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ORIGINAL PAPER

A new procedure for monitoring the range and standard deviation of a quality characteristic

M. Kiani · J. Panaretos · S. Psarakis

Abstract The Shewhart and the Bonferroni-adjustment R and S chart are usually applied to monitor the range and the standard deviation of a quality characteristic. These charts are used to recognize the process variability of a quality characteristic. The control limits of these charts are constructed on the assumption that the population follows approximately the normal distribution with the standard deviation parameter known or unknown. In this article, we establish two new charts based approximately on the normal distribution. The constant values needed to construct the new control limits are dependent on the sample group size (k) and the sample subgroup size (n). Additionally, the unknown standard deviation for the proposed approaches is estimated by a uniformly minimum variance unbiased estimator (UMVUE). This estimator has variance less than that of the estimator used in the Shewhart and Bonferroni approach. The proposed approaches in the case of the unknown standard deviation, give out-of-control average run length slightly less than the Shewhart approach and considerably less than the Bonferroni-adjustment approach.

Keywords Shewhart · Bonferroni-adjustment · Average run length · R chart · S chart

1 Introduction

The Shewhart and Bonferroni-adjustment control chart are common techniques for monitoring the process range and standard deviation of a quality characteristic. The Shewhart range and standard deviation control chart were introduced by Shewhart (1931). Ott (1975), Ryan (1989), Quesenberry (1997), Smith (1998) among others extended the Shewhart range and standard deviation control charts. The Shewhart procedure usually is based on sample group sizes (k) of at least 20–25 and on sample subgroup sizes (n) of at least 4–6. The Shewhart

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chart with known and unknown standard deviation parameter is based on a random variable that follows approximately the normal distribution.

In the case of the R chart, the values of the subgroup ranges (R_i) are plotted on a chart that includes the center line $E(R_i)$ and the following control limits

$$E(R_i) \pm Z_{\alpha/2}\sqrt{Var(R_i)}.$$

Here, the quality characteristics X_{ij} for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$ (jth observation in ith subgroup) are supposed to be identically independently distributed according to the normal distribution with mean μ and variance σ^2, and $R_i = X_{i(n)} - X_{i(1)}$. Here, $X_{i(1)}$ and $X_{i(n)}$ are order statistics of the random variable X_{ij} for the ith subgroup while $E(R_i)$ and $\sqrt{Var(R_i)}$ are the mean and standard deviation of R_i.

It is well known (see e.g. Johnson, et al. 1994), that the joint probability density function of $X_{i(1)}$ and $X_{i(n)}$ is given by

$$f_{1,n}(x, y) = \begin{cases} n(n-1)[F(y) - F(x)]^{n-2} f(x) f(y), & x < y \\ 0 & x \geq y. \end{cases}$$

Therefore, the joint probability density function of $X_{i(1)}$ and R would be

$$f_{1,R}(x, r) = n(n-1)[F(r+x) - F(x)]^{n-2} f(x) f(r+x).$$

As a result, the probability density function of R is obtained to be

$$f_R(r) = \int_{-\infty}^{+\infty} n(n-1)[F(r+x) - F(x)]^{n-2} f(x) f(r+x)\mathrm{d}x,$$

where the functions $f(x)$ and $F(x)$ are, respectively, the probability density function and the cumulative density function of the normal random variable X with parameters (μ, σ^2). The mean range ($E(R)$) can be evaluated to be

$$\begin{aligned} E(R) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r \times n(n-1)[F(r+x) - F(x)]^{n-2} f(x) f(r+x)\mathrm{d}x\mathrm{d}r \\ &= \int_{-\infty}^{+\infty} \left\{1 - (F(x))^n - (1 - F(x))^n\right\} \mathrm{d}x \qquad (1) \end{aligned}$$

(see, e.g. Johnson et al. 1994).

Let $Z_{ij} = \frac{X_{ij}-\mu}{\sigma} \overset{iid}{\sim} N(0, 1)$. Then the cumulative distribution function of $Z_{ij} = Z$ is

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} \mathrm{d}t, \quad -\infty < z < +\infty. \qquad (2)$$

Further, let R' denote the range of order statistics $Z_{h(1)}, Z_{h(2)}, \ldots, Z_{h(n)}$ for the hth subgroup ($h = 1, 2, \ldots, k$). Then, using Eqs. 1 and 2, the mean of R' is given by

$$E(R') = \int_{-\infty}^{+\infty} \left\{1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} \mathrm{d}t\right)^n - \left(1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} \mathrm{d}t\right)^n\right\} \mathrm{d}z.$$

The variance and the covariance of order statistics $Z_{h(1)}, Z_{h(2)}, \ldots Z_{h(n)}$ can be extended as (see Johnson et al. 1994)

$$\begin{aligned} Var\left(Z_{h(i)}\right) = \frac{p_i q_i}{n+2}\{(F^{-1})'_i\}^2 + \frac{p_i q_i}{(n+2)^2}\Big\{2(q_i - p_i)(F^{-1})'_i(F^{-1})''_i \\ + p_i q_i\{(F^{-1})'_i(F^{-1})'''_i + \frac{1}{2}\left[(F^{-1})''_i\right]^2\}\Big\} + \cdots, \end{aligned}$$

$$Cov\left(Z_{h(i)}, Z_{h(j)}\right) = \frac{p_i q_j}{n+2}\{(F^{-1})'_i (F^{-1})'_j\} + \frac{p_i q_i}{(n+2)^2}\Big\{(q_i - p_i)(F^{-1})''_i (F^{-1})'_j + (q_j - p_j)(F^{-1})''_j (F^{-1})'_i + \frac{1}{2} p_i q_i (F^{-1})'_j (F^{-1})'''_i + \frac{1}{2} p_j q_j (F^{-1})'_i (F^{-1})'''_j + \frac{1}{2} p_i q_j (F^{-1})''_i (F^{-1})''_j\Big\} + \cdots,$$

where, $p_i = i/(n+1)$; $q_i = 1 - p_i$; $(F^{-1})'_i = \mathrm{d}F^{-1}/\mathrm{d}y|_{y=r/(n+1)}$; $(F^{-1})''_i = \mathrm{d}^2F^{-1}/\mathrm{d}y^2|_{y=r/(n+1)}$, etc. Thus, the standard deviation of R' can be written

$$\sqrt{Var(R')} = \sigma\sqrt{Var(Z_{h(n)}) + Var(Z_{h(1)}) - 2Cov(Z_{h(n)}, Z_{h(1)})}.$$

Tables of the constant values $d_2 = E(R')$ and $d_3 = \sqrt{Var(R')}$ are given in Montgomery (2001), Ryan (1989), Quesenberry (1997) and others. Note that the parameters d_2 and d_3 are dependent only on the sample subgroup size (n). Furthermore, $R = X_{h(n)} - X_{h(1)}$ and $R' = Z_{h(n)} - Z_{h(1)}$ are related by $R = \sigma R'$. Consequently, the mean and standard deviation of R_i, $E(R_i)$ and $\sqrt{Var(R_i)}$, are obtained to be σd_2 and σd_3, respectively. If the standard deviation of the quality characteristic (σ) is unknown, the Shewhart and the Bonferroni-adjustment R chart, can be constructed using an unbiased estimate of σ that is given by the statistic $\bar{R}/d_2$, where $\bar{R}$ is the average range of the k preliminary samples.

In the case of the S chart, the values of the subgroup standard deviations $S_i = \sum_{j=1}^{n}(X_{ij} - \bar{X}_{i.})^2/(n-1)$, where $\bar{X}_{i.} = \sum_{i=1}^{n} X_{ij}/n$, are plotted on the chart. This chart includes the center line $E(S_i)$ and the control limits $E(S_i) \pm Z_{\alpha/2}\sqrt{Var(S_i)}$. Here, $E(S_i)$ and $\sqrt{Var(S_i)}$ are the mean and standard deviation of S_i, respectively.

Ryan (1989) introduced the Bonferroni-adjustment control limits as an alternative to the Shewhart approach. The Bonferroni-adjustment R and S control limits are given by $E(R_i) \pm Z_{\alpha/2k}\sqrt{Var(R_i)}$ and $E(S_i) \pm Z_{\alpha/2k}\sqrt{Var(S_i)}$. Here, the value α of the Shewhart control limits is replaced by the value α/k to construct these control limits.

The new range and standard deviation control chart with known standard deviation (σ) are established similarly to the Shewhart and the Bonferroni control chart. When the standard deviation is unknown the proposed chart is estimated using a statistic with variance less than that of the Shewhart and the Bonferroni-adjustment chart. Furthermore, the constant value for the new chart with unknown standard deviation is dependent on the sample subgroup and group sizes (n, k) whereas the constant value of the Shewhart and Bonferroni chart is depended only on the sample subgroup size (n).

In this article, the Shewhart and Bonferroni charts are presented in Sect. 2, while, in Sect. 3, the new charts are introduced. The in-control average run length (ARL_0) as well as the out-of-control average run length (ARL_1) for comparing these control charts are given in Sects. 4 and 5. Finally an example and some conclusions are presented in Sects. 6 and 7, respectively.

2 The Shewhart and Bonferroni R and S Charts

Suppose that the quality characteristics X_{ij} are identically independent distribution $N(\mu, \sigma^2)$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$. The statistics $R_i = X_{i(n)} - X_{i(1)}$ and $S_i = \sum_{j=1}^{n}(X_{ij} - \bar{X}_{i.})^2/(n-1)$ are used to construct the range and standard deviation charts, respectively. It is well known that $E(R_i) = \sigma d_2$, $\sqrt{Var(R_i)} = \sigma d_3$, $E(S_i) = \sigma c_4$ and $\sqrt{Var(S_i)} = \sigma\sqrt{1 - c_4^2}$, where

$$c_4 = \sqrt{\frac{2}{n-1}}\Gamma\left[\frac{n}{2}\right]\Big/\Gamma\left[\frac{(n-1)}{2}\right].$$

The Shewhart R chart with a known or unknown parameter σ are based on the random variables $(R_i - E(R_i))/\sqrt{Var(R_i)}$ and $(R_i - \hat{E}(R_i))/\sqrt{V\hat{a}r(R_i)}$, respectively, and for the Shewhart S chart, $(S_i - E(S_i))/\sqrt{Var(S_i)}$ and $(S_i - \hat{E}(S_i))/\sqrt{V\hat{a}r(S_i)}$.

Let $\hat{E}(R_i) = \bar{R}$, $\sqrt{V\hat{a}r(R_i)} = d_3\bar{R}/d_2$, $\hat{E}(S_i) = \bar{S}$ and $\sqrt{V\hat{a}r(S_i)} = \bar{S}\sqrt{1-c_4^2}/c_4$, where $\bar{R} = \sum_{i=1}^{k} R_i/k$ and $\bar{S} = \sum_{i=1}^{k} S_i/k$. These variables, for sample sizes as large as $k \geq 20$ and $n \geq 4$, follow approximately the standard normal distribution.

The Shewhart R and S control limits for a known parameter σ, with confidence $(1-\alpha)\%$, are given by $\sigma(d_2 \pm Z_{\alpha/2}d_3)$ and $\sigma(c_4 \pm Z_{\alpha/2}\sqrt{1-c_4^2})$, respectively. For these control limits, the center line is σd_2 and σc_4, where the constant values d_2, d_3 and c_4 depend only on the sample subgroup size (n).

If the standard deviation of the quality characteristic is unknown, then it is estimated by the unbiased statistics $\bar{R}/d_2$ and $\bar{S}/c_4$ for Shewhart R and S chart, respectively. Then, the center line and the control limits for the Shewhart R chart with unknown parameter (σ) take the form

$$U\hat{C}L = (\bar{R}/d_2)(d_2 + z_{\alpha/2}d_3); \quad \hat{C}L = \bar{R}; \quad L\hat{C}L = (\bar{R}/d_2)(d_2 - z_{\alpha/2}d_3), \tag{3}$$

while, for the Shewhart S chart,

$$U\hat{C}L = (\bar{S}/c_4)(c_4 + z_{\alpha/2}\sqrt{1-c_4^2}); \quad \hat{C}L = \bar{S}; \quad L\hat{C}L = (\bar{S}/c_4)(c_4 - z_{\alpha/2}\sqrt{1-c_4^2}). \tag{4}$$

The Bonferroni-adjustment R and S control chart were suggested by Ryan (1989) in order to improve the probability of detecting one or more false alarms of the Shewhart chart. The Bonferroni-adjustment R and S control limits for known standard deviation parameter are given below

$$\sigma(d_2 \pm Z_{\alpha/2k}d_3); \quad \sigma(c_4 \pm Z_{\alpha/2k}\sqrt{1-c_4^2}). \tag{5}$$

Furthermore, the Bonferroni R and S control limits with unknown parameter are

$$(\bar{R}/d_2)(d_2 \pm Z_{\alpha/2k}d_3) \tag{6}$$

$$(\bar{S}/c_4)(c_4 - z_{\alpha/2k}\sqrt{1-c_4^2}). \tag{7}$$

The center lines for the Bonferroni R chart with known and unknown standard deviation are σd_2 and $\bar{R}$, respectively, and for the Bonferroni S chart, σc_4 and $\bar{S}$.

3 The New R and S chart

When the standard deviation is unknown, for constructing the new range and standard deviation charts, we need a good estimator of the standard deviation σ of the normal distribution $N \sim (\mu, \sigma^2)$. A brief presentation of some estimators of the standard deviation is given in Subsect. 3.1. A uniformly minimum variance unbiased (UMVU) estimator is suggested in Sect. 3.2. The new R and S charts for both known and unknown standard deviation are presented in Subsect. 3.3.

3.1 A Brief overview on the estimation of the standard deviation

Markowitz (1968) suggested the use of the minimum mean-square-error estimator of σ given by $\hat{\sigma} = \sqrt{\sum_i^n (X_i - \bar{X})^2/k}$, where

$$k = 2\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 \Big/ \left[\Gamma\left(\frac{n}{2}\right)\right]^2.$$

Prescott (1971a) introduced a linear estimator for the standard deviation defined as $\hat{\sigma} = a_n \bar{W}$. Here, a_n is the unbiasing factor and $\bar{W}$ is given by

$$\bar{W} = \left(\sum_{j=n-r+1}^{n} X_{i(j)} - \sum_{j=1}^{r} X_{i(j)}\right) \Big/ (3r), \quad i = 1, 2, \ldots, k.$$

Furthermore, $r = n/6$ is rounded up to the nearest whole number if $n/6$ is not an integer and $X_{i(1)} \leq X_{i(2)} \leq \cdots \leq X_{i(n)}$ is an ordered sample of the normal distribution $N(\mu, \sigma^2)$. Prescott (1971b) proposed the use of another estimator for the standard deviation of the $N(\mu, \sigma^2)$ given by

$$\hat{\sigma} = \sum_{j=1}^{n} m_j X_{i(j)} \Big/ \sum_{j=1}^{n} m_j^2.$$

In this case, $m_j = E\left((X_{i(j)} - \mu)/\sigma\right)$.

Healy (1978) introduced the unbiased estimator of σ given by

$$\hat{\sigma} = \sqrt{\pi} \sum_{i=1}^{n} (2i - n - 1) X_i / \{n(n - 1/2)\}.$$

Vardeman (1999) considered using minimum mean-square-error estimator given, for the case of a single sample, by

$$\hat{\sigma} = \frac{R d_2}{d_2^2 + d_3^2}; \quad \hat{\sigma} = \frac{S}{c_4},$$

where R and S are the range and standard deviation of the single sample, and the constants d_2, d_3 and c_4 are as introduced in previous sections. He also introduced a combination of several estimators for the case of r samples of possibly different sizes $n_1, n_2, \ldots, n_r$ with ranges $R_1, R_2, \ldots, R_r$ defined by $\hat{\sigma} = \gamma_1 R_1 + \gamma_2 R_2 + \cdots + \gamma_r R_r$, where

$$\gamma_i = \left(\sum_{i=1}^{r} \frac{d_2^2(n_i)}{d_3^2(n_i)}\right)^{-1} \frac{d_2(n_i)}{d_3^2(n_i)}; \quad d_2(n_i) = E(R_i)/\sigma; \quad d_3(n_i) = \sqrt{Var(R_i)}/\sigma.$$

An analogous estimator was proposed for the case of r samples of possibly different sizes $n_1, n_2, \ldots, n_r$ with sample standard deviation estimators $S_1, S_2, \ldots, S_r$. The proposed estimators are $\hat{\sigma} = \gamma_1 S_1 + \gamma_2 S_2 + \cdots + \gamma_r S_r$ and $\hat{\sigma} = S_{\text{pooled}}/c_4(v+1)$. Here,

$$\gamma_i = \left(\sum_{i=1}^{r} \frac{c_4^2(n_i)}{c_5^2(n_i)}\right)^{-1} \frac{c_4(n_i)}{c_5^2(n_i)}; \quad c_4(n_i) = E(S_i)/\sigma; \quad c_5(n_i) = \sqrt{Var(S_i)}/\sigma$$

$$S^2_{\text{pooled}} = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2 + \cdots + (n_r-1)S_r^2}{(n_1-1)+(n_2-1)+\cdots+(n_r-1)};$$

$$v = (n_1-1)+(n_2-1)+\cdots+(n_r-1).$$

Some other estimators of the standard deviation have also been given by Glasser (1962), Khan (1968), Gurland and Tripathi (1971), Donatos (1989), Arnholt and Hebert (1995), Watson (1997). However, these authors did not employ UMVU estimators of the standard deviation of the normal distribution $N(\mu, \sigma^2)$.

3.2 An UMVU estimator of the standard deviation

As is well known, the random variable $k(n-1)S^2/\sigma^2$ is chi-square distributed. Let

$$S = \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{n}(X_{ij} - \bar{X}_{i.})^2/(k(n-1))}.$$

Then, the random variable $H = \sqrt{k(n-1)}S/\sigma$ follows the chi distribution with $k(n-1)$ degrees of freedom. The probability density function and the rth raw moment of H are

$$P_H(h) = \frac{1}{2^{(k(n-1)/2)-1}\Gamma\left(k(n-1)/2\right)} e^{-h^2/2} h^{k(n-1)-1}, \quad h > 0 \tag{8}$$

$$E(H^r) = \frac{\sqrt{2^r}\,\Gamma\left((k(n-1)+r)/2\right)}{\Gamma\left(k(n-1)/2\right)}. \tag{9}$$

Moreover, the standard chi distribution (8) is in fact a standard gamma distribution with probability density function

$$P_H(h) = \frac{1}{\eta^{\zeta}\Gamma(\zeta)} e^{-(h-\gamma)/\eta}(h-\gamma)^{\zeta-1}, \quad h > 0,$$

where, the values ζ, η, and γ are $k(n-1)/2$, 2, and 0, respectively, with H replacing H^2. Assume further a constant value ψ to be the unbiasing factor of the standard deviation estimator, where

$$\psi = \left(\sqrt{\tfrac{2}{k(n-1)}}\,\Gamma\left(\tfrac{k(n-1)+1}{2}\right) \Big/ \Gamma\left(\tfrac{k(n-1)}{2}\right)\right), \quad kn < 350$$

and

$$\psi \approx \frac{4k(n-1)}{4k(n-1)+1}, \quad kn \geq 350$$

The mean and variance of the statistic S are evaluated to be $\sigma\psi$ and $\sigma^2(1-\psi^2)$, respectively, using Eq. 9. Thus, the statistic S/ψ is an unbiased estimator of the standard deviation (σ). In this case, the constant value ψ depends on both the sample subgroup size (n) and the sample group size (k). The value of ψ with various sample sizes k and n is given in appendix C Table 10, for

$$k = 2(1)10,\ 15(5)30,\ 40,\ 100,\ 120$$

and

$$n = 2(1)11,\ 14,\ 15,\ 18,\ 20(5)50,\ 60(10)120.$$

Table 1 The variance of the statistics S/ψ, $\bar{R}/d_2$ and $\bar{S}/c_4$ with $\sigma^2 = 1$

	k								
	n	2	5	10	15	20	25	60	120
$\sigma^2\left(1-\psi^2\right)/\psi^2$	2	0.27323	0.10440	0.05118	0.03392	0.02527	0.02015	0.00838	0.00413
	5	0.06427	0.02527	0.01267	0.00838	0.00620	0.00499	0.00209	0.00104
	10	0.02819	0.01117	0.00557	0.00374	0.00280	0.00228	0.00093	0.00046
	20	0.01328	0.00531	0.00277	0.00172	0.00132	0.00105	0.00044	0.00022
	25	0.01048	0.00413	0.00209	0.00139	0.00104	0.00083	0.00035	0.00017
$\sigma^2 d_3^2/\left(kd_2^2\right)$	2	0.28592	0.11437	0.05718	0.03812	0.02859	0.02287	0.00953	0.00477
	5	0.06899	0.02760	0.01380	0.00920	0.00690	0.00552	0.00230	0.00115
	10	0.03352	0.01341	0.00670	0.00447	0.00335	0.00268	0.00112	0.00056
	20	0.01905	0.00762	0.00381	0.00254	0.00190	0.00152	0.00063	0.00032
	25	0.01622	0.00649	0.00324	0.00216	0.00162	0.00130	0.00054	0.00027
$\sigma^2\left(1-c_4^2\right)/\left(kc_4^2\right)$	2	0.28537	0.11415	0.05707	0.03805	0.02854	0.02283	0.00951	0.00476
	5	0.06587	0.02635	0.01317	0.00878	0.00659	0.00527	0.00220	0.00110
	10	0.02846	0.01138	0.00569	0.00379	0.00285	0.00228	0.00095	0.00047
	20	0.01336	0.00534	0.00267	0.00178	0.00134	0.00107	0.00045	0.00022
	25	0.01056	0.00423	0.00211	0.00141	0.00106	0.00085	0.00035	0.00018

The statistic S is an injective function of the complete sufficient statistic S^2 and the statistic S/ψ is an unbiased estimator of σ. Therefore according to the *Lehman-Scheffe* theorem, the statistic S/ψ is an UMVU estimator of σ (see Rohatgi 1984). Then, the UMVU estimator S/ψ can be used for constructing the new range and standard deviation control chart with unknown standard deviation. In the sequel, we compare the range and standard deviation control charts that are based on the statistic $\bar{R}/d_2$ (for the Shewhart and Bonferroni approach) to those based on the statistic S/ψ (for the new approach). Table 1 shows that the variance of the statistic S/ψ, $var(S/\psi) = \sigma^2(1-\psi^2)/\psi^2$, is less than that of the statistics $\bar{R}/d_2$ and $\bar{S}/c_4$, i.e., $var(\bar{R}/d_2) = \sigma^2 d_3^2/(kd_2^2)$ and $var(\bar{S}/c_4) = \sigma^2(1-c_4^2)/(kc_4^2)$.

3.3 The New R and S charts

The control limits for the average of a quality characteristic depend on the variability of the production process. While the process variability is outside the control limits, the control limits on the average quality characteristic will not have much meaning. Therefore, it is best that a range or standard deviation control limits is first set (see Montgomery 2001).

The quality characteristics X_{ij} for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$ are identically and independently normally distributed with mean μ and variance σ^2. The new range and standard deviation control charts with known standard deviation, like the Shewhart R and S control charts, are given by

$$UCL = \sigma(d_2 + z_{\alpha/2}d_3); \quad CL = \sigma d_2; \quad LCL = \sigma(d_2 - z_{\alpha/2}d_3)$$

$$UCL = \sigma\left(c_4 + Z_{\alpha/2}\sqrt{1-c_4^2}\right); \quad CL = \sigma c_4; \quad LCL = \sigma\left(c_4 - Z_{\alpha/2}\sqrt{1-c_4^2}\right).$$

To establish the proposed control charts with unknown standard deviation, we estimate $E(R_i) = \sigma d_2$, $\sqrt{Var(R_i)} = \sigma d_3$, $E(S_i) = \sigma c_4$ and $\sqrt{Var(S_i)} = \sigma\sqrt{1-c_4^2}$ using the

UMVU estimators Sd_2/ψ, Sd_3/ψ, Sc_4/ψ and $S\sqrt{1-c_4^2}/\psi$, respectively. The resulting control limits for the proposed R chart with unknown standard deviation would be

$$U\hat{C}L = (S/\psi)(d_2 + z_{\alpha/2}d_3); \quad \hat{C}L = (S/\psi)(d_2); \quad L\hat{C}L = (S/\psi)(d_2 - z_{\alpha/2}d_3) \tag{10}$$

and the control limits for the proposed S chart are

$$U\hat{C}L = (S/\psi)\left(c_4+z_{\alpha/2}\sqrt{1-c_4^2}\right); \quad \hat{C}L = Sc_4/\psi; \quad L\hat{C}L = (S/\psi)\left(c_4-z_{\alpha/2}\sqrt{1-c_4^2}\right). \tag{11}$$

In Sects. 4 and 5, the in-control and out-of-control average run length for the Shewhart, Bonferroni and new R and S control charts are examined.

4 In-control average run length

The in-control average run length (called ARL_0) is *the average number of subgroup ranges or standard deviations* that should be plotted before a subgroup range or standard deviation indicates an out-of-control condition. The ARL_0 can be calculated from $ARL_0 = 1/p$ under the condition that the process observations are uncorrelated. Here, p is the probability that any point exceeds the control limits. The in-control average run length can be used to evaluate the performance of the control chart.

In this section, the average run length is considered for *the initial group and groups* 2, 3, ... with known and unknown parameter σ when the process is in control. Let the individual events G_i denote the subgroup range R_i or standard deviation S_i exceeds the control limits of the in control process ($R = R_0$ or $\sigma = \sigma_0$).

For the initial group of observations with unknown parameter σ, the events G_i and $G_{i'}$ for $i \neq i' = 1, 2, \ldots, k$ are not independent, since the statistics $R_i - U\hat{C}L$ and $R_j - U\hat{C}L$, or $S_i - U\hat{C}L$ and $S_j - U\hat{C}L$, for the Shewhart, the Bonferroni and the new charts are based on the same observations of the initial group.

In case the events G_i are independent, the sequence of trials, comparing R_i with $U\hat{C}L$ or S_i with $U\hat{C}L$ will be a sequence of Bernoulli trials and the run length between occurrences of G_i will be a Geometric random variable with probability $\alpha = P(G_i)$. Additionally, the in-control average run length would be $1/P(G_i)$ or $1/\alpha$ such that,

$$P(G_i) = P(R_i \leq L\hat{C}L \text{ or } R_i \geq U\hat{C}L \,|R = R_0) \tag{12}$$

or

$$P(G_i) = P(S_i \leq L\hat{C}L \text{ or } S_i \geq U\hat{C}L \,|\sigma = \sigma_0). \tag{13}$$

However, the statistics $R_i - U\hat{C}L$ and $R_j - U\hat{C}L$ or $S_i - U\hat{C}L$ and $S_j - U\hat{C}L$ for *the initial group with unknown parameter* are not independent events. Therefore, the in-control ARL for the initial group with unknown parameter can be not calculated.

For the initial group with known parameter, the correlation between random variables $R_i - UCL$ and $R_j - UCL$ or $S_i - UCL$ and $S_j - UCL$ can be obtained to be 0. Here, the UCL with known parameter is a constant value and the subgroup ranges R_i and R_j or standard deviations S_i and S_j are independent. Thus, the events G_i and G_j for the initial group with known parameter are uncorrelated.

For the groups 2, 3, ... *with known and unknown parameter*, the events G_i and G_j are uncorrelated, since the UCL with known and unknown parameter are based on the observations of initial group, while the R_i or S_i belong to groups 2, 3, ... Thus, the correlation between random variables $R_i - UCL$ and $R_j - UCL$ or $S_i - UCL$ and $S_j - UCL$ can be obtained to be 0.

Based on the above, the sequence of the events $\{G_i\}$, for the initial group with known parameter and the groups 2, 3, ... with known and unknown parameter, would be Bernoulli trials and the run length between occurrences of G_i would be a Geometric random variable with probability $P(G_i)$. The probability $P(G_i)$ for both the Shewhart and the new approach is α, and for the Bonferroni-adjustment approach is α/k. As a result, the in-control average run length (ARL_0) would be $1/P(G_i) = 1/\alpha$ for the Shewhart and the new approach, and $1/P(G_i) = k/\alpha$ for the Bonferroni-adjustment approach. Thus the ARL_0 for the Shewhart and proposed chart ($1/\alpha$) is less than the ARL_0 for Bonferroni-adjustment chart (k/α, for $k \geq 2$). (See, also, Nedumaran and Pignatiello 2005, Tsai et al. 2005).

Now, we discuss the in-control average run length that based on *the average number of groups* before *a group* indicates an out-of-control condition. Here, the in-control average run length is called ARL'_0.

For the initial group with known parameter and the groups 2, 3, ... *with known and unknown parameter*, let the random variable Y denote the overall occurrences of events G_i for $i = 1, 2, \ldots, k$. Then, this random variable should follow the Binomial distribution with probability distribution (for the Shewhart and the new approach) given by

$$P(Y = y) = \binom{k}{y} \alpha^y (1-\alpha)^{k-y}, \quad y = 0, 1, 2, \ldots, k.$$

Therefore, the probability of one or more subgroup ranges or standard deviation falling out of the control limits (the probability of out-of-control condition for a group) for the Shewhart and the new approach is $P(Y \geq 1) = 1 - (1-\alpha)^k$. Ryan (1989) showed that the probability $1 - (1-\alpha)^k$ is approximately equal with $k\alpha$. Thus, Ryan (1989) suggested the Bonferroni-adjustment approach for the control limits. In this case, the probability of one or more false alarm is improved to $1 - (1 - \alpha/k)^k$ that is less than $1 - (1-\alpha)^k$. The probability distribution of Y, for the Bonferroni-adjustment approach, follows the Binomial distribution with parameters $(k, \alpha/k)$. Therefore, the probability of one or more subgroup ranges or standard deviations falling out of the control limits for the Bonferroni-adjustment approach is given by $P(Y \geq 1) = 1 - (1 - \alpha/k)^k$. As a result, the ARL'_0 is obtained to be $1/(1 - (1-\alpha)^k)$ for the Shewhart and the new approach, and $1/(1 - (1 - \alpha/k)^k)$ for the Bonferroni-adjustment approach. That means $1/(1-(1-\alpha)^k) < 1/(1-(1-\alpha/k)^k$. Thus, the ARL'_0 for the Shewhart and the new approach is less than that of the Bonferroni-adjustment approach. Consequently, the in-control average run length (ARL_0, ARL'_0) for the Bonferroni-adjustment approach is greater than the Shewhart and the new approach. In the next section, we illustrate that the out-of-control average run length for the Bonferroni-adjustment approach is not so satisfactory. In other words, the power of the Bonferroni-adjustment control limits is considerably less than the one of the Shewhart and new approach.

5 Out-of-control average run length

The ability of the range and standard deviation control charts to detect shifts in process quality is described by the out-of-control average run length (ARL_1). The probabilities of detecting a one or more false alarms when the process is in control were improved by using the Bonferroni-adjustment approach. In this section, the power of control limits for the usual approach (Shewhart and Bonferroni) and the new approach are compared using ARL_1.

If the in-control value of the standard deviation shifts from σ_0 to $\sigma_1 = \lambda\sigma_0 > \sigma_0, (\lambda > 1)$ the probability of not detecting the range or standard deviation shift (β) is calculated by

$$\beta = P\,(LCL \leq R_i \leq UCL\,|\sigma = \lambda\sigma_0\,) \quad \text{(R Chart)} \tag{14}$$

or

$$\beta = P\,(LCL \leq S_i \leq UCL\,|\sigma = \lambda\sigma_0\,) \quad \text{(S Chart)}. \tag{15}$$

These probabilities for both the Shewhart and proposed R and S charts with known parameter (σ) similarly are obtained to be

$$\beta = P\left(\frac{-Z_{\alpha/2}d_3 + d_2(1-\lambda)}{\lambda d_3} \leq Z_i \leq \frac{Z_{\alpha/2}d_3 + d_2(1-\lambda)}{\lambda d_3}\right) \quad \text{(R Chart)} \tag{16}$$

and

$$\beta = P\left(\frac{-Z_{\alpha/2}\sqrt{1-c_4^2} + c_4(1-\lambda)}{\lambda\sqrt{1-c_4^2}} \leq Z_i \leq \frac{Z_{\alpha/2}\sqrt{1-c_4^2} + c_4(1-\lambda)}{\lambda\sqrt{1-c_4^2}}\right) \quad \text{(S Chart)}. \tag{17}$$

Meanwhile, the probability β for the Bonferroni-adjustment R and S chart with known parameter (σ) is

$$\beta = P\left(\frac{-Z_{\alpha/2k}d_3 + d_2(1-\lambda)}{\lambda d_3} \leq Z_i \leq \frac{Z_{\alpha/2k}d_3 + d_2(1-\lambda)}{\lambda d_3}\right) \quad \text{(R Chart)} \tag{18}$$

and

$$\beta = P\left(\frac{-Z_{\alpha/2k}\sqrt{1-c_4^2} + c_4(1-\lambda)}{\lambda\sqrt{1-c_4^2}} \leq Z_i \leq \frac{Z_{\alpha/2k}\sqrt{1-c_4^2} + c_4(1-\lambda)}{\lambda\sqrt{1-c_4^2}}\right) \quad \text{(S Chart)}. \tag{19}$$

Usually, the parameter σ is unknown. In this case, we obtain the probability β for the control limits with unknown parameter. We already showed that S/ψ is a UMVU estimator of the standard deviation. Let us call S/ψ by S_t. In order to compute the type II error (β) for the Shewhart, the Bonferroni and the proposed approach with unknown parameter the standard deviation (σ_0) from Eqs. 14 and 15 is estimated by S_t. Thus, the probability β for the Shewhart approach with unknown parameter is calculated using Eqs. 3 and 14 for R chart and Eqs. 4 and 15 for S chart

$$\beta = P\left(\frac{(\bar{R}/d_2)(d_2 - Z_{\alpha/2}d_3) - \lambda S_t d_2}{\lambda S_t d_3} \leq Z_i \leq \frac{(\bar{R}/d_2)(d_2 + Z_{\alpha/2}d_3) - \lambda S_t d_2}{\lambda S_t d_3}\right) \tag{20}$$

and

$$\beta = P\left(\frac{(\bar{S}/c_4)(c_4 - Z_{\alpha/2}\sqrt{1-c_4^2}) - \lambda S_t c_4}{\lambda S_t\sqrt{1-c_4^2}} \leq Z_i \leq \frac{(\bar{S}/c_4)(c_4 + Z_{\alpha/2}\sqrt{1-c_4^2}) - \lambda S_t c_4}{\lambda S_t\sqrt{1-c_4^2}}\right). \tag{21}$$

Table 2 The constant values d_2, d_3 and c_4 to construct the OC curves

	n				
	2	5	10	20	25
d_2	1.128	2.326	3.078	3.735	3.931
d_3	0.853	0.864	0.797	0.729	0.708
c_4	0.7979	0.9400	0.9727	0.9869	0.9896

Similarly, the probability β for the Bonferroni-adjustment approach with unknown parameter is obtained by using Eqs. 6 and 14 for R chart and Eqs. 7 and 15 for S chart,

$$\beta = P\left(\frac{(\bar{R}/d_2)(d_2 - Z_{\alpha/2k}d_3) - \lambda S_t d_2}{\lambda S_t d_3} \leq Z_i \leq \frac{(\bar{R}/d_2)(d_2 + Z_{\alpha/2k}d_3/d_2) - \lambda S_t d_2}{\lambda S_t d_3}\right) \tag{22}$$

and

$$\beta = P\left(\frac{(\bar{S}/c_4)(c_4 - Z_{\alpha/2k}\sqrt{1-c_4^2}) - \lambda S_t c_4}{\lambda S_t\sqrt{1-c_4^2}} \leq Z_i \leq \frac{(\bar{S}/c_4)(c_4 + Z_{\alpha/2k}\sqrt{1-c_4^2}) - \lambda S_t c_4}{\lambda S_t\sqrt{1-c_4^2}}\right). \tag{23}$$

Also, the probability β for the new approach with unknown parameter is obtained using Eqs. 10 and 14 for R chart and Eqs. 11 and 15 for S chart as follow

$$\beta = P\left(\frac{(S/\psi)(d_2 - Z_{\alpha/2}d_3) - \lambda S_t d_2}{\lambda S_t d_3} \leq Z_i \leq \frac{(S/\psi)(d_2 + Z_{\alpha/2}d_3) - \lambda S_t d_2}{\lambda S_t d_3}\right) \tag{24}$$

and

$$\beta = P\left(\frac{(S/\psi)(c_4 - Z_{\alpha/2}\sqrt{1-c_4^2}) - \lambda S_t c_4}{\lambda S_t\sqrt{1-c_4^2}} \leq Z_i \leq \frac{(S/\psi)(c_4 + Z_{\alpha/2}\sqrt{1-c_4^2}) - \lambda S_t c_4}{\lambda S_t\sqrt{1-c_4^2}}\right). \tag{25}$$

The probability β with known standard deviation for various sample sizes n and coefficient λ is exhibited by the operating-characteristic (OC) curves. The OC curves are constructed according to the constant values of Table 2 and Eqs. 16 and 17 for the Shewhart and the new R and S charts, respectively, and Eqs. 18 and 19 for the Bonferroni-adjustment R and S chart.

Figures 1–4 indicate that the Shewhart and the new approach are considerably effective in detecting shifts greater than one standard deviation ($\lambda > 1$) on the first sample following the shift. Table 3 shows the probability β for various shifts and sample sizes from Figs. 1–4. This table indicates that the probability β for the Shewhart and the new approach is less than the probability in not detecting a shift for the Bonferroni-adjustment approach.

The probability β with unknown standard deviation for various sample sizes n and k, and coefficient λ is exhibited by Tables 4 and 5 for R chart, and Tables 6 and 7 for S chart. These tables are constructed using Monte Carlo simulation experiments (Appendix A for R chart and Appendix B for S chart) with constant values of Table 2 and Eqs. 20, 22, and 24 for the Shewhart, the Bonferroni-adjustment, and the new R chart, respectively, and Eqs. 21, 23,

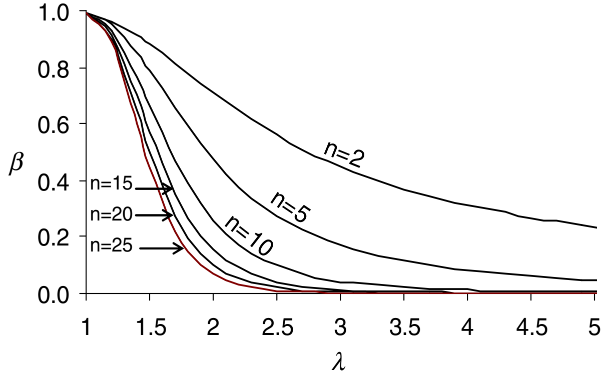

Fig. 1 Operating characteristic curves for the Shewhart and new R chart ($k = 20$, $a = 0.01$)

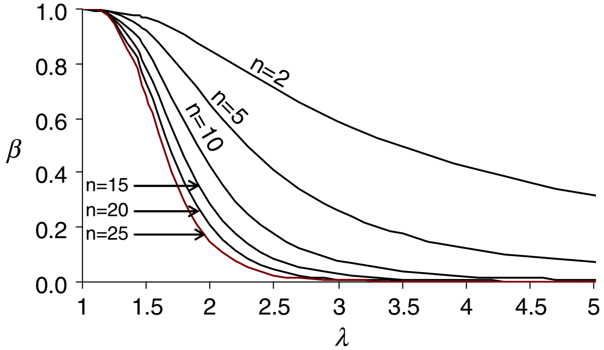

Fig. 2 Operating characteristic curves for the Bonferroni R chart ($k = 20$, $a = 0.01$)

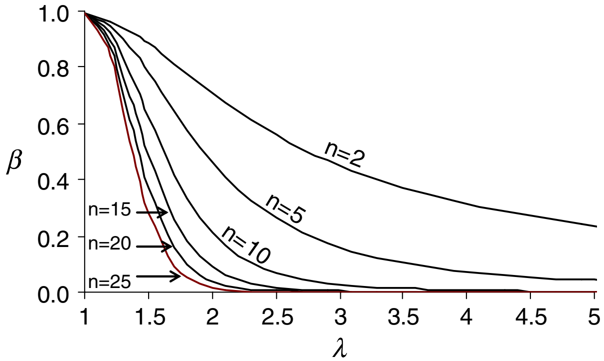

Fig. 3 Operating characteristic curves for the Shewhart and new S chart ($k = 20$, $a = 0.01$)

and 25 for the Shewhart, the Bonferroni-adjustment, and the new S chart, respectively. If the subgroup range or standard deviation falls out of the control limits, a counter for that was increased by one. This procedure was replicated 10,000 times, and then the probability β was estimated by dividing the number of replications in which points exceeded the control limits with the total number of replications. As a result, the estimated probability of not detecting a range or a standard deviation shift for the Shewhart, the Bonferroni-adjustment and the new R and S chart are shown in the following Tables 4–7.

The results in Tables 4–7 show that with unknown standard deviation the new R and S chart perform better than both the Shewhart and the Bonferroni-adjustment R and S chart.

Fig. 4 Operating characteristic curves for the Bonferroni S chart ($k = 20$, $a = 0.01$)

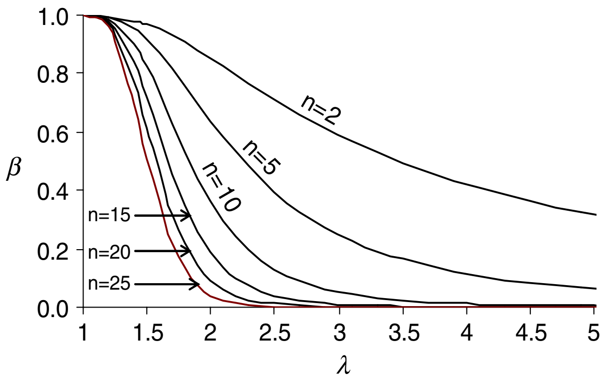

Table 3 The Probability β from Figures 1–4 ($k = 20$, $\alpha = 0.01$)

Approach	$n = 2$		$n = 10$		$n = 25$	
	$\lambda = 1.5$	$\lambda = 2.5$	$\lambda = 1.5$	$\lambda = 2.5$	$\lambda = 1.5$	$\lambda = 2.5$
Shewhart-New R chart	0.88363	0.55954	0.66503	0.09866	0.44671	0.01069
Bonferroni R chart	0.96705	0.71096	0.84909	0.17741	0.68073	0.02625
Shewhart-New S chart	0.88357	0.55930	0.62462	0.06863	0.28229	0.00098
Bonferroni S chart	0.96702	0.71073	0.82199	0.13076	0.51091	0.00312

To construct the ARL_1, let the events W_i denote that the subgroup range or the standard deviation fall out of control limits, while the standard deviation of the quality characteristic is shifted from σ_0 to $\lambda\sigma_0$. The events W_i for the initial group with known parameter and the groups $2, 3, \ldots$ with known and unknown parameter are uncorrelated. Therefore, the sequence of events $\{W_i\}$ would be a sequence of Bernoulli events and the run length between occurrences of W_i would be a Geometric random variable with parameter $P(W_i) = (1 - \beta)$. Consequently, the out-of-control average run length would be

$$ARL_1 = 1/(1 - \beta). \tag{26}$$

Here, the probability β for known and unknown parameter σ is replaced by Eqs. 16 and 20 for the Shewhart R chart and Eqs. 17 and 21 for the Shewhart S chart, Eqs. 18 and 22 for the Bonferroni-adjustment R chart and Eqs. 19 and 23 for the Bonferroni-adjustment S chart, Eqs. 16 and 24 for the new R chart and Eqs. 17 and 25 for the new R chart. Using the OC curves 1–4, and Table 3, for known parameter σ, the ARL_1 for the Shewhart and new approach is demonstrated to be less than the Bonferroni-adjustment approach for both R and S chart. For instance, using the probability β of Table 3 and the ARL_1 of Eq. 26, for $k = 20$, $\alpha = 0.01$, $n = 2$, and $\lambda = 1.5$, we have that the ARL_1 is equal 8.593 and 8.589 for the Shewhart and the proposed R and S chart, respectively, and equal 30.349 and 30.321 for the Bonferroni-adjustment R and S chart, respectively. Additionally, for unknown parameter σ, it can be demonstrated, trough Monte Carlo simulation experiments, that the ARL_1 for the new chart is slightly less than the Shewhart chart and considerably less than the Bonferroni-adjustment chart for both R and S chart. For instance, using the probability β of Tables 5 and 7 and the ARL_1 of Eq. 26, for $k = 20$, $\alpha = 0.01$, $n = 2$, and $\lambda = 1.5$, we have that the ARL_1 is equal 9.346, 16.949, and 8.928 for the Shewhart, the Bonferroni, and the proposed

Table 4 Estimated probability of not detecting a range shift (β) for $\alpha = 0.1$

k	Approach/λ	$n = 2$				$n = 5$				$n = 10$				$n = 25$			
		1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3
10	Shewhart	0.763	0.673	0.517	0.424	0.596	0.315	0.170	0.080	0.430	0.117	0.024	0.008	0.224	0.012	0.000	0.000
	Bonferroni	0.898	0.767	0.650	0.551	0.804	0.504	0.288	0.163	0.687	0.249	0.078	0.015	0.464	0.048	0.001	0.000
	New	0.747	0.633	0.509	0.421	0.594	0.314	0.171	0.080	0.412	0.122	0.023	0.009	0.226	0.012	0.000	0.000
15	Shewhart	0.768	0.615	0.523	0.444	0.589	0.325	0.166	0.088	0.435	0.123	0.026	0.009	0.223	0.009	0.000	0.000
	Bonferroni	0.906	0.774	0.675	0.561	0.843	0.527	0.315	0.181	0.709	0.292	0.099	0.023	0.514	0.055	0.006	0.000
	New	0.760	0.609	0.524	0.441	0.579	0.322	0.170	0.084	0.434	0.125	0.026	0.009	0.226	0.010	0.000	0.000
20	Shewhart	0.767	0.639	0.526	0.445	0.592	0.309	0.171	0.088	0.447	0.128	0.024	0.006	0.218	0.013	0.001	0.000
	Bonferroni	0.910	0.779	0.582	0.593	0.839	0.546	0.328	0.203	0.742	0.317	0.097	0.036	0.546	0.066	0.004	0.000
	New	0.766	0.635	0.524	0.438	0.586	0.306	0.170	0.088	0.450	0.128	0.024	0.006	0.211	0.013	0.001	0.000
30	Shewhart	0.766	0.622	0.521	0.440	0.608	0.316	0.182	0.089	0.447	0.124	0.034	0.013	0.234	0.016	0.001	0.000
	Bonferroni	0.914	0.795	0.699	0.611	0.857	0.565	0.353	0.204	0.751	0.337	0.124	0.041	0.574	0.079	0.005	0.000
	New	0.762	0.619	0.516	0.439	0.608	0.315	0.180	0.089	0.442	0.124	0.033	0.013	0.233	0.016	0.001	0.000
40	Shewhart	0.765	0.629	0.523	0.446	0.601	0.523	0.171	0.020	0.445	0.131	0.034	0.006	0.224	0.012	0.001	0.000
	Bonferroni	0.923	0.807	0.705	0.613	0.875	0.593	0.367	0.228	0.778	0.349	0.123	0.043	0.582	0.083	0.008	0.000
	New	0.758	0.627	0.519	0.446	0.594	0.323	0.171	0.010	0.440	0.132	0.034	0.006	0.224	0.011	0.001	0.000
100	Shewhart	0.764	0.628	0.527	0.441	0.601	0.320	0.173	0.095	0.449	0.123	0.033	0.009	0.233	0.015	0.000	0.000
	Bonferroni	0.937	0.832	0.730	0.645	0.892	0.639	0.414	0.257	0.823	0.412	0.157	0.056	0.661	0.125	0.009	0.001
	New	0.763	0.627	0.526	0.438	0.602	0.317	0.174	0.095	0.451	0.123	0.032	0.009	0.234	0.014	0.000	0.000
120	Shewhart	0.769	0.628	0.526	0.451	0.596	0.321	0.175	0.100	0.447	0.130	0.033	0.011	0.237	0.013	0.000	0.000
	Bonferroni	0.940	0.836	0.741	0.649	0.901	0.653	0.419	0.263	0.830	0.420	0.163	0.059	0.666	0.126	0.011	0.001
	New	0.767	0.627	0.524	0.451	0.598	0.321	0.175	0.100	0.445	0.129	0.033	0.010	0.235	0.013	0.000	0.000

Table 5 Estimated probability of not detecting a range shift (β) for $\alpha = 0.01$

k	Approach/λ	$n = 2$				$n = 5$				$n = 10$				$n = 25$			
		1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3
10	Shewhart	0.903	0.772	0.653	0.566	0.800	0.478	0.280	0.172	0.694	0.240	0.061	0.015	0.463	0.057	0.007	0.000
	Bonferroni	0.942	0.862	0.742	0.636	0.920	0.625	0.387	0.240	0.842	0.396	0.123	0.037	0.653	0.107	0.013	0.000
	New	0.901	0.745	0.649	0.552	0.798	0.474	0.277	0.175	0.690	0.230	0.050	0.015	0.460	0.048	0.004	0.000
15	Shewhart	0.898	0.742	0.658	0.572	0.820	0.490	0.289	0.166	0.701	0.271	0.081	0.020	0.471	0.076	0.002	0.000
	Bonferroni	0.957	0.833	0.737	0.647	0.908	0.631	0.405	0.241	0.845	0.402	0.164	0.038	0.640	0.135	0.005	0.002
	New	0.896	0.730	0.656	0.570	0.808	0.480	0.280	0.168	0.696	0.264	0.079	0.019	0.467	0.066	0.002	0.000
20	Shewhart	0.893	0.769	0.637	0.579	0.819	0.500	0.294	0.171	0.691	0.280	0.058	0.021	0.458	0.051	0.003	0.000
	Bonferroni	0.941	0.837	0.739	0.650	0.907	0.649	0.410	0.257	0.817	0.422	0.142	0.046	0.666	0.110	0.005	0.000
	New	0.888	0.766	0.628	0.575	0.809	0.497	0.291	0.169	0.689	0.279	0.060	0.020	0.459	0.052	0.003	0.000
30	Shewhart	0.895	0.767	0.643	0.558	0.809	0.491	0.312	0.170	0.690	0.266	0.087	0.038	0.465	0.044	0.003	0.000
	Bonferroni	0.949	0.839	0.727	0.637	0.898	0.645	0.420	0.253	0.828	0.407	0.158	0.061	0.672	0.102	0.007	0.000
	New	0.893	0.762	0.638	0.555	0.801	0.487	0.312	0.169	0.682	0.264	0.085	0.033	0.461	0.044	0.003	0.000
40	Shewhart	0.882	0.768	0.655	0.556	0.806	0.493	0.314	0.175	0.693	0.260	0.086	0.012	0.480	0.057	0.003	0.000
	Bonferroni	0.936	0.843	0.733	0.646	0.899	0.637	0.406	0.259	0.836	0.415	0.157	0.051	0.647	0.127	0.009	0.000
	New	0.880	0.766	0.652	0.556	0.801	0.497	0.315	0.174	0.694	0.255	0.086	0.012	0.480	0.054	0.003	0.000
100	Shewhart	0.885	0.762	0.649	0.568	0.802	0.514	0.311	0.182	0.691	0.279	0.087	0.028	0.476	0.054	0.003	0.000
	Bonferroni	0.937	0.836	0.732	0.647	0.897	0.639	0.417	0.264	0.819	0.418	0.163	0.054	0.655	0.121	0.008	0.001
	New	0.884	0.761	0.650	0.566	0.801	0.512	0.313	0.181	0.692	0.281	0.088	0.028	0.469	0.053	0.002	0.000
120	Shewhart	0.882	0.757	0.656	0.564	0.804	0.509	0.291	0.180	0.684	0.270	0.085	0.030	0.474	0.059	0.004	0.000
	Bonferroni	0.944	0.832	0.738	0.646	0.899	0.635	0.407	0.259	0.824	0.414	0.152	0.059	0.657	0.117	0.011	0.001
	New	0.881	0.756	0.654	0.562	0.801	0.509	0.291	0.178	0.682	0.267	0.084	0.029	0.474	0.058	0.004	0.000

Table 6 Estimated probability of not detecting a standard deviation shift (β) for $\alpha = 0.1$

k	Approach/λ	$n = 2$				$n = 5$				$n = 10$				$n = 25$			
		1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3
10	Shewhart	0.757	0.618	0.529	0.431	0.599	0.319	0.173	0.081	0.431	0.116	0.025	0.008	0.226	0.011	0.000	0.000
	Bonferroni	0.881	0.772	0.648	0.560	0.809	0.505	0.289	0.162	0.686	0.246	0.078	0.016	0.467	0.049	0.001	0.000
	New	0.745	0.598	0.523	0.425	0.597	0.317	0.172	0.080	0.419	0.116	0.023	0.009	0.222	0.010	0.000	0.000
15	Shewhart	0.749	0.627	0.532	0.447	0.591	0.328	0.175	0.086	0.437	0.120	0.029	0.009	0.223	0.010	0.000	0.000
	Bonferroni	0.904	0.773	0.669	0.567	0.844	0.529	0.319	0.180	0.711	0.291	0.099	0.025	0.514	0.056	0.006	0.000
	New	0.741	0.612	0.530	0.440	0.588	0.325	0.173	0.085	0.437	0.118	0.027	0.009	0.226	0.010	0.000	0.000
20	Shewhart	0.772	0.616	0.529	0.442	0.594	0.329	0.179	0.087	0.446	0.122	0.025	0.007	0.226	0.012	0.001	0.000
	Bonferroni	0.909	0.780	0.591	0.594	0.848	0.540	0.328	0.207	0.743	0.314	0.096	0.035	0.467	0.065	0.004	0.000
	New	0.768	0.602	0.527	0.437	0.589	0.328	0.176	0.085	0.445	0.120	0.023	0.007	0.222	0.011	0.001	0.000
30	Shewhart	0.763	0.623	0.530	0.444	0.606	0.330	0.181	0.080	0.448	0.126	0.035	0.012	0.234	0.015	0.001	0.000
	Bonferroni	0.916	0.804	0.703	0.617	0.861	0.569	0.352	0.208	0.754	0.339	0.121	0.043	0.573	0.080	0.005	0.000
	New	0.759	0.615	0.529	0.440	0.607	0.330	0.181	0.081	0.444	0.123	0.032	0.012	0.234	0.014	0.001	0.000
40	Shewhart	0.767	0.634	0.538	0.442	0.608	0.531	0.173	0.101	0.442	0.132	0.035	0.007	0.226	0.011	0.001	0.000
	Bonferroni	0.910	0.809	0.709	0.615	0.881	0.590	0.365	0.227	0.777	0.342	0.124	0.045	0.467	0.085	0.007	0.000
	New	0.761	0.630	0.521	0.441	0.599	0.329	0.172	0.098	0.441	0.132	0.034	0.006	0.222	0.012	0.001	0.000
100	Shewhart	0.766	0.631	0.531	0.445	0.609	0.329	0.172	0.098	0.447	0.127	0.031	0.009	0.233	0.016	0.000	0.000
	Bonferroni	0.936	0.841	0.741	0.648	0.899	0.641	0.418	0.253	0.823	0.419	0.159	0.058	0.661	0.127	0.009	0.001
	New	0.763	0.625	0.529	0.442	0.609	0.329	0.173	0.097	0.456	0.124	0.031	0.008	0.234	0.017	0.000	0.000
120	Shewhart	0.772	0.634	0.530	0.453	0.612	0.335	0.177	0.101	0.448	0.131	0.034	0.012	0.225	0.012	0.000	0.000
	Bonferroni	0.939	0.839	0.750	0.652	0.918	0.657	0.418	0.264	0.834	0.423	0.165	0.058	0.462	0.129	0.010	0.001
	New	0.769	0.632	0.530	0.453	0.611	0.333	0.177	0.100	0.447	0.129	0.032	0.011	0.221	0.013	0.000	0.000

Table 7 Estimated probability of not detecting a standard deviation shift (β) for $\alpha = 0.01$

k	Approach/λ	$n = 2$				$n = 5$				$n = 10$				$n = 25$			
		1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3	1.5	2	2.5	3
10	Shewhart	0.904	0.760	0.656	0.550	0.854	0.526	0.314	0.182	0.634	0.214	0.038	0.020	0.264	0.010	0.000	0.000
	Bonferroni	0.956	0.840	0.748	0.636	0.942	0.684	0.448	0.258	0.816	0.328	0.084	0.042	0.478	0.024	0.002	0.000
	New	0.896	0.752	0.652	0.548	0.848	0.516	0.298	0.178	0.628	0.214	0.037	0.020	0.258	0.011	0.000	0.000
15	Shewhart	0.908	0.761	0.654	0.553	0.852	0.528	0.318	0.185	0.638	0.219	0.040	0.023	0.268	0.013	0.000	0.000
	Bonferroni	0.959	0.842	0.743	0.638	0.942	0.685	0.449	0.259	0.819	0.332	0.088	0.044	0.483	0.026	0.003	0.000
	New	0.896	0.754	0.651	0.550	0.847	0.519	0.299	0.180	0.632	0.218	0.039	0.022	0.259	0.012	0.000	0.000
20	Shewhart	0.889	0.763	0.657	0.551	0.855	0.527	0.314	0.183	0.635	0.215	0.039	0.021	0.262	0.015	0.000	0.000
	Bonferroni	0.959	0.841	0.745	0.638	0.946	0.687	0.447	0.259	0.818	0.329	0.084	0.042	0.477	0.029	0.002	0.000
	New	0.886	0.755	0.655	0.549	0.851	0.519	0.299	0.181	0.629	0.214	0.039	0.021	0.254	0.016	0.000	0.000
30	Shewhart	0.895	0.762	0.655	0.556	0.853	0.525	0.319	0.184	0.632	0.218	0.038	0.027	0.261	0.010	0.000	0.000
	Bonferroni	0.957	0.845	0.743	0.639	0.941	0.682	0.451	0.261	0.815	0.332	0.085	0.049	0.477	0.026	0.002	0.000
	New	0.891	0.751	0.650	0.552	0.846	0.516	0.305	0.179	0.627	0.215	0.039	0.025	0.258	0.011	0.000	0.000
40	Shewhart	0.889	0.766	0.658	0.553	0.857	0.528	0.319	0.186	0.637	0.212	0.043	0.022	0.266	0.015	0.000	0.000
	Bonferroni	0.955	0.841	0.750	0.632	0.944	0.687	0.450	0.263	0.819	0.327	0.088	0.043	0.481	0.028	0.002	0.000
	New	0.887	0.756	0.656	0.549	0.852	0.520	0.303	0.183	0.634	0.214	0.042	0.023	0.265	0.013	0.000	0.000
100	Shewhart	0.890	0.762	0.652	0.556	0.855	0.529	0.313	0.181	0.634	0.216	0.035	0.028	0.269	0.018	0.000	0.000
	Bonferroni	0.961	0.842	0.744	0.641	0.946	0.689	0.448	0.255	0.819	0.329	0.080	0.043	0.480	0.030	0.003	0.000
	New	0.888	0.758	0.650	0.551	0.851	0.521	0.297	0.182	0.635	0.215	0.036	0.027	0.269	0.016	0.000	0.000
120	Shewhart	0.895	0.761	0.654	0.552	0.852	0.524	0.317	0.181	0.639	0.214	0.045	0.022	0.267	0.012	0.000	0.000
	Bonferroni	0.957	0.843	0.747	0.639	0.940	0.683	0.449	0.257	0.821	0.327	0.089	0.044	0.479	0.025	0.004	0.000
	New	0.892	0.759	0.654	0.551	0.848	0.515	0.307	0.180	0.639	0.214	0.043	0.023	0.267	0.012	0.000	0.000

Table 8 Inside diameter measurements (mm) on forged piston rings (Montgomery (2001))

Sa Nu	Observations				R_i	S_i
1	74.030	74.002	74.019	74.008	0.028	0.012
2	73.995	73.992	74.001	74.004	0.012	0.005
3	73.988	74.024	74.021	74.002	0.036	0.017
4	73.992	74.007	74.015	74.014	0.023	0.011
5	74.009	73.994	73.997	73.993	0.016	0.007
6	73.995	74.006	73.994	74.005	0.012	0.006
7	73.985	74.003	73.993	73.988	0.018	0.008
8	73.998	74.000	73.990	73.995	0.010	0.004
9	74.004	74.000	74.007	73.996	0.011	0.005
10	73.983	74.002	73.998	74.012	0.029	0.012
11	74.006	73.967	73.994	73.984	0.039	0.016
12	74.000	73.984	74.005	73.996	0.021	0.009
13	73.994	74.012	73.986	74.007	0.026	0.012
14	74.006	74.010	74.018	74.000	0.018	0.008
15	74.000	74.010	74.013	74.003	0.013	0.006
16	73.982	74.001	74.015	73.996	0.033	0.014
17	74.004	73.999	73.990	74.009	0.019	0.008
18	74.010	73.989	73.990	74.014	0.025	0.013
19	74.015	74.008	73.993	74.010	0.022	0.009
20	73.982	73.984	73.995	74.013	0.031	0.014

Table 9 Out-of-control average run length ($k = 20$, $n = 4$)

Approach/λ	$\alpha = 0.1$				$\alpha = 0.01$			
	1.5	2	2.5	3	1.5	2	2.5	3
Shewhart R chart	2.541	1.539	1.273	1.166	5.885	2.235	1.554	1.317
Bonferroni R chart	7.573	2.501	1.650	1.365	17.604	3.658	2.019	1.540
New R chart	2.463	1.517	1.263	1.161	5.538	2.174	1.531	1.305
Shewhart S chart	2.519	1.527	1.265	1.161	5.795	2.205	1.538	1.306
Bonferroni S chart	7.444	2.464	1.631	1.353	17.212	3.588	1.990	1.523
New S chart	2.444	1.507	1.256	1.156	5.467	2.148	1.517	1.296

R chart, respectively, and equal 9.009, 24.390, and 8.772 for the Shewhart, the Bonferroni, and the proposed S chart, respectively.

6 Example

Twenty samples ($k = 20$) each of size four ($n = 4$) of piston rings for an automotive engine are produced by a forging process, have been taken when the process is in control (Table 8). Using the data of inside diameter, data setting up S, $\bar{R}$ and $\bar{S}$, the values of these statistics are calculated to be 0.01055, 0.0221 and 0.00988, respectively.

Table 10 The constant value ψ for constructing the R and S control chart

n	K 2	3	4	5	6	7	8	9	10	15	20	25	30	40	100	120
2	0.8862	0.9213	0.9400	0.9516	0.9594	0.9650	0.9693	0.9726	0.9754	0.9835	0.9876	0.9901	0.9917	0.9937	0.9975	0.9979
3	0.9400	0.9594	0.9693	0.9754	0.9794	0.9823	0.9845	0.9862	0.9876	0.9917	0.9937	0.9950	0.9958	0.9969	0.9988	0.9990
4	0.9594	0.9726	0.9794	0.9835	0.9862	0.9882	0.9896	0.9908	0.9917	0.9945	0.9958	0.9967	0.9972	0.9979	0.9992	0.9993
5	0.9693	0.9794	0.9845	0.9876	0.9896	0.9911	0.9922	0.9931	0.9937	0.9958	0.9969	0.9975	0.9979	0.9984	0.9994	0.9995
6	0.9754	0.9835	0.9876	0.9901	0.9917	0.9929	0.9937	0.9945	0.9950	0.9967	0.9975	0.9980	0.9983	0.9988	0.9995	0.9996
7	0.9794	0.9862	0.9896	0.9917	0.9931	0.9941	0.9948	0.9954	0.9958	0.9972	0.9979	0.9983	0.9986	0.9990	0.9996	0.9997
8	0.9823	0.9882	0.9911	0.9929	0.9941	0.9949	0.9955	0.9961	0.9964	0.9977	0.9983	0.9986	0.9988	0.9991	0.9996	0.9997
9	0.9845	0.9896	0.9922	0.9937	0.9948	0.9955	0.9961	0.9965	0.9969	0.9979	0.9984	0.9988	0.9990	0.9992	0.9997	0.9997
10	0.9862	0.9908	0.9931	0.9945	0.9954	0.9961	0.9965	0.9969	0.9972	0.9981	0.9986	0.9989	0.9991	0.9993	0.9997	0.9998
11	0.9876	0.9917	0.9937	0.9950	0.9958	0.9964	0.9969	0.9972	0.9975	0.9983	0.9988	0.9990	0.9992	0.9994	0.9998	0.9998
14	0.9904	0.9937	0.9952	0.9962	0.9968	0.9972	0.9976	0.9979	0.9980	0.9987	0.9990	0.9992	0.9994	0.9995	0.9998	0.9998
15	0.9911	0.9941	0.9955	0.9964	0.9970	0.9974	0.9978	0.9980	0.9983	0.9988	0.9991	0.9993	0.9994	0.9996	0.9998	0.9999
18	0.9927	0.9951	0.9963	0.9971	0.9976	0.9979	0.9982	0.9984	0.9985	0.9990	0.9993	0.9994	0.9995	0.9996	0.9999	0.9999
20	0.9934	0.9956	0.9967	0.9974	0.9978	0.9981	0.9984	0.9985	0.9986	0.9991	0.9993	0.9995	0.9996	0.9997	0.9999	0.9999
25	0.9948	0.9965	0.9974	0.9979	0.9983	0.9985	0.9987	0.9989	0.9990	0.9993	0.9995	0.9996	0.9997	0.9997	0.9999	0.9999
30	0.9957	0.9971	0.9978	0.9983	0.9986	0.9988	0.9989	0.9990	0.9991	0.9994	0.9996	0.9997	0.9997	0.9998	0.9999	0.9999
35	0.9963	0.9976	0.9982	0.9985	0.9988	0.9989	0.9991	0.9992	0.9993	0.9995	0.9996	0.9997	0.9998	0.9998	0.9999	0.9999
40	0.9968	0.9979	0.9984	0.9987	0.9989	0.9991	0.9992	0.9993	0.9994	0.9996	0.9997	0.9997	0.9998	0.9998	0.9999	0.9999
45	0.9972	0.9981	0.9985	0.9988	0.9991	0.9992	0.9993	0.9994	0.9994	0.9996	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000
50	0.9974	0.9983	0.9987	0.9990	0.9991	0.9993	0.9994	0.9994	0.9995	0.9997	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000
60	0.9979	0.9986	0.9990	0.9991	0.9993	0.9994	0.9995	0.9995	0.9996	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000	1.0000
70	0.9982	0.9988	0.9991	0.9993	0.9994	0.9995	0.9995	0.9996	0.9996	0.9998	0.9998	0.9999	0.9999	0.9999	1.0000	1.0000
80	0.9984	0.9989	0.9992	0.9994	0.9995	0.9995	0.9996	0.9996	0.9997	0.9998	0.9998	0.9999	0.9999	0.9999	1.0000	1.0000

Table 10 continued

n	*K*															
	2	3	4	5	6	7	8	9	10	15	20	25	30	40	100	120
90	0.9986	0.9991	0.9993	0.9994	0.9995	0.9996	0.9996	0.9997	0.9997	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000
100	0.9987	0.9992	0.9994	0.9995	0.9996	0.9996	0.9997	0.9997	0.9997	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000
110	0.9989	0.9992	0.9994	0.9995	0.9996	0.9997	0.9997	0.9997	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000
120	0.9989	0.9993	0.9995	0.9996	0.9996	0.9997	0.9997	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000

The values of the out-of-control average run length (Table 9) are given using Eqs. 20, 22, and 24 into Eq. 26 for the Shewhart, the Bonferroni-adjustment, and the new R chart, respectively, and also Eqs. 21, 23, and 25 into Eq. 26 for the Shewhart, the Bonferroni-adjustment, and the new S chart, respectively. Here, the constant values ψ, d_2, d_3 and c_4 for $k = 20$ and $n = 4$, are 0.9958, 2.059, 0.88 and 0.9213, in the order mentioned.

Table 9 for fixed values $\alpha = 0.1, 0.01$ and $\lambda = 1.5, 2, 2.5, 3$ shows that the out-of-control average run length for the proposed approach is less than the Shewhart and Bonferroni-adjustment approach for both R and S chart. The power of the control limits is $1 - \beta$. As a result, it can be demonstrated that the new approach has maximum power of control limits compared with the Bonferroni-adjustment approach that has minimum power.

7 Conclusion

It has been shown that, for an unknown standard deviation parameter, the suggested new range and standard deviation control chart has three advantages over the Shewhart and the Bonferroni-adjustment range and standard deviation control charts.

- The first is that the constant values to construct the control chart for the new approach are based on both sample subgroup size and sample group size;
- The second advantage is that, for a fixed value α**,** the ARL_1 for the new approach is less than the ARL_1 for the Shewhart and Bonferroni-adjustment approach and
- The third advantage is that the new approach is based on a statistic with variance less than that of the Shewhart and Bonferroni-adjustment approach.

Therefore, practitioners are advised to use the new approach for monitoring the variability of a quality characteristic.

Appendix A

(S-Plus)

```
Time=10000 ; al=0.01 ; n=5 ; lam=2.5 ; k=10
if (al>=0.1 & al<=0.1) { z=1.6449 ; if (k>=10 & k<=10) z2= 2.576 ; if (k>=15 & k<=15) z2=2.713 ; if
(k>=20 & k<=20) z2= 2.807
if (k>=30 & k<=30) z2=2.935 ; if (k>=40 & k<=40) z2= 3.023 ; if (k>=100 & k<=100) z2=3.291 ; if
(k>=120 & k<=120) z2=3.341 }
if (al>=0.01 & al<=0.01) { z=2.5758 ; if (k>=10 & k<=10) z2= 3.291 ; if (k>=15 & k<=15) z2=3.403 ; if
(k>=20 & k<=20) z2= 3.481
if (k>=30 & k<=30) z2=3.588 ; if (k>=40 & k<=40) z2=3.662 ; if (k>=100 & k<=100) z2=3.891 ; if
(k>=120 & k<=120) z2=3.935 }
if ( n>=2 & n<=2) {
d2=1.128 ; d3=0.853 ; if (k>=10 & k<=10) psi= 0.9754 ; if (k>=15 & k<=15) psi= 0.9835 ; if (k>=20 &
k<=20) psi= 0.9876 ; if (k>=25 & k<=25) psi= 0.9901 ; if (k>=30 & k<=30) psi= 0.9917 ; if (k>=40 &
k<=40) psi= 0.9937 ; if (k>=100 & k<=100) psi= 0.9975 ; if (k>=120 & k<=120) psi= 0.9979}
if ( n>=5 & n<=5) {
d2=2.326 ; d3=0.864
if (k>=10 & k<=10) psi= 0.9937 ; if (k>=15 & k<=15) psi= 0.9958 ; if (k>=20 & k<=20) psi= 0.9969 ; if
(k>=25 & k<=25) psi= 0.9975
if (k>=30 & k<=30) psi= 0.9979 ; if (k>=40 & k<=40) psi= 0.9984 ; if (k>=100 & k<=100) psi= 0.9994 ;
if (k>=120 & k<=120) psi= 0.9995}
if ( n>=10 & n<=10) {
d2=3.078 ; d3=0.797 ; if (k>=10 & k<=10) psi= 0.9972 ; if (k>=15 & k<=15) psi= 0.9981 ; if (k>=20 &
k<=20) psi= 0.9986 ; if (k>=25 & k<=25) psi= 0.9989 ; if (k>=30 & k<=30) psi= 0.9991 ; if (k>=40 &
k<=40) psi= 0.9993 ; if (k>=100 & k<=100) psi= 0.9997 ; if (k>=120 & k<=120) psi= 0.9998 }
if ( n>=25 & n<=25) {
```

```
d2=3.931 ; d3=0.708 ; if (k>=10 & k<=10) psi= 0.9990 ; if (k>=15 & k<=15) psi= 0.9993 ; if (k>=20 &
k<=20) psi= 0.9995 ; if (k>=25 & k<=25) psi= 0.9996 ; if (k>=30 & k<=30) psi= 0.9997 ; if (k>=40 &
k<=40) psi= 0.9997 ; if (k>=100 & k<=100) psi= 0.9999 ; if (k>=120 & k<=120) psi= 0.9999 }
nn1=0 ; nn2=0 ; nn3=0 ; Vs=0
b_matrix(rnorm(20*5,0,1),nrow=20,ncol=5)
  for ( h in 1:20){
    Vs_Vs+ (stdev(b[h,])^2) }
  St_sqrt(Vs/20)/0.9969 ; Es_( sqrt(Vs/20)/0.9969)*d2 ; Ss_(sqrt(Vs/20)/0.9969)*d3
for ( i in 1:Time) {
  a_matrix(rnorm(k*n,0,1),nrow=k,ncol=n)
  Rs=0 ; Vs=0
  for ( h in 1:k){
    Rs_Rs+(max(a[h,])-min(a[h,]))
    Rbar_Rs/k
    Vs_Vs+ (stdev(a[h,])^2)
    Vbar_Vs/k }
    LCL1_((Rbar/d2)*(d2-z*d3)-(lam*St*d2))/(lam*St*d3);      UCL1_((Rbar/d2)*(d2+z*d3)-(lam*St*d2))/
     (lam*St*d3)
    LCL2_((Rbar/d2)*(d2-z2*d3)-(lam*St*d2))/(lam*St*d3); UCL2_((Rbar/d2)*(d2+z2*d3)-(lam*St*d2))/
     (lam*St*d3)
    LCL3_((    sqrt(Vbar)/psi)*(d2-z*d3)-(lam*St*d2))/(lam*St*d3);    UCL3_((sqrt(Vbar)/psi)*(d2+z*d3)-
     (lam*St*d2))/(lam*St*d3)
      for ( j in 1:k){
      Zi_((max(a[j,])-min(a[j,]))-(Es))/(Ss) ; if (Zi<=UCL1 & Zi>=LCL1) nn1=nn1+1 ; If (Zi<=UCL2 &
      Zi>=LCL2) nn2=nn2+1
      if (Zi<=UCL3 & Zi>=LCL3) nn3=nn3+1 } }
Beta1_(nn1/(k*Time)) ; Beta2_(nn2/(k*Time)) ; Beta3_(nn3/(k*Time))
print(Beta1) ; print(Beta2) ; print(Beta3)
```

Appendix B

(S-Plus)

```
Time=10000 ; al=0.01 ; n=5 ; lam=2.5 ; k=10
if (al>=0.1 & al<=0.1) { z=1.6449 ; if (k>=10 & k<=10) z2= 2.576 ; if (k>=15 & k<=15) z2=2.713 ; if
(k>=20 & k<=20) z2= 2.807
if (k>=30 & k<=30) z2=2.935 ; if (k>=40 & k<=40) z2= 3.023 ; if (k>=100 & k<=100) z2=3.291 ; if
(k>=120 & k<=120) z2=3.341 }
if (al>=0.01 & al<=0.01) { z=2.5758 ; if (k>=10 & k<=10) z2= 3.291 ; if (k>=15 & k<=15) z2=3.403 ; if
(k>=20 & k<=20) z2= 3.481
if (k>=30 & k<=30) z2=3.588 ; if (k>=40 & k<=40) z2=3.662 ; if (k>=100 & k<=100) z2=3.891 ; if
(k>=120 & k<=120) z2=3.935 }
if ( n>=2 & n<=2) {
c4=0.7979 ; c42=0.603 ; if (k>=10 & k<=10) psi= 0.9754 ; if (k>=15 & k<=15) psi= 0.9835 ; if (k>=20 &
k<=20) psi= 0.9876 ; if (k>=25 & k<=25) psi= 0.9901 ; if (k>=30 & k<=30) psi= 0.9917 ; if (k>=40 &
k<=40) psi= 0.9937 ; if (k>=100 & k<=100) psi= 0.9975 ; if (k>=120 & k<=120) psi= 0.9979}
if ( n>=5 & n<=5) {
c4=0.94 ; c42=0.381 ; if (k>=10 & k<=10) psi= 0.9937 ; if (k>=15 & k<=15) psi= 0.9958 ; if (k>=20 &
k<=20) psi= 0.9969 ; if (k>=25 & k<=25) psi= 0.9975
if (k>=30 & k<=30) psi= 0.9979 ; if (k>=40 & k<=40) psi= 0.9984 ; if (k>=100 & k<=100) psi= 0.9994 ;
if (k>=120 & k<=120) psi= 0.9995}
if ( n>=10 & n<=10) {
c4=0.9727 ; c42=0.232 ; if (k>=10 & k<=10) psi= 0.9972 ; if (k>=15 & k<=15) psi= 0.9981 ; if (k>=20 &
k<=20) psi= 0.9986 ; if (k>=25 & k<=25) psi= 0.9989 ; if (k>=30 & k<=30) psi= 0.9991 ; if (k>=40 &
k<=40) psi= 0.9993 ; if (k>=100 & k<=100) psi= 0.9997 ; if (k>=120 & k<=120) psi= 0.9998 }
if ( n>=25 & n<=25) {
c4=0.9895 ; c42=0.144 ; if (k>=10 & k<=10) psi= 0.9990 ; if (k>=15 & k<=15) psi= 0.9993 ; if (k>=20 &
k<=20) psi= 0.9995 ; if (k>=25 & k<=25) psi= 0.9996 ; if (k>=30 & k<=30) psi= 0.9997 ; if (k>=40 &
k<=40) psi= 0.9997 ; if (k>=100 & k<=100) psi= 0.9999 ; if (k>=120 & k<=120) psi= 0.9999 }
```

```
nn1=0 ; nn2=0 ; nn3=0 ; Vs=0
b_matrix(rnorm(20*5,0,1),nrow=20,ncol=5)
  for ( h in 1:20){
    Vs_Vs+ (stdev(b[h,])^2) }
  St_sqrt(Vs/20)/0.9969 ; Es_( sqrt(Vs/20)/0.9969)*c4 ; Ss_( sqrt(Vs/20)/0.9969)*c42
for ( i in 1:Time) {
  a_matrix(rnorm(k*n,0,1),nrow=k,ncol=n)
  Sbar1=0 ; Vs=0
  for ( h in 1:k){
    Sbar1_Sbar1+(stdev(a[h,]))
    Sbar_Sbar1/k
    Vs_Vs+ (stdev(a[h,])^2)
    Vbar_Vs/k }
    LCL1_((Sbar/c4)*(c4-z*c42)-(lam*St*c4))/(lam*St*c42);  UCL1_((Sbar/c4)*(c4+z*c42)-(lam*St*c4))/
     (lam*St*c42)
    LCL2_((Sbar/c4)*(c4-z2*c42)-(lam*St*c4))/(lam*St*c42); UCL2_((Sbar/c4)*(c4+z2*c42)-(lam*St*c4))
     /(lam*St*c42)
    LCL3_(( sqrt(Vbar)/psi)*(c4-z*c42)-(lam*St*c4))/(lam*St*c42); UCL3_(( sqrt(Vbar)/psi)*(c4+z*c42)-
     (lam*St*c4))/(lam*St*c42)
       for ( j in 1:k){
        Zi_((stdev(a[j,]))-(Es))/(Ss) ; if (Zi<=UCL1 & Zi>=LCL1) nn1=nn1+1 ; if (Zi<=UCL2 & Zi>=
        LCL2) nn2=nn2+1
        if (Zi<=UCL3 & Zi>=LCL3) nn3=nn3+1 } }
Beta1_(nn1/(k*Time)) ; Beta2_(nn2/(k*Time)) ; Beta3_(nn3/(k*Time))
print(Beta1) ; print(Beta2) ; print(Beta3)
```

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