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Abstract

We propose a complete version of the sequential equilibrium (CSE) and its alternative solution concept (WCSE) for general finite-period games with observed actions. The sequential equilibrium (SE) is not a complete solution concept in that it might not be a Nash equilibrium in the general games that allow a continuum of types and strategies. The CSE is always a Nash equilibrium and is equivalent to the SE in finite games. So, the CSE is a complete solution concept in the general games as a version of the SE. The WCSE is a weak, but simple version of the CSE. It is also a complete solution concept and functions as an alternative solution concept to the CSE. Their relation with converted versions of the perfect equilibrium and the perfect Bayesian equilibrium are discussed.

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1 Introduction

We propose a complete version of the sequential equilibrium and its alternative solution concept. Kreps and Wilson (1982) introduced the sequential equilibrium in the setting of finite games that allow only a finite number of types and strategies. As shown by them, the sequential equilibrium is appropriate for the finite games. However, it might be inappropriate for general games that can have a continuum of types and strategies. This inappropriateness makes the sequential equilibrium an incomplete solution concept in the general games[1] . In the present study, we attempt to develop a complete solution concept in the general games by improving the sequential equilibrium. Then, we simplify this complete solution concept to find its weak version as its alternative solution concept.

Section 2 illustrates the incompleteness of the sequential equilibrium in the general games with an example. Section 3 formulates a general finite-period games with observed actions. Section 4 introduces new concepts, complete beliefs and sequential convergency, and lays the foundations of a complete version of the sequential equilibrium. Section 5 defines the complete version of the sequential equilibrium and derives some results on it. Section 6 suggests an alternative solution concept to the complete version and analyzes their relation. Section 7 concludes with the discussion about how these two solution concepts are related to converted versions of the perfect equilibrium and the perfect Bayesian equilibrium.

[1] In this sense, the perfect Bayesian equilibrium by Fudenberg and Tirole (1991a) is also an incomplete solution concept in the general games.

2 Example: Incompleteness of the sequential equilibrium in general games

Consider the information transmission game introduced by Crawford and Sobel (1982). There are two players; a sender and a receiver. The sender is assigned a type θ that is a random variable from a uniform distribution on $[0, 1]$ and makes a signal $s \in [0, 1]$ to the receiver. Then, after observing the signal s, the receiver chooses his action $a \in [0, 1]$. The sender has a von Neumann-Morgenstern utility function $U^S(\theta, a, b) = -(\theta - (a + b))^2$ where $b > 0$ and the receiver has another von Neumann-Morgenstern utility function $U^R(\theta, a)$ $= -(\theta - a)^2$.

In this game, the sender's strategy $s(\theta) = \theta$ and the receiver's strategy $a(s) = \max\{s - b, 0\}$ are a sequential equilibrium together with the receiver's system of beliefs $\mu(\max\{s - b, 0\}; s) = 1$ which denotes that given a signal s, the type $\max\{s - b, 0\}$ would be assigned to the sender with probability one. This is because the strategies $s(\theta) = \theta$ and $a(s) = \max\{s-b, 0\}$ are sequentially rational with respect to the system of beliefs $\mu(\max\{s-b, 0\}; s)$ $= 1$. In addition, under the sender's strategy $s(\theta) = \theta$, each signal θ occurs with probability zero, and thus every system of beliefs does not violate Bayes' rule. As a result, the receiver's system of beliefs $\mu(\max\{s - b, 0\}; s) = 1$ is consistent with the sender's strategy $s(\theta) = \theta$ according to Bayes' rule, and therefore these strategies and the system of beliefs are a sequential equilibrium[2] .

This sequential equilibrium, however, is not an appropriate prediction of behavior in that

[2] Likewise, we can show that they are a perfect Bayesian equilibrium as well.

the receiver makes a systematic mistake in forming her beliefs, and moreover it is not even a Nash equilibrium. In the scenario of this equilibrium, the receiver constantly mistakes a true type θ for a wrong type $\max\{\theta - b, 0\}$ even though she expects the sender to signal truthfully $s(\theta) = \theta$. Furthermore, the receiver's strategy $a(s) = \max\{s - b, 0\}$ is not a best response to the sender's strategy $s(\theta) = \theta$, and so this sequential equilibrium is not a Nash equilibrium. Consequently, this sequential equilibrium is not an appropriate prediction, and therefore it is not an appropriate solution concept in this game.

This result of the game depends mainly on the setting that the sender has a continuum of types and signals. Accordingly, most games that have similar settings can testify that the sequential equilibrium might not be an appropriate solution concept. Since this setting represents a general situation, there are a large class of games in the general games that include similar settings. Therefore, we conclude that the sequential equilibrium is an incomplete solution concept in the general games.

3 General finite-period games with observed actions

We adopt the "finite-period games with observed actions" from Fudenberg and Tirole (1991a) and adapt it for general games that allow infinite actions and types, but finite players. Hence, in the general finite-period games with observed actions, there are a finite number of *players* denoted by $i = 1, 2, ..., I$. Each player i has a *type* $\theta_i \in \Theta_i$ and this type is her private information as in Harsanyi (1967–68). In addition, there exists a *state* $\theta_0 \in \Theta_0$ and the players do not know an actual state when they play. Thus, each player has information on her type θ_i, but no information on the other players' types and the state $\theta_{-i} \in \Theta_{-i} = \Theta_0 \times (\times_{i' \neq i} \Theta_{i'})$. We assume that $\Theta = \times_{i=0}^{I} \Theta_i$ is a non-empty metric space. All players have the same *prior distribution* η on Θ such that η is a probability measure on the class of the Borel subsets[3] $\times_{i=0}^{I} \beta(\Theta_i)$ of Θ. For simplicity, we assume $\eta(B) > 0$ for every open subset B in Θ.

The players play the game in *periods* $t = 1, 2, ..., T$. At each period t, all players simultaneously choose an action, and then their actions are revealed at the end of the period. We assume, for simplicity, that each player's available actions are independent of her type so that each player i's *action space* at period t is A_i^t regardless of her type. In addition, we assume that $A^t = \times_{i=1}^{I} A_i^t$ is a nonempty metric space[4] for each t. Finally, we consider only the perfect recall games introduced by Kuhn (1950).

A strategy is defined as follows. For each $i = 1, ..., I$ and $t = 1, ..., T$, let δ_i^t be a measure from $\Theta_i \times A^1 \times \cdots \times A^{t-1} \times \beta(A_i^t)$ to $[0, 1]$. Then, a behavioral *strategy* δ_i is an ordered set of measures $\delta_i = (\delta_i^1, ..., \delta_i^T)$ such that $i)$ for each $(\theta_i, a^1, ..., a^{t-1}) \in \Theta_i \times A^1 \times \cdots \times A^{t-1}$, $\delta_i^t(\theta_i, a^1, ..., a^{t-1}; \cdot)$ is a probability measure on $\beta(A_i^t)$ and $ii)$ for every $B \in \beta(A_i^t)$, $\delta_i^t(\cdot; B)$ is $\beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^{I} \beta(A_{i'}^{t'}))$ measurable. The condition $i)$ requires that each $\delta_i^t(\theta_i, a^1, ..., a^{t-1}; \cdot)$ specify what to play at each information set $(\theta_i, a^1, ..., a^{t-1})$. The condition $ii)$ requires that δ_i^t allow an well-defined expected utility functional defined later. Hereafter, we simply call a behavioral strategy a strategy. Let Π_i be the *set of strategies*

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[3] Given a metric space X, the class of the Borel sets $\beta(X)$ is the smallest class of subsets of X such that $i)$ $\beta(X)$ contains all open subsets of X and $ii)$ $\beta(X)$ is closed under countable unions and complements.

[4] Therefore, the space $\Theta \times A^1 \times \cdots \times A^T$ is a nonempty metric space. On this space, expected utility functionals are well defined according to Ash (1972, 2.6).

for player i and let Π be the *set of strategy profiles*, *i.e.* $\Pi = \times_{i=1}^{I}\Pi_i$. Note that these definitions originated from Milgrom and Weber (1985) and Balder (1988) and are adapted for the general finite-period games with observed actions[5] .

A *Von Neumann-Morgenstern utility function* for player i is defined as $U_i : \Theta \times A^1 \times \cdots \times A^T \longrightarrow \mathbb{R}$. We assume that each U_i is bounded and $\times_{i=0}^{I}\beta(\Theta_i) \times (\times_{t=1}^{T}\times_{i=1}^{I}\beta(A_i^t))$ measurable, which guarantees that U_i is integrable. In addition, we define an *expected utility functional* $E_i : \Pi \longrightarrow \mathbb{R}$ as

$$E_i(\delta_1, ..., \delta_n) = \int\limits_{\Theta}\int\limits_{A^1}\cdots\int\limits_{A^T} U_i(\theta, a)\delta^T(\theta, a^1, ..., a^{T-1}; da^T)\cdots\delta^1(\theta; da^1)\eta(d\theta)$$

where for each t, δ^t denotes the product measure of $\{\delta_1^t, ..., \delta_I^t\}$ on $\times_{i=1}^{I}\beta(A_i^t)$, *i.e.* $\delta^t = \delta_1^t \times \cdots \times \delta_I^t$. This definition of the expected utility functional makes sense according to Ash (1972, 2.6)[6] . First, since each δ_i^t is a probability measure and measurable, so is the product measure δ^t. Next, since $U_i(\theta, a)$ and $\delta^T(\theta, a^1, ..., a^{T-1}; da^T)$ are $\times_{i=0}^{I}\beta(\Theta_i)\times(\times_{t=1}^{T-1}\times_{i=1}^{I}\beta(A_i^t))$ measurable, so is the inner part of the integral $\int_{A^T} U_i(\theta, a)\delta^T(\theta, a^1, ..., a^{T-1}; da^T)$, and thus it is $\times_{i=1}^{I}\beta(A_i^{T-1})$ measurable. Furthermore, the inner integral is bounded, so it is integrable with respect to the measure δ^{T-1}. Finally, we can show each part of the integral is integrable likewise, and therefore the whole integral is well-defined.

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[5] For each i and $t \geq 2$, the measure $\delta_i^t(\cdot; \cdot)$ is known as a transition probability. For more information on the transition probability, please refer to Neveu (1965, III), Ash (1972, 2.6), and Uglanov (1997).

[6] Let $\mathcal{F}_j$ be a $\sigma - field$ of subsets of Ω_j for each $j = 1, ..., n$. Let μ_1 be a probability measure on $\mathcal{F}_1$, and , for each $(\omega_1, ..., \omega_j) \in \Omega_1 \times \cdots \times \Omega_j$, let $\mu(\omega_1, ..., \omega_j; B)$, $B \in \mathcal{F}_{j+1}$, be a probability measure on $\mathcal{F}_{j+1}$ $(j = 1, 2, ..., n-1)$. Assume that $\mu(\omega_1, ..., \omega_j; C)$ is measurable for each fixed $C \in \mathcal{F}_{j+1}$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ and $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$.

(1) There is a unique probability measure μ on $\mathcal{F}$ such that for each measurable rectangle $A_1 \times \cdots \times A_n \in \mathcal{F}$, $\mu(A_1 \times \cdots \times A_n) = \int_{A_1}\int_{A_2}\cdots\int_{A_n}\mu(\omega_1, ... , \omega_{n-1}; d\omega_n)\cdots\mu(\omega_2; d\omega_1)\mu_1(d\omega_1)$.

(2) Let $f : (\Omega, \mathcal{F}) \longrightarrow ([0,1], \beta([0,1]))$ and $f \geq 0$. Then, $\int_{\Omega} f d\mu = \int_{\Omega_1}\cdots\int_{\Omega_n} f(\omega_1, ... , \omega_n)\mu(\omega_1, ... , \omega_{n-1}; d\omega_n)\cdots\mu_1(d\omega_1)$.

Based on this expected utility functional, the *Nash equilibrium* by Nash (1951) is extended in the general games. The Nash equilibrium condition is used as a minimum requirement for a complete solution concept in the general games. Accordingly, given any solution concept, its complete version is required to satisfy at least the Nash equilibrium condition.

Definition 1 *A strategy profile $\delta = (\delta_1 \cdots \delta_I)$ is a* ***Nash equilibrium*** *if δ satisfies $E_i(\delta) \geq \max_{\delta_i' \in \Pi_i} E_i(\delta_i', \delta_{-i})$ for each $i \leq I$.*

4 Complete beliefs and Sequential convergency

In the setting of the general games, we develop a complete version of the sequential equilibrium. This complete version is called a complete sequential equilibrium. To be brief, a complete sequential equilibrium is a pair of a system of *complete beliefs* and a strategy profile such that *i*) the system of complete beliefs is consistent with the strategy profile through a *sequentially convergent* sequence of strategy profiles and *ii*) the strategy profile is sequentially rational with respect to the system of complete beliefs[7] . Thus, the complete sequential equilibrium bases on new concepts, complete beliefs and sequential convergency. Accordingly, this section is devoted to formulate these new concepts.

Complete beliefs are counterparts of beliefs in the sequential equilibrium. The sequential equilibrium consists of two components, a system of beliefs and a strategy profile. The problem with the sequential equilibrium in the general games, incompleteness, is caused by a weakness of the beliefs. The complete sequential equilibrium solves the problem with the

[7] Note that a sequential equilibrium is a pair of a system of beliefs and a strategy profile such that *i*) the system of beliefs is consistent with the strategy profile and *ii*) the strategy profile is sequentially rational with respect to the system of beliefs.

sequential equilibrium by improving the weakness of the beliefs. *Complete beliefs* are the improved version of the beliefs.

Definition 2 *For each i and t, a probability measure μ_i^t on $\times_{i=0}^{I} ß(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} ß(A_i^{t'}))$ is called a* ***complete belief*** *for player i at period t if $\mu_i^t(\Theta_{-i} \times B_i) > 0$ for every open set $B_i \subset \Theta_i \times A^1 \times \cdots \times A^{t-1}$. For each t, let μ^t denote $(\mu_1^t, ..., \mu_I^t)$, then a* ***system of complete beliefs*** *is an ordered set of complete beliefs $\mu = (\mu^1, ..., \mu^T)$.*

The property of a system of complete beliefs depends on the property of the space $\Theta \times A^1 \times \cdots \times A^T$. Since the space is metric, any system of complete beliefs consists of regular probability measures[8] according to Theorem 1.1 in Billingsley (1968). If the metric space is also separable and complete, then every system consists of tight measures[9] according to Theorem 1.4 in Billingsley (1968). In addition, the condition for a complete belief $\mu_i^t(\Theta_{-i} \times B_i) > 0$ for every open set B_i assures that with respect to a complete belief, we can check sequential rationality of a strategy profile in every open class of information sets off the equilibrium path as well as on the equilibrium path.

In the general games, particularly continuous games that have uncountably many types and strategies, the sequential equilibrium concept might not well-define a consistent relation between a strategy profile, which is defined on the whole class of information sets at each period, and a system of beliefs, which are defined on each information set. This is because some information sets might have probability zero with respect to the strategy profile, which implies those information sets are impossible to happen according to the strategy profile.

[8] By Billingsley (1968), a probability measure ρ on $ß(X)$ of a metric space X is defined to be regular if for any $B \in ß(X)$ and $\varepsilon > 0$, there exist a closed set G and an open set O such that $G \subset B \subset O$ and $\rho(O - G) < \varepsilon$.

[9] A probability measure ρ on $ß(X)$ of a metric space X is tight if for any $B \in ß(X)$, $\rho(B)$ is the supremum of $\rho(K)$ over the compact subsets K of A.

Therefore, the strategy profile cannot define how those information sets occur, and thus it cannot be consistent with any beliefs on those information sets. This explains why the sequential equilibrium sometimes fails to exclude an inconsistent system of beliefs in the general games. The system of complete beliefs, on the other hand, is defined on the whole class of information sets at each period. This change of the domains makes a consistent relation between a system of complete beliefs and a strategy profile well-defined, and as a result, an inconsistent system of complete beliefs would be excluded.

The complete sequential equilibrium indirectly defines the consistent relation between a system of complete beliefs and a strategy profile by using a sequence of strategy profiles that is related to both of them, as the sequential equilibrium does. Definition 3 presents conditions for such a sequence of strategy profiles that can show the consistent relation. For notational convenience, we define a probability measure ϕ with respect to a strategy profile δ as

$$\phi(B;\delta) = \int_{B^1} \int_{B^2(\theta)} \cdots \int_{B^t(\theta,a^1,...,a^{t-2})} \delta^{t-1}(\theta, a^1, ..., a^{t-2}; da^{t-1}) \cdots \delta^1(\theta; da^1)\eta(d\theta) \qquad (1)$$

for every set $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$ where $B^1 = \{\theta \in \Theta : (\theta, a^1, ..., a^{t-1}) \in B\}$, *i.e.* B^1 is the projection of B onto Θ, and for $t' = 2, ..., t$, $B^{t'}(\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t'-2}) = \{a^{t'-1} \in A^{t'-1} : (\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t'-2}, a^{t'-1}, ..., a^{t-1}) \in B\}$, *i.e.* $B^{t'}(\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t'-2})$ is the projection of B onto $\{(\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t'-2})\} \times A^{t'-1}$. The probability measure ϕ is well-defined according to Ash (1972, 2.6). Furthermore, given a sequence of pairwise disjoint Borel subsets $\{K_{i,j}^t\}$ in $\Theta_i \times A^1 \times \cdots$

$\cdot \times A^{t-1}$ and a sequence of strategy profiles $\{\delta_n\}$, define a measure $\nu_{j,n}$ as

$$\nu_{j,n}(B) = \frac{\phi(B \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)}{\phi(\Theta \times A^1 \times \cdots \times A^{t-1} \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)} \tag{2}$$

for each $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$. Then, for each i and t, if $\nu_{j,n}$ is well-defined, it will be a probability measure on $\times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$ since so is $\phi(\cdot; \delta_n)$. In detail, we have $\nu_{j,n}(\Theta \times A^1 \times \cdots \times A^{t-1}) = \frac{\phi(\Theta \times A^1 \times \cdots \times A^{t-1} \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)}{\phi(\Theta \times A^1 \times \cdots \times A^{t-1} \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)} = 1$ and $\nu_{j,n}(B) \geq 0$ for each $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$. In addition, for any disjoint countable union of Borel subsets $\cup_{e \in E} B_e$, we have $\nu_{j,n}(\cup_{e \in E} B_e) = \frac{\phi(\cup_{e \in E} B_e \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)}{\phi(\Theta \times A^1 \times \cdots \times A^{t-1} \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)}$ $= \sum_{e \in E} \frac{\phi(B_e \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)}{\phi(\Theta \times A^1 \times \cdots \times A^{t-1} \setminus \Theta_{-i} \times \cup_{e<j} K^t_{i,e}; \delta_n)} = \sum_{e \in E} \nu_{j,n}(B_e)$. Therefore, $\nu_{j,n}$ is a probability measure if it exists.

Definition 3 *A sequence of strategy profiles $\{\delta_n\}_{n=1}^{\infty}$ is **sequentially convergent** if for each i and t, there exists a sequence of pairwise disjoint Borel sets $\{K^t_{i,j}\}_{j \in J_i^t}$ in $\Theta_i \times A^1 \times \cdots \times A^{t-1}$ with an index set $J_i^t \subset \mathbb{N}$ such that for each j, i) the probability measure $\nu_{j,n}$ defined by (2) is well-defined for every n and converges weakly*[10] *to some probability measure ν_j on $\times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$; ii) $K^t_{i,j} = \bigcap \{G_i : G_i$ is relatively closed in $\Theta_i \times A^1 \times \cdots \times A^{t-1} \setminus \cup_{e<j} K^t_{i,e}$ and $\nu_j(\Theta_{-i} \times G_i) = 1\}$; and iii) $\cup_{j \in J_i^t} K^t_{i,j}$ is dense. Here, the sequence $\{K^t_{i,j}\}$ is called **sequential supports** of $\{\delta_n\}$ for player i at period t.*

To resolve the abstractness of this definition, we examine how a sequentially convergent sequence of strategy profiles $\{\delta_n\}$ operates. First, the sequence $\{\delta_n\}$ defines a sequence of probability measures $\{\nu_{1,n}\}$ and this sequence $\{\nu_{1,n}\}$ converges weakly to a probability measure ν_1. Then, the measure ν_1 has the smallest support $K^t_{i,1}$ that is closed. Next, the sequence $\{\delta_n\}$ and the support $K^t_{i,1}$ together define a sequence of probability measures $\{\nu_{2,n}\}$ and this sequence $\{\nu_{2,n}\}$ converges weakly to a probability measure ν_2. Then again,

[10] *A measure $\nu_{j,n}$ converges weakly to ν_j if $\lim_{n \longrightarrow \infty} \int_{\Theta \times A^1 \times \cdots \times A^{t-1}} f(\theta, a) d\nu_{j,n}(\theta, a) = \int_{\Theta \times A^1 \times \cdots \times A^{t-1}} f(\theta, a) d\nu_j(\theta, a)$ for every bounded and continuous real function f on $\Theta \times A^1 \times \cdots \times A^{t-1}$.*

the measure ν_2 has the smallest support $K^t_{i,2}$ that is relatively closed in $\Theta \times A^1 \times \cdots \times A^{t-1} \setminus \Theta_{-i} \times K^t_{i,1}$. Likewise, for each $j \geq 3$, the sequence $\{\delta_n\}$ and the supports $\{K^t_{i,e}\}_{e=1}^{j-1}$ determine a sequence of probability measures $\{\nu_{j,n}\}$, a probability measure ν_j, and the support $K^t_{i,j}$ until $\cup_{e \leq j} K^t_{i,e}$ becomes dense.

To sum up, a sequence of strategy profiles $\{\delta_n\}$ is sequentially convergent if $\{\delta_n\}$ together with its sequential supports $\{K^t_{i,j}\}$, whose union is dense, sequentially define a sequence of probability measures $\{\nu_{j,n}\}$ such that each sequence $\{\nu_{j,n}\}_{n=1}^{\infty}$ converges weakly to a probability measure ν_j and ν_j has the smallest and relatively closed support $K^t_{i,j}$. In particular, the condition i) requires that $\{\delta_n\}$ and $\{K^t_{i,j}\}$ sequentially well-define $\{\nu_{j,n}\}$ and $\{\nu_j\}$. The condition ii) requires that $K^t_{i,j}$ be the smallest and relatively closed support of ν_j so that $K^t_{i,j}$ is uniquely determined and every open set in $K^t_{i,j}$ has positive measure with respect to ν_j. Finally, the condition iii) requires that the union of the supports $\cup_{j \in J^t_i} K^t_{i,j}$ be dense so that $\{K^t_{i,j}\}$ fills the whole space fully enough.

The sequential equilibrium also uses a sequence of strategy profiles to show the consistent relation between a system of beliefs and a strategy profile. Concretely, it uses a convergent sequence of totally mixed strategy profiles. Here, a strategy profile $\mathring{\delta}$ is defined to be totally mixed if for each t, we have $\mathring{\delta}^t_i(\theta_i, a^1, ..., a^{t-1}; B) > 0$ for every $(\theta_i, a^1, ..., a^{t-1}) \in \Theta_i \times A^1 \times \cdots \times A^{t-1}$ and for every open set $B \subset A^t_i$. In finite games, a convergent sequence of totally mixed strategy profiles is sequentially convergent and well-defines the consistent relation as shown in Kreps and Wilson (1982). In general games, however, it might not be sequentially convergent and might not well-define the consistent relation as shown in Example 1.

Example 1 *Consider the game in Crawford and Sobel (1982) again. Define the sender's strategy* $\mathring{\delta}_n$ *as* $\mathring{\delta}_n(\theta; B) = \frac{1}{n+1} l(B) + \frac{n^2}{n+1} l(B \cap [0, \frac{1}{n}])$ *for every* $\theta \in [0,1]$ *and* $B \in \beta([0,1])$ *where* $l : \beta([0,1]) \longrightarrow [0,1]$ *is a Lebesgue measure. Then, the sequence* $\{\mathring{\delta}_n\}$ *is convergent and each* $\mathring{\delta}_n$ *is totally mixed. However, it is not sequentially convergent, and moreover, it does not well-define the consistent relation.*

Although a convergent sequence of totally mixed strategy profiles is not sufficient to be sequentially convergent, it is still useful to construct a sequentially convergent sequence in the general games. Example 2 demonstrates this construction. Consequently, Example 2 provides sufficient conditions for a sequence of strategy profiles to be sequentially convergent.

Example 2 *Suppose that a sequence of strategy profiles* $\{\mathring{\delta}_n\}_{n=1}^{\infty}$ *satisfies the following two conditions. First, each* $\mathring{\delta}_n$ *in the sequence is totally mixed. Second, there exist a set of strategy profiles* $\{\delta_e\}_{e \in E \cup \{\alpha\}}$ *where* $E \subset \mathbb{N}$ *and a sequence of positive real numbers* $\{\varepsilon_n\}$ *such that for each* n, $\mathring{\delta}_n = (1 - \sum_{e \in E} (\varepsilon_n)^e) \cdot \delta_\alpha + \sum_{e \in E} (\varepsilon_n)^e \cdot \delta_e$ *and* ε_n *converges to zero. Then, the sequence* $\{\mathring{\delta}_n\}$ *is sequentially convergent.*

The following Lemmas 1 and 2 reveal properties of the sequentially convergent sequence of strategy profiles. Specifically, Lemma 1 shows that every open class of information sets has positive measure with respect to probability measures induced by a sequentially convergent sequence. Lemma 2 proves that a linear combination of probability measures induced by a sequentially convergent sequence can be a probability measure itself. These two lemmas establish a way to define a system of complete beliefs with respect to a sequentially convergent sequence of strategy profiles.

Lemma 1 *If a sequence of strategy profiles* $\{\delta_n\}_{n=1}^{\infty}$ *is sequentially convergent, then given* i *and* t*, for any nonempty open set* $O_i \in \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^{I} \beta(A_{i'}^{t'}))$*, there exists a set* $\tilde{K}_{i,j}^t$ *among the sequential supports* $\{K_{i,j}^t\}$ *of* $\{\delta_n\}$ *such that* $\nu_j(\Theta_{-i} \times (\tilde{K}_{i,j}^t \cap O_i)) > 0$ *where the probability measure* $\nu_{j,n}$ *defined by (2) converges weakly to* ν_j*.*

Proof. Since $\cup_{j\in J_i^t} K_{i,j}^t$ is dense in $\Theta_i \times A^1 \times \cdots \times A^{t-1}$, there exists a set $\tilde{K}_{i,j}^t$ in $\{K_{i,j}^t\}$ such that $\tilde{K}_{i,j}^t \cap O_i \neq \varnothing$. It suffices to show that $\nu_j(\Theta_{-i} \times (\tilde{K}_{i,j}^t \cap O_i)) > 0$. Let a subspace X_i be $\Theta_i \times A^1 \times \cdots \times A^{t-1} \setminus \cup_{e<j} K_{i,e}^t$. From the definition, $\tilde{K}_{i,j}^t$ is relatively closed in X_i. Since $X_i \cap O_i$ is relatively open in X_i, the set $\tilde{K}_{i,j}^t \setminus (X_i \cap O_i)$ $(= \tilde{K}_{i,j}^t \setminus O_i)$ is relatively closed in X_i. Since $\tilde{K}_{i,j}^t \nsubseteq \tilde{K}_{i,j}^t \setminus O_i$ and ν_j is a probability measure, we have $\nu_j(\Theta_{-i} \times (\tilde{K}_{i,j}^t \setminus O_i))$ < 1 according to the condition *ii)* in Definition 3. Therefore, we have $\nu_j(\Theta_{-i} \times (\tilde{K}_{i,j}^t \cap O_i))$ $= 1 - \nu(\Theta_{-i} \times (\tilde{K}_{i,j}^t \setminus O_i)) > 0$. ■

Lemma 2 *Let a sequence of strategy profiles $\{\delta_n\}_{n=1}^{\infty}$ be sequentially convergent with sequential supports $\{K_{i,j}^t\}_{j\in J_i^t}$ and let a function $p : J_i^t \longrightarrow [0,1]$ be a probability mass function, i.e. $\sum_{j\in J_i^t} p(j) = 1$ and $p(j) \geq 0$ for each j. Given i and t, for each $n \in \mathbb{N}$, suppose that a set function ν_n on $\times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$ satisfies*

$$\nu_n(B) = \textstyle\sum_{j\in J_i^t} p(j)\nu_{j,n}(B)$$

for every B where $\nu_{j,n}$ is defined by (2). Then, each ν_n is a probability measure. Moreover, ν_n converges weakly to the probability measure $\nu = \sum_{j\in J_i^t} p(j) \cdot \nu_j$ such that for each j, $\nu_{j,n}$ converges weakly to ν_j.

Proof. The first result follows from the fact that each $\nu_{j,n}$ is a probability measure on $\times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$. That is, we have $\nu_n(\Theta \times A^1 \times \cdots \times A^{t-1}) = \sum_{j\in J_i^t} p(j)_i^t \nu_{j,n}(\Theta \times A^1 \times \cdots \times A^{t-1}) = 1$ and $\nu_n(B) = \sum_{j\in J_i^t} p(j)\nu_{j,n}(B) \geq 0$ for each $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$. To show countable additivity, let $\cup_{e\in E} B_e$ be a pairwise disjoint countable union of Borel sets. Then,

$$\nu_n(\cup_{e\in E} B_e) = \textstyle\sum_{j\in J_i^t} p(j)\nu_{j,n}(\cup_{e\in E} B_e) = \sum_{j\in J_i^t} \sum_{e\in E} p(j)\nu_{j,n}(B_e)$$

$$= \textstyle\sum_{e\in E} \sum_{j\in J_i^t} \nu_{j,n}(B_e) \text{ since the series is absolutely convergent, } = \sum_{e\in E} \nu_n(B_e).$$

For the second assertion, since a sequence of strategy profiles $\{\delta_n\}$ is sequentially convergent with the sequential supports $\{K_{i,j}^t\}$, there exists a probability measure ν_j such that

$\nu_{j,n}$ converges weakly to ν_j and $\nu_j(\Theta_{-i} \times K^t_{i,j}) = 1$. For notational convenience, let a space X be $\Theta \times A^1 \times \cdots \times A^{t-1}$. Then, for any arbitrary bounded and continuous real function f on X, we have

$$\lim_{n \longrightarrow \infty} \int_X f d\nu_n = \lim_{n \longrightarrow \infty} \int_X f d \textstyle\sum_{j \in J^t_i} p(j)\nu_{j,n}$$

$$= \textstyle\sum_{j \in J} p(j) \displaystyle\lim_{n \longrightarrow \infty} \int_X f d\nu_{j,n} + \lim_{n \longrightarrow \infty} \int_X f d \textstyle\sum_{j \in J^t_i \setminus J} p(j)\nu_{j,n} \text{ for any finite subset } J \subset J^t_i,$$

$$= \textstyle\sum_{j \in J} p(j) \displaystyle\int_{\Theta_{-i} \times K^t_{i,j}} f d\nu_j + \lim_{n \longrightarrow \infty} \int_X f d \textstyle\sum_{j \in J^t_i \setminus J} p(j)\nu_{j,n} = \sum_{j \in J^t_i} p(j) \displaystyle\int_{\Theta_{-i} \times K^t_{i,j}} f d\nu_j.$$

The last equality holds because $\lim_{n \longrightarrow \infty} \int_X f d \sum_{j \in J^t_i \setminus J} p(j)\nu_{j,n}$ converges to zero as J approaches to J^t_i. Therefore, the measure $\nu_n = \sum_{j \in J^t_i} p(j)\nu_{j,n}$ converges weakly to $\nu = \sum_{j \in J^t_i} p(j)\nu_j$. Finally, the result that $\sum_{j \in J^t_i} p(j)\nu_j$ is a probability measure on $\times^I_{i=0}\beta(\Theta_i) \times (\times^{t-1}_{t'=1} \times^I_{i=1} \beta(A^{t'}_i))$ comes from the observation that each ν_j is a probability measure. ■

Proposition 1 combines Lemmas 1 and 2 and concludes that given a probability mass function, a sequentially convergent sequence uniquely defines a system of complete beliefs.

Proposition 1 *Let a sequence of strategy profiles $\{\delta_n\}^{\infty}_{n=1}$ be sequentially convergent. Then, given i and t, $\{\delta_n\}$ has a unique sequence of sequential supports $\{K^t_{i,j}\}_{j \in J^t_i}$. In addition, let a function $p : J^t_i \longrightarrow [0,1]$ be a probability mass function. Then, $\{\delta_n\}$ and p together define exactly one probability measure ν on $\times^I_{i=0}\beta(\Theta_i) \times (\times^{t-1}_{t'=1} \times^I_{i=1} \beta(A^{t'}_i))$ according to the same way as in Lemma 2. Furthermore, the probability measure ν is a complete belief for player i at period t.*

Proof. The uniqueness of the sequence of sequential supports results from the condition *ii)* in Definition 3; that is, each $K^t_{i,j}$ is the smallest and relatively closed subset in $\Theta_i \times A^1 \times \cdots \times A^{t-1} \setminus \cup_{e<j} K^t_{i,e}$ such that $\nu_j(\Theta_{-i} \times K^t_{i,j}) = 1$. The other results directly follow from Lemmas 1 and 2. ■

5 Complete sequential equilibrium

In this section, we provide a formal definition of the complete sequential equilibrium based on the results from the previous section and examine its properties. First, we define a *complete assessment* that is a counterpart of the assessment in the sequential equilibrium.

Definition 4 *An ordered pair of a system of complete beliefs and a strategy profile (μ, δ) is called a* ***complete assessment****.*

The complete sequential equilibrium, as a complete version of the sequential equilibrium, preserves all the conditions for the sequential equilibrium. Thus, it requires a complete assessment to satisfy both *i*) consistency and *ii*) sequential rationality. The first condition, *consistency*, is formulated in Definition 5.

Definition 5 *A complete assessment (μ, δ) is* ***consistent*** *if there exists a sequentially convergent sequence of strategy profiles $\{\delta_n\}$ such that i) δ_n converges weakly*[11] *to δ and ii) each complete belief μ_i^t satisfies*

$$\mu_i^t(B) = \textstyle\sum_{j \in J_i^t} (\frac{1}{2})^{\#\{e \in J_i^t : e \leq j \text{ and } e < \sup J_i^t\}} \nu_j(B) \tag{3}$$

for every set $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$ where the index set J_i^t and each probability measure ν_j are defined in the same way as in Definition 3 and $\#\{e \in J_i^t : e \leq j$ and $e < \sup J_i^t\}$ denotes the number of elements e in J_i^t such that $e \leq j$ and $e < \sup J_i^t$.

Definition 5 makes sense according to Proposition 1. Intuitive explanation of this definition is presented later when we compare the complete sequential equilibrium with the sequential equilibrium. Note that Definition 5 designates the probability mass function p as $p(j) = (\frac{1}{2})^{\#\{e \in J_i^t : e \leq j \text{ and } e < \sup J_i^t\}}$ for each $j \in J_i^t$. This designation of the probability mass

[11] *δ_n converges weakly to δ if $\int_\Theta \int_{A^1} \cdots \int_{A^T} f(\theta, a) \delta_n^T(\theta, a^1, ..., a^{T-1}; da^T) \cdots \delta_n^1(\theta; da^1) d\eta(\theta)$ converges to $\int_\Theta \int_{A^1} \cdots \int_{A^T} f(\theta, a) \delta^T(\theta, a^1, ..., a^{T-1}; da^T) \cdots \delta^1(\theta; da^1) d\eta(\theta)$ for every bounded and continuous real function f on $\Theta \times A^1 \times \cdots \times A^T$.*

function guarantees that a sequentially convergent sequence defines a unique system of complete beliefs by Proposition 1, and consequently it simplifies the definition.

Next, the second condition of the complete sequential equilibrium, *sequential rationality*, is defined in Definition 6. Let Ψ be the set of all systems of complete beliefs. For notational convenience, given i and t, define a *conditional expected utility functional* $E_i^t : \Psi \times \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^{I} \beta(A_{i'}^{t'})) \times \Pi \longrightarrow \mathbb{R}$ as $\mu_i^t(\Theta_{-i} \times G_i) \cdot E_i^t(\mu, G_i, \delta) = \int_{\Theta_{-i} \times G_i} \int_{A^t} \cdots \int_{A^T} U_i(\theta, a) \delta^T(\theta, a^1, ..., a^{T-1}; da^T) \cdots \delta^t(\theta, a^1, ..., a^{t-1}; da^t) d\mu_i^t(\theta, a^1, ..., a^{t-1})$.

Definition 6 *A strategy profile δ is* ***sequentially rational*** *with respect to a system of complete beliefs μ if for each i and t, we have $E_i^t(\mu, G_i, \delta) \geq E_i^t(\mu, G_i, (\delta_i', \delta_{-i}))$ for every $\delta_i' \in \Pi_i$ and for every $G_i \in \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^{I} \beta(A_{i'}^{t'}))$ such that $\mu_i^t(\Theta_{-i} \times G_i) > 0$.*

Here, G_i denotes a class of player i's information sets at period t. Thus, the sequential rationality requires, responding to the other players' strategies δ_{-i}, each player i to play her best response δ_i, which induces the greatest expected utility conditional on reaching a class of information sets G_i that have positive measure with respect to the system of complete beliefs μ, *i.e.* $\mu_i^t(\Theta_{-i} \times G_i) > 0$. As a result, no player prefers to change her strategy at any open class of information sets. Note that Kreps and Wilson (1982) described the sequential rationality as the condition under which 'taking the beliefs as fixed, no player prefers at any information set to change his part of the strategy.' Therefore, Definition 6 adapts the sequential rationality from the sequential equilibrium for the general finite-period games with observed actions by replacing "any information set" with "any open class of information sets."

Definition 7 defines the *complete sequential equilibrium*.

Definition 7 *A complete assessment* (μ, δ) *is a* ***complete sequential equilibrium*** *if* (μ, δ) *is both* 1) *consistent and* 2) *sequentially rational.*

Here are the results on the complete sequential equilibrium.

Proposition 2 *Every complete sequential equilibrium is a Nash equilibrium.*

Proof. The result directly follows from the definitions. ■

Proposition 3 shows the relation between the complete sequential equilibrium and the sequential equilibrium in finite games. For notational simplicity, we define a system of beliefs $\dot{\mu}$[12] associated with a system of complete beliefs μ as $\dot{\mu}(\mu)$. That is, $\dot{\mu}(\mu)$ is a system of beliefs inducing the same distributions on each information set as μ does. We can see that in finite games, every system of complete beliefs uniquely determines the associated system of beliefs $\dot{\mu}(\mu)$.

Proposition 3 *In finite games, a complete assessment* (μ, δ) *is consistent if and only if the assessment* $(\dot{\mu}(\mu), \delta)$ *is consistent.*

Proof. The result directly follows from the definitions. ■

Proposition 3 implies that in finite games, the complete sequential equilibrium satisfies two intuitive notions of consistency introduced by Kreps and Wilson (1982); *Structural consistency* and *Lexicographic consistency.* According to them, the structural consistency is defined as a consistency criterion under which beliefs of the players should reflect the informational structure of the game. In addition, the lexicographic consistency is meant to be another consistency criterion under which all players should hold the same sequence

[12] Kreps and Wilson (1982) defined a system of beliefs $\dot{\mu}$ as a function from a set of all decision nodes to $[0, 1]$ such that $\sum_{x \in h} \dot{\mu}(x) = 1$ for each information set h.

of hypotheses to play a game and whenever they fail to apply the most likely hypothesis to their situation, they should apply the next most likely hypothesis. Kreps and Wilson showed the sequential equilibrium satisfies these two consistency criteria in finite games. Since the complete sequential equilibrium is equivalent to the sequential equilibrium in the finite games, the complete sequential equilibrium also satisfies these two consistency criteria in the finite games.

Theorem 1 is a corollary of Proposition 3. Moreover, Theorem 1 and Proposition 2 evidence that the complete sequential equilibrium is indeed a complete version of the sequential equilibrium. That is, the complete sequential equilibrium is a complete solution concept in the general games as a version of the sequential equilibrium.

Theorem 1 *In finite games, a complete assessment* (μ, δ) *is a complete sequential equilibrium if and only if the assessment* $(\dot{\mu}(\mu), \delta)$ *is a sequential equilibrium.*

6 Alternative solution concept: Weak complete sequential equilibrium

Next, we introduce an alternative solution concept to the complete sequential equilibrium. The complete sequential equilibrium has many advantages in that it is at least a Nash equilibrium and in finite games, it is equivalent to the sequential equilibrium. It is, however, rather complicated. Moreover, in practice, a solution concept that is weaker, but simpler than the complete sequential equilibrium is reasonable enough to make plausible predictions as Fudenberg and Tirole (1991a) indicated. Therefore, we attempt to develop a weak, but simple version of the complete sequential equilibrium as its alternative solution concept.

The complexity of the complete sequential equilibrium arises mainly from its consistency condition. Hence, we can develop a weak, but simple version of the complete sequential equilibrium by relaxing its consistency condition. At length, we impose restrictions only on the equilibrium path, and thus no restriction off the equilibrium path. This version of the consistency is called *weak consistency* and is formulated in Definition 8. For notational convenience, we use the same probability measure ϕ defined by (1)[13] .

Definition 8 *A complete assessment* (μ, δ) *is* ***weakly consistent*** *if i) for every* i*, we have* $\mu_i^1 = \eta$ *and ii) for each* i *and* $t \geq 2$*, there exists* $p_i^t \in (0, 1]$ *such that* $\mu_i^t(B) = p_i^t \cdot \phi(B; \delta)$ *for every Borel subset* B *in* $\Theta_{-i} \times K_i^t$ *where* $K_i^t = \bigcap\{G_i : G_i$ *is closed in* $\Theta_i \times A^1 \times \cdots \times A^{t-1}$ *and* $\phi(\Theta_{-i} \times G_i; \delta) = 1\}$.

Then, the alternative solution concept which is called a *weak complete sequential equilibrium* is defined as follows.

Definition 9 *A complete assessment* (μ, δ) *is a* ***weak complete sequential equilibrium*** *if* (μ, δ) *is both* 1) *weakly consistent and* 2) *sequentially rational.*

Here are the results on the weak complete sequential equilibrium. Theorem 2 reveals the relation among the three equilibria; the complete sequential equilibrium, the weak complete sequential equilibrium, and the Nash equilibrium. This theorem confirms that the weak complete sequential equilibrium is indeed a weak, but simple version of the complete sequential equilibrium.

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[13] That is, given any strategy profile δ and for each set $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$,

$$\phi(B;\delta) = \int_{B^1} \int_{B^2(\theta)} \cdots \int_{B^t(\theta, a^1, \ldots, a^{t-2})} \delta^{t-1}(\theta, a^1, ..., a^{t-2}; da^{t-1}) \cdots \delta^1(\theta; da^1)\eta(d\theta)$$

where $B^1 = \{\theta : (\theta, a^1, ..., a^{t-1}) \in B\}$ and $B^{t'}(\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t'-2}) = \{a^{t'-1} : (\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t'-2}, a^{t'-1}, ..., a^{t-1}) \in B\}$ for $t' = 2, ..., t$.

Theorem 2 *Every complete sequential equilibrium is a weak complete sequential equilibrium, and every weak complete sequential equilibrium is a Nash equilibrium.*

Proof. The results directly follow from the definitions. ■

Theorem 2 is especially useful in practice. In order to find a complete sequential equilibrium, we need to check only on weak complete sequential equilibria, which is simple, thus can be found easily. Consider the example by Crawford and Sobel (1982) again. We can see that partition equilibria found by them are weak complete sequential equilibria, and furthermore they are actually complete sequential equilibria. With the exception of a solution set of measure zero, there is no other weak complete sequential equilibria. Therefore, according to Theorem 2, these are all of the complete sequential equilibria in this game with the exception of a solution set of measure zero.

The next result concerns the relation between the weak complete sequential equilibrium and the weak sequential equilibrium introduced by Myerson (1991, 4.3). Recall that for each system of complete beliefs μ, $\dot{\mu}(\mu)$ denotes a system of beliefs inducing the same distributions on each information set as μ does.

Proposition 4 *In finite games, a complete assessment (μ, δ) is a weak complete sequential equilibrium if and only if the assessment $(\dot{\mu}(\mu), \delta)$ is a weak sequential equilibrium.*

Proof. The result directly follows from the definitions. ■

In a word, the weak complete sequential equilibrium is equivalent to the weak sequential equilibrium in finite games. Therefore, Proposition 4 together with Theorem 2, which proves a weak complete sequential equilibrium is a Nash equilibrium, show that the weak complete sequential equilibrium is in fact a complete version of the weak sequential equilibrium.

7 Complements and comments

In finite games, the perfect equilibrium, introduced by Selten (1975), and the perfect Bayesian equilibrium, formulated by Fudenberg and Tirole (1991a), are closely related to the sequential equilibrium in that every perfect equilibrium is a sequential equilibrium and every sequential equilibrium is a perfect Bayesian equilibrium[14] . These equilibrium concepts can be converted for the general games. Then, it is natural to ask whether their converted versions maintain their close relation in the general games. Hence, as a complement to the previous study, this section answers this question and shows they do not.

We first define a converted version of the *perfect equilibrium.*

Definition 10 *For* $\varepsilon > 0$ *and a totally mixed strategy profile* $\mathring{\delta}$*, an* $\varepsilon - \mathring{\delta}$***-constrained equilibrium***[15] *is a totally mixed strategy profile* $\delta^{\varepsilon}(\mathring{\delta})$ *such that for each* i*, the strategy* $\delta_i^{\varepsilon}(\mathring{\delta})$ *solves* $\max_{\delta_i} E_i(\delta_i, \delta_{-i}^{\varepsilon}(\mathring{\delta}))$ *subject to* $\delta_i = \varepsilon\mathring{\delta}_i + (1-\varepsilon)\delta_i'$ *for some* $\delta_i' \in \Pi_i$*. A strategy profile* δ *is a* ***perfect equilibrium*** *if there exists a sequence of* $\varepsilon_n - \mathring{\delta}_n$*-constrained equilibria* $\{\delta^{\varepsilon_n}(\mathring{\delta}_n)\}$ *such that i)* $\delta^{\varepsilon_n}(\mathring{\delta}_n)$ *converges weakly to* δ *and ii)* ε_n *converges to zero.*

That is, a strategy profile δ is a perfect equilibrium if there exists a sequence of totally mixed strategy profiles $\{\delta^{\varepsilon_n}(\mathring{\delta}_n)\}$ such that *i)* the sequence $\{\delta^{\varepsilon_n}(\mathring{\delta}_n)\}$ converges weakly to the strategy profile δ and *ii)* each strategy profile $\delta^{\varepsilon_n}(\mathring{\delta}_n)$ in the sequence constitutes mutual best responses under some constraint that disappears gradually. According to this definition, a perfect equilibrium might not be a complete sequential equilibrium or a weak complete sequential equilibrium. This is because in the continuous games, which belong to the general

[14] The second part of this statement is true only in perfect recall games. That is, in imperfect recall games, a sequential equilibrium might not be a perfect Baysian equilibrium.

[15] *This* $\varepsilon - \mathring{\delta}$*-constrained equilibrium is named after the "*ε*-constrained equilibrium" in Fudenberg and Tirole (1991b, 8.4.1)*

games, convergency of strategy profiles does not mean convergency of expected utilities if utility functions are unbounded or discontinuous. As a result, a perfect equilibrium could fail to be even a Nash equilibrium, and therefore it could fail to be a complete sequential equilibrium or a weak complete sequential equilibrium.

Next, we define a converted version of the perfect Bayesian equilibrium and call it a *complete equilibrium*.

Definition 11 *A complete assessment* (μ, δ) *is a* ***complete equilibrium*** *if* (μ, δ) *is both reasonably consistent and sequentially rational.*

In other words, the complete equilibrium is a complete assessment which satisfies $i)$ that the system of complete beliefs is reasonably consistent with the strategy profile and $ii)$ that the strategy profile is sequentially rational with respect to the system of complete beliefs. The definition of *reasonable consistency* is as follows. For notational convenience, we define a probability measure ψ with respect to a system of complete beliefs μ and a strategy profile δ as $\psi(B;\mu,\delta) = \int_{B^{t-1}} \int_{B^t(\theta,a^1,...,a^{t-2})} \delta^{t-1}(\theta, a^1, ..., a^{t-2}; da^{t-1})\mu_i^{t-1}(\theta, a^1, ..., a^{t-2})$ for every set $B \in \times_{i=0}^{I} \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i=1}^{I} \beta(A_i^{t'}))$ where $B^{t-1} = \{(\theta, a^1, ..., a^{t-2}) : (\theta, a^1, ..., a^{t-1}) \in B\}$ and $B^t(\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t-2}) = \{a^{t-1} : (\tilde{\theta}, \tilde{a}^1, ..., \tilde{a}^{t-2}, a^{t-1}) \in B\}$.

Definition 12 *A complete assessment* (μ, δ) *is* ***reasonably consistent*** *if* $i)$ *for every* i*, we have* $\mu_i^1 = \eta$ *and* $ii)$ *for each* i *and* $t \geq 2$*, there exists* $p_i^t \in (0, 1]$ *such that* $\mu_i^t(B) = p_i^t \cdot \psi(B;\mu,\delta)$ *for every Borel subset* B *in* $\Theta_{-i} \times \bar{K}_i^t$ *where* $\bar{K}_i^t = \bigcap\{G_i : G_i$ *is closed in* $\Theta_i \times A^1 \times \cdots \times A^{t-1}$ *and* $\psi(\Theta_{-i} \times G_i;\mu,\delta) = 1\}$.

That is, a system of complete beliefs (μ, δ) is reasonably consistent if each complete belief μ_i^t and the strategy profile δ together define the distribution of the complete belief of the next period μ_i^{t+1} over the class of information sets $\bar{K}_i^t$ that is the smallest and closed support of μ_i^t

and δ. According to this definition, the consistency in the complete sequential equilibrium does not means the reasonable consistency in the complete equilibrium, but the reasonable consistency means the weak consistency in the weak complete sequential equilibrium. This happens because the consistency and the reasonable consistency place different restrictions off the equilibrium path while the weak consistency places no restriction. Therefore, a complete sequential equilibrium might not be a complete equilibrium even though a complete equilibrium is always a weak complete sequential equilibrium.

Note that Definition 12 covers only the first condition out of the three consistency conditions for the perfect Bayesian equilibrium by Fudenberg and Tirole (1991a). Thus, it does not cover the "no-signaling-what-you-don't-know" conditions that restrict beliefs off the equilibrium path. Basically, these no-signaling-what-you-don't-know conditions function as restrictions to make the perfect Bayesian equilibrium close to the sequential equilibrium. In the general games, however, the reasonable consistency, which is a converted version of the first condition for the perfect Bayesian equilibrium, already differs from the consistency, which is a converted version of the consistency condition for the sequential equilibrium. Thus, no-signaling-what-you-don't-know conditions would not function as well as they do in the finite games. Accordingly, these conditions are excluded in Definition 12 for the sake of simplicity.

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