# Preempting versus Postponing: the Stealing Game

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**Abstract**

We present an endogenous timing game of action commitment in which players can steal from each other parts of a homogeneous and perfectly divisible pie (market). We show how the incentives to preempt or to follow the rivals radically change with the number of players involved in the game. In the course of the analysis we also introduce, discuss and apply the concept of $\mathbf{p}_u$-dominance, a generalization of the risk-dominance criterion to games with more than two players.

*Keywords*: stealing, endogenous timing games, $\mathbf{p}_u$-dominance.

*JEL Classification*: C72, C73.

## 1 Introduction

Ann owns part of a pie. Bob owns the remaining part, so there is no free pie. Both Ann and Bob want more; actually they both want as much pie as possible. The problem is exacerbated by the fact that trading or bargaining mechanisms are illegal, not enforceable, or simply not in the interest of the parts. Indeed, the only way Ann and Bob can try to meet their objective is by "stealing" part of the other's pie. This may seem to be the description of a highly specific environment but it actually fits many situations. For example, certain species of animals (humans included) steal from each other food or territory.[1] Similarly, although

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[1] In biology such a behavior is called intraspecific kleptoparasitism (see for instance Yates and Broom, 2007).

in a less physical way, firms compete to steal from each other customers and political parties struggle to conquer the opponents' voters. There is another feature which is common to these examples, namely the existence of a positive relationship between the amount of the pie that a player can steal and the player's strength, as measured by his current holding of the pie. Therefore, a predator with a larger territory can better feed himself such as to be more fit when facing a rival; a firm with a larger market share is likely to have higher revenues and so it can launch more expensive advertising campaigns; a party or a candidate with many supporters is able to raise more funds.

To capture a stylized version of situations of this kind we introduce what we call the Stealing Game. The Stealing Game is a timing game in which a few agents can steal from each other parts of a homogeneous and perfectly divisible pie. Agents must decide when and who to rob with the goal to finish the game being the player that holds the largest share.[2] The key assumption of the game is that the larger the share a player holds, the larger the portion that he can steal.

We model the Stealing Game as an endogenous timing game of action commitment that follows the structure introduced by Hamilton and Slutsky (1990). The rules are as follows. The game spans over two periods and each player has only one chance to move and must decide when to use it. Within each period, choices are simultaneous but players who decide to move in $t = 2$ are fully informed about the identity and the actions of those who moved in $t = 1$. Such a structure provides a simple but fruitful framework to study the issue of endogenous timing and the value of action commitment. For instance, it has been successfully applied in a series of papers that study the robustness of commitment equilibria (Hurkens and Van Damme, 1996) and the endogenous emergence of a Stackelberg leader in the presence of cost asymmetries (Hurkens and Van Damme, 1999) and of price competition (Hurkens and Van Damme, 2004) in duopolies. With respect to these papers, our game presents two main differences. First, we allow the game to be played by more than two agents. In particular, we analytically study the cases with two, three and four players. Second, the applications of our game do not solely belong to the realm of industrial organization.

Our main goal is to solve for the optimal timing strategies of the agents. We want to

[2] Referring again to the previous examples, the end of the game can be the mating season for what concerns animals or the election date for what concerns political parties or candidates.

find out when the best moment for a player to behave aggressively and steal part of the pie owned by the rivals will be. Such a decision is affected by the existence of an intuitive trade-off between preempting or postponing one's move. A player who moves in $t = 1$ eliminates the possibility of being preempted but he is then forced to passively suffer the potential retaliation of those who waited. On the other hand, a player who waits until the second period can play the best response but faces the risk of being preempted and robbed in $t = 1$. And because of the rules of the game, when a player is robbed his market share goes down and so does the amount he can steal in the second period.

Focusing on the existence of strict Nash equilibria in pure strategies, we show that this trade-off has different solutions depending on the number of agents involved in the game. No player postpones his move when the Stealing Game is played among two or four agents. At the opposite end, when $N = 3$, the game displays different Pareto equivalent equilibria. In some of them all the players are active in $t = 1$. In the remaining ones, all the players wait and make their move in $t = 2$. We refine these equilibria by introducing the concept of $\mathbf{p}_u$-dominance. This is a criterion for equilibrium selection which generalizes the concept of risk-dominance (Harsanyi and Selten, 1988) to games with more than two players. As such, it is closely related to the concept of $\mathbf{p}$-dominance (Morris *et al.*, 1995; Kajii and Morris, 1997). An equilibrium is $\mathbf{p}_u$-dominant with $\mathbf{p}_u = (p_1, ..., p_N)$ if, for each player $i \in N$, his equilibrium action is the unique best response to the conjecture that assigns probability at least $p_i$ on each opponent playing his component of the equilibrium profile and lets each of them uniformly randomize, with the remaining total probability of $(1 - p_i)$, on the other available actions. In case of multiple equilibria, the $\mathbf{p}_u$-dominance criterion selects the equilibrium which is $\mathbf{p}_u$-dominant for the smallest $\mathbf{p}_u$, according to standard vector algebra. The idea is that this is the less risky equilibrium, i.e., the equilibrium upon which players' expectations should coordinate. When applied to the Stealing Game with 3 players, the $\mathbf{p}_u$-dominance criterion selects the equilibria in which all the players postpone their move to $t = 2$.

The Stealing Game belongs therefore to different categories of timing games commonly studied in economics. The cases with two or four players belong to the class of preemption games. These are games in which it is better to anticipate the rivals; famous examples are the already mentioned Stackelberg quantity game (Von Stackelberg, 1934) and the centipede game (Rosenthal, 1981). The case with three players is instead more similar to a war of attrition (Maynard Smith, 1974), a strategic situation in which preempting the others hurts. Here lies the peculiarity of the Stealing Game. In fact, a general characteristic of timing games is that optimal timing strategies depend on the payoff structure and not on the number of participants.[3] The Stealing Game provides instead an example of a game in which, for a given payoff structure, optimal timing strategies change as a function of the number of players.

The remainder of the paper is organized as follows. Section 2 introduces the Stealing Game and frames it as an endogenous timing game of action commitment. Subsections are then devoted to the analysis of the game for the cases with two players, three players and four players. In studying the three players' game, we also introduce and discuss the concept of $\mathbf{p}_u$-dominance. Section 3 concludes.

# 2 The Stealing Game

The Stealing Game is a game in which a finite number of agents compete for the possession of a perfectly divisible resource whose size is constant and normalized to 1. We indicate with $\pi_i^t \in [0,1]$ the share of the resource that agent $i \in N$ holds at time $t$. Time is discrete, $\pi_i^0 = \frac{1}{N}$ for any $i$ (symmetric initial condition) and $\sum_i \pi_i^t = 1$ holds at any $t$. The goal of the players is to be the largest shareholder at the end of the game. The only way in which a player can increase his holdings is by stealing part of the resource from someone else. We make four assumptions about this "stealing": 1) each player can steal only once over the entire game; 2) each player can steal from a single opponent of his choice; 3) the amount $y^t$ that a player can steal at time $t$ is proportional to his holdings according to the relation $y^t = \alpha\pi_i^{t-1}$ with $\alpha \in \left(0, \frac{1}{N-1}\right)$;[4] and 4) the stealing is monetarily costless. Assumption 1 makes the Stealing Game a timing game because players have to decide when to be active. Assumption 2 adds some strategic considerations as players must also decide who to rob. Assumption 3 captures the mechanism presented in the introduction, i.e., the fact that the larger a player, the more he can steal. As will become clear, nothing would change if $y^t \in \left[0, \alpha\pi_i^{t-1}\right]$ because $\hat{y}^t = \alpha\pi_i^{t-1}$ would anyway emerge as a dominant strategy.

[3] More recent literature about timing games has focused in generalizing former results (Bulow and Klemperer, 1999), in providing a unified framework to study preemption games and wars of attrition (Park and Smith, 2006) or in testing experimentally some of the theoretical results (Brunnermeier and Morgan, 2006).

[4] The upper bound of the interval for the parameter $\alpha$ ensures that every player always gets the amount he wants to steal, i.e., there cannot be cases of excess demand.

Assumption 4 simplifies the analysis and allows a fortiori results.

For what concerns the timing structure, we model the Stealing Game as a two-stages game of action commitment with endogenous timing (Hamilton and Slutsky, 1990). The rules are as follows:

*Period 1*: players simultaneously choose to act or to wait until the second period. If a player decides to act then he steals the amount $y^1 = \alpha\pi_i^0$ from an opponent of his choice.
*Period 2*: players who did not act in $t = 1$ are fully informed about the actions taken by all the opponents. Then each one of them simultaneously moves and steals the amount $y^2 = \alpha\pi_i^1$ from an opponent of his choice.
*Payoffs*: the player that at the end of $t = 2$ holds the largest share of the resource gets $u_i = 1$. The others get $u_j = 0$. If there is more than a market leader then the prize is equally shared among the winners.

During the course of the analysis, we will often refer to the game in its normal form defined as $G = (N, A_i, u_i)$ with $u_i(a_i, a_{-i})$ for any $i \in N$ and $(a_i, a_{-i}) \in A = \times_{j\in N} A_j$. A player's action space is given by $A_i = \left\{\left\{N_{-i}^1\right\} \cup \left\{N_{-i}^2\right\}\right\}$ such that $|A_i| = 2(N-1)$ for any $i \in N$. Using a formulation that is similar to the one introduced in Van Damme and Hurkens (1996), $a_i = x^1 \in N_{-i}^1$ indicates the action "steal the amount $\alpha\pi_i^0$ from opponent $x$ in $t = 1$ and wait in $t = 2$" while $a_i = x^2 \in N_{-i}^2$ indicates the action "wait in $t = 1$ then in $t = 2$ play the unique best-response if the opponents moved; otherwise steal the amount $\alpha\pi_i^1$ from opponent $x$". We are interested in an equilibrium analysis and, as a consequence, we do not consider those strategies that tell player $i$ to wait in $t = 1$ and then do not prescribe him to play in $t = 2$ the unique best response. Strategies of this kind are strictly dominated; they cannot be part of any subgame perfect equilibrium and they will not even appear in the normal form game $G$. Notice also that if the best response is not unique because, for instance, player $i$ is indifferent between robbing $j$ or $k$, then action $a_i = j^2 \in N_{-i}^2$ still prescribes player $i$ to rob $j$. We do not consider mixed strategies.

Payoffs take the following form:

$$u_i(a_i, a_{-i}) = \begin{cases} \dfrac{1}{\sum\limits_j \left(\mathbf{1}_{\{\pi_j^2 = \pi_i^2\}}\right)} & \text{if } \pi_i^2 \geq \pi_j^2 \text{ for any } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

where $\pi_i^1 = \pi_i^0 + y_i^1 - \sum_{j:x_j=i} y_j^1$ and $\pi_i^2 = \pi_i^1 + y_i^2 - \sum_{j:x_j=i} y_j^2$ with $\pi_i^0 = \frac{1}{N}$ for any $i$. In other words, at any $t \in \{1,2\}$ a player's holdings are the result of three components: the share he had in the previous period plus the amount of the resource he steals from an opponent minus the amount he is stolen from the other players. In what follows, we analytically study the Stealing Game for the cases with two, three and four players.

## 2.1 The game with two players

The analysis of the Stealing Game is trivial when $N = 2$. In fact each player has only one opponent that he can possibly rob such that he must only decide when to be active. More precisely, $G$ is a 2x2 game where, for any $i$ and $j \neq i$, $A_i = \{j^1, j^2\}$, $u_i(j^1, i^1) = u_i(j^2, i^2) = \frac{1}{2}$, $u_i(j^1, i^2) = 1$ and $u_i(j^2, i^1) = 0$. Players share the prize if they simultaneously rob each other while a player that successfully preempts the opponent results as the unique winner. To see why this is so, consider the profile $(j^1, i^2)$. At $t = 1$ we have $\pi_i^1 = \frac{1}{2}(1 + \alpha)$ and $\pi_j^1 = \frac{1}{2}(1 - \alpha)$. Therefore, in $t = 2$, player $j$ can only steal the amount $\alpha\left(\frac{1}{2}(1-\alpha)\right)$ such that final shares are $\pi_i^2 = \frac{1}{2}(1 + \alpha^2)$ and $\pi_j^2 = \frac{1}{2}(1 - \alpha^2)$. Given that $\pi_i^2 > \pi_j^2$ we have that $u_i(j^1, i^2) = 1$ and $u_j(j^1, i^2) = 0$.

**Proposition 1** *With $N = 2$ the profile $\hat{a} = \left(j^1\right)_i$ is the unique equilibrium of the Stealing game.*

**Proof.** Given that $u_i(j^1, i^1) > u_i(j^2, i^1)$ and $u_i(j^1, i^2) > u_i(j^2, i^2)$ it follows that, for both players, action $a_i = j^1$ strictly dominates the alternative action $a_i = j^2$. ■

## 2.2 The game with three players

The Stealing Game among 3 players is characterized by the existence of numerous Nash equilibria. The analysis of the game in normal form (see the appendix) shows that these are actually 16 but only 4 of them are strict as well as subgame perfect. We restrict our attention to these 4 equilibria: the two "circles" (i.e., $A$ robs $B$, $B$ robs $C$, $C$ robs $A$ and $A$ robs $C$, $C$ robs $B$, $B$ robs $A$) with all the three players being active in $t = 1$ and the two "circles" with all the three players postponing their move to $t = 2$.

**Proposition 2** *With $N = 3$ the Stealing game has four strict equilibria:*

- *the two profiles $\hat{a} = \left(j^1\right)_i$ with $j^1 \in N^1_{-i}$ and such that $u_i = \frac{1}{3}$ for any $i$;*
- *the two profiles $\hat{a} = \left(j^2\right)_i$ with $j^2 \in N^2_{-i}$ and such that $u_i = \frac{1}{3}$ for any $i$.*

**Proof.** We check for profitable deviations over the two dimensions of the action space. First, consider the situation in which, from any of the four profiles $\hat{a} = (j^t)_i$, player $i$ robs $k$ instead of $j$: this would imply that $j$ is not robbed by anyone such that $\pi^2_j = \frac{1}{3}(1+\alpha) > \frac{1}{3} = \pi^2_i$ which implies $u_i = 0$. Then, consider possible deviations over the timing dimension. Start from $\hat{a} = \left(j^1\right)_i$ and let player $i$ postpone his move to $t = 2$ such that $\pi^1_i = \frac{1}{3}(1-\alpha)$ and $\pi^2_i = \frac{1}{3}(1-\alpha^2)$. It follows that $u_i = 0$ as the condition $\sum_i \pi^2_i = 1$ implies that there exists an agent $j$ such that $\pi^2_j > \pi^2_i$. Then consider $\hat{a} = \left(j^2\right)_i$. If player $i$ deviates to $a_i = j^1$ then $\pi^1_i = \frac{1}{3}(1+\alpha)$; in $t = 2$ player $k \neq i, j$ will then best respond by robbing $i$ such that $\pi^2_k > \pi^2_i$ and therefore $u_i = 0$. ■

With respect to the case with $N = 2$, the interesting feature of the three players' game is that now there exist equilibria in which all the players postpone their move. The reason is that the possibility of best responding in $t = 2$, even though potentially risky, is now worthwhile. To understand how the trade-off works in this case, consider the hypothetical situation in which player $A$ commits to be active in $t = 1$ while $B$ and $C$ wait. Player $A$ can rob either $B$ or $C$. Assume $a_A = B^1$ such that $\pi^1_A = \frac{1}{3}(1+\alpha)$, $\pi^1_B = \frac{1}{3}(1-\alpha)$ and $\pi^1_C = \frac{1}{3}$. In $t = 2$ player $B$ is indifferent about who to rob as he has been weakened by the stealing of $A$ and cannot catch up with his initial share: $\pi^2_B = \frac{1}{3}(1-\alpha^2)$ such that $u_B = 0$ no matter if $a_B = A^2$ or $a_B = C^2$. Player $C$ is instead sure to win the game as he can effectively best respond to what happened in $t = 1$ by robbing $A$. In fact, even assuming $a_B = C^2$, we would have $\pi^2_A = \frac{1}{3}$, $\pi^2_B = \frac{1}{3}(1-\alpha^2)$ and $\pi^2_C = \frac{1}{3}(1+\alpha^2)$ such that $u_A = u_B = 0$ and $u_C = 1$. The outcomes of agents $C$ and $B$ exemplify the advantages/disadvantages of postponing one's move.

### 2.2.1 The concept of $\mathbf{p}_u$-dominance

In order to discriminate among the four strict equilibria of the Stealing Game when $N = 3$ we introduce the concept of $\mathbf{p}_u$-dominance. $\mathbf{p}_u$-dominance is a generalization of risk-dominance (Harsanyi and Selten, 1988) to games with more than two players. It is inspired by, and closely related to, the concept of $\mathbf{p}$-dominance (Morris *et al.*, 1995, Kajii and Morris, 1997).

Indeed all these three criteria tackle the issue of equilibrium selection sharing the same intuition: if agents do not know which equilibrium will arise, they will compute the risk involved in playing each of these equilibria and they will coordinate expectations on the less risky one. We start by formally defining $\mathbf{p}_u$-dominance and then we will relate this concept to risk-dominance and **p**-dominance.

As a preliminary step, we define the vector $\mathbf{p}_u = \left(p_u^1, ..., p_u^N\right)$ that, given $N$ agents and $K$ alternative events $E = \{e_1, ..., e_K\}$, indicates a collection of $N$ probabilities distributions such that each agent $i \in N$ believes a certain event $e^* \in E$ will occur with probability $p_u^i$ while each of the remaining events $e_k \neq e^*$ will occur with probability $\frac{(1-p_u^i)}{K-1}$. Using a similar notation, an equilibrium $(\hat{a}_i, \hat{a}_{-i})$ is said to be $\mathbf{p}_u$-dominant for $\mathbf{p}_u = \left(p_u^1, ..., p_u^N\right)$ if, for any player $i \in N$ and any $j \in N_{-i}$, action $\hat{a}_i$ is the unique best response to any probability distribution $\lambda \in \Delta\left(A_{-i}\right)$ that assigns at least probability $p_u^i$ to the event of $j$ playing his equilibrium action $\hat{a}_j$ and lets $j$ uniformly randomize on his alternative actions with the remaining probability. In other words, $\mathbf{p}_u$-dominance mimics the process according to which an agent evaluates the likelihood of an equilibrium by focusing on the probability of the actions that sustain it while assuming a simplifying uniform distribution for what concerns the alternative actions. As such, the concept of $\mathbf{p}_u$-dominance can be rationalized and justified on the basis of behavioral arguments like salience, limited cognitive abilities and bounded rationality of the players.

**Definition 1** *Action profile* $(\hat{a}_1, ..., \hat{a}_N)$ *is a* $\mathbf{p}_u$*-dominant equilibrium with* $\mathbf{p}_u = \left(p_u^1, ..., p_u^N\right)$ *if for all* $i \in N$, $a_i \neq \hat{a}_i$ *and all* $\lambda \in \Delta\left(A_{-i}\right)$ *with* $\lambda\left(\hat{a}_j\right) \geq p_u^i$ *and* $\lambda\left(a_j\right) = \frac{(1-\lambda(\hat{a}_j))}{|A_j|-1}$ *for all* $a_j \neq \hat{a}_j$ *and* $j \in N_{-i}$,

$$\sum_{a_{-i} \in A_{-i}} \lambda\left(a_{-i}\right) u_i\left(\hat{a}_i, a_{-i}\right) \geq \sum_{a_{-i} \in A_{-i}} \lambda\left(a_{-i}\right) u_i\left(a_i, a_{-i}\right).$$

Some standard concepts of game theory can be formulated in terms of $\mathbf{p}_u$-dominance. For instance, an equilibrium in dominant strategies is a $\mathbf{p}_u$-dominant equilibrium with $\mathbf{p}_u = (0, ..., 0)$ while every Nash equilibrium is a $\mathbf{p}_u$-dominant equilibrium with $\mathbf{p}_u = (1, ..., 1)$. Notice also that if the profile $(\hat{a}_1, ..., \hat{a}_N)$ is a $\mathbf{p}_u$-dominant equilibrium then it is also a $\mathbf{p}_u'$-dominant equilibrium for any $\mathbf{p}_u' \geq \mathbf{p}_u$ (using the standard vector ordering). What characterizes an equilibrium is the smallest $\mathbf{p}_u$ for which the equilibrium is $\mathbf{p}_u$-dominant. This vector, which we indicate with $\mathbf{p}_u^*$, reports the minimum level of the beliefs $\lambda(\hat{a}_j)$ for which the equilibrium action under scrutiny dominates the alternatives. As such, $\mathbf{p}_u^*$ provides a measure of the riskiness of playing a certain equilibrium action as well as a tool to identify the equilibrium upon which players' expectations should coordinate. In particular, in the same spirit of what is suggested by Morris *et al.* (1995) for what concerns $\mathbf{p}$-dominance, the $\mathbf{p}_u$-dominance criterion selects the equilibrium characterized by the smallest $\mathbf{p}_u^*$. Notice that such an equilibrium may not exist as there may easily be situations in which it is not possible to unambiguously order the $\mathbf{p}_u^*$ vectors associated with the various equilibria. More precisely, in any generic game, the $\mathbf{p}_u$-dominance criterion selects at most one equilibrium while it selects exactly one equilibrium in symmetric games.

We now relate $\mathbf{p}_u$-dominance with risk-dominance (Harsanyi and Selten, 1988) and $\mathbf{p}$-dominance (Morris *et al.*, 1995; Kajii and Morris, 1997). In 2x2 coordination games an equilibrium $(\hat{a}_i, \hat{a}_j)$ is risk-dominant if it is the equilibrium characterized by the highest product of the deviation losses. The following lemma shows that such a requirement is implied by the condition that identifies $(\hat{a}_i, \hat{a}_j)$ as the $\mathbf{p}_u$-dominant equilibrium.

**Lemma 1** *In any 2x2 coordination game, if the* $\mathbf{p}_u$*-dominance criterion selects an equilibrium then this is the risk-dominant equilibrium.*

**Proof.** Consider the coordination game

| | $T$ | $B$ |
|---|---|---|
| $T$ | $a, e$ | $b, f$ |
| $B$ | $c, g$ | $d, h$ |

with $a > c$, $d > b$, $e > f$ and $h > g$.

The equilibrium $(T, T)$ is $\mathbf{p}_u$-dominant with $\mathbf{p}_u^* = \left(\frac{d-b}{a-c+d-b}, \frac{h-g}{e-f+h-g}\right)$ while $(B, B)$ is $\mathbf{p}_u$-dominant with $\mathbf{p}_u^* = \left(\frac{a-c}{a-c+d-b}, \frac{e-f}{e-f+h-g}\right)$. Therefore, $(T, T)$ emerges as the $\mathbf{p}_u$-dominant equilibrium if $\frac{d-b}{a-c+d-b} < \frac{a-c}{a-c+d-b}$ and $\frac{h-g}{e-f+h-g} < \frac{e-f}{e-f+h-g}$, i.e., if $a-c > d-b$ and $e-f > h-g$. But if these two conditions are valid then the condition $(a-c)(e-f) > (d-b)(h-g)$ also holds, i.e., $(T, T)$ is the risk-dominant equilibrium because it is characterized by the highest product of the deviation losses. The proof is analogous if $(B, B)$ emerges as the $\mathbf{p}_u$-dominant equilibrium. ■

As defined in Morris *et al.* (1995) for the two player case and extended by Kajii and Morris (1997) for what concerns the $N > 2$ case, an equilibrium $(\hat{a}_i, \hat{a}_{-i})$ is $\mathbf{p}$-dominant with $\mathbf{p} = (p_1, ..., p_N)$ if, for any agent $i$, action $\hat{a}_i$ is the unique best response to any probability distribution $\lambda \in \Delta(A_{-i})$ such that $\lambda(\hat{a}_j) \geq p_i$ for any $j \neq i$. In other words, action $\hat{a}_i$ is $\mathbf{p}$-dominant if it maximizes player $i$'s expected payoff whenever $i$ thinks that each one of the other players will play with probability not smaller than $p_i$ his component of the equilibrium profile. The difference with respect to $\mathbf{p}_u$-dominance is that $\mathbf{p}$-dominance does not require the remaining probability $(1 - p_i)$ to follow any particular distribution over the alternative actions $a_j \neq \hat{a}_j$.

**Lemma 2** *Any* $\mathbf{p}_u$*-dominant equilibrium is a* $\mathbf{p}$*-dominant equilibrium with* $\mathbf{p} = \mathbf{p}_u$*.*

**Proof.** A $\mathbf{p}$-dominant equilibrium with $\mathbf{p} = \mathbf{p}_u = \left(p_u^1, ..., p_u^N\right)$ is such that, for any $i$ and any $j$, any equilibrium action is a best response to the conjecture according to which $\lambda(\hat{a}_j) \geq p_u^i$ while the remaining probability can follow any distribution on any $a_j \neq \hat{a}_j$. Therefore, this conjecture comprises the case such that $\lambda(\hat{a}_j) \geq p_u^i$ and $\lambda(a_j) = \frac{(1-\lambda(\hat{a}_j))}{|A_j|-1}$ for any $a_j \neq \hat{a}_j$, which is the distribution that defines $\mathbf{p}_u$-dominance. ■

The concept of $\mathbf{p}_u$-dominance is thus less general than $\mathbf{p}$-dominance. Still it has the advantage of being much more easily computable, especially in games that have many players or actions and in which payoffs do not present much variability.

### 2.2.2 Equilibrium selection in the 3-players Stealing Game

We apply the $\mathbf{p}_u$-dominance criterion to refine the strict equilibria of the Stealing Game with $N = 3$ (see Proposition 2). Note that all four of these equilibria are Pareto equivalent such that there is no conflict between payoff realization and risk considerations. As a consequence, players should indeed coordinate on the less risky equilibrium, and $\mathbf{p}_u$-dominance is an appropriate criterion for identifying this equilibrium.

In what follows, we call the 3 players $A$, $B$ and $C$ and we refer to the game in normal form as it appears in the appendix. Given any strict equilibrium $(\hat{a}_A, \hat{a}_B, \hat{a}_C)$, we compute for each player $E_{p_u}(a_i)$, i.e., the expected payoff of each action $a_i \in A_i$ under the conjecture that each opponent plays action $\hat{a}_j$ with probability $p_u$ and each of his alternative actions with probability $\frac{1-p_u}{3}$. Then, by imposing the conditions $E_{p_u}(\hat{a}_i) > E_{p_u}(a_i)$ for any $a_i \neq \hat{a}_i$, we find the components of the vector $\mathbf{p}_u^*$ for which the equilibrium $(\hat{a}_A, \hat{a}_B, \hat{a}_C)$ is $\mathbf{p}_u$-dominant. Finally we will select the equilibrium characterized by the smallest $\mathbf{p}_u^*$.

For instance, starting from the equilibrium $\left(B^1, C^1, A^1\right)$ and focusing without loss of generality on player $A$, we have the following: $E_{p_u}(B^1) = -\frac{5}{9}p_u^2 + \frac{7}{9}p_u + \frac{1}{9}$, $E_{p_u}(C^1) = -\frac{11}{27}p_u^2 + \frac{4}{27}p_u + \frac{7}{27}$ and $E_{p_u}(B^2) = E_{p_u}(C^2) = -\frac{17}{27}p_u^2 + \frac{7}{27}p_u + \frac{10}{27}$. Therefore, the equilibrium action $B^1$ dominates action $C^1$ for any $p_u \geq 0.25$ and actions $B^2$ and $C^2$ for any $p_u \geq \frac{3}{2}\sqrt{7} - \frac{7}{2} \cong 0.47$. Given that similar relations also hold for players $B$ and $C$, the equilibrium $\left(B^1, C^1, A^1\right)$ is $\mathbf{p}_u$-dominant with $\mathbf{p}_u^* = \left(\frac{3}{2}\sqrt{7} - \frac{7}{2}, \frac{3}{2}\sqrt{7} - \frac{7}{2}, \frac{3}{2}\sqrt{7} - \frac{7}{2}\right)$. Not surprisingly, analogous computations show that also the other preempting equilibrium $\left(C^1, A^1, B^1\right)$ is $\mathbf{p}_u$-dominant for the same $\mathbf{p}_u^*$. Now consider one of the postponing equilibria, say $\left(B^2, C^2, A^2\right)$. Focusing again on player $A$, we have that $E_{p_u}(B^1) = -\frac{11}{27}p_u^2 + \frac{4}{27}p_u + \frac{7}{27}$, $E_{p_u}(C^1) = \frac{1}{27}p_u^2 - \frac{11}{27}p_u + \frac{10}{27}$, $E_{p_u}(B^2) = -\frac{1}{3}p_u^2 + \frac{1}{3}p_u + \frac{1}{3}$ and $E_{p_u}(C^2) = -\frac{17}{27}p_u^2 + \frac{7}{27}p_u + \frac{10}{27}$ such that the equilibrium action $B^2$ dominates $B^1$ for any $p_u \geq 0$, $C^1$ for any $p_u \geq 1 - \frac{3}{10}\sqrt{10} \cong 0.05$ and $C^2$ for any $p_u = 0.25$. Therefore, the equilibrium $\left(B^2, C^2, A^2\right)$ is $\mathbf{p}_u$-dominant with $\mathbf{p}_u^* = (0.25, 0.25, 0.25)$. The same $\mathbf{p}_u^*$ characterizes the equilibrium $\left(C^2, A^2, B^2\right)$.

**Proposition 3** *The equilibria in which all the players postpone their move, i.e.,* $\hat{a} = \left(j^2\right)_i$ *with* $j^2 \in N_{-i}^2$ *and such that* $u_i = \frac{1}{3}$ *for any* $i$*, are the* $\mathbf{p}_u$*-dominant equilibria of the Stealing Game with* $N = 3$*.*

**Proof.** The equilibria $\hat{a} = \left(j^1\right)_i$ with $j^1 \in N_{-i}^1$ and such that $u_i = \frac{1}{3}$ for any $i$ are $\mathbf{p}_u$-dominant with $\mathbf{p}_u^* = \left(\frac{3}{2}\sqrt{7} - \frac{7}{2}, \frac{3}{2}\sqrt{7} - \frac{7}{2}, \frac{3}{2}\sqrt{7} - \frac{7}{2}\right)$. The equilibria $\hat{a} = \left(j^2\right)_i$ with $j^2 \in N_{-i}^2$ and such that $u_i = \frac{1}{3}$ for any $i$ are $\mathbf{p}_u$-dominant with $\mathbf{p}_u^* = (0.25, 0.25, 0.25)$. Given that $0.25 < \frac{3}{2}\sqrt{7} - \frac{7}{2}$, the two equilibria in which all the players postpone their move are the equilibria selected by the $\mathbf{p}_u$-dominance criterion. ■

In other words, despite the possibility to be preempted, it is less risky to wait until $t = 2$ rather than move in $t = 1$. To have an intuition for this result, consider the case in which player $i \in \{A, B, C\}$ finds himself in the situation of being the only player who moved in $t = 1$. The payoff matrices in the appendix show that, if this occurs, player $i$ has no chance to win the game. At the opposite end, if $i$ happens to be the only agent who postpones his move then there is still a positive probability, associated with the event of the two other players robbing each other in $t = 1$, that $i$ wins the game. To sum up, in the three player Stealing game, no player wants to break the initial symmetric situation. This result is reminiscent of the analysis of so-called truels (gun duels among three players) which shows that, under certain conditions, the best strategy that a player can adopt is to postpone his shot or even to shoot in the air rather than against an opponent (see Kilgour, 1972 and Kilgour and Brams, 1997).

## 2.3 The game with four players

With 4 players the Stealing Game's unique strict equilibria are given by the action profiles in which all the players use their attack in $t = 1$ and share the prize. There are no equilibria in which players postpone their moves.

**Proposition 4** *With $N = 4$ the nine profiles $\hat{a} = \left(j^1\right)_i$ with $j^1 \in N^1_{-i}$ and such that $u_i = \frac{1}{4}$ for any $i$ are the unique strict equilibria of the game.*

**Proof.** Consider any of the profiles $\hat{a} = \left(j^1\right)_i$. If player $i$ robs a different opponent in $t = 1$ then $u_i = 0$ as there exists a player $j \neq i$ with $\pi^2_j = \frac{1}{4}(1+\alpha) > \frac{1}{4} = \pi^2_i$. Similarly if player $i$ postpones his attack to $t = 2$ then $\pi^2_i = \frac{1}{4}(1-\alpha^2)$ such that $u_i = 0$ because there exists a $j \neq i$ such that $\pi^2_j > \pi^2_i$. Now consider profiles of the kind $a = \left(j^2\right)_i$ with $j^2 \in N^2_{-i}$ and such that $u_i = \frac{1}{4}$ for any $i$ and assume that player $i$ deviates and robs agent $j$ in $t = 1$. In the subgame that takes place in $t = 2$ (see below) the two players $k \neq i, j$ will attack each other such that $u_i = 1$. Player $i$'s deviation is profitable and there are no strict equilibria in which all the players postpone their move. ■

In order to better understand why there cannot be "postponing" equilibria, start, without loss of generality, from the following candidate profile: all the players wait in $t = 1$ and then in $t = 2$ player $A$ robs $B$, $B$ robs $C$, $C$ robs $D$ and $D$ robs $A$ such that $\pi^2_i = u_i = \frac{1}{4}$ for any $i$. Now let agent $A$ deviate and rob an opponent (say $B$) in the first period. In $t = 2$ player $B$ cannot catch up with his initial share and $u_B = 0$ no matter what happens. Player $B$ is then indifferent as to who to rob such that his action cannot be predicted by the other players. In addition, having been weakened, $B$ can only steal a smaller amount of the good and his action cannot influence who will be the largest shareholder. Focusing on the players who still have a chance to win the game (agents $C$ and $D$), we can thus avoid to model $B$'s move. Moreover, for both $C$ and $D$ to rob $B$ is a dominated action. Therefore, we can simplify the subgame that follows $A$'s deviation by considering only the undominated actions of $C$ and $D$ (payoffs appear in the order $u_A, u_B, u_C, u_D$).

$t = 1$ $\qquad\qquad\qquad a_A = B^1$

$D$

| $t = 2$ | $C$ | | $A^2$ | $C^2$ |
|---|---|---|---|---|
| | | $A^2$ | $0, 0, \frac{1}{2}, \frac{1}{2}$ | $0, 0, 0, 1$ |
| | | $D^2$ | $0, 0, 1, 0$ | $1, 0, 0, 0$ |

The subgame that takes place at $t = 2$ has three Nash equilibria. Still, given that for both players $C$ and $D$ to rob $A$ is a weakly dominated strategy, the equilibrium in which $C$ and $D$ attack each other clearly emerges as the unique admissible, perfect (Selten, 1975), proper (Myerson, 1978) and $\mathbf{p}_u$-dominant equilibrium. In this equilibrium player $A$ wins the game, his deviation is profitable and the profiles in which all the players postpone their move cannot be equilibria of the Stealing Game with four players. The difference with the three players' case can be easily explained. In the three players' game there is only one agent who can take advantage of the situation in which a unique player is active in $t = 1$. With four players two are the agents who can exploit such a situation and a free riding problem arises. In the above example in fact players $C$ and $D$ steal from each other in the hope that the other robs $A$.

# 3 Summary and Discussion

The paper introduced what we called the Stealing Game. This is a game in which players must decide when to steal from each other parts of a homogeneous good, the amount that a player can steal is proportional to his actual holdings and the goal is to finish the game being the agent who owns the largest share. As such, we claimed that the Stealing Game captures a stylized version of strategic interactions that often occur in biology, business and politics. We framed the game as a two-period endogenous timing game of action commitment and we focused on solving for the optimal timing strategies of the players. In particular, we investigated how the choice to preempt or to follow the rivals changes in response to the number of players involved in the game. The main result is that agents always want to move in the first period when the game is played by two or four players while, in the three players' case, players prefer to postpone their move to the second period. This result has some obvious limitations (it would not hold under different payoffs structures or stealing technologies) but it nevertheless generalizes in some respects. For instance, it remains valid in the case in which there are more than just two periods where players can move. In particular agents will always move as soon as possible in the two and four players' game while they will wait until the final period in the situation with three players. And if a final period does not exist, or the players are not aware of it, then no agent involved in the three players game will be willing to behave aggressively and break the initial symmetric situation. Notice that we assumed no kind of monetary cost associated with the decision to rob an opponent. Therefore the above result holds a fortiori in the more realistic situation in which such a cost exists. Interpreting the amount of stealing as a specific form of competition, interactions among three agents may thus display a less competitive behavior with respect to duopolies. More in general, the Stealing Game provides an example of a timing game in which, for a given payoff structure, optimal timing strategies change according to the number of participants. This is an interesting aspect of timing games that has been so far neglected and that possibly requires further and more general research.

# 4 Appendix

The Stealing Game with $N = 3$ in normal form. Player $A$ chooses the matrix, player $B$ chooses the row, player $C$ chooses the column. For any player $i \in \{A, B, C\}$ and $j \neq i$, action $j^1$ indicates "steal the amount $\alpha\pi_i^0$ from opponent $j$ in $t = 1$ and wait in $t = 2$" while $j^2$ indicates the strategy "wait in $t = 1$ then in $t = 2$ play the unique best-response if the opponents moved; otherwise steal the amount $\alpha\pi_i^1$ from opponent $j$". In each cell payoffs appear in the order $u_A, u_B, u_C$.

$A = B^1$

$C$

| $B$ | | $A^1$ | $B^1$ | $A^2$ | $B^2$ |
|---|---|---|---|---|---|
| | $A^1$ | $0,0,1$ | $0,0,1$ | $0,0,1$ | $0,0,1$ |
| | $C^1$ | $\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | $1,0,0$ | $1,0,0$ | $1,0,0$ |
| | $A^2$ | $0,0,1$ | $0,0,1$ | $0,0,1$ | $0,0,1$ |
| | $C^2$ | $0,0,1$ | $1,0,0$ | $0,0,1$ | $0,0,1$ |

$A = C^1$

$C$

| $B$ | | $A^1$ | $B^1$ | $A^2$ | $B^2$ |
|---|---|---|---|---|---|
| | $A^1$ | $0,1,0$ | $\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | $0,1,0$ | $0,1,0$ |
| | $C^1$ | $0,1,0$ | $1,0,0$ | $0,1,0$ | $1,0,0$ |
| | $A^2$ | $0,1,0$ | $1,0,0$ | $0,1,0$ | $0,1,0$ |
| | $C^2$ | $0,1,0$ | $1,0,0$ | $0,1,0$ | $0,1,0$ |

$A = B^2$

$C$

| $B$ | | $A^1$ | $B^1$ | $A^2$ | $B^2$ |
|---|---|---|---|---|---|
| | $A^1$ | $0,0,1$ | $0,0,1$ | $0,0,1$ | $0,0,1$ |
| | $C^1$ | $0,1,0$ | $1,0,0$ | $1,0,0$ | $1,0,0$ |
| | $A^2$ | $0,1,0$ | $1,0,0$ | $0,0,1$ | $0,0,1$ |
| | $C^2$ | $0,1,0$ | $1,0,0$ | $\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | $1,0,0$ |

$A = C^2$

$C$

| $B$ | | $A^1$ | $B^1$ | $A^2$ | $B^2$ |
|---|---|---|---|---|---|
| | $A^1$ | $0,1,0$ | $0,0,1$ | $0,0,1$ | $0,0,1$ |
| | $C^1$ | $0,1,0$ | $1,0,0$ | $1,0,0$ | $1,0,0$ |
| | $A^2$ | $0,1,0$ | $1,0,0$ | $0,1,0$ | $\frac{1}{3},\frac{1}{3},\frac{1}{3}$ |
| | $C^2$ | $0,1,0$ | $1,0,0$ | $0,1,0$ | $1,0,0$ |

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