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A CHARACTERIZATION OF INEFFICIENCY IN STOCHASTIC OVERLAPPING GENERATIONS ECONOMIES

GAETANO BLOISE * AND FILIPPO L. CALCIANO **

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Abstract. In this paper, we provide a characterization of *interim* inefficiency in stochastic economies of overlapping generations under possibly sequentially incomplete markets. With respect to the established body of results in the literature, we remove the hypothesis of two-period horizons, by considering longer, though uniformly bounded, horizons for generations. The characterization exploits a suitably Modified Cass Criterion, grounded on the long-rung behavior of compounded safe interest rates and independent of the length of horizons of generations. Thus, the hypothesis of two-period horizons is purely heuristic in establishing a criterion for inefficiency. In addition, for sequentially incomplete markets, we adopt a suitable notion of *unambiguous* inefficiency, separating the inefficient intertemporal allocation of resources from incomplete risk-sharing. Unambiguous inefficiency reduces to inefficiency when markets are sequentially complete.

Keywords. Stochastic overlapping generations economies; inefficiency; competitive prices; Cass Criterion; social security; incomplete markets.

JEL Classification Numbers. D52, D61.

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1. Introduction

It is well established that competitive equilibrium might not achieve an optimal allocation of resources in economies of overlapping generations (Samuelson [19]). This inefficiency might occur even though competitive markets operate perfectly, as in the Arrow-Debreu abstraction. It is commonly understood as a lack of transversality condition, caused by the fact that generations act over short horizons compared with the infinite horizon of the economy, so that benefits from trade remain unexploited *at infinitum*. In order for this sort of market failure to justify active policy intervention (as social security), a suitable empirical criterion is needed for identifying unoptimality in the intertemporal allocation of resources, a criterion possibly grounded only on aggregate economic magnitudes. Does the mere observation of competitive prices fully reveal inefficiency?

This issue is of long tradition in general equilibrium. Intuitively, inefficiency requires that real interest rate (net of growth) be sufficiently negative in the long-run, an admittedly vague and unprecise statement when interest rate fluctuates. Inspired by the original studies of Cass [10] and Benveniste [6, 7] on capital theory, Balasko and Shell [3] and Okuno and Zilcha [18] initially proposed a precise necessary and sufficient Cass Criterion for inefficiency: the infinite sum of the reciprocals of (the norm of) present value prices of commodities over periods of trade converges. Their work was extended by Geanakoplos and Polemarchakis [15] to growing economies; by Aiyagari and Peled [1] (and, more recently, by Barbie and Kaul [5]) to recursive equilibria of economies with uncertainty; by Chattopadhyay and Gottardi [12] to economies with uncertainty; by Burke [9] and Molina-Abraldes and Pintos-Clapés [17] to economies with heterogeneous horizons for generations without uncertainty. Apart from these latter contributions, all of them adopt the simplifying hypothesis of two-period horizons for generations. In addition, with the only exception of Chattopadhyay and Gottardi [12] in part of their analysis, all of them assume sequentially complete markets. Optimality under sequentially incomplete markets was also studied by Chattopadhyay [11] and Henriksen and Spear [16].

In this note, we propose an extension of the analysis under uncertainty. In particular, differently from (most of) the literature, we allow for arbitrarily long, though uniformly bounded, horizons for generations and for possibly sequentially incomplete markets. Though these two extensions could be treated be means of a unified approach, it is worth separating the former from the latter, both for a more transparent presentation and because the former might be of interest independently of the latter. We first consider the case of sequentially complete markets.

For economies with two-period horizons for generations, Chattopadhyay and Gottardi [12] provides a necessary and sufficient condition for inefficiency, consisting in a sort of Weighted Cass Criterion: along any path, the weighted infinite sum of the reciprocals of prices converges. In addition, for recursive equilibria, elaborating on the criterion of Chattopadhyay and Gottardi [12], a recent work by Barbie and Kaul [5] presents an alternative characterization of inefficiency by means of a sort of First Order Condition, requiring the existence of bounded transfers whose value, in every state of nature, is less than their expected value in the following period.

In stochastic economies with two-period horizons for generations, when equilibrium allocation is inefficient, without loss of generality, a welfare improvement obtains by transferring resources from young individuals to old individuals. In economies with longer horizons for generations, instead, a welfare improvement might require a larger class of transfers. Thus, the extension to longer horizons of the characterization in the literature is not straightforward.

In order to carry out our extension and to provide a simpler Cass Criterion, we slightly modify the notion of inefficiency. An allocation is *robustly* inefficient if a welfare improvement exists even though a constant, however small, share of transferred resources is to be destroyed. This stronger notion of inefficiency permits to avoid the common assumption of a lower bound on the curvature of indifference curves, which is ecumenically adopted in the literature (for instance, Chattopadhyay and Gottardi [12, Definition 4]). In addition, it allows for more direct and straight arguments in the proofs. An interesting interpretation of robust inefficiency is that an hypothetical planner might achieve a welfare improvement even holding an unprecise knowledge of the (bounded) curvatures of indifference curves, that is, independently of (bounded) second order effects.

We show that, under acceptably restrictive assumptions, inefficiency of competitive equilibrium is equivalent to a suitably Modified Cass Criterion. The specific nature of welfare improving transfers is irrelevant, as only their long-run properties matter. More precisely, inefficiency is equivalent to the existence of *bounded* positive hypothetical transfers of commodities, $\{e_t\}$, satisfying, for some $1 > \rho > 0$,

$$\rho \mathbb{E}_t p_{t+1} e_{t+1} \geq p_t e_t,$$

where $\{p_t\}$ is the process of Arrow-Debreu prices, or contingent claim prices. These transfers are hypothetical as they need not coincide with those producing the welfare improvement, which might be largely more dispersed. Importantly, inefficiency can be equivalently characterized by means of the positive linear operator defined by

$$T(e)_t = \frac{1}{p_t} \mathbb{E}_t p_{t+1} e_{t+1}.$$

It exactly corresponds to the existence of a real eigenvalue, of such a linear operator, larger than the unity, with associated *bounded* eigenvector. This is reminiscence of the Dominant Root Characterization (Aiyagari and Peled [1]) for stochastic recursive equilibria (Perron-Frobenius Theorem). Hence, our characterization is tight, as it exploits the same criterion for possibly non-recursive equilibria.

Beyond generality, our Modified Cass Criterion bears two major advantages on the established criteria in the literature. First, on a theoretical ground, it is independent of the length of horizons of generations, thus showing that the hypothesis of two-period horizons is purely heuristic in identifying a criterion for inefficiency. Second, for practical purposes, comparing with the Weighted Cass Criterion, our Modified Cass Criterion appears of more direct application in empirical work, both because of its simpler formulation and because it allows for exploiting time series of prices of any time frequency (see lemmas 6 and 9 in appendix B).

The characterization is further extended to sequentially incomplete markets. When markets are incomplete, any equilibrium allocation is typically inefficient, as the lack of financial instruments inhibits complete insurance. Thus, a preliminary issue consists in establishing what sort, or which part, of inefficiency is to be revealed by observable asset prices, through a (Modified) Cass Criterion. The issue of separating different forms of inefficiency is conceptually deep and, perhaps, any of such separations is disputable and artificial, as those proposed by Chattopahyay and Gottardi [12] and Chattopadhyay [11]. We adopt a notion of unambiguous inefficiency, corresponding to the occurrence of a welfare improvement for all subjective evaluations of risk by individuals that are consistent with observable asset prices. Taking into account the multiplicity of implicit Arrow-Debreu prices, an identical Modified Cass Criterion characterizes inefficiency under sequentially complete and sequentially incomplete asset markets. More precisely, unambiguous inefficiency is equivalent to the existence of *bounded* positive hypothetical transfers of commodities, $\{e_t\}$, satisfying, for some $1 > \rho > 0$,

$$\rho \mathbb{E}_t p_{t+1} e_{t+1} \geq p_t e_t,$$

at *every* process of implicit Arrow-Debreu prices, $\{p_t\}$, consistent with the absence of arbitrage opportunities. Clearly, when markets are sequentially complete, such a process is uniquely determinate, up to one innocuous degree of multiplicity, so that the Modified Cass Criterion reduces to that previously established under sequentially complete markets.

The paper is organized as follows. In section 2, we introduce the hypotheses on the economy. In section 3, we provide the characterization under sequentially complete markets. In section 4, we extend the analysis to sequentially incomplete markets. Finally, appendix A presents a digression on budget constraints under sequentially incomplete markets, whereas appendix B collects all proofs.

2. Fundamentals

2.1. **Time and uncertainty.** Time and uncertainty are represented by an event-tree $\mathcal{S}$, a countably infinite set, endowed with ordering $\succeq$. For a date-event σ in $\mathcal{S}$, $t(\sigma)$ in $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$ denotes its date and

$$\sigma_+ = \{\tau \in \mathcal{S}(\sigma) : t(\tau) = t(\sigma) + 1\}$$

is the non-empty finite set of all immediate direct successors, where

$$\mathcal{S}(\sigma) = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}.$$

The initial date-event is ϕ in $\mathcal{S}$, with $t(\phi) = 0$, that is, $\sigma \succeq \phi$ for every σ in $\mathcal{S}$. This construction is canonical (Debreu [14, Chapter 7]).

2.2. **Vector space notation and terminology.** As far as notation and terminology for vector spaces are concerned, we shall basically follow Aliprantis and Border [2, Chapters 5-8]. Consider the vector space of all real maps on $\mathcal{S}$, $\mathbb{R}^{\mathcal{S}}$, endowed with the canonical (product) ordering. An element v of $\mathbb{R}^{\mathcal{S}}$ is positive (respectively, strictly positive) if $v_\sigma \geq 0$ for every σ in $\mathcal{S}$ (respectively, $v_\sigma > 0$ for every σ in $\mathcal{S}$). In addition, if an element v of $\mathbb{R}^{\mathcal{S}}$ is non-null and positive, we shall write $v > 0$. For an element v of $\mathbb{R}^{\mathcal{S}}$, v^+ in $\mathbb{R}^{\mathcal{S}}$ and v^- in $\mathbb{R}^{\mathcal{S}}$ are, respectively, its positive part and its negative part, so that $v = v^+ - v^-$ in $\mathbb{R}^{\mathcal{S}}$ and $|v| = v^+ + v^-$ in $\mathbb{R}^{\mathcal{S}}$. Also, for an arbitrary collection $\{v^j\}_{j \in \mathcal{J}}$ of elements of $\mathbb{R}^{\mathcal{S}}$, its supremum and its infimum in $\mathbb{R}^{\mathcal{S}}$, if they exist, are denoted, respectively, by

$$\bigvee_{j \in \mathcal{J}} v^j \text{ and } \bigwedge_{j \in \mathcal{J}} v^j.$$

To simplify presentation, we shall adopt some notational conventions. First, for an element $v = (v_\sigma)_{\sigma \in \mathcal{S}}$ of $\mathbb{R}^{\mathcal{S}}$, for every σ in $\mathcal{S}$, v_σ is regarded itself as an element of $\mathbb{R}^{\mathcal{S}}$, so that $v = \sum_{\sigma \in \mathcal{S}} v_\sigma$. Second, C is the (Riesz) vector subspace of $\mathbb{R}^{\mathcal{S}}$ consisting of all real maps on $\mathcal{S}$ vanishing at all but finitely many σ in $\mathcal{S}$. Third, L is the (Riesz) vector subspace of $\mathbb{R}^{\mathcal{S}}$ consisting of all *bounded* real maps on $\mathcal{S}$. Finally, the positive cone of any (Riesz) vector subspace F of $\mathbb{R}^{\mathcal{S}}$ is $\{v \in F : v \geq 0\}$.

2.3. **Commodity space.** The commodity space, L, is the vector space of all bounded real maps on $\mathcal{S}$. Thus, a single physical commodity is traded and consumed at every date-event σ in $\mathcal{S}$. The element u of L denotes the *unitary endowment* of commodities, that is, a *unit* of L. The supremum norm on L is given by

$$\|v\| = \inf\{\lambda > 0 : |v| \leq \lambda u\}.$$

2.4. **Overlapping generations.** The (countably infinite) set of individuals is $\mathcal{G}$. Such individuals are distributed across generations, each consisting of a single individual, whose economic activity extends over $(n+1)$ (consecutive) periods, with n in $\mathbb{N}$. In particular, every individual i in $\mathcal{G}$ is active over a finite subset $\mathcal{S}^i$ of $\mathcal{S}$. In addition, for every date event σ in $\mathcal{S}$, (a) there exists a single individual $i(\sigma)$ in $\mathcal{G}$ initiating her economic activity, that is, with

$$\mathcal{S}^{i(\sigma)} = \{\tau \in \mathcal{S}(\sigma) : t(\tau) \leq t(\sigma) + n\};$$

(b) there exist exactly $(n+1)$ individuals that are active, that is, for every σ in $\mathcal{S}$, the cardinality of $\{i \in \mathcal{G} : \sigma \in \mathcal{S}^i\}$ is $(n+1)$. Notice that the map $i : \mathcal{S} \to \mathcal{G}$ associates a new individual to every date-event. In addition, a finite set of individuals is inherited from the unrepresented past at the initial date-event.

2.5. **Preferences.** The consumption space of individual i in $\mathcal{G}$ is X^i, the positive cone of L^i, the set of real maps on $\mathcal{S}$ vanishing on $(\mathcal{S}/\mathcal{S}^i)$, regarded as a (Riesz) vector subspace of L. Preferences $\succeq^i$ on X^i of individual i in $\mathcal{G}$ are continuous, (strictly) monotone and (weakly) convex.

2.6. **Allocations.** The space of allocations is

$$X = \left\{ x \in \prod_{i \in \mathcal{G}} X^i : \sum_{i \in \mathcal{G}} x^i \in L \right\}.$$

So, an allocation x in X involves an aggregate endowment,

$$\sum_{i \in \mathcal{G}} x^i = \left(\sum_{i \in \mathcal{G}} x^i_\sigma \right)_{\sigma \in \mathcal{S}},$$

that is bounded, that is, $\sum_{i \in \mathcal{G}} x^i \leq \lambda u$ for some $\lambda > 0$. An allocation x in X is *interior* if there exists $\lambda > 0$ such that, for every individual i in $\mathcal{G}$, $x^i \geq \lambda u^i$, where u^i is the unit vector in L^i. Finally, allocation z in X Pareto dominates (respectively, strictly Pareto dominates) allocation x in X if, for every individual i in $\mathcal{G}$, $z^i \succeq^i x^i$ and, for some (respectively, for every) individual i in $\mathcal{G}$, $z^i \succ^i x^i$.

2.7. **Prices.** Under sequentially complete markets, commodities have well-defined Arrow-Debreu prices, or contingent claim prices. A price p is an element of

$$P = \{p \in \mathbb{R}^{\mathcal{S}} : p_\sigma > 0 \text{ for every } \sigma \in \mathcal{S}\},$$

the space of all strictly positive maps on $\mathcal{S}$. A price p in P defines a positive linear functional on C, the vector subspace of L consisting of all real maps on $\mathcal{S}$ vanishing at all but finitely many σ in $\mathcal{S}$, where, for an element v of C, the duality is given by

$$p \cdot v = \sum_{\sigma \in \mathcal{S}} p_\sigma v_\sigma.$$

Thus, as a matter of mere fact, it defines a positive linear functional on the commodity space L^i of every individual i in $\mathcal{G}$, though it might not be a well-defined linear functional on the aggregate commodity space L.

2.8. **Price support.** Competitive equilibrium is simply represented in terms of supporting prices. An allocation x in X is *supported* by a price p in P if, for every individual i in $\mathcal{G}$,

$$z^i \succ^i x^i \text{ implies } p \cdot (z^i - x^i) > 0.$$

Notice that, under the maintained assumptions on preferences, price support yields, for every individual i in $\mathcal{G}$,

$$z^i \succeq^i x^i \text{ implies } p \cdot (z^i - x^i) \geq 0.$$

The property of price support, which captures optimality of individual consumption plans, is strengthened in part of the analysis.

Smooth support. *An allocation x in X, with supporting price p in P, is* smoothly supported *by price p in P if, for every $1 > \rho > 0$, there exists $\lambda > 0$ such that, for every individual i in $\mathcal{G}$, provided that $\left\| z^i - x^i \right\| \leq \lambda$,*

$$\rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^- \text{ implies } z^i \succeq^i x^i.$$

Thus, smooth support requires that, locally, whenever the ρ-discounted value of positive net trades exceeds the value of negative net trades, a consumption plan be weakly welfare improving for an individual. Geometrically, this implies that the (translated) convex cone

$$\left\{z^i \in X^i : \rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^-\right\}$$

is locally contained in the weakly preferred set $\left\{z^i \in X^i : z^i \succeq^i x^i\right\}$. This notion, which is weaker than the common assumption of an upper bound on the curvature of indifference curves (see, for instance, Chattopadhyay and Gottardi [12, Definition 5]), was introduced, and discussed, in Bloise [8]. Smooth support basically prevents an arbitrarily small degree of substitutability among commodities. It is an acceptably restrictive hypothesis.

2.9. **Robust inefficiency.** An allocation x in X is *robustly inefficient* if it is Pareto dominated by an alternative allocation z in X satisfying, for some $1 > \epsilon > 0$,

$$\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^+ \leq (1 - \epsilon) \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^- .$$

Thus, at a robustly inefficient allocation, a welfare improvement obtains by a modification of consumption plans notwithstanding the destruction of a positive fraction of transferred resources. This definition is meant to capture the occurrence of robust Pareto improvements.

3. Characterization

We here provide an equivalent characterization of robust inefficiency in terms of supporting prices. In particular, we show that prices reveal inefficiency *independently* of the length of horizons of generations. Our argument is developed in two separate parts. First, we reduce the economy to a fictitious economy with generations operating over two-period horizons, as originally proposed by Balasko, Cass and Shell [4], so obtaining restrictions on supporting prices. Second, we retrace these constraints on prices to the original economy. For the second part of the argument, we need an additional restriction at equilibrium on the volatility of safe interest rates.

Non-vanishing (gross) interest rates. *A price p in P satisfies the* hypothesis of non-vanishing (gross) interest rates *if there exists $1 > \eta > 0$ such that, for every σ in $\mathcal{S}$,*

$$p_\sigma \geq \eta \sum_{\tau \in \sigma_+} p_\tau.$$

So, under the domain of this hypothesis, (gross) safe interest rates are uniformly bounded from below across periods of trade. Observe that uniformly positive (gross) interest rates imply that, for any m in $\mathcal{T}$, at every σ in $\mathcal{S}$,

$$p_\sigma \geq \left(\frac{1}{m+1}\right) \eta^m \sum_{\tau \in \mathcal{S}(\sigma)^{t(\sigma)+m}} p_\tau, \tag{†}$$

where, for a subset $\mathcal{F}$ of $\mathcal{S}$, $\mathcal{F}^t = \{\sigma \in \mathcal{F} : t(\sigma) \leq t\}$ for every t in $\mathcal{T}$. This straightly follows from solving forward the inequality and adding up terms.

In order to proceed with the characterization, we introduce some additional pieces of notation and preliminarily establish the equivalence among some restrictions concerning the long-run behavior of prices. First of all, given a price p in P, we define a linear operator $T_p : L \to \mathbb{R}^{\mathcal{S}}$ by setting, for every σ in $\mathcal{S}$,

$$T_p(e)_\sigma = \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau e_\tau.$$

This operator is positive and, under the hypothesis of non-vanishing (gross) interest rates, maps L into itself, as, for every σ in $\mathcal{S}$,

$$(\ddagger) \qquad |T_p(e)_\sigma| \leq \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau |e_\tau| \leq \|e\| \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau \leq \frac{1}{\eta} \|e\|.$$

Second, for a comparison with the literature, we use the notion of weight function, initially introduced in Chattopadhyay and Gottardi [12]. An element λ of L is a *super-martingale* if, for every σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} \lambda_\tau \geq \lambda_\sigma.$$

The set of such super-martingales is denoted by Λ.

Modified Cass Criterion. *Given a price p in P, the Modified Cass Criterion is given by one of the following equivalent conditions.*

(FOC) *There exists $e > 0$ in L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau e_\tau \geq p_\sigma e_\sigma.$$

(PLO) *There exists $e > 0$ in L satisfying, for some $1 > \rho > 0$,*

$$\rho T_p(e) \geq e.$$

Proposition 1 (Equivalent criteria)**.** *Under the hypothesis of non-vanishing (gross) interest rates, given a price p in P, the following conditions are equivalent to the Modified Cass Criterion.*

(WCC) *There exists $\lambda > 0$ in Λ satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$p_\sigma \geq \left(\frac{1}{\rho}\right)^{t(\sigma)} \lambda_\sigma.$$

(DRC) *The linear operator $T_p : L \to L$ admits a real eigenvalue greater than the unity.*

Remark 1 (Spectral radius)**.** *The spectral radius of a positive linear operator T from L into itself is defined as*

$$r(T) = \lim_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}.$$

As L is a Banach lattice and T is a positive linear operator, it is well-known that $r(T)$ belongs to the spectrum $\sigma(T)$, where

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

This means that condition (DRC) *is satisfied only if the spectral radius is greater than the unity, $r(T) > 1$.*

Remark 2 (Comparison with the literature). *Notice that condition* (WCC) *implies that there exists a super-martingale* $\lambda > 0$ *in* Λ *such that, for every path* $\{\sigma_t\}_{t\in\mathcal{T}}$ *of* $\mathcal{S}$*,*

$$\sum_{t\in\mathcal{T}} \frac{\lambda_{\sigma_t}}{p_{\sigma_t}} \leq 1,$$

where a path $\{\sigma_t\}_{t\in\mathcal{T}}$ *of* $\mathcal{S}$ *is an element of* $\mathcal{S}^{\mathcal{T}}$ *such that, for every* t *in* $\mathcal{T}$*,* σ_{t+1} *belongs to* σ_{t+}*. This is the necessary and sufficient Weighted Cass Criterion established in the literature (Gottardi and Chattopadhyay* [12]*). In addition, condition* (FOC) *is similar to the First Order Condition Criterion presented by Barbie and Kaul* [5] *for recursive equilibria and condition* (DRC) *is basically the Dominant Root Characterization in Aiyagari and Peled* [1] *(see also Chattopadhyay and Gottardi* [12]*) for stationary equilibria.*

Remark 3 (Independence of the length of horizons). *Notice that the Modified Cass Criterion is independent of the length of horizons of generations. Thus, the characterization below implies that longer horizons do not affect the criterion for inefficiency obtained under the simplifying assumption of two-period horizons, unless safe (gross) interest rates vanish along some branches of the event-tree.*

The characterization exploits the Modified Cass Criterion. This basically requires that, in every period of trade, the (present) value of a positive transfer be below a constant share of the (present) value of the positive transfers in the following period. For two-period horizons, such quantities coincide with physical transfers of resources from young to old individuals. For longer horizons, a welfare improvement might obtain through a more sophisticated structure of physical transfers, which nevertheless correspond to values in any given horizon of constant length (n) falling below a constant share of values in the following horizons of constant length (n). When safe (gross) interest rates are non-vanishing, such values can be transformed into physical quantities in the original economy.

For two-period horizons, the argument for necessity is extremely straight. Indeed, assuming robust inefficiency, at every date-event σ in $\mathcal{S}$, let the transfer correspond to the negative net trade of the young individual, that is,

$$e_\sigma = \left(z^{i(\sigma)} - x^{i(\sigma)}\right)^-_\sigma.$$

As the feasible welfare improvement is to be initiated at some date-event, e is a non-null positive element of L. In addition, at every date-event σ in $\mathcal{S}$, by feasibility,

$$x^{i(\sigma)} - e_\sigma + \rho \sum_{\tau\in\sigma_+} e_\tau \geq z^{i(\sigma)},$$

as the consumption of the individual, who is young at that date-event, in the second period of activity cannot exceed a given share, ρ, of the negative net trades of young individuals at the following date-events. It is for the existence of such a given share that *robust* inefficiency plays its major role in the argument. Thus, at every date-event σ in $\mathcal{S}$, price support yields

$$\rho \sum_{\tau\in\sigma_+} p_\tau e_\tau \geq p_\sigma e_\sigma,$$

which is the Modified Cass Criterion. The extension of this simple argument to longer horizons requires a transformation of values into quantities. This transformation can be carried out exploiting the uniform positivity of (gross) interest rates, for, otherwise, it would yield unbounded physical quantities.

The argument for sufficiency, under smooth support, is even more elementary and, in addition, is independent of the length of horizons. Indeed, an alternative allocation is constructed by means of feasible transfers fulfilling, at every date-event σ in $\mathcal{S}$,

$$z^{i(\sigma)} = x^{i(\sigma)} - e_\sigma + \sum_{\tau \in \sigma_+} e_\tau.$$

By the Modified Cass Criterion, at every date-event σ in $\mathcal{S}$,

$$\rho p \cdot \left(z^{i(\sigma)} - x^{i(\sigma)}\right)^+ \geq p \cdot \left(z^{i(\sigma)} - x^{i(\sigma)}\right)^-,$$

so that smooth support yields a welfare improvement.

Proposition 2 (Characterization)**.** *Under the hypothesis of non-vanishing (gross) interest rates, an interior allocation x in X, with smoothly supporting price p in P, is robustly inefficient* if and only if *the Modified Cass Criterion is satisfied.*

Remark 4 (Extensions)**.** *The hypotheses of a single commodity* per *period and a single individual* per *generation are not really used, so that they could be possibly removed. In addition, as in Molina-Abraldes and Pintos-Clapés* [17]*, the equal length of horizons of generations is not at all exploited, apart from its role in simplifying notation. It would suffice to postulate the existence of an upper bound.*

4. Sequentially Incomplete Markets

4.1. **Conceptual separation.** We have provided a characterization of inefficiency under the hypothesis of sequentially complete markets. When markets are sequentially incomplete, two independent sources of inefficiency emerge: on the one side, the overlapping generations structure might cause a lack of transversality *at infinitum*; on the other side, the absence of financial instruments might inhibit complete insurance. The former cause of inefficiency pertains to the mere intertemporal allocation of resources, independently of risk. The latter cause of inefficiency regards the distribution of risk among individuals and it is not peculiar to overlapping generations economies, as it is exhibited by finite-horizon economies as well under incomplete financial markets. Can these two distinct phenomena be separated in principle?

Under incomplete markets, at equilibrium, individuals might have different subjective evaluations of risk, or subjective prices for risk. This disparity in evaluations allows an hypothetical planner to generate a welfare improvement, by redistributing from individuals with low evaluation to individuals with high evaluation. Such a sort of welfare improving reallocations necessarily requires the knowledge of subjective prices for risk of individuals. Undisputably, the observation of asset prices alone does not fully reveal subjective prices for risk under sequentially incomplete markets.

A failure of efficiency in the intertemporal allocation of resources, instead, might be revealed by asset prices alone. That is, an hypothetical planner might be able to Pareto improve independently of subjective prices for risk by individuals or, equivalently, for all subjective prices for risk that are consistent with the absence of arbitrage opportunities at observable asset prices. This sort of welfare improvement, ignoring unexploited insurance opportunities, cannot occur over finite horizons. Thus, in a sense, it captures a lack of transversality *at infinitum*.

Moving from the above considerations, we here propose a notion of unambiguous inefficiency, consisting in the occurrence of a welfare improvement for all subjective evaluations of risk. Asset prices fully reveal unambiguous inefficiency by means of a sort of Modified Cass Criterion. Importantly, this criterion corresponds to that for sequentially complete markets, evaluated at all implicit Arrow-Debreu prices, consistent with the absence of arbitrage opportunities. Thus, the same criterion reveals (unambiguous) inefficiency under sequentially complete and sequentially incomplete markets.

4.2. **Comparison with the literature.** In the literature, apart from Gottardi and Chattopadhyay [12] and Chattopadhyay [11], to the best of our knowledge, there are no other contributions proposing a separation of inefficiency in the intertemporal allocation of resources from inefficiency in the allocation of risk for stochastic economies of overlapping generations. We briefly discuss the approaches presented in these mentioned essays. A comparison with Henriksen and Spear [16], which instead avoids any issue of separation, is presented in remark 6.

For two-period horizons, Chattopadyhay and Gottardi [12] adopt a notion of *ex post* efficiency, corresponding to an evaluation of welfare of individuals conditional on the realization of uncertainty in the second period of their lives. Such a criterion requires separability in preferences (see their assumption 2 and their definition 6) and the existence of only a single safe asset at every date-event (see their definition 7). In addition, whereas our criterion for incomplete markets coincides with the established criterion for complete markets, their criterion is stronger when applied to complete markets (see their remark 5), as stronger is *ex post* inefficiency with respect to canonical inefficiency. Finally, their formulation of the necessary and sufficient criterion for *ex post* inefficiency exploits the observation of subjective prices of individuals, that is, the criterion is expressed in terms of weighted sum of reciprocals of subjective prices of some individuals.

For two-period horizons, Chattopadhyay [11] proposes a notion of constrained inefficiency, corresponding to a welfare improving modification of consumption plans by means of a *feasible* redistribution of the holdings of assets in positive net supply, along with direct transfers only in the first period of activity of individuals. This construction requires to take as given asset prices in order to determine the deliveries of long-term assets and, hence, the space of allowed redistributions. A major difference with our formulation is that inefficiency depends on the particular set of traded infinite-maturity assets. To understand this, consider a simple stationary economy under uncertainty, with sequentially complete markets, and suppose that, at a fully stationary equilibrium, the safe interest rate is negative. This allocation is necessarily constrained efficient according to Chattopadhyay's [11] notion, because of the absence of assets in positive net supply. However, it is unambiguously inefficient in our formulation.

4.3. **Price support.** To the only purpose of simplifying presentation, we assume the existence of a single one-period riskless asset that is traded at every date-event. That is, at every date-event, one unit of such an asset delivers one unit of the single commodity at every immediately succeeding date-event. An asset price is q in $Q = \left\{q \in \mathbb{R}^{\mathcal{S}} : q_\sigma > 0 \text{ for every } \sigma \in \mathcal{S}\right\}$. The absence of arbitrage opportunities implies the existence of *state prices*, or *contingent claim prices*, that is, a set of implicit Arrow-Debreu prices,

$$P(q) = \left\{ p \in P : p_\sigma q_\sigma = \sum_{\tau \in \sigma_+} p_\tau \text{ for every } \sigma \in \mathcal{S} \right\}.$$

Clearly, $P(q)$ is a non-empty convex (open) cone. More general asset structures might be straightly encompassed in our analysis, as, in the proofs, arguments are presented in greater generality. The hypothesis of sequentially complete asset markets, in the general case, obtains when $P(q)$ is, up to a normalization, a singleton.

As for sequentially complete markets, competitive equilibrium is represented only in terms supporting prices. An allocation x in X is *supported* by an asset price q

in Q if, for every individual i in $\mathcal{G}$,

$$z^i \succ^i x^i \text{ implies } \bigvee_{p \in P(q)} p \cdot \left(z^i - x^i\right) > 0.$$

This is a condition for optimality of consumption plans subject to budget constraints, under sequentially incomplete markets. We take it as a primitive notion, though a clarifying digression on the dual representation of sequential budget constrains is presented in appendix A.

Remark 5 (Subjective prices)**.** *An interior allocation x in X is supported by an asset price q in Q if and only if, for every individual i in $\mathcal{G}$, there exists p^i in $P(q)$ satisfying*

$$z^i \succ^i x^i \text{ implies } p^i \cdot \left(z^i - x^i\right) > 0.$$

Smooth support. *An allocation x in X, with supporting asset price q in Q, is* smoothly supported *by asset price q in Q if, for every $1 > \rho > 0$, there exists $\lambda > 0$ satisfying, for every individual i in $\mathcal{G}$, provided that $\left\|z^i - x^i\right\| \leq \lambda$,*

$$\rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^-, \text{ for every } p \in P(q), \text{ implies } z^i \succeq^i x^i.$$

This notion of smooth support is no more restrictive than the corresponding hypothesis for sequentially complete markets, as, for every individual i in $\mathcal{G}$, there exists a subjective price p^i in $P(q)$.

4.4. **Unambiguous inefficiency.** We now introduce the notion of unambiguous inefficiency, with a digression on its economic content and implications. An allocation x in X, with supporting asset price q in Q, is *robustly unambiguously inefficient* if it is Pareto dominated by an allocation z in X satisfying, for some $1 > \epsilon > 0$,

$$\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^+ \leq (1 - \epsilon) \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^-$$

and, for every individual i in $\mathcal{G}$,

$$(\S) \qquad \bigwedge_{p \in P(q)} p \cdot \left(z^i - x^i\right) \geq 0.$$

Notice that, provided that preferences are locally non-satiated, robust unambiguous inefficiency coincides with robust inefficiency under sequentially complete markets, as the condition (§) for an unambiguous welfare improvement becomes redundant (*i.e.*, it becomes a consequence of price support).

The unambiguous feature of inefficiency, being the major contribution of our analysis, requires adequate justification. In order to understand the implications of restriction (§), consider a feasible redistribution z in X for which it fails. Thus, for some individual i in $\mathcal{G}$, there exists p^* in $P(q)$ satisfying

$$p^* \cdot \left(z^i - x^i\right) < 0.$$

Such a redistribution might be welfare improving only if p^* in $P(q)$ is *not* a subjective price for individual i in $\mathcal{G}$. Consequently, the notion of unambiguous inefficiency rules out such situations, which are ambiguous in the sense that to ascertain welfare improvement requires a precise knowledge of unobservable subjective prices.

On a purely analytical ground, unambiguous inefficiency might be interpreted as an artificial concept that precisely identifies a particular class of welfare improving redistributions whose occurrence is fully revealed by the observation of equilibrium asset prices alone. Some other sources of inefficiency remain, but their exact identification requires more information than the mere observation of asset prices.

Alternatively, we might propose a positive interpretation of unambiguous inefficiency in terms of instruments available to an hypothetical planner. Indeed, robust unambiguous inefficiency might be interpreted as a situation in which, under the validity of the hypothesis of smooth support, an hypothetical planner might Pareto improve upon the initial allocation by simply exploiting the information on preferences that are revealed by asset prices. In other terms, a feasible welfare improving reallocation is consistent with all arbitrary specifications of preferences, consistent with optimality of consumption plans, given asset prices, that fulfill smooth support. More precisely, robust unambiguous inefficiency corresponds to the existence of a feasible redistribution guaranteeing, for every individual, that, evaluated at all possible consistent state prices, a constant share of the value of positive net trade be above the value of negative net trade.

Lemma 1 (Unambiguous inefficiency)**.** *An allocation x in X, with smoothly supporting asset price q in Q, is robustly unambiguously inefficient* if and only if *there exists an allocation z in X (not coinciding with x in X) satisfying*

$$\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^+ \leq \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^-$$

and, for some $1 > \rho > 0$, for every individual i in $\mathcal{G}$,

$$\rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^-\ , \text{ for every } p \in P(q),$$

which obviously implies

$$\bigwedge_{p \in P(q)} p \cdot \left(z^i - x^i\right) \geq 0.$$

Unambiguous inefficiency does not occur over a finite horizon, notwithstanding missing financial markets. So, in this sense, this notion captures a robust failure of efficiency at equilibrium that is not caused by an unoptimal allocation of risk across individuals.

Lemma 2 (Long-run nature of unambiguous inefficiency)**.** *If an allocation x in X, with supporting asset price q in Q, is Pareto dominated by an alternative allocation z in X satisfying*

$$\sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^+ \leq \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^-$$

and, for every individual i in $\mathcal{G}$,

$$\bigwedge_{p \in P(q)} p \cdot \left(z^i - x^i\right) \geq 0,$$

then $\mathcal{G}^* = \left\{i \in \mathcal{G} : \left|z^i - x^i\right| > 0\right\}$ is a countably infinite set.

Unambiguous inefficiency requires an infinite value of the aggregate endowment for all consistent evaluations of risk. This is of relevance for economies in which productive assets are traded in the asset market.

Lemma 3 (Summable state prices)**.** *An allocation x in X, with supporting asset price q in Q, is robustly unambiguously inefficient only if*

$$\bigwedge_{p \in P(q)} p \cdot \sum_{i \in \mathcal{G}} x^i \text{ is infinite.}$$

Remark 6 (Productive asset)**.** *The result of lemma 3 shows that, whenever a sufficiently productive infinite-maturity asset is traded in the asset market, competitive equilibrium is* not *robustly unambiguously inefficient. Similarly, in a straightforward extension of Chattopadhyay and Gottardi's* [12] *analysis, an equilibrium allocation is* ex post *efficient in the presence of a sufficiently productive infinite-maturity asset. This, in particular, implies that all equilibrium allocations, that are considered by*

Henriksen and Spear [16], *are* not *robustly unambiguously inefficient, as a productive asset yields a non-negligible share of the aggregate endowment. Nevertheless, as Henriksen and Spear's* [16] *analysis shows, by a mere reallocation of risk, a substantial welfare improvement might exist, even leading to the socially optimal stationary allocation of full insurance.*

4.5. **Characterization.** As for sequentially complete markets, in order to extend the characterization to longer horizons, we impose an additional restriction about non-vanishing (gross) interest rates at equilibrium.

Non-vanishing (gross) interest rates. *An asset price q in Q satisfies the* hypothesis of non-vanishing interest rates *if there exists $1 > \eta > 0$ satisfying, for every σ in $\mathcal{S}$,*

$$p_\sigma \geq \eta \sum_{\tau \in \sigma_+} p_\tau, \text{ for every } p \in P(q).$$

Given an asset price q in Q, under the hypothesis of non-vanishing (gross) interest rates, the (non-linear) operator $\bigwedge_{p \in P(q)} T_p : L \to \mathbb{R}^{\mathcal{S}}$ is well-defined, where, for every σ in $\mathcal{S}$,

$$\left(\bigwedge_{p \in P(q)} T_p(e) \right)_\sigma = \bigwedge_{p \in P(q)} T_p(e)_\sigma .$$

This operator is positive and, as a matter of fact, it maps L into itself, as condition (‡) establishes bounds that are independent of the particular price p in $P(q)$.

Modified Cass Criterion. *Given a asset price q in Q, the Modified Cass Criterion is given by one of the following equivalent conditions.*

(FOC) *There exists $e > 0$ in L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau e_\tau \geq p_\sigma e_\sigma, \text{ for every } p \in P(q).$$

(PLO) *There exists $e > 0$ in L satisfying, for some $1 > \rho > 0$,*

$$\rho \bigwedge_{p \in P(q)} T_p(e) \geq e.$$

The characterization exploits the Modified Cass Criterion. The argument for sufficiency is a straightforward adaptation of that in the case of sequentially complete markets. For two-period horizons, the argument for necessity requires a minimal modification. In particular, exactly reproducing the construction for sequential complete markets, price support simply implies

$$\rho \sum_{\tau \in \sigma_+} p_\tau e_\tau \geq p_\sigma e_\sigma, \text{ for } \textit{some } p \in P(q),$$

which is insufficient to prove the claim. However, *unambiguous* inefficiency (restriction (§)) yields

$$\rho \sum_{\tau \in \sigma_+} p_\tau e_\tau \geq p_\sigma e_\sigma, \text{ for } \textit{every } p \in P(q),$$

which exactly corresponds to the Modified Cass Criterion. The extension of this simple argument to longer horizons again requires a transformation of values into quantities, which is not as straight as for sequentially complete markets.

Proposition 3 (Characterization)**.** *Under the hypothesis of non-vanishing (gross) interest rates, an interior allocation x in X, with smoothly supporting asset price q in Q, is robustly unambiguously inefficient* if and only if *the Modified Cass Criterion is satisfied.*

Remark 7 (Extensions). *Though we have assumed the existence of a single riskless asset, our arguments can be straightly generalized to encompass alternative structures of incomplete markets. Notice, however, that the hypothesis of non-vanishing (gross) interest rates requires the availability, at every date-event, of some financial instrument to transfer wealth at the following date-events.*

5. Conclusion

We have provided a characterization of inefficiency in stochastic economies of overlapping generations operating over long horizons, under both sequentially complete and sequentially incomplete markets. The proposed Modified Cass Criterion is simple and suitable for empirical work. It is basically a generalization of the well-known Dominant Root Characterization for stationary equilibria.

The conceptual separation of long-run inefficiency from limited insurance, under incomplete markets, might be of application also in other economies with incomplete markets, or borrowing constraints, under the hypothesis of a finite set of patient individuals. This might serve to assess benefits from active policy intervention even though market imperfections cannot be removed.

As a final remark, we observe that the methodology might be appropriate for providing an empirical (local) *measure* of inefficiency, along the lines of Debreu [13], inversely proportional to the smallest coefficient, ρ, fulfilling the Modified Cass Criterion.

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Appendix A. Incomplete markets and budget constraints

Given an asset price q in Q, a portfolio h in C, the vector space of all real maps on $\mathcal{S}$ vanishing at all but finitely many σ in $\mathcal{S}$, generates a revenue $r(q,h)$ in C by means of

$$r_\phi(q,h) = -q_\phi h_\phi$$

and, for every σ in $\mathcal{S}$,

$$(r_\tau(q,h))_{\tau\in\sigma_+} = (h_\sigma - q_\tau h_\tau)_{\tau\in\sigma_+}.$$

Notice that, given any h in C, for every p in $P(q)$,

$$p\cdot r(q,h) = \sum_{\sigma\in\mathcal{S}}\left(-p_\sigma q_\sigma + \sum_{\tau\in\sigma_+} p_\tau\right) h_\sigma = 0.$$

This makes sense as h belongs to C and, thus, the above sum involves only finitely many non-null terms.

The sequential budget constraint of individual i in $\mathcal{G}$, for an initial endowment of commodities e^i in X^i, is given by

$$B^i\left(q,e^i\right) = \left\{x^i\in X^i : x^i \le e^i + r(q,h) \text{ for some } h\in C\right\}.$$

This budget set admits an equivalent characterization in terms of Arrow-Debreu prices.

Lemma 4 (Dual representation of budget constrains)**.** *For every x^i in X^i,*

$$x^i\in B^i\left(q,e^i\right) \text{ if and only if } \bigvee_{p\in P(q)} p\cdot\left(x^i-e^i\right)\le 0.$$

Proof of lemma 4. Necessity is trivial. For sufficiency, endow C with the inductive limit topology (see Burbaki, Topological Vector Spaces), that is, the finest locally convex topology on C such that the natural embedding of any finite dimensional subspace K of C is continuous, where K is endowed with its unique Hausdorff linear topology. Relevantly, a subset F of C is closed if and only if, for every fnite dimensional subspace K of C, $K\cap F$ is closed in the unique Hausdorff linear topology of K. The topological dual of C coincides with its algebraic dual, and can be identifed with $\mathbb{R}^{\mathcal{S}}$.

Assume that there exists x^i in X^i such that

$$\bigvee_{p\in P(q)} p\cdot\left(x^i-e^i\right)\le 0 \text{ and } x^i\notin B^i\left(q,e^i\right).$$

This implies that the closed convex set $K = r(q,C) - C^+$ in C does not contain $\left(x^i-e^i\right)$ in C. By the Strong Separating Hyperplane Theorem (see Corollary 5.59 in Aliprantis and Border [2]), there exists a non-null p^i in $\mathbb{R}^{\mathcal{S}}$ such that, for every (h,v) in $C\times C^+$,

$$p^i\cdot\left(x^i-e^i\right) > p^i\cdot r(q,h) - p\cdot v.$$

By canonical arguments, this implies that $p^i>0$ in $\mathbb{R}^{\mathcal{S}}$ and that, for every h in C,

$$p^i\cdot r(q,h) = \sum_{\sigma\in\mathcal{S}}\left(-p^i_\sigma q_\sigma + \sum_{\tau\in\sigma_+} p^i_\tau\right) h_\sigma = 0,$$

that is, for every σ in $\mathcal{S}$,

$$p^i_\sigma q_\sigma = \sum_{\tau \in \sigma_+} p^i_\tau.$$

Therefore, pegging any p in $P(q)$, for every $\epsilon > 0$, $p^i + \epsilon p$ is an element of $P(q)$. For every $\epsilon > 0$, this yields

$$p^i \cdot \left(x^i - e^i\right) + \epsilon p \cdot \left(x^i - e^i\right) \leq \left(p^i + \epsilon p\right) \cdot \left(x^i - e^i\right) \leq 0,$$

a contradiction as $p^i \cdot \left(x^i - e^i\right) > 0$. □

Thus, optimality of consumption plan requires

$$\bigvee_{p \in P(q)} p \cdot \left(x^i - e^i\right) \leq 0$$

and

$$z^i \succ^i x^i \text{ implies } \bigvee_{p \in P(q)} p \cdot \left(z^i - e^i\right) > 0.$$

In turn, optimality implies price support, as, for every p in $P(q)$,

$$p \cdot \left(z^i - e^i\right) \leq \bigvee_{p \in P(q)} p \cdot \left(z^i - x^i\right) + \bigvee_{p \in P(q)} p \cdot \left(x^i - e^i\right).$$

Lemma 5 (Subjective prices). *For every x^i in the interior of X^i,*

$$z^i \succ^i x^i \textit{ implies } \bigvee_{p \in P(q)} p \cdot \left(z^i - x^i\right) > 0$$

if and only if, for some p^i in $P(q)$,

$$z^i \succ^i x^i \textit{ implies } p^i \cdot \left(z^i - x^i\right) > 0.$$

Proof of lemma 5. One implication is obvious. For the other implication, the (non-empty) convex sets $Z^i = \left\{z^i \in X^i : z^i \succ^i x^i\right\}$ and

$$B^i = \left\{ y^i \in X^i : \bigvee_{p \in P(q)} p \cdot \left(y^i - x^i\right) \leq 0 \right\}$$

have an empty intersection. A canonical argument of separation yields $p^i > 0$ in L^i separating Z^i from B^i, that is, for every $\left(z^i, y^i\right)$ in $Z^i \times B^i$,

$$p^i \cdot z^i \geq p^i \cdot y^i.$$

This, in particular, exploiting the interiority of consumption plan x^i in X^i and strict monotonicity of preferences, ensures that p^i is a strictly positive element of L^i and that

$$z^i \succ^i x^i \text{ implies } p^i \cdot \left(z^i - x^i\right) > 0.$$

Let $P^i(q)$ be the restriction of $P(q)$ to L^i, that is,

$$P^i(q) = \left\{(p_\sigma)_{\sigma \in \mathcal{S}^i} \in L^i : p \in P(q)\right\}.$$

If p^i is not an element of $P^i(q)$, being strictly positive, then it cannot be an element of the closure of $P^i(q)$. By strong separation, if p^i is not an element of the closure of $P^i(q)$, there exists a non-null h^i in L^i such that, for every p in $P^i(q)$,

$$p^i \cdot h^i > p \cdot h^i.$$

As $P^i(q)$ is an open cone in L^i, it follows that $p^i \cdot h^i > 0$ and that $p \cdot h^i \leq 0$ for every p in $P(q)$. By interiority of the consumption plan, at no loss of generality, one might assume that $x^i + h^i$ is an element of B^i, whereas, by strict monotonicity of preferences, $x^i + \epsilon u^i$ is an element of Z^i for every $\epsilon > 0$, where u^i is the unit of L^i. However, by the first application of the separation theorem,

$$\epsilon p^i \cdot u^i \geq p^i \cdot h^i.$$

As $\epsilon > 0$ is arbitrarily small, this reveals a contradiction. □

Appendix B. Proofs

Proof of proposition 1. Let $e > 0$ in L satisfy condition (FOC). Pick any γ in $\mathcal{S}$ such that $e_\gamma > 0$. Consider the tree with root at γ obtained by taking the successors σ of γ with $e_\sigma > 0$; that is, denoting σ^- the immediate predecessor of a (non-initial) date-event σ in $\mathcal{S}(\gamma)$, consider the tree

$$\mathcal{F} = \{\sigma \in \mathcal{S}(\gamma) : e_\sigma > 0, e_{\sigma^-} > 0\}.$$

Note that, for every σ in $\mathcal{F}$, by (FOC), for some τ in σ^+, $e_\tau > 0$. Hence, by (FOC), for every σ in $\mathcal{F}$,

$$\rho \frac{1}{p_\sigma e_\sigma} \geq \frac{1}{\sum_{\tau \in \sigma_+} p_\tau e_\tau}.$$

Consider a $\lambda > 0$ in $\mathbb{R}^{\mathcal{S}}$ defined, for the initial date-event γ in $\mathcal{F}$, by some $\lambda_\gamma > 0$; for every σ in $\mathcal{F}$, at every τ in $\sigma_+ \cap \mathcal{F}$, by

$$\lambda_\tau = \left(\frac{p_\tau e_\tau}{\sum_{\tau \in \sigma_+} p_\tau e_\tau} \right) \lambda_\sigma;$$

for every σ in $(\mathcal{S}/\mathcal{F})$, by $\lambda_\sigma = 0$. It can be easily verified that $\lambda > 0$ is an element of Λ. In addition, it follows that, given any σ in $\mathcal{F}$, for every τ in $\sigma_+ \cap \mathcal{F}$,

$$\rho^{t(\sigma)-t(\gamma)+1} \frac{\lambda_\gamma}{p_\gamma e_\gamma} \geq \rho \frac{\lambda_\sigma}{p_\sigma e_\sigma} \geq \frac{\lambda_\sigma}{\sum_{\tau \in \sigma_+} p_\tau e_\tau} \geq \frac{\lambda_\tau}{p_\tau e_\tau} \geq \frac{1}{\|e\|} \frac{\lambda_\tau}{p_\tau}.$$

This suffices to prove that (WWC) is satisfied.

Assuming that $\lambda > 0$ in Λ satisfies condition (WCC), for every σ in $\mathcal{S}$, define

$$e_\sigma = \frac{\lambda_\sigma}{\rho^{t(\sigma)} p_\sigma}.$$

This delivers an element $e > 0$ of L, because of (WCC). In addition, for every σ in $\mathcal{S}$,

$$p_\sigma e_\sigma = \left(\frac{1}{\rho^{t(\sigma)}} \right) \lambda_\sigma \leq \rho \left(\frac{1}{\rho^{t(\sigma)+1}} \right) \sum_{\tau \in \sigma_+} \lambda_\tau = \rho \sum_{\tau \in \sigma_+} p_\tau e_\tau,$$

so proving that condition (FOC) is satisfied.

Clearly, if $e > 0$ in L satisfies condition (FOC), without loss of generality, by the simple argument we sketch below, it can be assumed that, at every σ in $\mathcal{S}$,

$$\rho \sum_{\tau \in \sigma_+} p_\tau e_\tau = p_\sigma e_\sigma.$$

Indeed, there exists t in $\mathcal{T}$ such that $\sum_{\sigma \in (\mathcal{S}/\mathcal{S}^t)} e_\sigma > 0$ in L. By proceeding forward, possibly contracting the element e of L, the equality can be assumed for every σ in $(\mathcal{S}/\mathcal{S}^t)$. This modification, involving contractions, does not affect boundedness. By proceeding backward, possibly modifying the element e of L, the equality can be satisfied at every σ in $\mathcal{S}^t$. This latter modification, involving finitely many terms, does not affect boundedness as well. Thus, after both modifications, e remains an element of L. This suffices to prove that the operator $T_p : L \to L$ admits an eigenvalue larger than the unity, so that condition (DRC) holds.

Assume that condition (DRC) is satisfied, that is, for some $1 > \rho > 0$, there exists a non-null element e of L satisfying $\rho T_p(e) = e$. As T_p is a positive linear operator,

$$\rho T_p(|e|) \geq \rho |T_p(e)| = |e|,$$

which proves that condition (FOC) is satisfied. □

Proof of proposition 2. Necessity follows, through lemma 6, from lemma 7, whereas sufficiency is proved by lemma 8. □

Lemma 6. *Given $1 > \rho > 0$, there exists $v > 0$ in L satisfying $\rho^n T_p^n(v) \geq v$ only if there exists $e > 0$ in L satisfying $\rho T_p(e) \geq e$.*

Proof of lemma 6. Define

$$e = v + \rho T_p(v) + \cdots + \rho^{n-1} T_p^{n-1}(v).$$

By positivity of operator T_p, $e \geq v > 0$. It follows that

$$\rho T_p(e) \geq \rho T_p(v) + \cdots + \rho^{n-1} T_p^{n-1}(v) + \rho^n T_p^n(v) \geq e + \rho^n T_p^n(v) - v \geq e.$$

This proves the claim. □

Lemma 7. *Under the hypothesis of non-vanishing (gross) interest rates, an allocation x in X, with supporting price p in P, is robustly inefficient only if condition* (FOC) *is satisfied.*

Proof of lemma 7. The proof requires the preliminary introduction of some pieces of notation. The underlying logic is as sort of reduction to a two-period economy in order to obtain a necessary criterion, which is then retraced to the original economy through lemma 6.

For every σ in $\mathcal{S}$, define

$$\langle\sigma\rangle = \{\nu \in \mathcal{S}(\sigma) : t(\nu) \leq t(\sigma) + n - 1\}$$

and

$$\langle\sigma\rangle_+ = \{\nu \in \mathcal{S}(\sigma) : t(\nu) = t(\sigma) + n\}.$$

In addition, for every σ in $\mathcal{S}$,

$$\mathcal{G}_\sigma = \left\{i \in \mathcal{G} : \langle\sigma\rangle \cap \mathcal{S}^i \neq \emptyset\right\}$$

is the set of individuals acting at some date-event ν in $\langle\sigma\rangle$;

$$\mathcal{G}_\sigma^+ = \{i \in \mathcal{G} : i = i(\tau) \text{ for some } \tau \in \langle\sigma\rangle\}$$

is the set of individuals initiating their activity at some date-event ν in $\langle\sigma\rangle$; finally,

$$\mathcal{G}_\sigma^- = \left(\mathcal{G}_\sigma / \mathcal{G}_\sigma^+\right)$$

is the set of individuals acting at some date-event ν in $\langle\sigma\rangle$ that are inherited from previous date-events (or from the unrepresented past at the initial date-event) and exhaust their economic activity over date-events $\langle\sigma\rangle$. Notice that, by construction, for every date-event σ in $\mathcal{S}$,

$$\mathcal{G}_\sigma^+ = \bigcup_{\tau \in \langle\sigma\rangle_+} \mathcal{G}_\tau^-.$$

Finally, we use the following notational convention: given an element $v = (v_\sigma)_{\sigma \in \mathcal{S}}$ of L, for every σ in $\mathcal{S}$, v_σ is itself regarded as an element of L, so that $v = \sum_{\sigma \in \mathcal{S}} v_\sigma$; in addition, given a (non-empty) subset $\mathcal{F}$ of $\mathcal{S}$, for an element $v = (v_\sigma)_{\sigma \in \mathcal{S}}$ of L, $v_{\mathcal{F}} = \sum_{\sigma \in \mathcal{S}} v_\sigma$ is a well-defined element of L (that is, it is the algebraic projection of v in L onto $\{v \in L : v_\sigma = 0 \text{ for every } \sigma \in (\mathcal{S}/\mathcal{F})\}$).

Assume robust inefficiency, that is, that allocation x in X is Pareto dominated by an alternative allocation z in X satisfying, for some $\epsilon > 0$,

$$-\epsilon \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^- \geq \sum_{i \in \mathcal{G}} \left(z^i - x^i\right).$$

Notice that this implies that, for every σ in $\mathcal{S}$,

$$\begin{aligned} \sum_{i \in \mathcal{G}_\sigma} \left(z^i - x^i\right)^-_{\langle\sigma\rangle} &\geq \\ -\epsilon \sum_{i \in \mathcal{G}_\sigma} \left(z^i - x^i\right)^-_{\langle\sigma\rangle} + \sum_{i \in \mathcal{G}_\sigma} \left(z^i - x^i\right)^-_{\langle\sigma\rangle} &\geq \sum_{i \in \mathcal{G}_\sigma} \left(z^i - x^i\right)^+_{\langle\sigma\rangle} \\ &\geq \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)^+_{\langle\sigma\rangle}. \end{aligned}$$

The middle inequality directly follows from robust inefficiency, as only individuals in $\mathcal{G}_\sigma$ are active at date-events in $\langle\sigma\rangle$. The last inequality holds true because individuals in $\mathcal{G}^-_\sigma$ form a subset of individuals in $\mathcal{G}_\sigma$. Therefore, at every σ in $\mathcal{S}$,

$$\begin{aligned} -\epsilon \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle} &\geq \\ -\epsilon \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)^+_{\langle\sigma\rangle} &\geq \\ -\epsilon \sum_{i \in \mathcal{G}_\sigma} \left(z^i - x^i\right)^-_{\langle\sigma\rangle} &\geq \sum_{i \in \mathcal{G}_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle} \\ &\geq \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle} + \sum_{i \in \mathcal{G}^+_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle}, \end{aligned}$$

where the last inequality (in fact, an equality) is implied by the partition of individuals in $\mathcal{G}_\sigma$ in the two subsets $\mathcal{G}^-_\sigma$ and $\mathcal{G}^+_\sigma$. Hence, for every σ in $\mathcal{S}$,

$$-\left(\frac{1}{1+\epsilon}\right) \sum_{i \in \mathcal{G}^+_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle} \geq \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle}.$$

In addition, for every σ in $\mathcal{S}$,

$$\begin{aligned} p \cdot \sum_{i \in \mathcal{G}^+_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle} + \sum_{\tau \in \langle\sigma\rangle_+} p \cdot \sum_{i \in \mathcal{G}^-_\tau} \left(z^i - x^i\right)_{\langle\tau\rangle} &= \\ p \cdot \sum_{i \in \mathcal{G}^+_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle} + \sum_{\tau \in \langle\sigma\rangle_+} p \cdot \sum_{i \in \mathcal{G}^+_\sigma} \left(z^i - x^i\right)_{\langle\tau\rangle} &= \\ p \cdot \sum_{i \in \mathcal{G}^+_\sigma} \left(z^i - x^i\right) &\geq 0, \end{aligned}$$

where the initial equalities follow from the decomposition of net trades, whereas the last inequality is implied by price support. This establishes that, at every σ in $\mathcal{S}$,

$$(*) \qquad \left(\frac{1}{1+\epsilon}\right) \sum_{\tau \in \langle\sigma\rangle_+} p \cdot \sum_{i \in \mathcal{G}^-_\tau} \left(z^i - x^i\right)_{\langle\tau\rangle} \geq p \cdot \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle}.$$

Let v in $\mathbb{R}^{\mathcal{S}}$ be defined, for every σ in $\mathcal{S}$, by

$$v_\sigma = \left(\frac{1}{p_\sigma} p \cdot \sum_{i \in \mathcal{G}^-_\sigma} \left(z^i - x^i\right)_{\langle\sigma\rangle}\right)^+.$$

In fact, as the hypothesis of non-vanishing (gross) interest rates guarantees boundedness (see (†)), v is a positive element of L and, by inequality (*), it satisfies, at every σ in $\mathcal{S}$,

$$\left(\frac{1}{1+\epsilon}\right)\sum_{\tau\in\langle\sigma\rangle_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

That is, $\rho^n T_p^n(v) \geq v$, where

$$\rho = \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{n}}.$$

Hence, applying lemma 6 would prove the claim, provided that v is a non-null element of L.

To obtain a contradiction, assume not, that is, at every σ in $\mathcal{S}$,

$$-p\cdot\sum_{i\in\mathcal{G}_\sigma^-}\left(z^i - x^i\right)_{\langle\sigma\rangle} \geq 0.$$

Feasibility implies that, at every t in $\mathcal{T}$,

$$\begin{aligned}
-\epsilon\sum_{i\in\mathcal{G}}\left(z^i - x^i\right)^-_{\mathcal{S}^{t+n-1}} &\geq \sum_{i\in\mathcal{G}}\left(z^i - x^i\right)_{\mathcal{S}^{t+n-1}} \\
&= \sum_{i\in\mathcal{G}_t^-}\left(z^i - x^i\right) + \sum_{\sigma\in\mathcal{S}_t}\sum_{i\in\mathcal{G}_\sigma^+}\left(z^i - x^i\right)_{\langle\sigma\rangle},
\end{aligned}$$

where $\mathcal{S}^t = \{\sigma\in\mathcal{S} : t(\sigma)\leq t\}$, $\mathcal{S}_t = \{\sigma\in\mathcal{S} : t(\sigma) = t\}$ and

$$\mathcal{G}_t^- = \bigcup_{\sigma\in\mathcal{S}^t}\mathcal{G}_\sigma^-.$$

Taking values, by price support, yields, for every t in $\mathcal{T}$,

$$\begin{aligned}
-\epsilon p\cdot\sum_{i\in\mathcal{G}}\left(z^i - x^i\right)^-_{\mathcal{S}^{t+n-1}} &\geq p\cdot\sum_{i\in\mathcal{G}_-^t}\left(z^i - x^i\right) + \sum_{\sigma\in\mathcal{S}_t} p\cdot\sum_{i\in\mathcal{G}_\sigma^+}\left(z^i - x^i\right)_{\langle\sigma\rangle} \\
&\geq -\sum_{\sigma\in\mathcal{S}_t}\sum_{\tau\in\langle\sigma\rangle_+} p\cdot\sum_{i\in\mathcal{G}_\sigma^+}\left(z^i - x^i\right)_{\langle\tau\rangle} \\
&\geq -\sum_{\sigma\in\mathcal{S}_t}\sum_{\tau\in\langle\sigma\rangle_+} p\cdot\sum_{i\in\mathcal{G}_\tau^-}\left(z^i - x^i\right)_{\langle\tau\rangle} \\
&\geq 0,
\end{aligned}$$

where the last inequality holds true by hypothesis. That is, by strict positivity of price p in P,

$$\sum_{i\in\mathcal{G}}\left(z^i - x^i\right)^- = 0.$$

This reveals a contradiction, as any feasible Pareto improvement requires the transfer of some resources. □

Lemma 8. *An interior allocation x in X, with smoothly supporting price p in P, is robustly inefficient if condition* (FOC) *is satisfied.*

Proof of lemma 8. At no loss of generality, it can be assumed that there exists $1 > \epsilon > 0$ such that, at every σ in $\mathcal{S}$,

$$(1-\epsilon)\,\rho\sum_{\tau\in\sigma_+} p_\tau e_\tau \geq p_\sigma e_\sigma.$$

Let $\lambda > 0$ be given by the hypothesis of smooth support at $1 > \rho > 0$ and, at no loss of generality, assume that $\|e\| \leq 1$ and that $x^i \geq \lambda u^i$ for every individual i

in $\mathcal{G}$, where u^i is the unit of L^i. Thus, define an alternative allocation z in X by setting, for every σ in $\mathcal{S}$,

$$z^{i(\sigma)} = x^{i(\sigma)} - \lambda e_\sigma + (1-\epsilon) \sum_{\tau \in \sigma_+} \lambda e_\tau .$$

As feasibility is obviously satisfied, along with a positive destruction of transferred resources, smooth support suffices to prove the claim, redistributing resources made available by the above transfers. □

Proof of lemma 1. Suppose that allocation x in X is Pareto dominated by allocation y in X satisfying, for some $1 > \epsilon > 0$,

$$\sum_{i \in \mathcal{G}} \left(y^i - x^i\right)^+ \leq (1-\epsilon) \sum_{i \in \mathcal{G}} \left(y^i - x^i\right)^-$$

and, for every individual i in $\mathcal{G}$,

$$p \cdot \left(y^i - x^i\right) \geq 0, \text{ for every } p \in P(q).$$

Define an alternative allocation z in X by setting, for every individual i in $\mathcal{G}$, $z^i = y^i + \epsilon \left(y^i - x^i\right)^-$, so that $\left(z^i - x^i\right)^- \leq \left(y^i - x^i\right)^-$. It follows that, given any p in $P(q)$, for every individual i in $\mathcal{G}$,

$$p \cdot \left(z^i - x^i\right) \geq p \cdot \left(y^i - x^i\right) + \epsilon p \cdot \left(y^i - x^i\right)^- \geq \epsilon p \cdot \left(z^i - x^i\right)^-,$$

that is,

$$\left(\frac{1}{1+\epsilon}\right) p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^- .$$

In addition, feasibility might be straightly verified. As far as the reverse implication is concerned, at no loss of generality, it can be assumed that there exists $1 > \epsilon > 0$ such that, for every individual i in $\mathcal{G}$,

$$(1-\epsilon) \rho p \cdot \left(z^i - x^i\right)^+ \geq p \cdot \left(z^i - x^i\right)^-, \text{ for every } p \in P(q).$$

Let $\lambda > 0$ be given by the assumption of smooth support at $1 > \rho > 0$. At no loss of generality, by convexity of preferences, assume that $\left\|z^i - x^i\right\| \leq \lambda$ for every individual i in $\mathcal{G}$. Define the alternative allocation y in X by letting, for every individual i in $\mathcal{G}$, $y^i = z^i - \epsilon \left(z^i - x^i\right)^+$. Smooth support ensures a Pareto improvement. In addition,

$$\begin{aligned} \sum_{i \in \mathcal{G}} \left(y^i - x^i\right)^+ &\leq \\ (1-\epsilon) \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^+ &\leq (1-\epsilon) \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^- \\ &\leq (1-\epsilon) \sum_{i \in \mathcal{G}} \left(y^i - x^i\right)^-, \end{aligned}$$

so proving the claim. □

Proof of lemma 2. This is basically the proof of the First Welfare Theorem. Indeed, suppose that $\mathcal{G}^*$ is a finite set. By welfare improvement, there exists p^* in $P(q)$ such that, for some individual i in $\mathcal{G}^*$, $p^* \cdot \left(z^i - x^i\right) > 0$. In addition, by unambiguous inefficiency (condition (§)), for every individual i in $\mathcal{G}^*$, $p^* \cdot \left(z^i - x^i\right) \geq 0$. Hence, summing up, as p^* in $P(q)$ is positive,

$$0 < \sum_{i \in \mathcal{G}^*} p^* \cdot \left(z^i - x^i\right) \leq p^* \cdot \sum_{i \in \mathcal{G}^*} \left(z^i - x^i\right) \leq p^* \cdot \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \leq 0,$$

a contradiction. □

Proof of lemma 3. Supposing not, there exists p in $P(q)$ such that $p \cdot \sum_{i \in \mathcal{G}} x^i$ is finite and, hence, p in P defines a sequential (*i.e.*, order continuous) linear functional on

$$\left\{ v \in L : |v| \leq \lambda \sum_{i \in \mathcal{G}} x^i \text{ for some } \lambda > 0 \right\}.$$

By unambiguous Pareto dominance (condition (§)), for every individual i in $\mathcal{G}$, $p \cdot \left(z^i - x^i\right) \geq 0$. Summing up and using feasibility, for some $1 > \epsilon > 0$,

$$0 \leq p \cdot \sum_{i \in \mathcal{G}} \left(z^i - x^i\right) \leq -\epsilon p \cdot \sum_{i \in \mathcal{G}} \left(z^i - x^i\right)^- < 0,$$

a contradiction. □

Proof of proposition 3. Necessity follows, through lemma 9, from lemma 10, whereas sufficiency is proved by lemma 11. □

Lemma 9. *Given $1 > \rho > 0$, there exists $v > 0$ in L satisfying $\rho^n \bigwedge_{p \in P(q)} T_p^n(v) \geq v$ only if there exists $e > 0$ in L satisfying $\rho \bigwedge_{p \in P(q)} T_p(e) \geq e$.*

Proof of lemma 9. Preliminarily, observe that the cone of prices satisfies the following decomposition property: for every σ in $\mathcal{S}$, one might define a map $f_\sigma : P(q) \times P(q)^{\mathcal{S}} \to P(q)$ by setting

$$f_\sigma\left(p, (p^\nu)_{\nu \in \mathcal{S}}\right) = p - \sum_{\tau \in \sigma_+} p_{\mathcal{S}(\tau)} + \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\tau^\tau}\right) p_{\mathcal{S}(\tau)}^\tau,$$

where, by notational convention, given an element v of $\mathbb{R}^{\mathcal{S}}$, for every σ in $\mathcal{S}$, $v_{\mathcal{S}(\sigma)}$ is the element of $\mathbb{R}^{\mathcal{S}}$ coinciding with v on $\mathcal{S}(\sigma)$ and vanishing on $(\mathcal{S}/\mathcal{S}(\sigma))$. Price $f_\sigma\left(p, (p^\nu)_{\nu \in \mathcal{S}}\right)$ follows price p up to date-events in σ_+ and (normalized) price p^τ beginning from date-event τ in σ_+.

Peg any m in $\mathbb{N}$. We shall show that, given any p in $P(q)$, for every e in L,

$$T_p\left(\bigwedge_{p^* \in P(q)} T_{p^*}^m(e)\right) \geq \bigwedge_{p^* \in P(q)} T_{p^*}^{m+1}(e).$$

To this purpose, observe that, for every ν in $\mathcal{S}$,

$$\left(\bigwedge_{p^* \in P(q)} T_{p^*}^m(e)\right)_\nu = \bigwedge_{p^* \in P(q)} T_{p^*}^m(e)_\nu.$$

Thus, given $\eta > 0$, for every ν in $\mathcal{S}$, there exists p^ν in $P(q)$ such that

$$\bigwedge_{p^* \in P(q)} T_{p^*}^m(e)_\nu + \eta \geq T_{p^\nu}^m(e)_\nu.$$

Given any σ in $\mathcal{S}$, exploiting the positivity of T_p, by direct computation, one verifies that

$$\begin{aligned}
T_p\left(\bigwedge_{p^*\in P(q)} T^m_{p^*}(e)\right)_\sigma + \eta T_p(u)_\sigma &= \\
T_p\left(\bigwedge_{p^*\in P(q)} T^m_{p^*}(e) + \eta u\right)_\sigma &= \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau \left(\bigwedge_{p^*\in P(q)} T^m_{p^*}(e)_\tau + \eta\right) \\
&\geq \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau T^m_{p^\tau}(e)_\tau \\
&= T^{m+1}_{f_\sigma\left(p,(p^\nu)_{\nu\in\mathcal{S}}\right)}(e)_\sigma \\
&\geq \bigwedge_{p^*\in P(q)} T^{m+1}_{p^*}(e)_\sigma .
\end{aligned}$$

As $\eta > 0$ is arbitrarily small, this proves the claim.

Define

$$e = v + \rho \bigwedge_{p^*\in P(q)} T_{p^*}(v) + \cdots + \rho^{n-1} \bigwedge_{p^*\in P(q)} T^{n-1}_{p^*}(v) .$$

By positivity of operators $(T_{p^*})_{p^*\in P(q)}$, $e \geq v > 0$. It follows that

$$\begin{aligned}
\rho T_p(e) &= \rho T_p(v) + \cdots + \rho^n T_p\left(\bigwedge_{p^*\in P(q)} T^{n-1}_{p^*}(v)\right) \\
&\geq \rho \bigwedge_{p^*\in P(q)} T_{p^*}(v) + \cdots + \rho^{n-1}\bigwedge_{p^*\in P(q)} T^{n-1}_{p^*}(v) + \rho^n \bigwedge_{p^*\in P(q)} T^n_{p^*}(v) \\
&= e + \rho^n \bigwedge_{p^*\in P(q)} T^n_{p^*}(v) - v \\
&\geq e,
\end{aligned}$$

which proves the claim. □

Lemma 10. *Under the hypothesis of non-vanishing (gross) interest rates, an allocation x in X, with supporting asset price q in Q, is unambiguously robustly inefficient only if condition* (FOC) *is satisfied.*

Proof of lemma 10. Using the notation of lemma 7, by an analogous argument, one obtains that, for every price p in $P(q)$, at every σ in $\mathcal{S}$,

$$\text{(**)} \qquad \left(\frac{1}{1+\epsilon}\right)\sum_{\tau\in\langle\sigma\rangle_+} p\cdot\sum_{i\in\mathcal{G}^-_\tau}\left(z^i - x^i\right)_{\langle\tau\rangle} \geq p\cdot\sum_{i\in\mathcal{G}^-_\sigma}\left(z^i - x^i\right)_{\langle\sigma\rangle} .$$

Let a positive element v of $\mathbb{R}^{\mathcal{S}}$ be defined, for every σ in $\mathcal{S}$, by

$$v_\sigma = \left(\bigwedge_{p^*\in P(q)} \frac{1}{p^*_\sigma} p^*\cdot\sum_{i\in\mathcal{G}^-_\sigma}\left(z^i - x^i\right)_{\langle\sigma\rangle}\right)^+ .$$

In fact, as the hypothesis of non-vanishing (gross) interest rates guarantees boundedness (see (†)), v is a positive element of L. As in the proof of lemma 9, for every σ in $\mathcal{S}$, one might define a map $f^n_\sigma : P(q)\times P(q)^{\mathcal{S}} \to P(q)$ by setting

$$f^n_\sigma\left(p,(p^\nu)_{\nu\in\mathcal{S}}\right) = p - \sum_{\tau\in\langle\sigma\rangle_+} p_{\mathcal{S}(\tau)} + \sum_{\tau\in\langle\sigma\rangle_+}\left(\frac{p_\tau}{p^\tau_\tau}\right) p^\tau_{\mathcal{S}(\tau)} .$$

As a matter of fact, price $f_\sigma^n\left(p,(p^\nu)_{\nu\in\mathcal{S}}\right)$ follows price p up to date-events in $\langle\sigma\rangle_+$ and (normalized) price p^τ beginning from date-event τ in $\langle\sigma\rangle_+$. Notice that, given $\eta>0$, for every ν in $\mathcal{S}$, there exists p^ν in $P(q)$ such that

$$v_\nu+\eta\geq\left(\frac{1}{p_\nu^\nu}p^\nu\cdot\sum_{i\in\mathcal{G}_\nu^-}\left(z^i-x^i\right)_{\langle\nu\rangle}\right)^+.$$

Given p in $P(q)$, for every σ in $\mathcal{S}$, evaluating inequality (**) at price $f_\sigma^n\left(p,(p^\nu)_{\nu\in\mathcal{S}}\right)$ in $P(q)$ yields

$$\begin{aligned}\rho^n\sum_{\tau\in\langle\sigma\rangle_+}p_\tau v_\tau+\eta\rho^n\sum_{\tau\in\langle\sigma\rangle_+}p_\tau &=\\ \rho^n\sum_{\tau\in\langle\sigma\rangle_+}p_\tau\left(v_\tau+\eta\right) &\geq\\ \rho^n\sum_{\tau\in\langle\sigma\rangle_+}p_\tau\left(\frac{1}{p_\tau^\tau}p^\tau\cdot\sum_{i\in\mathcal{G}_\tau^-}\left(z^i-x^i\right)_{\langle\tau\rangle}\right)^+ &\geq p_\sigma\left(\frac{1}{p_\sigma}p\cdot\sum_{i\in\mathcal{G}_\sigma^-}\left(z^i-x^i\right)_{\langle\sigma\rangle}\right)^+\\ &\geq p_\sigma v_\sigma,\end{aligned}$$

where

$$\rho=\left(\frac{1}{1+\epsilon}\right)^{\frac{1}{n}}.$$

That is, as $\eta>0$ is arbitrarily small, there exists a positive element v of L satisfying, for every p in $P(q)$, $\rho^nT_p^n(v)\geq v$. Exploiting lemma 9, the claim is proved, provided that v is a non-null element of L. This follows from an argument that is analogous to that in the last part of the proof of lemma 6. □

Lemma 11. *An interior allocation x in X, with smoothly supporting asset price q in Q, is robustly inefficient if condition* (FOC) *is satisfied.*

Proof of lemma 11. At no loss of generality, it can be assumed that there exists $1>\epsilon>0$ such that, at every σ in $\mathcal{S}$,

$$(1-\epsilon)\,\rho\sum_{\tau\in\sigma_+}p_\tau e_\tau\geq p_\sigma e_\sigma,\ \text{for every } p\in P(q).$$

Let $\lambda>0$ be given by the hypothesis of smooth support at $1>\rho>0$ and, at no loss of generality, assume that $\|e\|\leq 1$ and that $x^i\geq\lambda u^i$ for every individual i in $\mathcal{G}$, where u^i is the unit of L^i. Thus, define an alternative allocation z in X by setting, for every σ in $\mathcal{S}$,

$$z^{i(\sigma)}=x^{i(\sigma)}-\lambda e_\sigma+(1-\epsilon)\sum_{\tau\in\sigma_+}\lambda e_\tau.$$

Feasibility is obviously satisfied, along with a positive destruction of transferred resources, and smooth support guarantees a welfare improvement, redistributing resources made available by the above transfers. In addition, condition (§) is obviously satisfied. □