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Abstract.

Balanced (exponential) growth cannot be generalized to a concept which would not require knife-edge conditions to be imposed on dynamic models. Already the assumption that a solution to a dynamical system (i.e. time path of an economy) satisfies a given functional regularity (e.g. quasi-arithmetic, logistic, etc.) imposes at least one knife-edge assumption on the considered model. Furthermore, it is always possible to find *divergent* and *qualitative* changes in dynamic behavior of the model – strong enough to invalidate its long-run predictions – if a certain parameter is infinitesimally manipulated. In this sense, dynamics of all growth models are fragile and "unstable".

Keywords and Phrases: knife-edge condition, balanced growth, regular growth, bifurcation, growth model, long run, long-run dynamics

JEL Classification Numbers: C62, O40, O41

1 Introduction

One of the aspects present in the debate on sources and limitations of long-run growth is the prevalence of knife-edge conditions in certain classes of growth models. According to Uzawa (1961), technical change must be purely labor-augmenting in neoclassical growth models if balanced growth is to be obtained. Much more recently, the fact that endogenous growth models rely on linear differential equations for the existence of a balanced growth path (BGP) has sparked the "linearity critique" (cf. Jones, 2005a), according to which there is no *a priori* reason to believe that in a given equation of form:

$$\dot{X} = \alpha X^{\phi}, \tag{1}$$

the parameter ϕ would be exactly equal to 1, guaranteeing the existence of a BGP. Indeed, sufficiently small deviations from $\phi = 1$ will never be rejected on purely statistical premises, no matter what type of real-world data is used in the empirical work. But it is the *exact* linearity of (1), or *purely* labor-augmenting technical change in the case of neoclassical growth models, which is conducive to balanced (exponential) growth.

This argument was further developed by Li (2000), Christiaans (2004), and Growiec (2007a), eventually indicating that in fact, a generalized version of the linearity critique holds for any growth model which is capable of generating exponential growth: it is the assumption of exponential growth itself which gives rise to knife-edge requirements. In the current paper, we provide a significant generalization of this result: we demonstrate that knife-edge conditions are necessary if *any* type of (sufficiently smooth) pre-determined growth regularity is going to be derived. We also add a further amplification of this finding by proving that even infinitesimal departures from the benchmark parametrization of a given growth model – if sufficiently smartly designed – could result in qualitatively different, divergent dynamics of the model, thereby ruining the pre-defined long-run growth regularity.

Let us clarify the conceptual base first. We shall build upon the following definition (cf. Growiec, 2007a).

Definition 1 *A* knife-edge condition *is a condition imposed on parameter values such that the set of values satisfying this condition has an empty interior in the space of all possible values. Parameter values that are requested to satisfy a particular knife-edge condition would also be referred to as* non-typical.

There are, in principle, two ways of dealing with the problem of knife-edge assumptions in growth models. First, one may stick to the BGP requirement and try to find growth-driving knife-edge conditions of form which is most plausible empirically. This path has been followed, among others, by Jones (2003) who judged that a linear equation of population growth to be the most plausible one and proceeded to build a semi-endogenous growth model with endogenous fertility.[1] A similar approach has been taken by Connolly and Peretto (2003). Recent empirical evidence shows that it could also be plausible that, even more so than in the population equation, the crucial knife-edge condition should be placed in the knowledge production function, following the Schumpeterian formulation (Ha and Howitt, 2007; Madsen, 2008). In the light of these results, Schumpeterian R&D-based growth models provide an accurate representation of the growth process, and the knife-edge assumptions they make are (at least approximately) empirically relevant.

[1]Solow (2003) casts doubt on the Jones' (2003) *bon-mot*: "it is a biological fact of nature that people reproduce in proportion to their number". He writes: "I am doubtful about this, for two reasons. The first is that birth rates can and probably do depend on population size, and that is a nonlinearity. Fertility is surely a social phenomenon in rich societies. (...) Furthermore, there are various environmental and social factors that lead to logistic curves." Indeed, population growth for animal species in isolation is best modeled by logistic equations; are people really so different?

The apparent second way of dealing with knife-edge assumptions in growth models is to generalize the concept of exponential growth to allow more general and flexible forms of temporal evolution of variables. Perhaps the most prominent idea in this field is the concept of regular (quasi-arithmetic, less-than-exponential) growth. This idea, put forward by Mitra (1983) and developed by Asheim et al. (2007) and Groth, Koch, and Steger (2008), will be discussed in more detail in the following sections.

One of the statements made in works dealing with regular growth is that generalizing exponential growth helps get rid of knife-edge assumptions. This is not true. As we shall see shortly, such step can only change the type of knife-edge assumptions imposed on the model. Of course, this alone could be a significant development since the new knife-edge assumptions may be markedly more plausible empirically.[2] Extending the concept of exponential growth cannot eliminate the need for knife-edge assumptions, however, no matter how many consecutive generalizations are applied.

The primary objective of this paper is to show that balanced (exponential) growth cannot be generalized to a concept which would not require knife-edge conditions to be imposed on growth models. Indeed, making the assumption that a solution to a dynamical system (i.e. the time path of the economy) satisfies a given (non-trivial and sufficiently smooth) functional regularity necessarily imposes at least one knife-edge assumption on the considered model. It is true *regardless* of the type of regularity we would like to impose; what matters is that the presumed functional form must be given in advance.

The second substantive result of this paper is a proof that it is always possible to extend the formulation of a given model in a way that infinite divergence in results appears over the long run if a certain parameter is infinitesimally manipulated. Furthermore, if the given

[2] Generalizing exponential growth may also help eliminate *some of* the required knife-edge conditions if the original formulation featured multiple ones.

model predicts unbounded growth, *qualitative* changes in dynamic behavior of the model in response to infinitesimal shifts in that parameter are also necessarily observed and infinite divergence follows already in finite time.

One well-known example of such unstable and bifurcative behavior is the one of equation (1): if $\phi > 1$, X diverges to infinity in finite time (no matter how tiny the difference between ϕ and 1 is); if $\phi < 1$, however, then growth is less-than-exponential and growth rates gradually fall down to zero.[3] Only for $\phi = 1$ can balanced growth be sustained. In the light of our results, however, exponential growth is not special at all in giving rise to so enormous changes in the dynamic behavior of the model when a certain parameter is infinitesimally manipulated. This in fact happens for *all* possible functional forms of the considered model, as long as it predicts unbounded growth. Moreover, these changes are generically qualitative, giving rise to bifurcations in the modes of dynamic behavior.

All relevant theorems will be proven in Section 2. In Section 3 we will refer to regular, less-than-exponential growth as an important application of the theorems. We will also generalize that concept, proposing a specification which nests regular growth as a special case. We will then show how to extend this procedure *ad infinitum*, allowing ever larger classes of functions but never getting rid of knife-edge assumptions. We will also discuss the important cases of logistic growth as well as more-than-exponential growth. Section 4 concludes with a discussion of our results and their methodological consequences for modeling long-run growth.

[3]The equation $\dot{X} = \alpha X^{\phi}$ with $\phi < 1$ gives rise to regular (quasi-arithmetic) growth as discussed e.g. by Groth, Koch, and Steger (2008). As we shall see shortly, regular growth is subject to such bifurcative behavior as well.

2 The theorems

This section is devoted to proving the principal results of this paper. We shall first deal with models set up in continuous time, then we shall switch to discrete time. Finally, having returned to continuous time, we will show why knife-edge conditions should always be associated with instabilities and bifurcations once manipulations in model parameters are allowed, even if these manipulations were arbitrarily small.

2.1 Continuous time

Let us consider a very general form of a continuous-time model of economic growth. Its dynamics are ruled by a system of autonomous differential equations of order m:

$$F(X, \dot{X}, ..., X^{(m)}) = 0, \qquad X(0), \dot{X}(0), ..., X^{(m-1)}(0) \text{ given.} \tag{2}$$

By $X = (X_1, X_2, ..., X_n)$ we denote a vector of n state variables. Each i-th variable X_i is assumed to be at least m times continuously differentiable with respect to time. By $\dot{X}$ we denote a vector of X_i's first order time derivatives, and by $\hat{X} = \dot{X}/X$ we denote a vector of their growth rates.[4] It is assumed that all X_i's are strictly positive; m and n are arbitrary positive integers. It is also assumed that $F \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$. We shall concentrate on autonomous differential equations only, since it is natural for economists to look for general laws that are valid irrespective of time. We assume that all solutions to (2) are well defined for all $t \geq 0$.

A further remark is that in (2), we ignore control (choice, decision) variables. Although these are vital ingredients of economic models which include optimization – as most con-

[4]Provided that $X > 0$, the vector $\hat{X}$ is also a vector of their first order log-time derivatives. The definition of $\hat{X}$ which we consider here is however more general since it applies to negative X's as well. In fact, we will frequently refer to negative X's in this paper.

temporary growth models do – they can be ruled out from present considerations, since we are interested in the long-run dynamics only.

We shall also pose another function, $G \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$, capturing the *predefined growth regularity*. Precisely, the condition $G(X, \dot{X}, ..., X^{(m)}) = 0$ is the particular regularity imposed on the solution $\{X(t)\}_{t=0}^{\infty}$ to the model (2). We shall assume that G is locally Lipschitz continuous for all arguments $(X, \dot{X}, ..., X^{(m)})$ satisfying the eqality $G(X, \dot{X}, ..., X^{(m)}) = 0$.

Under the above assumptions, the following theorem holds.

Theorem 1 (Continuous time version) *The set $\mathcal{F}$ of functions $F \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$ such that $G(X(t), \dot{X}(t), ..., X^{(m)}(t)) = 0$ for some solution $\{X(t)\}_{t=0}^{\infty}$ to $F(X, \dot{X}, ..., X^{(m)}) = 0$ has an empty interior in $C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$.*

Proof. Let $\{X(t)\}_{t=0}^{\infty}$ solve the system of differential equations: $G(X, \dot{X}, ..., X^{(m)}) = 0$. Since G is locally Lipschitz continuous at $X(t), \dot{X}(t), ..., X^{(m)}(t)$, we know that such a time path exists and is locally unique. Since it is locally unique for all $t \geq 0$, it is also globally unique.

Since this time path $\{X(t)\}_{t=0}^{\infty}$ is also a particular solution of the considered growth model, we obtain:

$$\Phi(t) \equiv F(X(t), \dot{X}(t), ..., X^{(m)}(t)) = 0, \qquad \forall t \geq 0. \tag{3}$$

To show that the set of functions $\mathcal{F}$ satisfying (3) has an empty interior, consider a family of functions F_ε such that $F_\epsilon(X, \dot{X}, ..., X^{(m)}) = F(X, \dot{X}, ..., X^{(m)}) + \varepsilon \mathbf{e}_1$ for $\varepsilon > 0$. Of course, $||F_\varepsilon - F||_{C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)} = \varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand, for all $\varepsilon > 0$,

$$\Phi_\varepsilon(t) \equiv F_\varepsilon(X(t), \dot{X}(t), ..., X^{(m)}(t)) = \varepsilon \mathbf{e}_1 \neq 0, \qquad \forall t \geq 0. \tag{4}$$

Thus, $F_\varepsilon \notin \mathcal{F}$ for all $\varepsilon > 0$ so $\mathcal{F}$ has an empty interior. ■

When put in plain English, Theorem 1 states that if one requires the solution of her model to satisfy a predefined functional regularity, then one must impose some knife-edge restriction on her model, regardless of the type of regularity.[5] The parameter values and functional forms assumed in the model must be non-typical for the predefined growth regularity to hold.

Please note that the restriction that F and G are both functions of X's up to their m-th derivatives is not restrictive: if F would take as arguments p derivatives of X, and G would take r, one could simply define $m = \max\{p, r\}$ and the same proof would follow.

Corollary 1 (Exponential growth) *The set $\mathcal{F}$ of functions $F \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$ such that $\dot{\hat{X}} = 0$ (so that the growth rates of all state variables are constant) for some solution $\{X(t)\}_{t=0}^{\infty}$ to $F(X, \dot{X}, ..., X^{(m)}) = 0$ has an empty interior in $C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$.*

Please note that Corollary 1 replicates the result presented in Growiec (2007a). The above proof of this result is simpler because it does not require the differentiation of F.

2.2 Discrete time

A result analogous to Theorem 1 holds also for models set up in discrete time. Let us now consider a very general form of a discrete-time model of economic growth. Its dynamics are ruled by a system of autonomous difference equations of order m:

$$F(X_t, X_{t-1}, ..., X_{t-m}) = 0, \qquad X_{-m+1}, X_{-m+2}, ..., X_0 \text{ given.} \tag{5}$$

This time, we do not even have to impose any particular restriction on the class of functions F and G applicable here. The space of all mappings $F : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ is thus going to be

[5] Our argument is not completely general. Please note that the proof of Theorem 1 requires the regularity $G \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$ to be locally Lipschitz continuous for all arguments $(X, \dot{X}, ..., X^{(m)})$ satisfying the eqality $G(X, \dot{X}, ..., X^{(m)}) = 0$.

considered our "parameter space" and denoted by $\mathcal{P}$. We shall endow the space $\mathcal{P}$ with the usual supremum metric but without ruling out functions that are divergent with respect to this metric. We shall assume that all solutions to (5) are well defined for all $t = 0, 1, 2, ...$

Theorem 2 (Discrete time version) *The set $\mathcal{F}$ of functions $F : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ such that $G(X_t, X_{t-1}, ..., X_{t-m}) = 0$ for some solution $\{X_t\}_{t=0,1,2,...}$ to $F(X_t, X_{t-1}, ..., X_{t-m}) = 0$ has an empty interior in $\mathcal{P}$.*

Proof. Let $\{X_t\}_{t=0,1,2,...}$ solve the system of difference equations: $G(X_t, X_{t-1}, ..., X_{t-m}) = 0$. Since this time path $\{X_t\}_{t=0,1,2,...}$ is also a particular solution of the considered growth model, we obtain:

$$\Phi(t) \equiv F(X_t, X_{t-1}, ..., X_{t-m}) = 0, \qquad \forall t = 0, 1, 2, ... \tag{6}$$

To show that the set of functions $\mathcal{F}$ satisfying (6) has an empty interior, consider a family of functions $F_\varepsilon \in \mathcal{P}$ such that $F_\varepsilon(Y_0, Y_1, ..., Y_m) \equiv F(Y_0, Y_1, ..., Y_m) + \varepsilon \mathbf{e}_1$ for $\varepsilon > 0$. Of course, $||F_\varepsilon - F||_{C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)} = \varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand, for all $\varepsilon > 0$,

$$\Phi_\varepsilon(t) \equiv F_\varepsilon(X_t, X_{t-1}, ..., X_{t-m}) = \varepsilon \mathbf{e}_1 \neq 0, \qquad \forall t = 0, 1, 2, ... \tag{7}$$

Thus, $F_\varepsilon \notin \mathcal{F}$ for all $\varepsilon > 0$ so $\mathcal{F}$ has an empty interior. ■

2.3 Instability and bifurcations

One of the aspects of the debate on knife-edge conditions in growth economics is their relation to bifurcations and instabilities. As is apparent in a number of examples discussed in the literature (e.g. Li, 2000; Jones, 2001, 2003, 2005a), in the long run (that is, as $t \to \infty$), even smallest deviations in values of certain (appropriately chosen) parameters may give rise to qualitatively different modes of dynamic behavior, completely ruining the presupposed growth regularities.

In line with the previous findings of the current paper, it turns out that all models which are built in order to replicate a predefined long-run growth regularity, give rise to bifurcations with respect to certain parameters.[6]

Let us first discuss a complementary theorem, however: in the long run, even tiniest changes in parameter values might be infinitely magnified. This does not imply qualitative differences in the model behavior yet, but signifies that those differences are *quantitatively* divergent. Thus, it strongly indicates the fragility of maintaining any presupposed growth regularity over the long run.

Theorem 3 (Divergence) *Let $\{X(t)\}_{t=0}^{\infty}$ be a time path of a dynamic model economy summarized by (2). Assume that either there exists $i = 1, 2, ..., n$ such that $X_i(t) \to \infty$ or there exists $i = 1, 2, ..., n$ such that $X_i(t) \to \bar{X}_i$. Under these assumptions, there exists a more general class of functions $F_\phi(X, \dot{X}, ..., X^{(m)})$, $F_\phi \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$, such that $F_\phi = F$ for $\phi = 0$, but for all $\phi \neq 0$,*

$$\begin{aligned} & \sup_{t\geq 0} ||F_\phi(X(t), \dot{X}(t), ..., X^{(m)}(t)) - F(X(t), \dot{X}(t), ..., X^{(m)}(t))|| = \\ = & \sup_{t\geq 0} ||F_\phi(X(t), \dot{X}(t), ..., X^{(m)}(t))|| = +\infty. \end{aligned} \tag{8}$$

Proof. In case $X_i(t) \to \infty$ with $t \to \infty$ for some $i = 1, 2, ..., n$, it suffices to take

$$F_\phi(X, \dot{X}, ..., X^{(m)}) = F(X, \dot{X}, ..., X^{(m)}) + \phi X.$$

Clearly, $F_\phi = F$ for $\phi = 0$, but for all $\phi \neq 0$, $\sup_{t\geq 0} ||F_\phi(X(t), \dot{X}(t), ..., X^{(m)}(t))|| = \sup_{t\geq 0} \phi ||X|| = +\infty$.

[6] A special case of this result has been proven and illustrated in phase diagrams by Growiec (2007b).

If however there exists a finite-valued vector $\tilde{X} > 0$ such that $X_i(t) \leq \tilde{X}_i$ for all $t \geq 0$ and $i = 1, 2, ...n$, and $\exists(i = 1, 2, ..., n)X_i(t) \to \bar{X}_i$ then one can use

$$F_\phi(X, \dot{X}, ..., X^{(m)}) = F(X, \dot{X}, ..., X^{(m)}) + \frac{\phi}{|\bar{X}_p - X|}$$

where $p = \arg\min_{i=1,2,...,n} \bar{X}_i$ among those variables which converge to steady state values. Then $F_\phi = F$ for $\phi = 0$ but for all $\phi \neq 0$, $\sup_{t \geq 0} ||F_\phi(X(t), \dot{X}(t), ..., X^{(m)}(t))|| = \sup_{t\geq 0} \phi||\frac{1}{|\bar{X}_p - X|}|| = +\infty$. ■

It follows that in the long run, no matter how tiny $\phi \neq 0$ is, it is sufficiently large to generate infinite divergence of the manipulated model from the benchmark model with $\phi = 0$, as long as the benchmark model implies unbounded growth or convergence to a steady state.

Theorem 3 does not imply qualitative changes in the behavior of variables because infinite divergence predicted by this theorem could also be generated with quantitative differences only, e.g. by two cases of exponential growth, albeit with different growth rates.

The changes in model dynamics following infinitesimal manipulations in values of certain parameters are indeed qualitative, though. In fact, all knife-edge assumptions in growth models should be associated with certain *bifurcations*. We find that if the original model, specified as (2), is able to generate unbounded growth – that is, to have $||X(t)|| \to \infty$ as $t \to \infty$ which makes at least one economic variable grow unboundedly – then by infinitesimal manipulations, one can turn her model either into (i) a model which implies convergence to a bounded set, or (ii) a model which generates explosive growth rendering infinite levels of variables in finite time. This finding is stated formally as the following Theorem:[7]

[7]Please note that the theorem is stated in continuous time. It cannot be replicated directly in discrete time

Theorem 4 (Bifurcations) *Let $\{X(t)\}_{t=0}^{\infty}$ be a time path of a dynamic model economy summarized by (2). Assume further that there exists $i = 1, 2, ..., n$ such that $X_i(t) \to \infty$. Under these assumptions, there exists a more general class of functions $F_\phi(X, \dot{X}, ..., X^{(m)})$, $F_\phi \in C^1(\mathbb{R}^{(m+1)n}, \mathbb{R}^n)$ such that $F_\phi = F$ for $\phi = 0$, such that there exists a solution for the equality $F_\phi(X, \dot{X}, ..., X^{(m)}) = 0$ in the time domain $t \in [0, T_\phi)$ with $T_\phi > 0$ and possibly $T_\phi = +\infty$ – which we denote $\{X_\phi(t)\}_{t=0}^{T_\phi}$ – and finally, such that for all $\phi \neq 0$:*

$$\exists(0 < T_\phi < +\infty) \quad \exists(i = 1, 2, ..., n) \qquad \lim_{t \to T_\phi} X_{\phi,i}(t) = +\infty \quad \textit{for} \quad \phi > 0,$$
$$\exists(\bar{X}_\phi \in \mathbb{R}^n) \quad \forall(t > 0) \qquad 0 < X_\phi(t) < \bar{X}_\phi \quad \textit{for} \quad \phi < 0.$$

Proof. It is sufficient to consider the case $m = 1$ because for $m > 1$, one could use the theorem fundamental to ordinary differential equations (cf. Arnold, 1975), substitute $Y_i = X^{(i)}$ for all $i = 1, 2, ..., m-1$, arrange these variables in a common vector $Y_\Sigma \equiv [X, Y_1, ..., Y_{m-1}]'$ and write the resultant system of equations:

$$\begin{aligned}
\dot{X} &= Y_1, \\
\dot{Y}_1 &= Y_2, \\
&\vdots \\
F(X, Y_1, ..., Y_{m-1}, \dot{Y}_{m-1}) &= 0
\end{aligned}$$

as $F_\Sigma(Y_\Sigma, \dot{Y}_\Sigma) = 0$. Thus, sticking to the original notation, we can consider the simplest case of $F(X, \dot{X}) = 0$ with $X(0)$ given without any loss of generality.

Now, using the Implicit Function Theorem and the assumptions that (i) a solution $\{X(t)\}_{t=0}^{\infty}$ to $F(X, \dot{X}) = 0$ exists and (ii) F is continuously differentiable, we find that an explicit form $\dot{X} = \Phi(X)$ exists almost everywhere. Let us denote the (dense) set of points where such form exists as $\mathcal{A} \subset \mathbb{R}^n_+$.

because divergence to infinity in finite time is not well-defined in discrete time.

We will now posit a function $F_\phi(X, \dot{X})$ such that for all $X \in \mathcal{A}$, the equality $F_\phi(X, \dot{X}) = 0$ is equivalent to:

$$\dot{X} = \Phi(X) + \phi X^\psi, \qquad \psi > 1,$$

and such that $F_\phi = F$ for all $X \notin \mathcal{A}$. The solution to $F_\phi(X, \dot{X}) = 0$ will be denoted as $\{X_\phi(t)\}$.

Clearly, $F_\phi = F$ if $\phi = 0$.

If $\phi > 0$ then for all $i = 1, 2, ..., n$, it holds that $0 < \Phi_i(X) < \phi X_i^\psi$ provided that X_i is sufficiently large (otherwise the benchmark model would imply either explosive dynamics or bounded dynamics, neither of which is allowed). Let us pick p such that $p = \arg\max_{i=1,2,...,n} X_{\phi,i}$. From the model specification we are sure that this double inequality will hold for some coordinate of X_ϕ at some time $t_0 > 0$. Then from t_0 on, we have that

$$X_{\phi,p}(t) > \left((1-\psi)\phi t + X_{\phi,p}(0)^{1-\psi}\right)^{\frac{1}{1-\psi}}, \tag{9}$$

where the right-hand side of (9) is the solution to the differential equation $\dot{X}_{\phi,p} = \phi X_{\phi,p}^\psi$. Since $\psi > 1$, from the RHS we find that $X_{\phi,p}$ will reach infinity at or before $T_{\max,\phi} = \frac{X_{\phi,p}(0)^{1-\psi}}{\phi(\psi-1)}$. In conclusion, $\exists (0 < T_\phi < T_{\max,\phi}) \lim_{t \to T_\phi} X_p(t) = +\infty$ for all $\phi > 0$.

If $\phi < 0$ then for all $i = 1, 2, ..., n$, $\dot{X}_{\phi,i} < 0$ for $X_{\phi,i}$ sufficiently large (otherwise the original model would imply explosive dynamics which is not allowed). Since also $X_{\phi,i} > 0$ for all i by definition, it follows that for all i, $X_{\phi,i}$ must be confined to a bounded interval in $\mathbb{R}_+$. ■

Intuitively speaking, the idea behind Theorem 4 is to construct two "ϕ-variations" of the benchmark model which nevertheless give rise to qualitatively different modes of dynamic behavior. The benchmark model is the one with $\phi = 0$ which gives rise to the predefined growth regularity. The first type of variation has $\phi > 0$ and implies explosive growth yielding infinite X's in finite (arbitrarily short) time. The second type of variation has $\phi < 0$ and implies convergence to a bounded set – possibly (but not necessarily) a steady state.

Please note that Theorem 4 does not apply to models whose benchmark formulations already imply bounded dynamics such as convergence to a steady state.

Let us now present a few simple applications of Theorem 4. For a start, consider a case of regular (quasi-arithmetic) growth with $\dot{x} = \alpha x^{\gamma}$, $\gamma < 1$. It is obtained that $x(t) \to \infty$ as $t \to \infty$. If one adds constant-rate depreciation to this picture, though, so that $\dot{x} = \alpha x^{\gamma} + \phi x$, $\phi < 0$, she gets that $x(t)$ converges to a finite steady state. This result holds for all $\phi < 0$. On the other hand, if $\phi > 0$ we get a case where growth ceases to be quasi-arithmetic but becomes instead exponential in the limit; in result, dynamics à la Jones and Manuelli (1990) follow. Clearly, the depreciation rate of factor x, denoted as $(-\phi)$, or equivalently, the constant-returns-to-scale production rate ϕ, is a source of bifurcation here: the dynamic behavior of $x(t)$ is qualitatively different in the case $\phi = 0$ compared to the cases where $\phi > 0$ or $\phi < 0$.

A markedly more general example refers to any growth pattern summarized by $\dot{X} = Q(X)$ and implying that $X(t) \to \infty$ as $t \to \infty$. If we rule out explosions to infinity in finite time (that is, finite-time singularities, cf. Johansen and Sornette, 2001), adding a quadratic term as in $\dot{X} = Q(X) + \phi X^2$ will for sure guarantee that (i) there will be convergence to a bounded set instead of unbounded growth whenever $\phi < 0$, or that (ii) there will be a finite-time explosion whenever $\phi > 0$. This is again a bifurcation around $\phi = 0$.

Clearly, examples like these can be easily multiplied. Exponential growth generated by linear differential equations is thus not special at all in giving rise to spectacular explosions or growth decays if a smallest, but sufficiently smartly designed, nonlinearity is added. In fact, the same result follows for models capturing any other predefined (sufficiently smooth) growth regularity.

In the following section, we will provide one more illustration of this point by finding an interesting bifurcation in the case of regular growth.

3 Applications of the theorems

All special cases included below can be summarized in short corollaries akin to Corollary 1: the knife-edge character of each particular type of growth regularity follows directly from Theorem 1. We feel, however, that since the economic role of each of this examples is potentially large, they should be elaborated in more detail.

We shall first limit the scope of our analysis to a case of a single state variable. This restriction will be relaxed afterwards.

3.1 Regular growth

Regular (quasi-arithmetic) growth is defined (e.g. Asheim et al., 2007; Groth, Koch, and Steger, 2008) as a time path of the economy, such that a variable x satisfies the following differential equation:

$$\hat{\hat{x}} = -\beta\hat{x}, \quad \forall t \geq 0. \tag{10}$$

The parameter $\beta \geq 0$ is called the *damping* coefficient since it indicates the rate of damping in the growth process. The above specification nests as special cases: (i) exponential growth (in the limit case of no damping, $\beta = 0$), (ii) arithmetic growth ($\beta = 1$) as well as (iii) stagnation, $x \equiv \text{const}$ ($\beta = +\infty$).

Simple calculus shows that the solution to (10) is given by

$$x(t) = x(0)(1 + \hat{x}(0)\beta t)^{1/\beta}. \tag{11}$$

The concept of regular growth is certainly an important concept worth further investigation and development: apart from the notable field of environmental and resource economics (e.g. Mitra, 1983; Asheim et al., 2007) and the recent contribution of Groth, Koch, and Steger (2008), very little has been said yet about economies which exhibit less-than-exponential growth.

To see that, despite the claims present in some works, the requirement of regular growth imposes knife-edge restrictions on the presumed model, it is enough to apply theorem 1 to $G(x, \dot{x}, \ddot{x}) = \hat{\hat{x}} + \beta\hat{x} = \frac{\ddot{x}x - \dot{x}^2}{\dot{x}^2} + \beta\frac{\dot{x}}{x}$.

Alternatively, one could also use the function $\varphi_R : \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$\varphi_R(x) = c_1 \exp\left(c_2 x^\beta\right), \qquad c_1, c_2 > 0. \tag{12}$$

The function φ_R is continuously differentiable, strictly increasing, and such that $\varphi_R(x) \to \infty$ when $x \to \infty$.

The trick inherent in using φ_R is that when $y = \varphi_R(x)$, then x follows regular growth with a coefficient β if and only if y grows exponentially at a rate $g = c_2 x(0)^\beta \beta \hat{x}(0)$. φ_R is thus a smooth transformation of regular growth paths into exponential growth paths. The smoothness of φ_R implies that the knife-edge character of exponential growth in y is automatically inherited by regular growth in x. Any model which gives rise to regular growth with a coefficient β must involve at least one knife-edge condition.

It must also be noted that β does *not* have to be fixed *a priori* for our result to hold. In fact, the regular growth pattern has the knife-edge property regardless of whether we know β beforehand or this parameter is free. To see this, differentiate (10) sidewise and obtain

$$\hat{\hat{\hat{x}}} = \hat{\hat{x}}. \tag{13}$$

This is, of course, an equality restriction of form $G(x, \dot{x}, \ddot{x}, x^{(3)}) = 0$. The only difference between (10) and (13) is that (13) is formulated at the level of third instead of second derivatives.

Equation (13) indicates the way in which regular growth may be generalized. In the following subsection, we shall replace the factor of unity multiplying $\hat{\hat{x}}$ on the right hand side of (13), with an arbitrary parameter $\phi > 0$ and demonstrate that such a growth regularity has the same knife-edge property despite nesting (13) as its special case.

3.2 Generalized regular growth

The concept of regular growth can be easily generalized to allow one more degree of freedom and yet to give rise to equally smooth a growth pattern. The proposed generalization consists in allowing the parameter $\phi > 0$ in

$$\hat{\hat{x}} = -\beta \hat{x}^{\phi} \tag{14}$$

to deviate from unity. Obviously, the special case $\phi = 1$ brings us back to regular growth. Furthermore, if β is not known *a priori*, equation (14) can be expressed more generally, at the level of third derivatives, as

$$\hat{\hat{\hat{x}}} = \phi \hat{\hat{x}}, \tag{15}$$

thereby generalizing equation (13). Solving (14) for the explicit time path $x(t)$, we obtain:

$$x(t) = x(0) \exp\left(\frac{\left(\beta\phi t + \hat{x}(0)^{-\phi}\right)^{\frac{\phi-1}{\phi}}}{\beta(\phi-1)} - \frac{\hat{x}(0)^{1-\phi}}{\beta(\phi-1)} \right). \tag{16}$$

Generalized regular growth has been illustrated graphically in Figure 1.

Two qualitatively different cases of dynamic behavior of x are found here. If $\phi \geq 1$ then $x(t) \to \infty$ as $t \to \infty$. If $\phi < 1$, however, then $x(t)$ is uniformly bounded from above, converging from below to the finite value of $\bar{x}$:

$$\forall(\phi \in (0,1)) \quad \lim_{t\to\infty} x(t) = \bar{x} = x(0) \exp\left(\frac{\hat{x}(0)^{1-\phi}}{\beta(1-\phi)} \right). \tag{17}$$

It must be pointed out that if $\phi < 1$ then $x(t)$ is bounded regardless of the value of β. Hence, the condition $\phi = 1$ assumed in the regular growth case sets up a bifurcation in the sense that it delineates two cases of qualitatively different behavior of $x(t)$ (the cases of $\phi < 1$ and $\phi > 1$). This is precisely the bifurcation property of regular growth announced above.

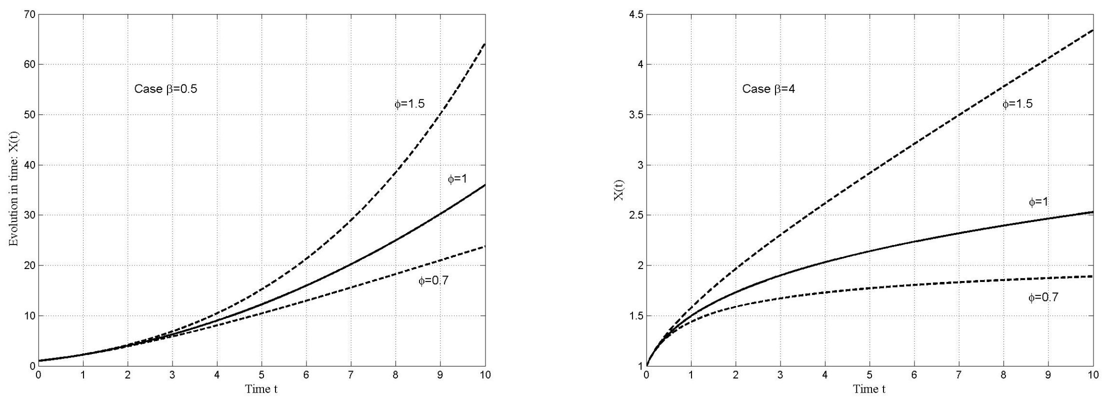

Figure 1: Generalized regular growth. Time paths of variables satisfying (14). We assumed $x(0) = \hat{x}(0) = 1$ in all cases. Left panel: case $\beta = 0.5$ (more-than-arithmetic growth). Right panel: case $\beta = 4$ (less-than-arithmetic growth). Please note that $x(t)$ is bounded from above if $\phi < 1$.

Equation (14) imposes a growth regularity of form $G(x, \dot{x}, \ddot{x}) = \hat{\hat{x}} + \beta \hat{x}^{\phi} = 0$. It thus places a knife-edge condition on the class of models capable of capturing this regularity (Theorem 1).

To see the correspondence between generalized regular growth and exponential growth, one could use the function $\varphi_G : \mathbb{R}_+ \to \mathbb{R}_+$ (case $\phi > 1$) or $\varphi_G : \left(0, x(0) \exp\left(\frac{\hat{x}(0)^{1-\phi}}{\beta(1-\phi)}\right)\right) \to \mathbb{R}_+$ (case $\phi < 1$), given by the uniform formula:

$$\varphi_G(x) = c_1 \exp\left(c_2 (\ln(x/C))^{\frac{\phi}{\phi-1}}\right), \qquad c_1, c_2 > 0, \tag{18}$$

where $C = x(0) \exp\left(-\frac{\hat{x}(0)^{1-\phi}}{(\phi-1)\beta}\right)$. The function φ_G is a continuously differentiable and strictly increasing bijection.[8] It is easily found that x grows according to generalized regular growth with parameters (β, ϕ) if and only if $y = \varphi_G(x)$ grows exponentially at a rate

$$g = c_2 \beta \phi \left(\frac{\beta(\phi - 1)}{C} \right)^{\frac{\phi}{1-\phi}}. \tag{19}$$

The smoothness of the transformation φ_G implies that the knife-edge character of exponential growth in y is inherited by generalized regular growth in x. The knife-edge property of exponential or regular growth is thus shared by generalized regular growth as well, even though the current specification is markedly more general.

3.3 Nested specifications

By construction, generalized regular growth nests regular growth which in turn nests exponential and arithmetic growth as special cases. How come that all these growth regularities require knife-edge conditions despite the obvious relation of inclusion?

The crucial reason for this outcome is that relaxing a particular knife-edge restriction is always a partial solution: it is not about eradicating restrictions but about pushing them "one level deeper". In the cases discussed above, this clearly applied to consecutive derivatives of the imposed growth regularities: for exponential growth, the second log-derivative[9] must be zero (Growiec, 2007a); for regular growth, the *third* log-derivative must be equal to the second log-derivative (Eq. (13)); for generalized regular growth, the *fourth* log-derivative must be equal to the third log-derivative, etc. It is easy to invent further generalizations in this manner, involving fifth, sixth, seventh derivatives, etc., so forth *ad infinitum*. It must be noted, however, that despite introducing an additional degree of freedom at each

[8] In the case $\phi < 1$, this finding follows from the fact that $x(t) < \bar{x}$ for all $t \geq 0$ (see Eq. (17)).

[9] Recall that dubbing $\hat{X}$ a "log-derivative" is only a convention used for simplicity. The exact definition of what we call the "log-derivative" here is $\hat{X} = \dot{X}/X$ which applies to both positive and negative X's. The exact log-derivative $\frac{d \ln X}{dt}$ is equal to $\hat{X}$ wherever it exists; it is however well-defined for positive X's only.

consecutive level of extra generality, some knife-edge condition must still be imposed on the mapping F in order for the model to deliver a solution which would replicate the imposed regularity.

One intuition for this result is the following. By generalizing the imposed growth regularity, we capture one more dimension of the parameter space. The whole parameter space is, however, infinite dimensional, so its entirety cannot be covered by any iterative procedure of this sort.

3.4 Logistic growth

Set aside exponential growth and stagnation, the logistic growth pattern would probably be the one most often mentioned in the literature. The concept comes from natural sciences where the simple logistic law is a very accurate tool for describing growth of natural populations as it incorporates both proportional multiplication when the population is small and the limiting impact of the finite environmental carrying capacity when the population is large (Smith, 1974). In economics, logistic laws have been used relatively rarely; the few notable exceptions include Brida, Mingari Scarpello and Ritelli (2006) as well as Brida and Accinelli (2007) who incorporate logistic population laws in the Solow and the Ramsey growth models, respectively.

Furthermore, in the important class of growth models dealing with the Demographic Transition and the transition from the Malthusian stagnation regime to the modern balanced growth regime, population dynamics could be arguably well approximated by logistic-type curves provided that we assume population to stabilize asymptotically (see e.g. Jones, 2001).

The logistic law is characterized by

$$\dot{x} = Ax(B - x), \qquad A, B > 0, \qquad \text{with } x(0) \in (0, B). \tag{20}$$

It is easily solved as:

$$x(t) = \frac{B}{1 + Ce^{-At}}, \quad \text{with } C = \frac{B}{x(0)} - 1. \tag{21}$$

As it was indicated above for the case of generalized regular growth with $\phi < 1$, also here is the variable $x(t)$ bounded from above: $\lim_{t\to\infty} x(t) = B$. The parameter B is thus straightforwardly interpreted as the environmental carrying capacity (or the level of satiation).

The knife-edge character of logistic growth follows by application of Theorem 1 to (20). There exists however also an intriguing mutual correspondence between logistic and exponential growth paths. Following the lines of examples presented above, let us now define a function $\varphi_L : (0, B) \to \mathbb{R}_+$ as:

$$\varphi_L(x) = c_1 \left(\frac{x}{B - x} \right)^{g/A}. \tag{22}$$

φ_L is continuously differentiable, strictly increasing, and such that $\varphi(x) \to \infty$ when $x \to B_-$. It is obtained that x follows logistic growth with coefficients A and B if and only if $y = \varphi_L(x)$ grows exponentially at a rate g. The smoothness of φ_L implies that the knife-edge character of exponential growth in y is directly inherited by logistic growth in x. Hence, perhaps a little surprisingly, logistic growth is also subject to the critique of knife-edge conditions.

3.5 Double exponential growth

It is sometimes counterfactually presumed by economists that if the growth rate of some variable falls down to zero with time, the variable itself must converge to a finite constant.

The concept of regular growth is a perfect counterexample to such an assertion. Analogously, there also exists a fallacious belief that, under continuous time, if the growth rate of a variable explodes to infinity, the variable itself will reach infinity *in finite time* (there will be a finite-time singularity). This belief comes as an extrapolation of the often discussed functional specification (1) with $\phi > 1$, being the standard quantification of increasing returns to scale. This result is usually referred to as puzzling, cognitively unattractive, and having empirically implausible implications (see Solow, 1994). Historical time series of several demographic and economic variables observed over last two centuries can be fitted by functions leading to a finite-time singularity with astonishingly good accuracy, though (Johansen and Sornette, 2001).[10]

Growth can nevertheless be faster than exponential and yet not lead to finite-time singularities. One example of such a growth regularity, predicting the growth rate to diverge to infinity, is the pattern of *double exponential* growth, summarized by the differential equation:

$$\dot{x} = gx \ln x, \qquad g > 0, \quad x(0) > 1. \tag{23}$$

Straightforward integration yields:

$$x(t) = x(0)^{e^{gt}} \tag{24}$$

which is, of course, well defined for all $t \geq 0$, and thus no finite-time singularity occurs.

By Theorem 1, the growth regularity imposed by (23) gives rise to knife-edge requirements. This could also be illustrated with the use of the logarithmic function $\varphi_M : (1, +\infty) \to \mathbb{R}_+$: $\varphi_M(x) = \ln x$. Obviously, φ_M is continuously differentiable, strictly increasing, and such that $\varphi(x) \to \infty$ when $x \to \infty$. Hence, it is obtained that x fol-

[10]Curiously, Johansen and Sornette's (2001) estimations uniformly indicate that if no transition to a new dynamic regime occurs, the singularity will take place at 2052 ± 10 years.

lows double exponential growth (with $\hat{x}(t) = g \ln x(0) e^{gt} \to \infty$ as $t \to \infty$) if and only if $y = \varphi_M(x)$ grows exponentially at a rate g. The smoothness of φ_M implies that the knife-edge character of exponential growth in y is directly inherited by double exponential growth in x.

As a side remark, we note that by replacing $\ln x$ in φ_M by $\ln(\ln x)$, $\ln(\ln(\ln x))$, etc., we can easily generate triple, quadruple, etc. exponential growth paths generating ever faster growth without implying finite-time singularities, and thus being an attractive compromise between the functional forms estimated by Johansen and Sornette (2001) and the common intuition on economic plausibility.

3.6 Multiple variables

The above examples have been, for the sake of clarity, presented in the simplest case of a single variable $x(t)$. There is, however, no difficulty at all to extend these results to n variables by putting all x's in an n-dimensional vector $X(t)$ and applying all required transformations φ_z, where $z \in \{R, G, L, M\}$, to the particular coefficients of the vector, $X_i(t)$. As long as we impose particular growth patterns on each variable separately and thus rule out inter-equation restrictions, the properties of $Y = \varphi(X)$ are inherited directly from the properties of each separate coefficient $Y_i = \varphi_{z_i}(X_i)$. It is also straightforward to allow different variables X_i to follow different growth regularities, as long as all these regularities are well defined *a priori*.

For multi-dimensional regularities with inter-equation restrictions, the method of specifying smooth transformations φ_i, $i = 1, 2, ..., n$ which we used above does not work but the knife-edge character of each growth regularity still follows by the virtue of Theorems 1 and 2.

4 Discussion

In the history of modeling growth regularities, the first notice that balanced growth requires models to rely on restrictive assumptions is probably due to Uzawa (1961).[11] His steady-state growth theorem[12] indicates that for a simple neoclassical model to deliver balanced growth, the production function must be Cobb-Douglas or technical change must be purely labor-augmenting. The obvious knife-edge character of both requirements was recently supplemented by theoretical arguments why technical change could be endogenously purely labor-augmenting in equilibrium (Acemoglu, 2003; Jones, 2005b). These works do not solve the Uzawa's fundamental problem of highly restrictive knife-edge conditions, though (cf. Jones, 2005a; Growiec, 2008).[13]

Another milestone in the development of this line of discussion is the linearity critique of endogenous growth models (Jones, 2005a). The crux of this argument is that if the vital growth-driving linearity (a knife-edge assumption) is relaxed, exponential growth ceases to be obtained unless exponential population growth is additionally assumed. Exponential population growth is, however, just another knife-edge assumption. Otherwise, growth rates gradually fall to zero with time.

[11] "Growth on the knife edge" is also a well-known property of the Harrod–Domar growth model (Harrod, 1939; Domar, 1946) which laid the first foundations for modern economic growth theory. Knife-edge conditions (taken in the form of constant marginal returns to physical capital) were not in the focus of those two important early contributions, though.

[12] The Uzawa's steady-state growth theorem has been recently proved again by Schlicht (2006) who completed the proof by markedly simpler means than Uzawa (1961) did in his original contribution. A discussion of the theorem and both proofs has been provided by Jones and Scrimgeour (2008).

[13] The objective of Acemoglu (2003) and Jones (2005b) was, of course, not to get rid of knife-edge assumptions but to provide sound economic explanations why purely labor-augmenting technical change could indeed be an equilibrium outcome.

The linearity critique has been extended to allow for cross-equation parameter restrictions in multi-sector growth models by Li (2000) and Christiaans (2004). Recently, a general argument that balanced growth requires knife-edge conditions to be imposed on growth models has been formulated and proved by Growiec (2007a).

One type of conclusion following from this literature is that in order to get rid of knife-edge conditions, one should generalize the very restrictive concept of balanced (exponential) growth. We have however shown in this paper that this idea is, in fact, misguided: whatever number of generalizations of balanced growth (e.g. regular growth, generalized regular growth; logistic growth, generalized logistic growth, etc.) is allowed, there will always remain some knife-edge assumption necessary to obtain the particular growth regularity. Even more worryingly, there will always remain some exogenous parameter which could not be altered, even by tiniest amounts, under the threat of blowing the model up, both qualitatively and quantitatively.

In the end of the day, it turns out that the problem of knife-edge conditions in growth models is, in principle, *methodological*. This paper has shown that whatever type of long-run growth regularity is to be reproduced by the model (it may be arbitrarily general, allowing an arbitrary number of free parameters), one has to impose some specific knife-edge restrictions on the assumed parameter values and/or functional forms in the model. Thus, if the model is constructed by "reverse engineering", i.e. designed to fit empirically observed macro-scale regularities, knife-edge conditions – which are by Theorems 3 and 4 so restrictive that even slightest deviations from them would overturn both qualitative and quantitative features of the model – are inevitable. In other words: if we start out with some empirical growth regularity which we would like to be reproduced as an equilibrium outcome of some model, that model would have to be non-typical, i.e. so specific that a slightest deviation from the required functional form, if sufficiently smartly designed, would completely ruin its predictions.

We can think of three possible, mutually exclusive, interpretations for this result. Since the first two are somewhat self-critical, and the last one is probably overly revolutionary, we suppose that for pragmatical reasons, neither of them would prevail over the long run. They might, however, be used as interesting starting points for further discussion. These interpretations are as follows:

1. *The long run with $t \to \infty$ is irrelevant to growth economics; only finite time spans should be analyzed instead.* It seems that this approach is favored by Temple (2003) who proposes not to over-emphasize long-run properties of growth models: "restrictive assumptions are useful precisely because they allow us to abstract from matters not directly relevant to the problem at hand, and to carry out experiments holding certain variables constant. (...) [U]sing models for this purpose casts a rather different light on the role of knife-edge assumptions."(p. 500) For Temple (2003), exponential growth (or any other presupposed growth pattern) is an assumption of convenience rather than a potentially significant result. One fact favoring this interpretation is that for t bounded, Theorem 3 does not hold and deviations from the required growth regularity may be kept within "reasonable" bounds when model parameters are manipulated. These bounds are strongly and non-linearly dependent on the time span in question, though, becoming the less reasonable the longer is the considered time perspective. Most worryingly, by increasing the exogenous parameter $\psi > 1$ in the proof of Theorem 4, we can construct "ϕ-deviations" from the benchmark model able to blow the model up to infinity not only in finite time, but also in an *arbitrarily short* interval of time.

2. *The concept of knife-edge conditions is useless as means of criticizing economic*

models. Knife-edge conditions are inevitable in modeling empirically observed phenomena and so are qualitative changes in dynamic behavior of the model if some parameters are manipulated; this should not be questioned. Hence, the associated "instability" result should be ignored with the hope that the type of distortions mentioned in Theorems 3 and 4 will never occur in reality. Some other criterion such as the *relation of inclusion* could be used instead for discriminating among economic models: inclusion makes it clear which functional form is more restrictive than the other. The downside of using inclusion as a means of discriminating between models is that a vast multiplicity of modeling assumptions are not nested and thus cannot be compared. This could possibly open up the possibility to use Bayesian testing procedures to discriminate between non-nested models using real-world data.

3. *All dynamic models designed to reproduce empirically observed macro-scale regularities are methodologically flawed,* because infinitesimal deviations in parameter settings will always be able to change their predictions strongly enough to invalidate them. This interpretation suggests that the only way to avoid this methodological problem would be to gather micro-level rather than macro-level data, plug these findings directly into the model's low-level mechanisms, and deal with cumbersome aggregation procedures in order to obtain meaningful and *robust* predictions at the macro scale.[14]

The current article does not provide any formal means for discriminating between the three above interpretations of the main results contained herein. While $t \to \infty$ might not be a reasonable time perspective, there remains significant uncertainty if the qualitative and

[14]This interpretation provides an argument in favor of the agent-based modeling (ABM) methodology which has however rarely been used in macroeconomics yet (see the remarkable exception due to Axtell, 1999, though).

quantitative divergence results presented in Theorems 3 and 4 will manifest themselves in 5 or in 555 years. In the first case, one could probably conclude that her model is methodologically flawed while in the other case it is probably not. Similarly, while the concept of knife-edge conditions might be too general to discriminate between candidate explanations of a certain economic phenomenon, at the same time it might be useful as means of assessment where the fundamental "growth engine" of a model is located and what type of distortions could be most threatening for the sustainment of the current growth regime.

Finally, one should ask oneself one important question: Could it be that we are living in a world where none of the distortions to the growth mechanism mentioned in Theorems 1–4 can ever appear? In such case, the methodological issues discussed above would be void. But are we able to construct an empirical test able to assess whether such distortions have indeed ever appeared, given the long-standing problem of model uncertainty? For now, this question remains open.

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