbf{s}' + (1-\varepsilon)\mathbf{s}''\right) \geq v_2\left(\varepsilon \mathbf{s}' + (1-\varepsilon)\mathbf{s}''\right)\right\}$$

and notice that ε^* is well defined, since 0 belongs to the set over which the maximum is taken. If $\varepsilon^* = 1$, there is nothing to prove, so suppose $\varepsilon^* < 1$, and define $\mathbf{s} = \varepsilon^* \mathbf{s}' + (1-\varepsilon^*)\mathbf{s}''$. We then have: $v_1(\mathbf{s}) = v_2(\mathbf{s})$ and for all $\overline{s}_1$ such that $s_1' > \overline{s}_1 > s_1$, $v_1(\overline{s}_1, s_2) < v_2(\overline{s}_1, s_2)$ obtains. This implies that for all $\overline{s}_1 > s_1$

$$v_1(\overline{s}_1, s_2) - v_1(\mathbf{s}) < v_2(\overline{s}_1, s_2) - v_2(\mathbf{s}) \Rightarrow \frac{\partial v_1(\mathbf{s})}{\partial s_1} \leq \frac{\partial v_2(\mathbf{s})}{\partial s_1}$$

which contradicts the SC condition, and therefore proves that if SC holds, so does the OEP.

We will now show that if the Maskin Single Crossing holds, so does the OEP. As before, assume $\varepsilon^* < 1$, so that

$$v_1\left(s_1', s_2\right) < v_2\left(s_1', s_2\right) \tag{1}$$

and define $\mathbf{s} = \varepsilon^* \mathbf{s}' + (1-\varepsilon^*)\mathbf{s}''$ which implies $v_1(\mathbf{s}) = v_2(\mathbf{s})$. This last equality and equation (1) contradict Maskin's Single Crossing since ,

$$\begin{aligned} \frac{\partial v_1(\overline{s}_1, s_2)}{\partial s_1} &\geq \frac{\partial v_2(\overline{s}_1, s_2)}{\partial s_1} \forall \overline{s}_1 \in \left[s_1', s_1\right] \Rightarrow \int_{s_1}^{s_1'} \frac{\partial v_1(\overline{s}_1, s_2)}{\partial s_1} ds_1 \geq \int_{s_1}^{s_1'} \frac{\partial v_2(\overline{s}_1, s_2)}{\partial s_1} ds_1 \Rightarrow \\ v_1\left(s_1', s_2\right) - v_1(s_1, s_2) &\geq v_2\left(s_1', s_2\right) - v_2(s_1, s_2) \Leftrightarrow v_1\left(s_1', s_2\right) \geq v_2\left(s_1', s_2\right) \end{aligned}$$

as was to be shown. ■

[4] Different versions of the statement "the single crossing is necessary for an efficient equilibrium" can be found in Maskin (1992), Birulin and Izmalkov (2003) and Dasgupta and Maskin (2000).

3.2 The OEP, Average and Cyclical Crossing

In this section I show that the two properties used by Krishna (Average and Cyclical Crossing Conditions) imply an "equal increments" condition, which in turn implies the OEP. Therefore, this section shows the connection between both of Krishna's existence results.

For any $P \subset N$, let I_P denote the vector in $\mathbf{R}^n$ with 1 in the jth coordinate iff $j \in P$ and 0 otherwise and let ∇v_k denote the gradient of v_k.

Definition. The set of functions v satisfies:

(a) the *Equal Increments Condition* if for all $P \subset N$ there exists $j \in P$ such that for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \notin P$, $I_P \nabla v_j > I_P \nabla v_i$.

(b) Krishna's *Average Crossing Condition (ACC)* if for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \neq j$

$$\sum_{k=1}^{n} \frac{\partial v_k}{\partial s_j} > n \frac{\partial v_i}{\partial s_j}.$$

(c) Krishna's *Cyclical Crossing Condition (CCC)* if for all j

$$\frac{\partial v_j}{\partial s_j} > \frac{\partial v_{j+1}}{\partial s_j} \geq \frac{\partial v_{j+2}}{\partial s_j} \geq \cdots \frac{\partial v_{j-1}}{\partial s_j}$$

holds at every $\mathbf{s}$ with $|W(\mathbf{s})| > 1$, where $j + k \equiv (j + k)$ modulo n.

We now prove a simple Lemma that will help us show that both the Average Crossing Condition and the Cyclical Crossing Condition imply the OEP. The key to showing that these conditions imply the OEP is making the connection between the effect of one signal on all valuations (as stated in the ACC and CCC) and the effect of several signals on the valuations of two players.

Lemma A. *If v satisfies the ACC or the CCC then it satisfies the Equal Increments Condition.*

The previous Lemma asserts that when one increases the signals of a set of winners (by the same small amount) then the total growth of the valuation of one of the players whose signal increased is larger than the growth of any of those whose signals did not increase. This is just an arm's length away from the OEP.

Theorem 4. OEP is weaker than Average Crossing and Cyclical Crossing. *If v satisfies the Equal Increments Condition, then it satisfies the OEP.*

3.3 Birulin & Izmalkov

In this Section I show that the OEP is weaker than the assumptions used in Birulin and Izmalkov. To do so, I first describe the equilibrium used in BI, and show that the existence of such an equilibrium (as implied by the assumptions in BI) implies the OEP.

The proof of Theorem 1 is based on the construction of an equilibrium with certain properties. This kind of equilibrium was previously used in Milgrom and Weber (1982), Maskin (1992), Krishna (2003) and Birulin and Izmalkov (2003). It is based on the following simple idea. Since exits in English auctions are public, one player's quitting conveys information to the other players about the quitter's signal. Suppose there is an increasing function $\sigma(p)$ mapping prices into profiles of signals such that $v_i(\sigma(p)) = v_j(\sigma(p)) = p$ for all i and j. Suppose that no player has quit, and the price is p. Then, in the proposed equilibrium player i stays in the auction as long as $s_i \geq \sigma_i(p)$ and quits when $s_i = \sigma_i(p)$. Therefore, when a player quits, his signal $s_i = \sigma_i(p)$ becomes known. This is a reasonable strategy since, as long as nobody quits, players know that $\mathbf{s} \geq \sigma(p)$ and therefore $v_i(\mathbf{s}) \geq p$ for all i. In any sub-auction in which the set of active players is B, let us call $\mathbf{y}^{N\backslash B}$ the vector of known signals of the players who have already quit. The informal description of the strategies that will be used in the efficient ex-post equilibrium are the following:

- in the empty history, player i remains in the auction as long as $b \geq s_i > \sigma_i^{\mathbf{0}^N}(p)$ (the profile of signals $\mathbf{0}$ satisfies (a) and (b) of Lemma 1, so the function σ exists); all players know this; player i drops at the lowest price p such that $s_i = \sigma_i^{\mathbf{0}^N}(p)$; let the price of the first drop be p^1, let i^* be the player who drops at p^1 and at the time of his drop, player i^* signal becomes known, so let $y_{i^*} = \sigma_{i^*}^{\mathbf{0}^N}\left(p^1\right)$;
- let $A = N\backslash\{i^*\}$ and $\mathbf{y}^{N\backslash A} \equiv y_{i^*}$ and notice that since $\sigma^{\mathbf{0}^N}\left(p^1\right)$ satisfies $V_j^{\mathbf{0}^N}\left(\sigma^{\mathbf{0}^N}\left(p^1\right)\right) = p$ for all j, the profile $\mathbf{y}^A = \sigma_{-i^*}^{\mathbf{0}^N}\left(p^1\right)$ satisfies the conditions of Lemma 1, so that a function $\sigma^{\mathbf{y}^{N\backslash A}}$ satisfying (i)-(iii) in that Lemma exists. Then, player $j \in A$ remains in the auction as long as $s_j \geq \sigma^{\mathbf{y}^{N\backslash A}}(p)$, and drops at the lowest p such that $s_j = \sigma^{\mathbf{y}^{N\backslash A}}(p)$.
- the process continues in this fashion.

The formal description of the strategies just mentioned is as follows: in a subgame in which types $\mathbf{y}^{N\backslash A}$ are known and active players are A, $\beta_i^A\left(s_i, \mathbf{y}^{N\backslash A}\right) = \beta_i^{\mathbf{y}^{N\backslash A}}(s_i) = \min\left\{p : \sigma^{\mathbf{y}^{N\backslash A}}(p) \geq s_i\right\}$. Notice that since σ is continuous and weakly increasing, β is strictly increasing and well defined.

Theorem 5. *Suppose that the assumptions of BI are satisfied. That is, s is drawn from a density, v's are twice differentiable, regular, $\nabla v_j(\mathbf{s}) \geq \mathbf{0}$ for all j and $\mathbf{s}$, and satisfy the Generalized Single Crossing: for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and any $A \subset W(\mathbf{s})$,*

$$\max_{j \in A} \mathbf{u}\nabla v_j(\mathbf{s}) \geq \mathbf{u}\nabla v_k(\mathbf{s})$$

for all $k \in W(\mathbf{s}) \setminus A$ and any $\mathbf{u}$ such that $u_i > 0$ for $i \in A$ and $u_j = 0$ otherwise. Then, v satisfies the OEP.

The previous Theorem shows that Theorem 1 is indeed a generalization of Proposition 1 in Birulin and Izmalkov (2003): they assume differentiability, densities, regularity, positive gradients and the GSC; I drop differentiability, densities, regularity, valuations in this paper are only increasing at ties (and not $\nabla v_j(\mathbf{s}) \geq \mathbf{0}$ for all j and $\mathbf{s}$), and the OEP is weaker than the GSC in the presence of the other assumptions. Moreover, as the next example shows, it would be "unfair" to compare just the GSC and the OEP, since the GSC is too weak a property in the absence of the other assumptions (in particular, regularity).

Example. GSC not sufficient in the absence of regularity. Let there be two players, 1 and 2, whose signals s_1 and s_2 are drawn independently from a density on $[0,1]$. Let

$$z_1(s_1) = (2s_1 - 1)^5 \text{ and } z_2(s_1) = \begin{cases} (2s_1 - 1)^3 & s_1 \leq \frac{1}{2} \\ 2(2s_1 - 1)^3 & s_1 \geq \frac{1}{2} \end{cases}$$

It is easy to check that if valuations are $v_i(\mathbf{s}) = s_1 + s_2 + z_i(s_1) + 1$, then all of BI's assumptions are satisfied, except for regularity. Also, there is no efficient equilibrium, since we would need that for all $s_1 < \frac{1}{2}$, $\beta_1(s_1) > \beta_2(s_2)$ for all s_2 and for all $s_1 > \frac{1}{2}$, $\beta_1(s_1) < \beta_2(s_2)$ for all s_2. But then, when player 1 has a signal of 1, he is strictly better off bidding as if he had a signal of 1/4, showing that there is no efficient equilibrium. ■

4 Appendix

The following Lemma proves the existence of a σ function as described in Section 3.3 for any (relevant) sub auction. For any set $A \subseteq N$, any player $i \in A$, and any $\mathbf{y}$, let $V_i^{\mathbf{y}^{N\setminus A}} : [0,b]^{|A|} \to \mathbf{R}$ be defined by $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = v_i\left(\mathbf{s}, \mathbf{y}^{N\setminus A}\right)$.

Lemma 1. Fix any $B \subseteq N$, with $|B| > 1$, and fix a profile of types $\mathbf{y}^{N\setminus B}$ such that there exists $\mathbf{y}^B \neq \mathbf{b}$ for which for all $i \in B$, $y_i < b$ implies $v_i(\mathbf{y}) = \max_{j\in N} v_j(\mathbf{y})$. If v is increasing and satisfies the OEP, there exists a $p_{\mathbf{y}}^B > \max_i v_i(\mathbf{y})$ and a weakly increasing function $\sigma^{\mathbf{y}^{N\setminus B}} : \left[\max_i v_i(\mathbf{y}), p_{\mathbf{y}}^B\right] \to \prod_{i\in B}[y_i, b]$ mapping prices into types of active players, such that:

(i) $\sigma_j^{\mathbf{y}^{N\setminus B}}\left(p_{\mathbf{y}}^B\right) = b$ for some j with $y_j < b$ and for all $i \in B$, $p = p_{\mathbf{y}}^B$ and $y_i < b$ imply the break even condition

$$V_i^{\mathbf{y}^{N\setminus B}}\left(\sigma^{\mathbf{y}^{N\setminus B}}(p)\right) = p. \tag{2}$$

holds;

(ii) for all $p < p_{\mathbf{y}}^B$, if $y_i < b$ then $\sigma_i^{\mathbf{y}^{N\setminus B}}(p) < b$ and the break even condition (2) hold;

(iii) for all $p \leq p^B_{\mathbf{y}}$, and all $k \in N$, $v_k\left(\sigma^{\mathbf{y}^{N\backslash B}}(p), \mathbf{y}^{N\backslash B}\right) \leq p$.

The previous Lemma provides the basis for the existence of an ex-post and efficient equilibrium. The proof is based on a method first introduced in Echenique and Manelli, which does not require differentiability of v, or the existence of a density for the distribution of types. Although this proof is more involved than theirs, beause it is based on a weaker property and does not use a border condition that they had assumed, the basic idea is the same.

The proof of Lemma 1 is based on the following Lemma.

Lemma 2. Fix any $A \subseteq N$, with $|A| > 1$, and fix a profile of types $\mathbf{y}^{N\backslash A}$ such that there exists a $\mathbf{y}^A$ for which:

(a) for all $i, j \in A$, $v_i(\mathbf{y}) = v_j(\mathbf{y})$ and $y_i < b$
(b) for all $k \notin A$, and $i \in A$, $v_k(\mathbf{y}) \leq v_i(\mathbf{y})$.

If v satisfies the OEP, there exists a $p^A_{\mathbf{y}} > v_i(\mathbf{y}) = V_i^{\mathbf{y}^{N\backslash A}}\left(\mathbf{y}^A\right)$ (for $i \in A$) and a weakly increasing function $\sigma^{\mathbf{y}^{N\backslash A}} : \left[V_i^{\mathbf{y}^{N\backslash A}}\left(\mathbf{y}^A\right), p^A_{\mathbf{y}}\right] \to \prod_{i\in A}[y_i, b]$ mapping prices into types of active players, such that:

(i) $\sigma_j^{\mathbf{y}^{N\backslash A}}\left(p^A_{\mathbf{y}}\right) = b$ for some j, and for all $i \in A$, $p = p^A_{\mathbf{y}}$ implies that the condition (2) holds.

(ii) for all $p < p^A_{\mathbf{y}}$, $\sigma^{\mathbf{y}^{N\backslash A}}(p) \ll \mathbf{b} = (b, ..., b)$ and the break even condition (2) holds for all $i \in A$.

(iii) for all $p \leq p^A_{\mathbf{y}}$, and all $k \in N$, $v_k\left(\sigma^{\mathbf{y}^{N\backslash A}}(p), \mathbf{y}^{N\backslash A}\right) \leq p$

Proof of Lemma 2. Fix any A and $\mathbf{y}$ that satisfy conditions (a) and (b). Let $\left(b, \mathbf{y}^A_{-i}\right)$ denote the vector $\mathbf{y}^A$ with the ith component replaced by a b. Since $V_i^{\mathbf{y}^{N\backslash A}}$ is strictly increasing in s_i and $y_i < b$ (by (a)) we get for all i, $V_i^{\mathbf{y}^{N\backslash A}}\left(\mathbf{y}^A\right) < V_i^{\mathbf{y}^{N\backslash A}}\left(b, \mathbf{y}^A_{-i}\right)$.

Defining a nonempty set X. For any $i \in A$, let $\pi = \left[V_i^{\mathbf{y}^{N\backslash A}}\left(\mathbf{y}^A\right), \min_i V_i^{\mathbf{y}^{N\backslash A}}\left(b, \mathbf{y}^A_{-i}\right)\right]$ (which is independent of i) and let $\mathcal{P}$ be the set of nonempty subsets of π that contain $V_i^{\mathbf{y}^{N\backslash A}}\left(\mathbf{y}^A\right) = v_i(\mathbf{y})$. The set $\mathcal{P}$ has typical elements P and P', and each is a set of prices. Let $Y = \left\{(P, \sigma) : P \in \mathcal{P},\ \sigma : P \to \prod\limits_A \left[y_i^A, b\right]\right\}$ and $X \subset Y$ be

$$X = \left\{(P, \sigma) : \sigma \text{ is weakly increasing, } \sigma(v_i(\mathbf{y})) = \mathbf{y}^A \text{ and } V_i^{\mathbf{y}^{N\backslash A}}(\sigma(p)) = p,\ \forall i \in A, \forall p \in P\right\}.$$

Notice that by condition (a) $P = \{p : p = v_i(\mathbf{y}) \text{ for some } i \in A\} \in \mathcal{P}$ is a singleton and the function σ defined by $\sigma(v_i(\mathbf{y})) = \mathbf{y}^A$ satisfies $V_i^{\mathbf{y}^{N\backslash A}}(\sigma(p)) = p$. Therefore, X is nonempty.

Defining a partial order on X. Define a partial order on X by $(P', \sigma') \succeq (P, \sigma)$ if and only if $P' \supseteq P$ and $\sigma'(p) = \sigma(p)$ for all $p \in P$.

Showing that every chain in X has an upper bound. Take any totally ordered set in X (a chain) $\{(P_\alpha, \sigma_\alpha)\}_\alpha$ in X and define $P \equiv \cup_\alpha P_\alpha$ and $\sigma : P \to \prod_A \left[y_i^A, b\right]$ through $\sigma(p) = \sigma_\alpha(p)$ for any α such that $p \in P_\alpha$. Notice that the definition of σ does not depend on the specific α chosen, since if p belongs to two different P_α and $P_{\alpha'}$, we still get $\sigma_\alpha(p) = \sigma_{\alpha'}(p)$. I will first show that $(P, \sigma) \in X$, and then that (P, σ) is an upper bound for $\{(P_\alpha, \sigma_\alpha)\}_\alpha$.

It is easy to check that σ is weakly increasing. Also, for any $p \in P$, there is some α for which: $p \in P_\alpha$ and $\sigma_\alpha(p) = \sigma(p)$. Then, since $(P_\alpha, \sigma_\alpha) \in X$, we get

$$V_i^{\mathbf{y}^{N\setminus A}}(\sigma_\alpha(p)) = p \Rightarrow V_i^{\mathbf{y}^{N\setminus A}}(\sigma(p)) = p$$

showing that $(P, \sigma) \in X$.

To see that (P, σ) is an upper bound, note that for any α we have $P \supseteq P_\alpha$ and $\sigma(p) = \sigma_\alpha(p)$ for all $p \in P_\alpha$.

Showing that the maximal element implied by Zorn's Lemma must have $P = \pi$. Zorn's lemma then ensures that there exists a maximal element $\left(P^M, \sigma^M\right)$ in X. We now show that $P^M = \pi$. Suppose $p' \notin P^M$, notice that we have $v_i(\mathbf{y}) \in P^M$ and $v_i(\mathbf{y})$ is a lower bound for P^M, so $\left\{\widetilde{p} \in P^M : \widetilde{p} < p'\right\}$ is nonempty, so define $p_* = \sup_{\widetilde{p}} \left\{\widetilde{p} \in P^M : \widetilde{p} < p'\right\}$. If there is some $p \in P^M$ such that $p > p'$ let

$$p^* = \inf_{\widetilde{p}} \left\{\widetilde{p} \in P^M : \widetilde{p} > p'\right\}.$$

Case A, $p_* \notin P^M$. Consider first the case in which $p_* \notin P^M$. We set $P' = P^M \cup \{p_*\}$ and letting $\{p_n\}$ be an increasing sequence in P^M that converges to p_*, define σ' on P' through

$$\sigma'(p) = \begin{cases} \sigma'(p) = \sigma^M(p) & \text{for all } p \neq p' \\ \sigma'(p_*) = \lim_n \sigma^M(p_n) & \end{cases}.$$

Since σ^M is increasing, the limit is well defined. Moreover, it is easy to check that σ' is increasing. For all $p \in P^M$, we already know that $V_i^{\mathbf{y}^{N\setminus A}}(\sigma'(p)) = V_i^{\mathbf{y}^{N\setminus A}}\left(\sigma^M(p)\right) = p$ holds, and for p_*, we also have that, by continuity of $V_i^{\mathbf{y}^{N\setminus A}}$,

$$V_i^{\mathbf{y}^{N\setminus A}}\left(\sigma'(p_*)\right) = V_i^{\mathbf{y}^{N\setminus A}}\left(\lim_n \sigma^M(p_n)\right) = \lim_n V_i^{\mathbf{y}^{N\setminus A}}\left(\sigma^M(p_n)\right) = \lim_n p_n = p_*$$

establishing that $(P', \sigma') \in X$. Since $(P', \sigma') \succ \left(P^M, \sigma^M\right)$ by construction, this contradicts $\left(P^M, \sigma^M\right)$ being maximal.

Case B, $p_* \in P^M$ **and** $\exists p \in P^M$ **such that** $p > p'$. Consider now the case in which $p_* \in P^M$, so that $p_* < p'$. If there is some $p \in P^M$ such that $p > p'$, one can follow the same steps as in Case A to discard the case in which $p^* \notin P^M$, so assume that $p^* \in P^M$. Let $\underline{\mathbf{s}} = \sigma(p_*)$, and $\overline{\mathbf{s}} = \sigma(p^*)$ and fix any p with

$$p_* < p < \min V_i^{\mathbf{y}^{N\setminus A}}\left(\overline{s}_i, \sigma_{-i}(p_*)\right) \leq p^* \tag{3}$$

Assume, without loss of generality, that $\overline{s}_i > \underline{s}_i$ for all i (when they are equal, the signal of player i just becomes a fixed "parameter" in the V functions, and thus plays no role).

Let $g : \mathbf{R} \to (-1, 1)$ be any strictly decreasing function with $g(0) = 0$. Let $\underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) - p$ and for $i \in A$, let

$$h_i(\mathbf{s}) = \begin{cases} s_i + g\left(\underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s})\right)(s_i - \underline{s}_i) & \text{if} \quad \underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) > 0 \\ s_i & \text{if} \quad \underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = 0 \\ s_i + g\left(\underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s})\right)(\overline{s}_i - s_i) & \text{if} \quad \underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) < 0 \end{cases}$$

The function

$$h : \prod_A [\underline{s}_i, \overline{s}_i] \to \prod_A [\underline{s}_i, \overline{s}_i]$$

satisfies hypothesis of Brouwer, so there is a fixed point $\mathbf{s}^f$. We will now show that for all i,

$$V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = p. \tag{4}$$

1. Suppose that for some i, $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$. If $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$, we get $\underline{V}_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > 0$, and since $h_i\left(\mathbf{s}^f\right) = s_i^*$, we must have $s_i^f = \underline{s}_i$ (otherwise, $g\left(\underline{V}_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)\right)$ would be subtracting something from s_i^f). We then get $V_i^{\mathbf{y}^{N\setminus A}}\left(\underline{s}_i, \mathbf{s}_{-i}^f\right) > p$ and since equation (3) ensures

$$V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}) = V_i^{\mathbf{y}^{N\setminus A}}(\sigma(p_*)) = p_*,$$

we must have $s_j^f > \underline{s}_i$ for some j. Let k be the player in $P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)$ for whom $V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = \max_{i \in P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)} V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)$. By applying the OEP we see that that for player i with $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$ and $s_i^f = \underline{s}_i$,

$$V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) \geq \max_{j \notin P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)} V_j^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) \geq V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p.$$

Then, player k is such that $V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$, but since $k \in P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)$, we must have $s_k^f > \underline{s}_k$ and

$$h_k\left(\mathbf{s}^f\right) = s_k^f = s_k^f + g\left(\underline{V}_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)\right)\left(s_k^f - \underline{s}_k\right)$$

with $g\left(\underline{V}_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)\right) < 0$ (because $V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$ and this means $\underline{V}_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > 0$) which contradicts $\mathbf{s}^f$ being a fixed point.

2. If $V_m^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) < p$, for some m, then, since $h_m\left(\mathbf{s}^f\right) = s_m^f$, we must have $s_m^f = \overline{s}_m$, because otherwise $g\left(\underline{V}_m^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)\right)$ would be adding something strictly positive to s_m^*. Because we can use that v is Increasing at Ties with $\mathbf{s}' = \left(\overline{s}_m, \mathbf{s}_{-m}^f, \mathbf{y}^{N\setminus A}\right)$ and $\mathbf{s} = \left(\sigma(p_*), \mathbf{y}^{N\setminus A}\right)$, we obtain

$$\begin{aligned} p &> V_m^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = V_m^{\mathbf{y}^{N\setminus A}}\left(\overline{s}_m, \mathbf{s}_{-m}^f\right) \geq V_m^{\mathbf{y}^{N\setminus A}}\left(\overline{s}_m, \underline{\mathbf{s}}_{-m}\right) = V_m^{\mathbf{y}^{N\setminus A}}\left(\overline{s}_m, \sigma_{-m}(p_*)\right) \\ &\geq \min V_i^{\mathbf{y}^{N\setminus A}}\left(\overline{s}_i, \sigma_{-i}(p_*)\right) > p \end{aligned}$$

which is a contradiction. That is, we had chosen a small p, so that a large increase in the signal of m from $\underline{s}_m$ to $\overline{s}_m$ increases $V_m^{\mathbf{y}^{N\setminus A}}$ above p.

Items 1 and 2 have established that $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = p$ for all i, so that $P' = P^M \cup \{p\}$ and

$$\sigma'(\widetilde{p}) = \begin{cases} \sigma'(\widetilde{p}) = \sigma^M(\widetilde{p}) & \text{for all } \widetilde{p} \neq p \\ \sigma'(p) = \mathbf{s}^f \end{cases}$$

satisfy $(P', \sigma') \succ \left(P^M, \sigma^M\right)$ which contradicts $\left(P^M, \sigma^M\right)$ being maximal. We conclude that $P^M = \pi$, and that σ^M maps $\pi = \left[V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{y}^A\right), \min_i V_i^{\mathbf{y}^{N\setminus A}}\left(b, \mathbf{y}_{-i}^A\right)\right]$ into $\prod_A \left[y_i^A, b\right]$, is increasing and $V_i^{\mathbf{y}^{N\setminus A}}\left(\sigma^M(p)\right) = p$, $\forall i \in A, \forall p \in \pi$.

Case C, $p_* \in P^M$ **and** $\nexists p \in P^M$ **such that** $p > p'$. Recall $\underline{s} = \sigma(p_*)$ and fix any p with

$$p_* = V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}) < p \leq \min V_i\left(b, \mathbf{y}_{-i}^A\right). \tag{5}$$

Let $g : \mathbf{R} \to (-1, 1)$ be any strictly decreasing function with $g(0) = 0$. Let $\underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) - p$ and for $i \in A$, let

$$h_i(\mathbf{s}) = \begin{cases} s_i + g\left(\underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s})\right)(s_i - \underline{s}_i) & \text{if } \underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) > 0 \\ s_i & \text{if } \underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = 0 \\ s_i + g\left(\underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s})\right)(b - s_i) & \text{if } \underline{V}_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) < 0 \end{cases}$$

The function h has a fixed point $\mathbf{s}^f$, so we will show that for all i, $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = p$.

1. Suppose that for some i, $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$, so that $s_i^f = \underline{s}_i$. We then get $V_i^{\mathbf{y}^{N\setminus A}}\left(\underline{s}_i, \mathbf{s}_{-i}^f\right) > p$ and since $V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}) < p$, we must have $s_j^* > \underline{s}_j$ for some j. Let k be the player in $P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)$ for whom $V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = \max_{i \in P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)} V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)$. By applying the OEP we see that that for player i with $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$ and $s_i^f = \underline{s}_i$,

   $$V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) \geq \max_{j \notin P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)} V_j^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) \geq V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p.$$

   Then, player k is such that $V_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) > p$, but since $k \in P\left(\mathbf{s}^f, \underline{\mathbf{s}}\right)$, we must have $s_k^f > \underline{s}_k$ and

   $$h_k\left(\mathbf{s}^f\right) = s_k^f = s_k^f + g\left(\underline{V}_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)\right)\left(s_k^f - \underline{s}_k\right)$$

   with $g\left(\underline{V}_k^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right)\right) < 0$ which contradicts $\mathbf{s}^f$ being a fixed point.

2. If $V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) < p$, for some i, then, since $h_i\left(\mathbf{s}^f\right) = s_i^f$, we must have $s_i^f = b$. Then, using the choice of p in equation (5) and that v is Increasing at Ties, with $\mathbf{s}' = \left(b, \mathbf{s}_{-i}^f, \mathbf{y}^{N\setminus A}\right)$ and $\mathbf{s} = \mathbf{y}$, we obtain

   $$p > V_i^{\mathbf{y}^{N\setminus A}}\left(\mathbf{s}^f\right) = V_i^{\mathbf{y}^{N\setminus A}}\left(b, \mathbf{s}_{-i}^f\right) \geq V_i^{\mathbf{y}^{N\setminus A}}\left(b, \mathbf{y}_{-i}^A\right) \geq \min V_i^{\mathbf{y}^{N\setminus A}}\left(b, \mathbf{y}_{-i}^A\right) \geq p$$

   which is a contradiction.

Items 1 and 2 have established for all i, so that $P' = P^M \cup \{p\}$ and

$$\sigma'(\widetilde{p}) = \begin{cases} \sigma'(\widetilde{p}) = \sigma^M(\widetilde{p}) & \text{for all } \widetilde{p} \neq p \\ \sigma'(p) = \mathbf{s}^f \end{cases}$$

satisfy $(P', \sigma') \succ (P^M, \sigma^M)$ which contradicts (P^M, σ^M) being maximal. We conclude that $P^M = \pi$, and that σ^M maps $\pi = \left[V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A), \min_i V_i^{\mathbf{y}^{N\setminus A}}(b, \mathbf{y}_{-i}^A)\right]$ into $\prod_A [y_i^A, b]$, is increasing and $V_i^{\mathbf{y}^{N\setminus A}}(\sigma^M(p)) = p$, $\forall i \in A, \forall p \in \pi$.

So far we have established that for all p in $\left[V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A), \min_i V_i^{\mathbf{y}^{N\setminus A}}(b, \mathbf{y}_{-i}^A)\right]$ there exists of a profile of signals $\sigma^{\mathbf{y}^{N\setminus A}}(p) \equiv \mathbf{s}^f$ such that $V_i^{\mathbf{y}^{N\setminus A}}\left(\sigma^{\mathbf{y}^{N\setminus A}}(p)\right) = V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) = p$ for all i, for all $p \leq \min V_i(b, \mathbf{y}_{-i}^A)$, and $\sigma^{\mathbf{y}^{N\setminus A}}$ is increasing. Since $\mathbf{y}$ and A are fixed throughout the proof, we will let $\sigma(p)$ stand for $\sigma^{\mathbf{y}^{N\setminus A}}(p)$.

Let $p^1 = \min V_i(b, \mathbf{y}_{-i}^A)$ and fix $\mathbf{s}^1 = \sigma(p^1)$. If $s_i^1 = b$ for some i, the proof is complete by letting $p_{\mathbf{y}}^A = p^1$ since for all $p < p^1$ we have that $\sigma(p) \ll b$, for if $\sigma_i(p)$ was equal to b, we would get the following contradiction

$$p = V_i(\sigma(p)) \geq \min V_i(\sigma(p)) \geq \min V_i(b, \mathbf{y}_{-i}^A) = p_1 > p.$$

So assume $s_i^1 < b$ for all i. Then, we have that

$$p^1 = \min V_i(b, \mathbf{y}_{-i}^A) = V_i(\sigma(p^1)) = V_i(\mathbf{s}^1)$$

and $s_i^1 < b$ imply that $p^1 < \min V_i(b, \mathbf{s}_{-i}^1) \equiv p^2$. Fix any $p^1 < p \leq p^2$. We can now repeat exactly the same steps as we have done so far (with $\mathbf{s}^1$ in place of $\mathbf{y}^A$) and show that in the domain $\left[V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A), \min_i V_i(b, \mathbf{s}_{-i}^1)\right]$ one has an increasing function $\sigma(\cdot)$ such that $V_i(\sigma(p)) = p$ for all i. Fix any $\mathbf{s}^2 = \sigma(p^2)$, and notice again that if $\sigma_i(p^2) = b$ for some i, the proof is complete by letting $p_{\mathbf{y}}^A = p^2$.

Continuing in this fashion, we get an increasing sequence of $\mathbf{s}^t$ and p^t with the properties that for all i,

$$V_i(\mathbf{s}^t) = p^t < p^{t+1} = \min_i V_i(b, \mathbf{s}_{-i}^t).$$

In the limit $p^\infty, \mathbf{s}^\infty$ we obtain for all i

$$V_i(\mathbf{s}^\infty) = p^\infty = \min_i V_i(b, \mathbf{s}_{-i}^\infty)$$

and so, for some i, $V_i(\mathbf{s}^\infty) = p^\infty = V_i(b, \mathbf{s}_{-i}^\infty)$. Since V_i is increasing in s_i this means that $s_i^\infty = b$, so that we can set $p_{\mathbf{y}}^A = p^\infty$. This completes the proof of (i) and (ii).

To establish (iii) set $\mathbf{s}' = \left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right)$ and $\mathbf{s} = \mathbf{y}$. If $\mathbf{s}' = \mathbf{s}$ conditions (a) and (b) yield the desired result, so assume $\mathbf{s}' \neq \mathbf{s}$. Note that: $k \in A$ implies $p = v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right)$; $k \notin A$ implies that $k \notin P(\mathbf{s}', \mathbf{s})$ so that the OEP and $P(\mathbf{s}', \mathbf{s}) \subseteq A$ ensure

$$p = \max_{k \in P(\mathbf{s}', \mathbf{s})} v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right) \geq \max_{k \notin P(\mathbf{s}', \mathbf{s})} v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right) \geq v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right)$$

for all $k \notin A$ as was to be shown. ■

The previous Lemma establishes the existence of a σ function that maps prices into signals, the resulting profile of signals being the "presumption" that other players will have about a players' signal, if he quits at a certain price. The set A is the set of "active" players at a certain moment, and the profile of signals $\mathbf{y}$ is decomposed in the set of signals of inactive players $\mathbf{y}^{N\backslash A}$ and the set of signals such that all active players have signals greater than $\mathbf{y}^{A}$. Lemma 1 describes the presumption of other players about a certain player's signal, when he should have quit, but he didn't (in the sense that his presumed signal is b, but he didn't quit). The difference with the previous Lemma is that we allow some elements of $\mathbf{y}^{B}$ to be equal to b (whereas in Lemma 2 we had $y_i^B < b$ for all i in B).

Proof of Lemma 1. Let B and $\mathbf{y}$ be as in the statement of this Lemma. Consider first the case in which $y_k < b$ for $k = i, j \in B$, $i \neq j$. Defining $A = B \backslash \{j \in B : y_j = b\}$ and applying Lemma 2 yields the desired result. So assume there is a unique $i \in B$ such that $y_i < b$. Let $p_{\mathbf{y}}^{B} = v_i\left(b, \mathbf{y}_{-i}\right)$ and let $v_i^{-1}\left(p; \mathbf{y}_{-i}\right)$ be the "inverse" of v_i, defined by

$$v_i\left(v_i^{-1}\left(p; \mathbf{y}_{-i}\right), \mathbf{y}_{-i}\right) \equiv p.$$

Then, it is easy to check that $\sigma^{\mathbf{y}^{N\backslash B}}$ defined by

$$\sigma_j^{\mathbf{y}^{N\backslash B}}(p) = \begin{cases} b & j \in B\backslash\{i\} \\ v_i^{-1}\left(p; \mathbf{y}_{-i}\right) & j = i \end{cases}.$$

satisfies conditions (i) and (ii). To check condition (iii), two cases must be considered.

(I) If $|W(\mathbf{y})| > 1$, we have that for

$$\mathbf{s}' = \left(\sigma_i^{\mathbf{y}^{N\backslash B}}(p), \mathbf{y}_{-i}\right) = \left(\sigma_i^{\mathbf{y}^{N\backslash B}}(p), b, ..., b, \mathbf{y}^{N\backslash B}\right) = \left(\sigma^{\mathbf{y}^{N\backslash B}}(p), \mathbf{y}^{N\backslash B}\right)$$

and $\mathbf{s} = \mathbf{y}$, $i = P(\mathbf{s}', \mathbf{s})$ the OEP implies that for all p,

$$p = v_i\left(\mathbf{s}'\right) = \max_{j \in P(\mathbf{s}', \mathbf{s})} v_j\left(\mathbf{s}'\right) \geq \max_{j \neq i} v_j\left(\mathbf{s}'\right) = \max_{j \neq i} v_j\left(\sigma^{\mathbf{y}^{N\backslash B}}(p), \mathbf{y}^{N\backslash B}\right)$$

as was to be shown.

(II) If $|W(\mathbf{y})| = 1$, we have that for $p_* = \max_j v_j(\mathbf{y}) = v_i(\mathbf{y})$,

$$v_i\left(\sigma^{\mathbf{y}^{N\backslash B}}(p_*), \mathbf{y}^{N\backslash B}\right) = \max_{j \in N} v_j(\mathbf{y}) > \max_{j \neq i} v_j(\mathbf{y}) = \max_{j \neq i} v_j\left(\sigma^{\mathbf{y}^{N\backslash B}}(p_*), \mathbf{y}^{N\backslash B}\right). \tag{6}$$

Suppose that contrary to what we want to show, there was some $\overline{p}$ such that for some $j \neq i$

$$v_j\left(\sigma^{\mathbf{y}^{N\backslash B}}(\overline{p}), \mathbf{y}^{N\backslash B}\right) > \overline{p} = v_i\left(\sigma^{\mathbf{y}^{N\backslash B}}(\overline{p}), \mathbf{y}^{N\backslash B}\right). \tag{7}$$

Given equations (6) and (7), continuity of $\sigma^{\mathbf{y}^{N\backslash B}}(p)$ (ensured by construction) and Bolzano's Theorem, there exists a p^* such that $\max_{j\neq i} v_j\left(\sigma^{\mathbf{y}^{N\backslash B}}(p^*), \mathbf{y}^{N\backslash B}\right) = v_i\left(\sigma^{\mathbf{y}^{N\backslash B}}(p^*), \mathbf{y}^{N\backslash B}\right)$. Then, letting $\mathbf{s}' = \left(\sigma_i^{\mathbf{y}^{N\backslash B}}(\overline{p}), \mathbf{y}_{-i}\right)$ and $\mathbf{s} = \left(\sigma_i^{\mathbf{y}^{N\backslash B}}(p^*), \mathbf{y}_{-i}\right)$ the OEP implies

$$v_i(\mathbf{s}') \geq \max_{k\neq i} v_k(\mathbf{s}') \geq v_j(\mathbf{s}') \Leftrightarrow v_i\left(\sigma_i^{\mathbf{y}^{N\backslash B}}(\overline{p}), \mathbf{y}_{-i}\right) \geq v_j\left(\sigma^{\mathbf{y}^{N\backslash B}}(\overline{p}), \mathbf{y}^{N\backslash B}\right)$$

which contradicts (7), and therefore completes the proof. ■

The next Lemma gives the connection between one set of functions σ^B and the set of functions σ^A when $A = B\backslash\{l\}$ for some $l \in B$. This gives the relation between the bidding strategies in a sub-auction with active players B, and the one that follows after player l has dropped out. If various players drop out at the same price, one only needs to apply the Lemma repeatedly at the price of the drops ($\widetilde{p}$ in the Lemma).

Lemma 3. Fix any $B \subseteq N$, with $|B| > 2$, and fix a set of types $\mathbf{y}^{N\backslash B}$ such that there exists a $\mathbf{y}^B \neq \mathbf{b}$ for which for all $i \in B$, $y_i < b$ implies $v_i(\mathbf{y}) = \max_{j\in N} v_j(\mathbf{y})$. Assume that v is increasing and satisfies the OEP, and fix a $p_{\mathbf{y}}^B$ and $\sigma^{\mathbf{y}^{N\backslash B}}$ as in the statement of Lemma 1. Fix any $l \in B$ and let $A = B\backslash\{l\}$. For any $\widetilde{p} \leq p_{\mathbf{y}}^B$, if $\mathbf{s}^B = \sigma^{\mathbf{y}^{N\backslash B}}(\widetilde{p})$ then for $z \equiv \left(\mathbf{y}^{N\backslash B}, s_l\right)$ there exists $p_z^A \geq v_i\left(\mathbf{s}^B, \mathbf{y}^{N\backslash B}\right) = V_i^z\left(\mathbf{s}^A\right)$ (for i with $y_i < b$) and a weakly increasing function $\sigma^z : \left[V_i^z\left(\mathbf{s}^A\right), p_z^A\right] \rightarrow \prod_{i\in A}[s_i, b]$ mapping prices into types of active players, such that:

(i) $\sigma_j^z\left(p_z^A\right) = b$ for some j with $y_j < b$ and for all $i \in A$, $p = p_{\mathbf{y}}^A$ and $y_i < b$ imply the break even condition

$$V_i^z(\sigma^z(p)) = p. \tag{8}$$

(ii) for all $p < p_z^A$, if $y_i < b$ then $\sigma_i^z(p) < b$ and the break even condition (8) holds for all $i \in A$.

(iii) for all $p \leq p_{\mathbf{y}}^A$, and all $k \in N$, $v_k\left(\sigma^{\mathbf{y}^{N\backslash A}}(p), \mathbf{y}^{N\backslash A}\right) \leq p$.

(iv) for all $j \in A$

$$\sigma_j^z(\widetilde{p}) = \sigma_j^{\mathbf{y}^{N\backslash B}}(\widetilde{p}).$$

Proof of Lemma 3. Items (i), (ii) and (iii) follow as a direct application of Lemma 1. Then, item (iv) follows because for all i, $\sigma_i^z(\widetilde{p}) \geq s_i^A$, and if $\sigma_j^z(\widetilde{p}) > s_j^A$ we would get (using $\mathbf{s}' = \left(s_j^A, \sigma_{-j}^z(p), z\right)$ and $\mathbf{s} = \left(\mathbf{s}^A, z\right)$ and that v is Increasing at Ties)

$$\widetilde{p} = V_j^z(\sigma^z(\widetilde{p})) > V_j^z\left(s_j^A, \sigma_{-j}^z(p)\right) \geq V_j^z\left(\mathbf{s}^A\right) = v_j\left(\mathbf{s}^B, \mathbf{y}^{N\backslash B}\right) = V_j^{\mathbf{y}^{N\backslash B}}\left(\mathbf{s}^B\right) = V_j^{\mathbf{y}^{N\backslash B}}\left(\sigma^{\mathbf{y}^{N\backslash B}}(\widetilde{p})\right) = \widetilde{p}$$

which is a contradiction. ■

We now show that the σ function is continuous.

Lemma 4. Continuity. For every A and $\mathbf{y}^{N\backslash A}$ satisfying the conditions of Lemma 1, the function $\sigma^{\mathbf{y}^{N\backslash A}}$ is continuous.

Proof of Lemma 4. Suppose that σ is discontinuous at p^*. It must be either not continuous from the right, or from the left, so assume without loss of generality that it is discontinuous from the left: there is an ε such that for all δ there is some p with $p^* - p < \delta$ but $\sigma(p^*) - \sigma(p) \geq \varepsilon$ (we have used σ non decreasing). Fix then $\delta_1 = 1$ and $p_1 < p^*$ such that $p^* - p_1 < \delta$ but $\sigma(p^*) - \sigma(p_1) \geq \varepsilon$. Pick then, by induction, $0 < \delta_n < p^* - p_{n-1}$ and $p^* - p_n < \delta_n$ but $\sigma(p^*) - \sigma(p_n) \geq \varepsilon$. We then obtain: $p_n \to p^*$, p_n is increasing, $\sigma(p_n)$ is increasing and therefore has a limit (since its bounded above by b) $\mathbf{s}^\infty$ and $\mathbf{s}^\infty \neq \sigma(p^*)$, $\mathbf{s}^\infty \leq \sigma(p^*)$ (because $\sigma(p_n) \leq \sigma(p^*)$ for all n).

Since for all n and for all i, $V_i(\sigma(p_n)) = p_n$ we obtain by continuity of V_i,

$$p^* = \lim p_n = \lim V_i(\sigma(p_n)) = V_i(\lim \sigma(p_n)) = V_i(\mathbf{s}^\infty).$$

But then, $\mathbf{s}^\infty \neq \sigma(p^*)$ and $\mathbf{s}^\infty \leq \sigma(p^*)$ imply that for some i, $s_i^\infty < \sigma_i(p^*)$. This, in turn, means that since V_i is strictly increasing in s_i and Increasing at ties (at $\mathbf{s}^\infty$ all are tied), $V_i(\mathbf{s}^\infty) < V_i(\sigma(p^*)) = p^*$. This is a contradiction, and shows that σ is continuous. ■

Proof of Theorem 1. Efficiency. We will prove that the profile of strategies that in any auction with active players A and signals of inactive players $\mathbf{y}^{N\backslash A}$ calls for a player with signal s_i to quit at a price $\beta_i^{\mathbf{y}^{N\backslash A}}(s_i) = \min\left\{p : \sigma^{\mathbf{y}^{N\backslash A}}(p) \geq s_i\right\}$, for σ as in Lemma 1, is an ex post equilibrium. We will then show that it is also efficient.

The first part of the proof (ex-post equilibrium) follows Krishna's Lemma 1 closely, but does not use the fact that σ is unique or strictly increasing. Consider bidder 1 and suppose that all bidders $i > 1$ are following the strategy β_i. We will now show that player 1 does not have a profitable deviation.

Consider first the case in which following β_1 player 1 wins when active players are A and signals are $\mathbf{s}$: this can only happen if players in $A\backslash\{1\}$ drop at the same price, say p^*. We will now show that he earns a profit, so that no deviations are profitable: quitting before earns him 0, and he can never change the price he pays. Without loss of generality, let $A = \{2, 3, ..., a\}$. Since all strategies β are increasing, all bidders in A can infer the signals $\mathbf{s}^{N\backslash A}$ of inactive bidders from the prices at which they dropped. Also, since player $i = 2, ..., a$ drop at p^* and

$$\beta_i^{\mathbf{s}^{N\backslash A}}(s_i) = \min\left\{p : \sigma^{\mathbf{s}^{N\backslash A}}(p) \geq s_i\right\} = p^*$$

we obtain $s_i = \sigma_i^{\mathbf{s}^{N\backslash A}}(p^*)$. Moreover, $s_1 > \sigma_1^{\mathbf{s}^{N\backslash A}}(p^*)$ and therefore $V_1^{\mathbf{s}^{N\backslash A}}\left(\sigma^{\mathbf{s}^{N\backslash A}}(p^*)\right) = p^*$ implies

$$v_1(\mathbf{s}) = v_1\left(s_1, \sigma_{-1}^{\mathbf{s}^{N\backslash A}}(p^*), \mathbf{s}^{N\backslash A}\right) > v_1\left(\sigma^{\mathbf{s}^{N\backslash A}}(p^*), \mathbf{s}^{N\backslash A}\right) = V_1^{\mathbf{s}^{N\backslash A}}\left(\sigma^{\mathbf{s}^{N\backslash A}}(p^*)\right) = p^*$$

which means that player 1 makes a profit, as was to be shown.

As a second alternative, consider the case in which β_1 calls for bidder 1 to drop at some price p_1^* in some sub-auction with active bidders $A = \{1, 2, ..., a\}$, when the other players quit at signals $\mathbf{s}^{N \setminus A}$, and suppose that bidder 1 considers staying longer until he wins the object. Suppose he stays until winning and that bidders quit in the order $a, a-1, a-2, ..., 2$ at prices $p_a \leq ..., \leq p_2$, so that 1 wins at a price p_2. We will show that by doing this he can't make a profit.

For p_2, the price at which player 2 quits, $s_2 = \sigma_2^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2)$ so (iii) of Lemma 1 implies that

$$p_2 \geq v_1 \left(\sigma_1^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2), \mathbf{s}_{-1} \right). \tag{9}$$

Then, since for each fixed pair $\left(B, \mathbf{s}^{N\setminus B}\right)$ the function $\sigma^{\mathbf{s}^{N\setminus B}}$ is increasing and when a bidder $j \in B$ drops out at p_j, we get $\sigma^{\mathbf{s}^{N\setminus B}}(p_j) = \sigma^{\mathbf{s}^{N\setminus\{B\setminus\{j\}\}}}(p_j)$ (by (iv) of Lemma 3), we obtain

$$\begin{aligned} \sigma_1^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2) &\geq \sigma_1^{\mathbf{s}^{N\setminus\{1,2\}}}(p_3) = \sigma_1^{\mathbf{s}^{N\setminus\{1,2,3\}}}(p_3) \geq \sigma_1^{\mathbf{s}^{N\setminus\{1,2,3\}}}(p_4) = \sigma_1^{\mathbf{s}^{N\setminus\{1,2,3,4\}}}(p_4) \geq ... \\ &\geq \sigma^{\mathbf{s}^{N\setminus\{A\setminus\{a\}\}}}(p_a) = \sigma_1^{\mathbf{s}^{N\setminus A}}(p_a) \geq \sigma_1^{\mathbf{s}^{N\setminus A}}(p_1^*) = s_1. \end{aligned} \tag{10}$$

(the last equality follows from the fact that player 1 was supposed to quit at p_1^*). Equations (9) and (10) imply that $p_2 \geq v_1(\mathbf{s})$ so that player 1 can't make a profit by staying longer than what his strategy calls for.

We have already shown that it is not profitable to quit when β_1 calls for staying, and it is not profitable to stay when β_1 calls for quitting. We will now show that if in some off equilibrium path, player 1 is still active at price p when he should have quit at price $p_1^* < p$, then quitting is a best response (in particular, it is better than winning at p). Let the set of active bidders at p be $J = \{1, ..., j\}$. Then, as in equation (10),

$$\sigma_1^{\mathbf{s}^{N\setminus J}}(p) \geq \sigma_1^{\mathbf{s}^{N\setminus J}}(p_{j+1}) = \sigma_1^{\mathbf{s}^{N\setminus\{J\cup\{j+1\}\}}}(p_{j+1}) \geq ... \geq \sigma_1^{\mathbf{s}^{N\setminus A}}(p_a) \geq \sigma_1^{\mathbf{s}^{N\setminus A}}(p_1^*) = s_1$$

so that $p \geq v_1\left(\sigma^{\mathbf{s}^{N\setminus J}}(p), \mathbf{s}^{N\setminus J}\right)$ implies $p \geq v_1\left(s_1, \sigma_{-1}^{\mathbf{s}^{N\setminus J}}(p), \mathbf{s}^{N\setminus J}\right)$. This means quitting, as his strategy prescribes, is optimal. This completes the proof that the profile of strategies defined by σ is an ex-post equilibrium.

Proof of Theorem 1. Efficiency. Without loss of generality, suppose that at a profile of signals $\mathbf{s}$ the winner is player 1 and that the last to quit is player 2 at price p_2. Then, we have that $s_1 > \sigma_1^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2)$, $s_2 > \sigma_2^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2)$ and $v_1\left(\sigma^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2), \mathbf{s}^{N\setminus\{1,2\}}\right) = v_2\left(\sigma^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2), \mathbf{s}^{N\setminus\{1,2\}}\right)$. The OEP then tells us that for

$$P = P\left(\mathbf{s}, \left(\sigma^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2), \mathbf{s}^{N\setminus\{1,2\}}\right)\right)$$

we must have

$$v_1(\mathbf{s}) = \max_{i \in P} v_i(\mathbf{s}) \geq \max_{j \notin P} v_j(\mathbf{s})$$

establishing efficiency. ■[5]

Before proceeding to the proof of Theorem 4, we prove Lemma A, which in turn uses this simple result.

Lemma 5. If v satisfies the ACC, then for all $P \subset N$ such that $j \in P$ we have that for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \neq j$

$$\sum_{k \in P} \frac{\partial v_k(\mathbf{s})}{\partial s_j} > |P| \frac{\partial v_i(\mathbf{s})}{\partial s_j}$$

Proof of Lemma 5. The proof proceeds by induction on the size of P. We already know that the result is true for $P = N$, so assume it is true for all P' with $|P'| = m + 1$. In order to obtain a contradiction, suppose that for some P with $|P| = m$, $j \in P$, and some $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \neq j$ we had $\sum_P \frac{\partial v_k}{\partial s_j} \leq |P| \frac{\partial v_i}{\partial s_j}$. In such a case, we must have $i \in P$, since otherwise, for $P' = P \cup \{i\}$ we would have $\sum_{P'} \frac{\partial v_k}{\partial s_j} \leq |P'| \frac{\partial v_i}{\partial s_j}$, contradicting the induction hypothesis. We must also have $\partial v_i / \partial s_j > \partial v_h / \partial s_j$ for all $h \notin P$, since otherwise, for some $h \notin P$ with $\partial v_i / \partial s_j \leq \partial v_h / \partial s_j$ we would have that for $P' = P \cup \{h\}$

$$\sum_{k \in P} \frac{\partial v_k}{\partial s_j} \leq |P| \frac{\partial v_i}{\partial s_j} \leq |P| \frac{\partial v_h}{\partial s_j} \Rightarrow \sum_{P'} \frac{\partial v_k}{\partial s_j} \leq |P'| \frac{\partial v_h}{\partial s_j}$$

contradicting the induction hypothesis. But then $\partial v_i / \partial s_j > \partial v_h / \partial s_j$ for all $h \notin P$, implies that

$$\sum_{k \in P} \frac{\partial v_k}{\partial s_j} \leq |P| \frac{\partial v_i}{\partial s_j} \Rightarrow \sum_{k \in P} \frac{\partial v_k}{\partial s_j} + \sum_{h \notin P} \frac{\partial v_h}{\partial s_j} < |P| \frac{\partial v_i}{\partial s_j} + |N \backslash P| \frac{\partial v_i}{\partial s_j} \Leftrightarrow \sum_{k \in N} \frac{\partial v_k}{\partial s_j} < |N| \frac{\partial v_i}{\partial s_j}$$

which contradicts the ACC. This concludes the proof. ■

Proof of Lemma A. Let us start with the claim that the ACC implies the Equal Increments Condition. For all $h \in P$, and $i \notin P$, by Lemma 5, $\sum_P \frac{\partial v_k(\mathbf{s})}{\partial s_h} > |P| \frac{\partial v_i(\mathbf{s})}{\partial s_h}$. Keeping i fixed, and adding over all $h \in P$, we obtain

$$\sum_{h \in P} \sum_{k \in P} \frac{\partial v_k(\mathbf{s})}{\partial s_h} > \sum_{h \in P} |P| \frac{\partial v_i(\mathbf{s})}{\partial s_h}.$$

We can write the previous equation as

$$\sum_{k \in P} I_P \nabla v_k > |P| I_P \nabla v_i.$$

This implies that for some $j \in P$, $I_P \nabla v_j > I_P \nabla v_i$ as was to be shown.

Now assume that v satisfies the CCC, and pick any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and any $i \notin P$. We must show that there exists $j \in P$ such that $I_P \nabla v_j > I_P \nabla v_i$. Suppose first that there is some $k \in P$ with

[5] Vijay: in the proof of your Lemma 4, don't you need to ensure that after a player has quit his value does not surpass that of the players that remain in the auction?

$k < i$ and let j be the largest k in P which is still smaller than i. That is, if $P = \{1, 2, 5\}$ and $i = 4$, pick $j = 2$. In order to show that $I_P \nabla v_j > I_P \nabla v_i$, it will suffice to show that for all $k \in P \backslash \{j\}$, $\partial v_j / \partial s_k \geq \partial v_i / \partial s_k$, since then $\partial v_j / \partial s_j > \partial v_i / \partial s_j$ will make the desired inequality strict. Notice that for all $k < i$, we have $k < j < i$, so by the CCC, we have $\partial v_j / \partial s_k \geq \partial v_i / \partial s_k$. For $k > i$, we have that the CCC tells us that

$$\frac{\partial v_k}{\partial s_k} > \frac{\partial v_n}{\partial s_k} \geq \frac{\partial v_1}{\partial s_k} \geq \frac{\partial v_j}{\partial s_k} \geq \frac{\partial v_i}{\partial s_k}.$$

Suppose now that for the chosen i there is no $k < i$ in P. For $j \equiv \max_{k \in P} P$ we will show, as before, that for all $k \in P \backslash \{j\}$, $\partial v_j / \partial s_k \geq \partial v_i / \partial s_k$. Notice that for all $k \in P \backslash \{j\}$ we have $i > k > j$, so the CCC tells us that

$$\frac{\partial v_k}{\partial s_k} > \frac{\partial v_j}{\partial s_k} \geq \frac{\partial v_n}{\partial s_k} \geq \frac{\partial v_1}{\partial s_k} \geq \frac{\partial v_i}{\partial s_k}$$

as was to be shown. ■

Proof of Theorem 3. Suppose the OEP is violated, so that there exists an $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and an $\mathbf{s}' \geq \mathbf{s}$ such that $s'_j > s_j$ if and only if $j \in P \subset W(\mathbf{s})$, and that for all $j \in P$, $v_j(\mathbf{s}') < v_i(\mathbf{s}')$ for some i. Without loss of generality, suppose $P = \{1, ..., m\}$ and assume also without loss of generality, that $s'_1 - s_1 \leq s'_2 - s_2 \leq ... \leq s'_m - s_m$. Define

$$\alpha_1 = \max \{\alpha : \exists j \in P, v_j(\mathbf{s} + I_P \alpha) \geq v_i(\mathbf{s} + I_P \alpha) \forall i \notin P\}.$$

Note that $\alpha_1 \leq s'_1 - s_1$ would imply that there are $j \in P$ and $i \notin P$ such that $v_j(\mathbf{s} + I_P \alpha_1) = v_i(\mathbf{s} + I_P \alpha_1)$ and that for all $\varepsilon > 0$, $v_k(\mathbf{s} + I_P[\alpha_1 + \varepsilon]) < v_i(\mathbf{s} + I_P[\alpha_1 + \varepsilon])$ for all $k \in P$. Taking derivatives with respect to ε and evaluating at $\varepsilon = 0$, we obtain that for $\mathbf{s}' = \mathbf{s} + I_P \alpha_1$ we have $|W(\mathbf{s}')| > 1$, and that $I_P \nabla v_k(\mathbf{s} + I_P \alpha_1) \leq I_P \nabla v_i(\mathbf{s} + I_P \alpha_1)$ for all $k \in P$, which contradicts Lemma A, and would therefore conclude the proof. Assume then $\alpha_1 > s'_1 - s_1$.

Define then $P_2 = P \backslash \{1\}$, and

$$\alpha_2 = \max \left\{\alpha : \exists j \in P_2, v_j\left(\left(s'_1, \mathbf{s}_{-1}\right) + I_{P_2} \alpha\right) \geq v_i\left(\left(s'_1, \mathbf{s}_{-1}\right) + I_{P_2} \alpha\right) \forall i \notin P_2\right\}.$$

That is, we have "replaced" $\mathbf{s}$ by $(s'_1, \mathbf{s}_{-1})$ and we will now show that we can't have $\alpha_2 \leq s'_2 - s_2$. Since $\alpha_1 > s'_1 - s_1$, the set on which α_2 is defined is non-empty, so α_2 is well defined. If we had $\alpha_2 \leq s'_2 - s_2$, we would obtain that there are $j \in P_2$ and $i \notin P_2$ such that $v_j((s'_1, \mathbf{s}_{-1}) + I_{P_2} \alpha_2) = v_i((s'_1, \mathbf{s}_{-1}) + I_{P_2} \alpha_2)$ and that for all $\varepsilon > 0$, $v_k((s'_1, \mathbf{s}_{-1}) + I_{P_2}[\alpha_2 + \varepsilon]) < v_i((s'_1, \mathbf{s}_{-1}) + I_{P_2}[\alpha_2 + \varepsilon])$ for all $k \in P_2$. Taking derivatives with respect to ε and evaluating at $\varepsilon = 0$, we obtain that for $\mathbf{s}' = (s'_1, \mathbf{s}_{-1}) + I_{P_2} \alpha_2$ we have $|W(\mathbf{s}')| > 1$, and that for all $k \in P_2$

$$I_{P_2} \nabla v_k\left(\mathbf{s}'\right) \leq I_{P_2} \nabla v_i\left(\mathbf{s}'\right)$$

which contradicts Lemma A, and would therefore conclude the proof.

Fix some $l \leq m$ and define $\widetilde{\mathbf{s}} = \left(s'_1, ..., s'_{l-1}, s_l, s_{l+1}, ..., s_n\right)$ and $P_l = P \backslash \{1, ..., l-1\}$. As an induction hypothesis, suppose that for some $j \in P_l$, $v_j(\widetilde{\mathbf{s}}) \geq v_i(\widetilde{\mathbf{s}})$ for all $i \notin P_l$ (we have already proved this for $l = 1$ and $l = 2$) and define

$$\alpha_l = \max\left\{\alpha : \exists j \in P_l, v_j\left(\widetilde{\mathbf{s}} + I_{P_l}\alpha\right) \geq v_i\left(\widetilde{\mathbf{s}} + I_{P_l}\alpha\right) \forall i \notin P_l\right\}.$$

Again, if we had $\alpha_l \leq s'_l - s_l$ we would obtain that there are $j \in P_l$ and $i \notin P_l$ such that $v_j\left(\widetilde{\mathbf{s}} + I_{P_l}\alpha_l\right) = v_i\left(\widetilde{\mathbf{s}} + I_{P_l}\alpha_l\right)$ and that for all $\varepsilon > 0$, $v_k\left(\widetilde{\mathbf{s}} + I_{P_l}\left[\alpha_l + \varepsilon\right]\right) < v_i\left(\widetilde{\mathbf{s}} + I_{P_l}\left[\alpha_l + \varepsilon\right]\right)$ for all $k \in P_l$. Taking derivatives with respect to ε and evaluating at $\varepsilon = 0$, we obtain that for $\mathbf{s}' = \widetilde{\mathbf{s}} + I_{P_l}\alpha_l$ we have $|W(\mathbf{s}')| > 1$, and that $I_{P_l}\nabla v_k(\mathbf{s}') \leq I_{P_l}\nabla v_i(\mathbf{s}')$ for all $k \in P_l$. This contradiction concludes the proof. ■

Before proceeding to the proof of Theorem 2, we state and prove two simple facts that will help in the proof.

Fact 1. For $|P| = 1$**.** Assume the Hypothesis of Theorem 2 (although undominated is not needed). If $s''_1 > s^*_1; s''_i = s^*_i$ for $i = 2, 3; W(\mathbf{s}^*) = \{1\}$ and $1, j \notin W(\mathbf{s}'')$ for $j \neq 1$, then there is a μ such that no efficient equilibrium exists. To see so, set $\mu(\mathbf{s}^*) = \mu(\mathbf{s}'') = 1/2$, suppose without loss of generality that $3 \in W(\mathbf{s}'')$ (i.e. $j = 2$). Suppose there is an efficient equilibrium. The quitting price of 2 is irrelevant, so let β_3 be the quitting price of player 3 when player 1 is active. We must have, by efficiency, $\beta_1(s''_1) < \beta_3 < \beta_1(s^*_1)$. Since $\beta_1(s^*_1)$ is a best response, player 1 wants the object at a price of β_3 and we must therefore have $v_1(s^*_1, s_2, s_3) \geq \beta_3$. But then $v_1(s''_1, s_2, s_3) > v_1(s^*_1, s_2, s_3)$ implies that $\beta_1(s'_1)$ is not a best response, and 1 would like to pretend that his signal is s^*_1 when it is s'_1. This concludes the proof of Fact 1.

Fact 2. For $|P| = 2$. Assume the Hypothesis of Theorem 2. If $s''_i > s^*_i$ for $i = 1, 2$; $s''_3 = s^*_3; W(\mathbf{s}^*) = \{1, 2\}$ and $W(\mathbf{s}'') = 1$, then there is a μ such that no efficient equilibrium in undominated strategies exists. For a proof, let $\mu(\mathbf{s}^*) = \mu(\mathbf{s}'') = 1/2$ and let β_3 be the quitting price of player 3 when all players are active. Notice that, given the perfect correlation, it is a dominant strategy for players 1 and 2 to bid their valuations, so letting $\beta_i(\cdot)$ denote the bidding strategy of player i in the empty history, we obtain $\beta_i(s^*_i) = v_j(\mathbf{s}^*)$ for $i, j = 1, 2$. Efficiency requires that in state $\mathbf{s}^*$ player 3 quits before 1 or 2, and hence

$$\beta_i(s^*_i) = v_i(\mathbf{s}^*) > \beta_3. \tag{11}$$

But in state $\mathbf{s}''$ player 3 can't be the first to quit, so we must have $\beta_3 > \min_{i<3}\beta_i(s''_i) = \min_{i<3} v_i(\mathbf{s}'')$, so joining this with equation (11) we obtain

$$v_1(\mathbf{s}^*) = v_2(\mathbf{s}^*) > \beta_3 > \min_{i<3} v_i(\mathbf{s}'')$$

which contradicts $v_i(\mathbf{s}'') > v_i(\mathbf{s}^*)$ for $i = 1, 2$. This concludes the proof of Fact 2.

Proof of Theorem 2. Suppose that for some $\mathbf{s}$ with $s_i > 0$ for all i and $|W(\mathbf{s})| > 1$ we have $\mathbf{s}' \geq \mathbf{s}$, $s'_j > s_j$ if and only if $j \in P \subseteq \{1,2\} = W(\mathbf{s})$ but that $\max_{k \notin P} v_k(\mathbf{s}') > v_j(\mathbf{s}')$ for all $j \in P$.

Consider now the four cases in which there is only one winner at $\mathbf{s}$ and call him player 1, $P = \{1\}$.

a. $P = \{1\}$ and $v_2(\mathbf{s}') \neq v_3(\mathbf{s}')$ and $v_2(\mathbf{s}) > v_3(\mathbf{s})$. Since $v_1(\mathbf{s}) = v_2(\mathbf{s})$, the regularity assumption ensures that there is a $(\widetilde{s}_1, \widetilde{s}_2)$ close to $\mathbf{s}$ (in particular, we need $s'_1 > \widetilde{s}_1$) such that: for $\mathbf{s}^* = (\widetilde{s}_1, \widetilde{s}_2, s_3)$ we obtain $v_1(\mathbf{s}^*) > v_2(\mathbf{s}^*) > v_3(\mathbf{s}^*)$; for $\mathbf{s}'' = (s'_1, \widetilde{s}_2, s_3)$, it is still true that $v_2(\mathbf{s}'') \neq v_3(\mathbf{s}'')$ and since $\widetilde{s}_2$ is close to s_2 it will still be true that $1 \notin W(\mathbf{s}'')$. Formally, this is just an application of the inverse function theorem: we pick any values of v_1^* and v_2^* that we want in the open set around $(v_1(\mathbf{s}), v_2(\mathbf{s}))$, and the inverse function theorem tells us that there exists an $\mathbf{s}^*$ in a neighborhood of $\mathbf{s}$ such that $v_1(\mathbf{s}^*) = v_1^*$ and $v_2(\mathbf{s}^*) = v_2^*$. Now apply Fact 1.

b. $P = \{1\}$ and $v_2(\mathbf{s}') = v_3(\mathbf{s}')$ and $v_2(\mathbf{s}) > v_3(\mathbf{s})$. First, as before, choose $(\widetilde{s}_1, \widetilde{s}_2)$ $(s'_1 > \widetilde{s}_1)$, so that: for $\mathbf{s}^{**} = (\widetilde{s}_1, \widetilde{s}_2, s_3)$ we obtain $v_1(\mathbf{s}^{**}) > v_2(\mathbf{s}^{**}) > v_3(\mathbf{s}^{**})$; for $\mathbf{s}''' = (s'_1, \widetilde{s}_2, s_3)$, since $\widetilde{s}_2$ is close to s_2, it is still true that $1 \notin W(\mathbf{s}''')$. If $v_2(\mathbf{s}''') \neq v_3(\mathbf{s}''')$, set $\mathbf{s}^* = \mathbf{s}^{**}$ and $\mathbf{s}'' = \mathbf{s}'''$ and apply Fact 1. If $v_2(\mathbf{s}''') = v_3(\mathbf{s}''')$, apply the regularity assumption to obtain $(\overline{s}_2, \overline{s}_3)$ close to $(\widetilde{s}_2, s_3)$ that preserve the inequalities

$$v_1(\widetilde{s}_1, \widetilde{s}_2, s_3) > v_2(\widetilde{s}_1, \widetilde{s}_2, s_3) > v_3(\widetilde{s}_1, \widetilde{s}_2, s_3) \quad \text{and} \quad v_1\left(s'_1, \widetilde{s}_2, s_3\right) < v_2\left(s'_1, \widetilde{s}_2, s_3\right)$$

but for which $v_2(s'_1, \overline{s}_2, \overline{s}_3) \neq v_3(s'_1, \overline{s}_2, \overline{s}_3)$. Then apply Fact 1 with $\mathbf{s}'' = (s'_1, \overline{s}_2, \overline{s}_3)$ and $\mathbf{s}^* = (\widetilde{s}_1, \overline{s}_2, \overline{s}_3)$.

c. $P = \{1\}$ and $v_2(\mathbf{s}') \neq v_3(\mathbf{s}')$ and $v_2(\mathbf{s}) = v_3(\mathbf{s})$. Using the regularity assumption, slightly perturb $\mathbf{s}$ to $\mathbf{s}^*$ to obtain $v_1(\mathbf{s}^*) > \max_{i \neq 1} v_i(\mathbf{s}^*)$, while preserving $s'_1 > s_1^*$, $v_2(s'_1, s_2^*, s_3^*) \neq v_3(s'_1, s_2^*, s_3^*)$ and one of them greater than $v_1(s'_1, s_2^*, s_3^*)$. Then apply Case a, or Case b, depending on whether $v_2(\mathbf{s}^*)$ is different or equal to $v_3(\mathbf{s}^*)$.

d. $P = \{1\}$ and $v_2(\mathbf{s}') = v_3(\mathbf{s}')$ and $v_2(\mathbf{s}) = v_3(\mathbf{s})$. Using the regularity assumption, slightly perturb $\mathbf{s}$ to $\mathbf{s}^{**}$ to obtain $v_1(\mathbf{s}^{**}) > v_2(\mathbf{s}^{**}) > v_3(\mathbf{s}^{**})$, while preserving $s'_1 > s_1^{**}$ and preserving, for $\mathbf{s}''' = (s'_1, s_2^{**}, s_3^{**})$, $v_1(\mathbf{s}''') < \max_{i \neq 1} v_i(\mathbf{s}''')$. If $v_2(\mathbf{s}''') \neq v_3(\mathbf{s}''')$, set $\mathbf{s}^* = \mathbf{s}^{**}$ and $\mathbf{s}'' = \mathbf{s}'''$ and apply Fact 1. If $v_2(\mathbf{s}''') = v_3(\mathbf{s}''')$, apply the regularity assumption to obtain $(\overline{s}_2, \overline{s}_3)$ close to (s_2^{**}, s_3^{**}) that preserve the inequalities

$$v_1(\widetilde{s}_1, s_2^{**}, s_3^{**}) > v_2(\widetilde{s}_1, s_2^{**}, s_3^{**}) > v_3(\widetilde{s}_1, s_2^{**}, s_3^{**}) \quad \text{and} \quad v_1\left(s'_1, s_2^{**}, s_3^{**}\right) < v_2\left(s'_1, s_2^{**}, s_3^{**}\right)$$

but for which $v_2(s'_1, \overline{s}_2, \overline{s}_3) \neq v_3(s'_1, \overline{s}_2, \overline{s}_3)$. Then apply Fact 1 with $\mathbf{s}'' = (s'_1, \overline{s}_2, \overline{s}_3)$ and $\mathbf{s}^* = (\widetilde{s}_1, \overline{s}_2, \overline{s}_3)$.

Consider now the two cases in which $P = \{1, 2\}$: either $v_3(\mathbf{s}) = v_1(\mathbf{s}) = v_2(\mathbf{s})$ or $v_3(\mathbf{s}) < v_1(\mathbf{s}) = v_2(\mathbf{s})$. If the latter is the case, apply Fact 2 with $\mathbf{s}'' = \mathbf{s}'$ and $\mathbf{s}^* = \mathbf{s}$. If $v_3(\mathbf{s}) = v_1(\mathbf{s}) = v_2(\mathbf{s})$, the inverse function theorem (and the regularity assumption) ensure that there is an open set around $v(\mathbf{s}) = (v_1(\mathbf{s}), v_2(\mathbf{s}), v_3(\mathbf{s}))$ such that one can pick any $v^* = (v_1^*, v_2^*, v_3^*)$ in that neighborhood and there will exist an $\mathbf{s}^*$ such that $v(\mathbf{s}^*) = v^*$. We then pick $v_1^* = v_2^* > v_3^*$ close enough to $v(\mathbf{s})$, and s^* will satisfy $s_i' > s_i^*$ for $i = 1, 2$ and also, for $\mathbf{s}'' = (s_1', s_2', s_3^*)$, we still get $v_3(\mathbf{s}'') > \max_{i<3} v_i(\mathbf{s}'')$. Then, apply Fact 2. This concludes the proof of Theorem 2. ■

Proof of Theorem 5. Pick any $\mathbf{s}$ such that $|W(\mathbf{s})| > 1$ and suppose that $\mathbf{s}' \geq \mathbf{s}$ and $s_j' > s_j$ if and only if $j \in P(\mathbf{s}', \mathbf{s}) \subseteq W(\mathbf{s})$. We will now show that $\max_{j \in P(\mathbf{s}', \mathbf{s})} v_j(\mathbf{s}') \geq \max_{k \notin P(\mathbf{s}', \mathbf{s})} v_k(\mathbf{s}')$. To obtain a contradiction, suppose that for some player $i \notin P(\mathbf{s}', \mathbf{s})$ we have $v_i(\mathbf{s}') = \max_{k \notin P(\mathbf{s}', \mathbf{s})} v_k(\mathbf{s}') > \max_{j \in P(\mathbf{s}', \mathbf{s})} v_j(\mathbf{s}')$. In the equilibrium proposed by BI, all players not in $P(\mathbf{s}', \mathbf{s})$ are inactive at $p = v_j(\mathbf{s})$ for $j \in P(\mathbf{s}', \mathbf{s}) \subseteq W(\mathbf{s})$ (either they had quit before p or quit at p) and so can't win the auction when types are $\mathbf{s}'$. Since the winners at $\mathbf{s}'$ are not in $P(\mathbf{s}', \mathbf{s})$, the equilibrium can't be efficient and this contradicts Proposition 1 in BI, which asserts that under their assumptions, the proposed equilibrium is efficient. ■

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