Unifying Contests: from Noisy Ranking to Ratio-Form Contest Success Functions

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(very preliminary)

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Abstract

This paper proposes a multi-winner noisy-ranking contest model. Contestants are ranked in a descending order by their perceived outputs, and rewarded by their ranks. A contestant's perceivable output increases with his/her autonomous effort, but is subject to random perturbation. We establish, under plausible conditions, the equivalence between our model and the family of (winner-take-all and multi-winner) lottery contests built upon ratio-form contest success functions. Our model thus provides a micro foundation for this family of often studied contests. In addition, our approach reveals a common thread that connects a broad class of seeming disparate competitive activities and unifies them in the nutshell of ratio-form success functions.

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1 Introduction

A wide class of competitive activities can be viewed as contests, in which all participants forfeit scarce resources, regardless of winning or losing.[1] Due to the ubiquity of such phenomenon, an enormous amount of scientific literature has been developed to uncover the strategic nature of contests.

It has long been recognized in the literature that contestants' incentives and behaviors could sensitively respond to the rules of the competitive events. Central to the rules of a contest is the mechanism that picks the winners and distributes the prizes. The selection mechanism can be technically described in formal modelling as contest success functions that map contestants' effort entries into the likelihood of every contestant winning each prize. A lion's share of the existing literature concerns itself with winner-take-all contests where all tangible reward goes to a single winner. To model this type of contests, a handful of theoretical frameworks have been independently proposed and studied. One of (perhaps the most) widely adopted approaches is the lottery contest model that assumes a ratio-form contest success function,[2] with the Tullock contest model as its most popular special case.[3] This framework provides an intuitive and tractable abstraction of a complex selection process in the presence of randomness, and has been axiomized by Skaperdas (1996).

The prevalence of multiple-winner contests in reality naturally generates the demand for theories on such phenomenon. A growing literature has emerged to fill in this gap and has supplied important results on strategic behaviors in multiple-winner contests. Clark and Riis (1996 and 1998a) modify the basic Tullock contest framework, and propose a multiple-winner nested contest model to allow a block of prizes to be distributed. Assuming ratio-form success functions as its building block, this model conducts a series of conditionally independent (single-winner) "lotteries", and lets each of them "draws" one recipient for a prize until all prizes are given away.[4] The

[1] Illustrative examples include college admissions, influence politics, sports, war and conflicts, internal labor market competition, etc.

[2] The winning likelihood P_i of a contestant i is given by the ratio of the output of his/her effort to the total outputs contributed by the entire pool of competitors, i.e., $P_i = g_i(x_i)/\sum_{j=1}^{n} g_j(x_j)$, where the output production function $g_i(x_i)$ is an increasing function of effort x_i.

[3] The Tullock model assumes a contestant's output is a power function of his/her effort outlay, i.e., $P_i = x_i^r/\sum_{j=1}^{n} x_j^r$, with $r > 0$.

[4] As a result, the conditional probability of a remaining contestant to be selected in the next "draw" is independent

nested contest model offers so far the most reasonable and the most prominent alternative to determine multiple winners based on ratio-form contest success functions.[5,6] However, the sequential implementation of independent lotteries runs in contrast to the presumption of the model that contestants simultaneously commit to their one-shot effort entries. The legitimacy and plausibility of this framework have yet to be justified when it is applied to contests that distribute multiple prizes as one batch.[7] More importantly, a micro foundation still lacks to support the sequential lottery contest. The selection mechanism underlying each single lottery, as well as the entire prize distribution process, remains in a "black box".

As Konrad (2007) points out in his thorough survey of economic studies on contests, a contest can be naturally regarded as a competitive event where contestants expend costly effort in order to "get ahead of their rivals" (quoted from Konrad, 2007). By this natural definition, a contest, regardless of the number of winners, requires the contest organizer to (at least partially) "order" these contestants based on a rational preference relation. A ranking rule is therefore indispensable to link the participants to the prizes, while it remains obscure for the family of lottery contests.

In this paper, we propose a multi-winner contest model that selects prize recipients through a noisy ranking of contestants. We start from this framework to explore the logic that underlies the family of lottery contest models (the winner-take-all lottery contest model and its multiple-winner variant). In particular, this model involves a fixed number of economic agents (contestants) who produce their outputs out of their inputs (effort entries) and contribute the outputs to a rational decision maker (the contest organizer). The decision maker, who strictly prefers higher outputs, ranks these contestants by their perceived outputs in a descending order. More specifically, following the idea of McFadden (1973 and 1974), we model one's perceivable output as the sum of a deterministic component (a strictly increasing function of his/her effort) and a noise term that follows a particular distribution. As a result, given a set of effort entries of contestants, and any

of the effort entries of contestants selected in previous "draws".

[5] Besides Clark and Riis (1996, 1998a), the application of lottery contest models in multiple-winner settings can be seen in the studies by Amegashie (2000), Yates and Heckelman (2001), Szymanski and Vallettiand (2005), and Fu and Lu (2007).

[6] Another approach to model multiple-winner contests is multiple-prize all-pay auction model. A handful of studies have contributed to this research agenda, which include Barut and Kovenock (1998), Moldovanu and Sela (2001), and Moldovanu, Sela, and Shi (2006).

[7] One natural analogy would be the prize distribution rule in Gymnastics competition. Athletes are ranked purely by their scores, while the score one receives depend on his/her effort as well as many other factors. Medals are awarded to the three top ranked athletes.

realization of noise terms, a complete ranking immediately arises. Each agent is therefore accorded a prize of his rank. That is, the l-th ranked contestant (the agent who is associated the l-th highest perceived output) wins the l-th prize. This type of rank-order prize allocation rule could be seen in the studies by Glazer and Hassin (1988), Barut and Kovenock (1998), Moldovanu and Sela (2001), etc.

We show that this model generates an outcome that is stochastically identical to that of a lottery contest: For any given effort entries and production functions, (1) one's probability of being top ranked coincides with his/her winning odd in a winner-take-all lottery contest with the same set of production functions; (2) when more than one prize is available, the ex ante likelihood of every possible prize allocation plan perfectly corresponds to that in a sequential lottery process, as described by the multiple-winner nested contest.[8]

These results establish the strategic equivalence between the two seemingly disparate frameworks. This isomorphism therefore makes it possible to uncover the microeconomic underpinning of the family of lottery contest models. It implies that a rational "preference relation" (ranking system) can be "recovered" under the disguise of a ratio-form contest success function. In other words, a complete, transitive and strictly monotonic ranking system indeed exists that generates and rationalizes the popular ratio-form stochastic selection rule.

Our paper represents a continuing effort in the literature that attempts to bridge differing modelling approaches and to illuminate the "black-box" of lottery contests. The pioneering study of Baye and Hoppe (2003) reveals the strategic equivalence among research tournament models (Fullerton and McAfee, 1999), patent race model (Dasgupta and Stiglitz, 1980), and winner-take-all Tullock contests. We advance this line of research by investigating multiple-winner contests, which demands a complete and transitive ordering of participants' outputs in the presence of randomness. These discoveries, however, naturally lead to more fundamentally important questions: (1) Why could different classes of contests be unified under the same umbrella (ratio-form contest success functions)?; (2) To what extent could such isomorphism continue to hold? In other words, what kind of competitive activities can be abstracted as lottery contests? For this purpose, we introspectively scrutinize our noisy-ranking model. A rationale for these issues unfolds as the economic interpretation of our technical approach (McFadden, 1973, 1974) develops. The micro foundation of lottery contests uncovered in our analysis enables us not only to connect a wide variety

[8] In other words, the ex ante likelihood that a contestant is ordered on a l-th rank equates to the probability a contestant is selected for the l-th draw in a multiple-winner nested contest.

of observationally detached competitive activities, but also to explore in depth the unobserved common thread that runs through these contests and imposes a conceptual limit on the scope of this unity.

The rest of the paper is organized as follows. In Section 2 we set up the model, complete the analysis, and briefly discuss the implication of this model. In Section 3, we reinforce our argument by presenting the "dual" problem to our original model. A concluding remark is provided in Section 4.

2 A Multi-Winner Noisy-Ranking Contest Model

2.1 Setup

We propose a multi-winner noisy-ranking contest model. $I \geq 2$ contestants, indexed by $i \in \mathbf{I} \triangleq \{1, 2, ..., I\}$, simultaneously submit their effort entries $\mathbf{x} = (x_1, \ldots x_I)$, to compete for $L \in \{1, 2, ..., I\}$ prizes. Their effort outlays are not directly observable to the contest organizer. Nevertheless, the contest organizer perceives a noisy signal (y_i) about contestant i's output and evaluates their performance through this signal. Following McFadden (1973, 1974), we assume that the noisy signal (y_i) is described through

$$\log y_i = \log g_i(x_i) + \varepsilon_i, \quad \forall i \in \mathbf{I}, \tag{1}$$

where the deterministic strictly increasing function $g_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ measures the impact of contestant i's effort x_i,[9] and the additive noise term ε_i reflects the randomness in the production process or the imperfection of the observation and evaluation process. We name $g_i(\cdot)$ the production function of contestant i. We define $\mathbf{g} \triangleq \{g_i(\cdot), i \in \mathbf{I}\}$, which denotes the set of technologies. The idiosyncratic noises $\boldsymbol{\varepsilon} \triangleq \{\varepsilon_i(\cdot), i \in \mathbf{I}\}$ are independently and identically distributed. It is worth noting that the additive-noise ranking model (1) is equivalent to a multiplicative-noise ranking model

$$y_i = g_i(x_i)\tilde{\varepsilon}_i, \ \forall i \in \mathbf{I}, \tag{2}$$

where the noise term $\tilde{\varepsilon}_i$ is defined as $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$.

The L prizes are ordered by their values, with $V_1 \geq V_2 \geq \ldots \geq V_L$. We assume that each contestant is eligible for at most one prize. As contestants' outputs accrue to the benefits of the

[9] We define $\log g_i(x_i) = -\infty$ if $g_i(x_i) = 0$.

contest organizer, the contest organizer thus rank these contestants by their performance evaluations (i.e., perceivable output $\log y_i$) in a descending order. Prizes are allocated among contestants by their ranks, given the availability of the prize. That is, the contestant who contributes the highest perceivable output y_i receives V_1, the one who contributes the second highest perceivable output then receives V_2, and so on until all prizes are given away.

When $L = 1$, the model degenerates to a winner-take-all contest, with the top-ranked contestant to be the only winner. When $L \geq 2$, a multi-winner contest would follow, which requires a more complete ranking among contestants to implement its prize allocation rule. For any given effort entries $\mathbf{x}$, a complete ranking among contestants immediately result from any realization of the noise terms $\boldsymbol{\varepsilon}$. We assume a fair and random tie breaking rule. The probability of a contestant i winning a prize V_l is simply given by the probability that he/she is ranked at the l-th position. This setup therefore embraces the notion that a contest is a competitive event where contestants compete to "get ahead of others" (Konrad, 2007).[10]

In this paper, we impose virtually no restriction on the technology $g_i(\cdot)$, and the number of prizes L. However, we follow McFadden (1973, 1974), and let the random component ε_i be drawn from a type I extreme-value (maximum) distribution. Denote the cumulative distribution function of ε_i by $F(\cdot)$, then we have

$$F(\varepsilon_i) = e^{-e^{-\varepsilon_i}}, \varepsilon_i \in (-\infty, +\infty), \quad \forall i \in \mathbf{I}, \tag{3}$$

and the density function is

$$f(\varepsilon_i) = e^{-\varepsilon_i - e^{-\varepsilon_i}}, \ \forall i \in \mathbf{I}. \tag{4}$$

The performance evaluation mechanism underlying this formulation will be discussed in Section 2.4, which reveals the economic implication of this seemingly peculiar distribution. Note that when ε_i follows a type I extreme-value (maximum) distribution, then $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$ must follow a Weibull (maximum) distribution.

[10] This family of contest models include Lazear and Rosen (1981), Glazer and Hassin (1988), Fullerton and McAfee (1999), and etc. All these models link the top ranked contestant to a unique prize, while they differ in the output technology, the formulation of the random component, and therefore the probability of a contestant winning a prize given the effort entries.

2.2 The Equivalence to Lottery Contests

In this part, we will show that this noisy-ranking model is stochastically equivalent to the family of lottery contests (winner-take-all lottery contests and multiple-winner nested contests).

In our setting, given the effort entries $\mathbf{x}$, a contestant i is ranked ahead of another j, if and only if

$$\begin{aligned} \log g_i(x_i) + \varepsilon_i &\geq \log g_j(x_j) + \varepsilon_j \\ &\Leftrightarrow \varepsilon_j \leq \varepsilon_i + \log \frac{g_i(x_i)}{g_j(x_j)}. \end{aligned}$$

A contestant i would be top ranked if and only if

$$\varepsilon_j \leq \varepsilon_i + \log \frac{g_i(x_i)}{g_j(x_j)}, \forall j \in \mathbf{I} \backslash \{i\}.$$

In the setup of McFadden (1973, 1974), the decision maker cares about the top-ranked choice. We therefore adapt the result established by McFadden (1973, 1974) to our contest setting.

Lemma 1 *For any given $\mathbf{x} \geq 0$ such that $\sum_{j \in \mathbf{I}} g_j(x_j) > 0$, the ex ante likelihood that a contestant i achieves the top rank is*

$$p(i|\mathbf{x}) = \frac{g_i(x_i)}{\sum_{j \in \mathbf{I}} g_j(x_j)}, \quad \forall i \in \mathbf{I}. \tag{5}$$

The proof is omitted as it is available from McFadden (1973, 1974). By Lemma 1, the probability of a contestant being top ranked can be written as a ratio between his/her output $g_i(x_i)$, and the sum of outputs contributed by all contestants. This winning probability exactly coincides with the popularly assumed ratio-form contest success function of winner-take-all lottery contests, provided that each contestant i produces his/her output through a technology $g_i(x_i)$. Denote such a lottery contest with contestants $\mathbf{I}$ and technology $\mathbf{g}$ by $C(\mathbf{I}, \mathbf{g}, V)$, where V represents the unique prize available to contestants. Lemma 1 immediately leads to the following result.

Theorem 1 *When $L = 1$ and ε_i follows the type I extreme-value (maximum) distribution, the noisy-ranking model (1) is strategically equivalent to a winner-take-all lottery contest $C(\mathbf{I}, \mathbf{g}, V)$.*

When $L > 1$, the model evolves into a multi-winner contest. In this case, we have to completely characterize the probability of each contestant winning each prize. To this end, we need to explore the probabilities of all possible complete (when $L \geq I - 1$) or partial (when $L \leq I - 2$) rankings. To cover all these possibilities, we consider the complete ranking of all contestants.

Suppose K $(1 \leq K \leq I-2)$ contestants are ranked from top 1 to top K by the amount of y_i. Let i_k indicate the index of the k-th ranked contestant. Define $\mathbf{I}_K = \{i_k, k = 1, ..., K\}$, which is the index set of the top K contestants. We thus have $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_K} \geq y_j$, $\forall j \in \Omega_{K+1} \triangleq \mathbf{I} \backslash \mathbf{I}_K$. We next calculate the conditional probability of a contestant $n \in \Omega_{K+1}$ being the $(K+1)$-th ranked. We denote this probability by $p(n|\mathbf{N}_K, \mathbf{x}, Y_K)$, where $\mathbf{N}_K = (i_1, ..., i_K)$ denotes the sequence of the top K-ranked contestants, $Y_K = (y_{i_1}, ..., y_{i_K})$ denotes the sequence of the perceived outputs of the top K-ranked contestants.

Since ε_i are i.i.d., the conditional cumulative distribution function of ε_j, $\forall j \in \Omega_{K+1}$ is described by

$$\begin{aligned} F(\varepsilon_j|\mathbf{N}_K, \mathbf{x}, Y_K) &= F(\varepsilon_j|y_j < y_{n_K}) \\ &= e^{-e^{-\varepsilon_j}}/e^{-e^{-\bar{\varepsilon}_j}}, \varepsilon_j \in (-\infty, \bar{\varepsilon}_j),\ \forall j \in \Omega_{K+1}, \end{aligned} \tag{6}$$

where $\bar{\varepsilon}_j \equiv \log y_{i_K} - \log g_j(x_j)$, $\forall j \in \Omega_{K+1}$. It therefore yields the density function

$$f(\varepsilon_j|\mathbf{N}_K, \mathbf{x}, Y_K) = e^{-\varepsilon_j - e^{-\varepsilon_j}}/e^{-e^{-\bar{\varepsilon}_j}}, \varepsilon_j \in (-\infty, \bar{\varepsilon}_j),\ \forall j \in \Omega_{K+1}. \tag{7}$$

As implied by (6) and (7), the conditional distribution of $\varepsilon_j, \forall j \in \Omega_{K+1}$, only depends on the minimum of $\{y_{i_k}, k = 1, ..., K\}$, i.e., y_{i_K}, because y_i are ranked in a descending order. We have the following result.

Lemma 2 *For any given effort entries $\mathbf{x} \geq 0$ such that $\sum_{j \in \mathbf{N}} g_j(x_j) > 0$, the probability that a contestant $n \in \Omega_{K+1}$ is the $(K+1)$-th ranked, conditioning on that contestants $i_1, i_2, ..., i_K$ are respectively ranked from top 1 to top K, can be written as*

$$p(n|\mathbf{N}_K, \mathbf{x}) = \frac{g_n(x_n)}{\sum_{j \in \Omega_{K+1}} g_j(x_j)}, \quad \forall n \in \Omega_{K+1}. \tag{8}$$

Proof. We first calculate $p(n|\mathbf{N}_K, \mathbf{x}, Y_K)$, which denotes the probability that a contestant $n \in \Omega_{K+1}$ is the $(K+1)$-th ranked conditioning on that contestants $\mathbf{N}_K = (i_1, i_2, ..., i_K)$ are respectively ranked from top 1 to top K and their perceived outputs are Y_K. Note that $\varepsilon_n + \log g_n(x_n) - \log g_j(x_j) \leq$

$\bar{\varepsilon}_j, \forall \varepsilon_n \in (-\infty, \bar{\varepsilon}_n)$, $\forall j, n \in \Omega_{K+1}, j \neq n$. We thus have

$$\begin{aligned}
& p(n|\mathbf{N}_K, Y_K, \mathbf{x}) \\
= \; & \Pr(\varepsilon_j \leq \varepsilon_n + \log g_n(x_n) - \log g_j(x_j),\ \forall j \in \Omega_{K+1}, j \neq n.) \\
= \; & \int_{-\infty}^{\bar{\varepsilon}_n} [\Pi_{j \in \Omega_{K+1}, j \neq n} F(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j)|\mathbf{N}_K, \mathbf{x}, Y_K)] f(\varepsilon_n|\mathbf{N}_K, \mathbf{x}, Y_K) d\varepsilon_n \\
= \; & \int_{-\infty}^{\bar{\varepsilon}_n} [\Pi_{j \in \Omega_{K+1}, j \neq n} e^{-e^{-(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j))}} / e^{-e^{-\bar{\varepsilon}_j}}] e^{-\varepsilon_n - e^{-\varepsilon_n}} / e^{-e^{-\bar{\varepsilon}_n}} d\varepsilon_n \\
= \; & (\Pi_{j \in \Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \int_{-\infty}^{\bar{\varepsilon}_n} [\Pi_{j \in \Omega_{K+1}, j \neq n} e^{-e^{-(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j))}}] e^{-\varepsilon_n - e^{-\varepsilon_n}} d\varepsilon_n \\
= \; & (\Pi_{j \in \Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \int_{-\infty}^{\bar{\varepsilon}_n} \exp[-\varepsilon_n - e^{-\varepsilon_n} \cdot (1 + \sum_{j \in \Omega_{K+1}, j \neq n} \frac{g_j(x_j)}{g_n(x_n)})] d\varepsilon_n. \qquad (9)
\end{aligned}$$

Let $\lambda_{n,\Omega_{K+1}} = \log(1 + \sum_{j \in \Omega_{K+1}, j \neq n} \frac{g_j(x_j)}{g_n(x_n)}) = \log(\sum_{j \in \Omega_{K+1}} \frac{g_j(x_j)}{g_n(x_n)})$, then

$$\begin{aligned}
& p(n|\mathbf{N}_K, Y_K, \mathbf{x}) \\
= \; & (\Pi_{j \in \Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \int_{-\infty}^{\bar{\varepsilon}_n} \exp[-\varepsilon_n - e^{-(\varepsilon_n - \lambda_{n,\Omega_{K+1}})}] d\varepsilon_n \\
= \; & (\Pi_{j \in \Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \exp(-\lambda_{n,\Omega_{K+1}}) \int_{-\infty}^{\bar{\varepsilon}_n - \lambda_{n,K}} \exp[-\varepsilon'_n - e^{-\varepsilon'_n}] d\varepsilon'_n \\
= \; & (\Pi_{j \in \Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \exp(-\lambda_{n,\Omega_{K+1}}) \exp[-e^{-(\bar{\varepsilon}_n - \lambda_{n,K})}] \\
= \; & [g_n(x_n) \diagup \sum_{j \in \Omega_{K+1}} g_j(x_j)] \cdot \{(\Pi_{j \in \Omega_{K+1}} \exp[e^{-\bar{\varepsilon}_j}]) \exp[-e^{-(\bar{\varepsilon}_n - \lambda_{n,\Omega_{K+1}})}]\} \\
= \; & [g_n(x_n) \diagup \sum_{j \in \Omega_{K+1}} g_j(x_j)] \cdot \exp\{(\sum_{j \in \Omega_{K+1}} e^{-\bar{\varepsilon}_j}) - e^{-(\bar{\varepsilon}_n - \lambda_{n,\Omega_{K+1}})}\}. \qquad (10)
\end{aligned}$$

Note that

$$\begin{aligned}
& (\sum_{j \in \Omega_{K+1}} e^{-\bar{\varepsilon}_j}) - e^{-(\bar{\varepsilon}_n - \lambda_{n,\Omega_{K+1}})} \\
= \; & (\sum_{j \in \Omega_{K+1}} e^{-(\varepsilon_{n_K} + \log g_{n_K}(x_{n_K}) - \log g_j(x_j))}) \\
& - \exp\{-[\varepsilon_{n_K} + \log g_{n_K}(x_{n_K}) - \log g_n(x_n) - (\log(\sum_{j \in \Omega_{K+1}} g_j(x_j)) - \log g_n(x_n))]\} \\
= \; & \frac{e^{-\varepsilon_{n_K}}}{g_{n_K}(x_{n_K})} \{\sum_{j \in \Omega_{K+1}} g_j(x_j) - \sum_{j \in \Omega_{K+1}} g_j(x_j)\} \\
= \; & 0. \qquad (11)
\end{aligned}$$

(10) and (11) give

$$p(n|\mathbf{N}_K, Y_K, \mathbf{x}) = g_n(x_n) \diagup \sum_{j \in \Omega_{K+1}} g_j(x_j). \qquad (12)$$

(12) is a very strong result as it says that $p(n|\mathbf{N}_K, Y_K, \mathbf{x})$ does not depend on Y_K. Aggregating over all possible Y_K, we must have

$$p(n|\mathbf{N}_K, \mathbf{x}) = g_n(x_n) \diagup \sum_{j\in\Omega_{K+1}} g_j(x_j), n \in \Omega_{K+1}, K = 1, ..., I-2, \tag{13}$$

which completes the proof.

Q.E.D. ■

Lemma 2 is important. Firstly, it reveals that given the top K ranked contestants, the conditional probability of a contestant to be ranked as the next is completely independent of $(x_{i_1}, \ldots, x_{i_K})$, the effort entries of these top K ranked contestants. Secondly, it shows that the conditional probability $p(n|\mathbf{N}_K, \mathbf{x})$ can be conveniently written as a ratio-form contest success function $g_n(x_n) \diagup \sum_{j\in\Omega_{K+1}} g_j(x_j)$, which mimics a lottery among the set of contestants who are ranked worse than level K.

Let the sequence $\{i_k\}_{k=1}^{I}$ denote a complete ranking among the I contestants, where i_k is the index of the $k-$th ranked contestant. Combine Lemma 1 and Lemma 2, we therefore conclude the following.

Theorem 2 *For any given effort entries $\mathbf{x} \geq 0$ such that $g_i(x_i) > 0, \forall i \in \mathbf{I}$, the ex ante likelihood of any complete ranking outcome $\{i_k\}_{k=1}^{I}$ can be written as*

$$p(\{i_k\}_{k=1}^{I}) = \Pi_{k=1}^{I} \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}. \tag{14}$$

Theorem 2 states that the ex ante likelihood of a complete ranking can be written as the cumulative product of the conditional probability $p(i_k|\mathbf{N}_{k-1}, \mathbf{x}) = g_{i_k}(x_{i_k}) / \sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})$ that contestant i_k is ranked as the top among all contestants $\{i_k, i_{k+1}, ..., i_I\}$. As aforementioned, the L prizes are awarded to the L contestants who contribute the highest y_is, respectively, by their ranks. Thus, a prize allocation outcome is therefore represented by the subsequence $\{i_k\}_{k=1}^{L}$ of $\{i_k\}_{k=1}^{I}$, where i_k denotes the index of the contestant who is ranked at the k-th position and receives V_k. The probability of a prize allocation plan $\{i_k\}_{k=1}^{L}$ is therefore determined in light of Theorem 2.

Corollary 1 *For any given effort entries $\mathbf{x} \geq 0$ such that $g_i(x_i) > 0, \forall i \in \mathbf{I}$, the ex ante likelihood* *of any prize allocation outcome* $\{i_k\}_{k=1}^L$ *can be written as*

$$p(\{i_k\}_{k=1}^L) = \Pi_{k=1}^L \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}. \tag{15}$$

Note that Corollary 1 does not cover the case where $g_i(x_i) = 0$ for some contestants. Since ties are randomly broken fairly in our noisy ranking model, these contestants will be ranked among the last ones with the same probability. By these results, we can conclude the strategic equivalence between our noisy-ranking model and a generalized multiple-winner nested contest model as proposed by Clark and Riis (1996 and 1998a). Clark and Riis (1996 and 1998a) extends winner-take-all Tullock contests to allow a block of prizes to be allocated among contestants. The selection mechanism is illustrated as a sequential lottery process. Contestants simultaneously submit their one-shot effort entries $\mathbf{x}$. The recipient of each prize is selected through a lottery among all remaining candidates represented by a ratio-form contest success function. As each contestant is eligible for at most one prize, the recipient of a prize is immediately removed from the pool of candidates who are eligible for the next draw. This procedure is repeated until all prizes are given away. If we use Ω_m to represent the index set of all remaining contestants for the m-th draw for the m-th prize V_m, then for any contestant $j \in \Omega_m$, he/she wins prize V_m with a probability $\frac{f_j(x_j)}{\sum_{i\in\Omega_m} f_i(x_i)}$ if $\sum_{i\in\Omega_m} f_i(x_i) > 0$. Here $f_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is the output function of contestant i, which is assumed to be strictly increasing with effort outlay x_i. To the extent that $\sum_{i\in\Omega_m} f_i(x_i) = 0$, i.e., $f_i(x_i) = 0, \forall i \in \Omega_m$, prizes are randomly given away. Thus, the prize allocation plan of this nested contest is determined by a series of M independent lotteries if M prizes are available. This nested contest reduces to a standard winner-take-all lottery contest when only one prize is available.

Let $C(\mathbf{I},\ \mathbf{g}(\cdot),\ \mathbf{V})$ denote a multi-winner nested Tullock contest with contestants $\mathbf{I}$, output functions $\mathbf{g}(\cdot)$ and prizes $\mathbf{V}$. The vector $\mathbf{V} = (V_1, \ldots, V_L)$ represents the ordered set of L prizes with $V_1 \geq V_2 \geq \ldots \geq V_L$. Each contestant i is endowed with an output production technology $f_i(x_i) = g_i(x_i)$. Corollary 1 immediately leads to the following result.

Theorem 3 *When* $L \geq 1$ *and* ε_i *follows the type I extreme-value (maximum) distribution, the noisy-ranking model (1) is strategically equivalent to a generalized multiple-winner nested contest* $C(\mathbf{I},\ \mathbf{g}(\cdot),\ \mathbf{V})$ *.*

Thus, Theorem 3 establishes the strategic equivalence between the noisy-ranking contest model and a generalized multiple-winner nested contest that is built upon ratio-form contest success functions. Clark and Riis (1998a), among others, provide a complete solution for the multiple-winner nested contests when contestants are symmetric and the winning odd in each lottery takes a Tullock success function, i.e., the output function $f_i(x_i) = x_i^r$. These results, by Theorem 3, also solve for the equilibrium of the noisy-ranking model (1) when contestants are assumed to be identical.

2.3 Discussion

In Section 2.2, we establish the equivalence between our noisy ranking contest model and the family of lottery contests that build upon ratio-form contest success functions. The decision maker (contest organizer), who has a strictly monotonic preference, evaluates and ranks contestants by the perceivable output y_i in the presence of randomness, and awards these contestants by their ranks. When only one prize is available, and therefore only the top rank is of concern, the ex ante likelihood that a contestant i achieves the top rank and wins the prize can be written in a ratio form $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$, which is identical to the success function of a generalized lottery contest with production functions $g_i(x_i)$. Hence, we have established the strategic equivalence between these two types of contests in winner-take-all settings.

A more intriguing duality is detected when more than one winner is allowed to receive a prize. The multiple-winner nested contest model (Clark and Riis, 1996, 1998a) uses ratio-form contest success function as its building block, but allows a block of prizes to be given away. This multiple-winner nested contest model provides a reasonable and relatively tractable framework to analyze contess with more than one winner. However, this model seems to require a series of independent lotteries to allocate the prizes one by one, which could severely narrow the scope of its application. As visualized as a sequential lottery process, the legitimacy of this model is immediately left in doubt when it is applied to situations where prizes are allocated as a batch.[11]

Nevertheless, our results directly resolve this concern. In our framework, for given effort entries $\mathbf{x}$, a complete ranking of contestants, as well as the corresponding prize allocation plan, immediately results from any realization of the noise terms $\boldsymbol{\varepsilon}$. Corollary 1 reveals that a noisy-ranking contest model and a nested contest model generate an identical probability for every prize allocation rule. We therefore establish the equivalence between the seemingly sequential lottery process and a simultaneous noisy ranking scheme. Consequently, the sequential lottery process should only be

[11] For example, one could imagine that a college admits a batch of top scoring students using a cutoff standard, instead of making acceptance decision sequentially.

understood as a convenient "visualization" of the selection procedure that underlies a multiple-winner nested contest, instead of revealing the "panorama" inside the "black box". Indeed, a noisy ranking system hidden under the disguise of the lottery process has been "recovered" by Theorem 3!

It is well known in contest literature that a **winner-take-all** all-pay auction is a limiting case of a Tullock contest. Assume that the output function takes the commonly adopted form $g_i(x_i) = x_i^r, r > 0$. When r approaches infinity, a slight increment in effort makes a sure win. As a result, the Tullock contest converges to an all-pay auction. By the same token, one would conclude that a multiple-winner nested contest converges to a **multiple-winner** all-pay auction (Clark and Riis, 1998b, Barut and Kovenock, 1998, Moldovanu and Sela, 2001) when $g_i(x_i) = x_i^r, r > 0$. This convergence looks even more intuitive once we recover the noisy-ranking system underlying ratio-form contest success functions. As r (the measure of productivity) increases, effort would contribute more to one's perceivable output, while the noise does less. As r approaches infinity, the impact of noise is completely overshadowed by that of substantive effort, and the complete ranking of contestants is purely determined by their effort entries.[12]

2.4 A Micro Foundation of Ratio-Form Contest Success Functions

In this part, we further explore the implications of our results. As we mentioned in Introduction, this paper has been inspired by and is closely linked to Fullerton and McAfee (1999) and Baye and Hoppe (2003). Both of these papers establish the strategic equivalence between a research tournament (Fullerton and McAfee, 1999) and a winner-take-all Tullock contest. They show that the likelihood of a firm i winning a research tournament can be written as a standard Tullock contest success function $\frac{x_i}{\sum_j x_j}$, where x_i is the number of parallel experiments a firm i conducts. A more fundamental question naturally arises: Why could these seemingly disparate models be unified? Or in other words, what is the common thread that connects them? In light of the duality we have established in Section 2.2, our model bears a strong tie to this research tournament model as well. Our results follow the effort of these pioneering studies, and allow us to further expand the family of competitive activities that can be unified within an integral framework.

[12] We can immediately realize that our model covers the allocation scheme of all single-period auctions, regardless of the payment scheme or the number of objects on sale. Any meaningful auction rule requires a ranking of the bids, and the highest bidder(s) win the object(s). Bids are assumed to be perfectly observable so that no exogenous noise plays a role.

The tie reveals itself as we closely scrutinize the setup of our noisy ranking model. We assume the perceivable output ($\log y_i$) contains a deterministic component and a random component, which follows an extreme value type I (maximum) distribution. Following Lazear and Rosen (1981), one could interpret the perceivable output ($\log y_i$) of a contestant as the sum of its expectation and a random shock. Type I Extreme value distributions (Gumbel) are the limiting distributions of the maximum or the minimum of a large collection of i.i.d. random observations from a same arbitrary continuous distribution on support $(-\infty, \infty)$. The type I extreme value (maximum) distribution is pertinent to a circumstance where (only) the maximum value of a collection of random shocks is of interest. By assuming this distribution, the model therefore depicts a selection mechanism where the decision maker (the contest organizer) honors the "best shot" of each contestant's repeated attempts. For instance, weight lifters are ranked by their most successful tries. Alternatively, our model images the situations where only the best performance is observable to the decision maker. An architect submits only his/her best idea to a design competition. A participant of Olympic Physics Competition puts down only the most satisfactory solution that occurs to him/her to any given problem. Indeed, in many occasions, only the best performance of a contestant is perceived. Thus, the seemingly peculiar type I extreme value (maximum) distribution adopted in Section 2.1 in fact captures the essence of a broad class of competition activities. This specification is at least a good approximation for a performance evaluation scheme that is based on the best luck of contestants. These observations rationalize our adoption of this distribution.

In light of the isomorphism between our model and contests built upon ratio-form success functions, this argument directly sheds light on the hidden mechanism in the black box of the family of lottery contests. One could also understand that a lottery contest model represents a noisy ranking system that orders contestants by their best performance. This insight immediately reveals the logic underlying the equivalence established by Fullerton and McAfee (1999) and Baye and Hoppe (2003). In a research tournament, each firm hires a number of scientists to conduct research. Each of them could run an experiment and come up with an idea with a randomly distributed value. Each firm then submits the best it obtains to compete with others, and a firm that submits "the best among the best" wins. A research tournament naturally exemplifies this winning mechanism. It is such a performance evaluation mechanism that underpins and unites these models! Additional evidence is provided in next section to support this argument.

3 Extensions

In this section, we dialectically elaborate the argument we have proposed. We "test" this argument by examining the "dual" problem to our original model, i.e., a multi-winner race model. We establish its equivalence to a multiple-winner nested contest. A hidden ranking rule is intuitively "recovered" that confirms and reinforces our hypothesis. In addition, we present "the antithesis" to our argument. We provide a model that cannot be abstracted as a standard lottery contest, as it hosts a different winning mechanism.

3.1 The "Dual" Problem: A Multi-Winner Race Model

Baye and Hoppe (2003) establish the strategic equivalence between a patent race and a standard Tullock contest. In this part, we show that the arguments proposed in Section 2.4 also shed light on this isomorphism. A noisy ranking that honors the most favorable shock could also be "recovered" underneath models on racing competitions, i.e., the type of competitive events where contestants are better rewarded by accomplishing a specific task faster than others.

We first propose a generalized race model that allows for more than one prize. We adopt the framework of Dasgupta and Stiglitz (1979). When each of I contestants $\mathbf{I}$ chooses a lump-sum effort x_i, a contestant i would accomplish a task (e.g. making a scientific discovery) by the time t_i with a probability (i.e. a Weibull minimum distribution)

$$\Psi(t_i|x_i) = 1 - e^{-z_i(x_i)t_i}, \quad x_i, t_i \geq 0, \tag{16}$$

where $z_i(x_i)$ represents the hazard rate of contestant i, i.e., the conditional probability of accomplishing this task between t_i and time $t_i + \Delta t_i$. Conditional on effort entry $\mathbf{x}$, t_is are i.i.d. The hazard rate $z_i(x_i)$ is a strictly increasing function of the expenditure x_i. We define $\mathbf{z}(\cdot) \triangleq (z_i(\cdot))$.

Diverging from Dasgupta and Stiglitz (1979), we allow for multiple winners. We assume $L \in \{1, 2, ..., I\}$ prizes (denoted by $\mathbf{V} = (V_1, V_2, ..., V_L)$) to be awarded to contestants. That is, the contestant who finishes the first receives prize V_1, the second receives V_2, so on and so forth.[13] Given effort entries $\mathbf{x}$, each conditional realization of (t_i) determines a ranking of contestants and accordingly the prize allocation rule. By the nature of a race, we may also intuitively interpret a race as a noisy-ranking contest: contestants are ranked in an ascending order by the time they

[13] One could imagine a number of firms are engaged in process R&D competition. The earlier a firm discovers the secret of a cost-reduction technology, the higher its accumulated profit.

spend on accomplishing the task, and a contestant is better rewarded for the realization of smaller t_i.

Denote by $R(\mathbf{I},\, \mathbf{z}(\cdot),\, \mathbf{V})$ this multi-winner race model and $C(\mathbf{I},\, \mathbf{z}(\cdot),\, \mathbf{V})$ a multiple-winner nested Tullock contest with contestants $\mathbf{I}$, technology $\mathbf{z}(\cdot)$ and prizes $\mathbf{V}$. We first present the following result and we will build its micro foundation at a later point.

Theorem 4 *A multiple-winner race $R(\mathbf{I},\, \mathbf{z}(\cdot),\, \mathbf{V})$ is strategically equivalent to a multiple-winner nested Tullock contest $C(\mathbf{I},\, \mathbf{z}(\cdot),\, \mathbf{V})$.*

Proof. One may directly obtain that for given expenditure entries $\mathbf{x}$ such that for $\sum\limits_{j\in\mathbf{I}} z_j(x_j) > 0$, a firm i could leapfrog all others with a probability

$$\begin{aligned}\Pr(t_j \;\; &\geq \;\; t_i,\; j\in\mathbf{I},\; j\neq i) = \int_0^\infty z_i(x_i) e^{-t_i \sum\limits_{j\in\mathbf{I}} z_j(x_j)} dt_i \\ &= \;\; \frac{z_i(x_i)}{\sum\limits_{j\in\mathbf{I}} z_j(x_j)}, \quad \forall i\in\mathbf{I}, \end{aligned} \tag{17}$$

which perfectly mimics one's winning odd in a generalized Tullock lottery contest with (increasing) output functions $z_i(x_i)$.

Suppose $\tilde{K}$ $(1\leq\tilde{K}\leq I-2)$ contestants are ranked as the first to the $\tilde{K}$-th in ascending order according to (t_i), with contestant i_k ranked at k-th one. Define $\tilde{\mathbf{I}}_{\tilde{K}} = \{i_k, k=1,...,\tilde{K}\}$. We thus have $t_{i_1}\leq t_{i_2}\leq\cdots\leq t_{i_{\tilde{K}}}\leq t_j$, $\forall j\in\tilde{\Omega}_{\tilde{K}+1} = \mathbf{I}\backslash\tilde{\mathbf{I}}_{\tilde{K}}$. We next consider the conditional probability of a contestant $n\in\tilde{\Omega}_{\tilde{K}+1}$ being the $(\tilde{K}+1)$-th ranked. We denote this probability by $q(n|\tilde{\mathbf{N}}_{\tilde{K}}, \mathbf{x}, T_{\tilde{K}})$, where $T_{\tilde{K}} = (t_{i_1},...,t_{i_{\tilde{K}}})$, $\tilde{\mathbf{N}}_{\tilde{K}} = (i_1,...,i_{\tilde{K}})$. This conditional probability is simply

$$\begin{aligned} q(n|\tilde{\mathbf{N}}_{\tilde{K}}, \mathbf{x}, T_{\tilde{K}}) &= \Pr(t_n\leq t_j,\; j\in\tilde{\Omega}_{\tilde{K}+1}\,,\; j\neq n | t_n\geq t_{i_{\tilde{K}}}) \\ &= \int_{t_{i_{\tilde{K}}}}^\infty z_n(x_n)\exp(-t_n \sum_{j\in\tilde{\Omega}_{\tilde{K}+1}} z_j(x_j)) dt_n \diagup \exp(-t_{i_{\tilde{K}}} \sum_{j\in\tilde{\Omega}_{\tilde{K}+1}} z_j(x_j)) \\ &= \frac{z_n(x_n)}{\sum\limits_{j\in\tilde{\Omega}_{\tilde{K}+1}} z_j(x_j)}, \forall n\in\tilde{\Omega}_{\tilde{K}+1}. \end{aligned} \tag{18}$$

This strong result says that $q(n|\tilde{\mathbf{N}}_{\tilde{K}}, \mathbf{x}, T_{\tilde{K}})$ does not depend on $T_{\tilde{K}}$. Aggregating over all possible $T_{\tilde{K}}$, we must have that conditioning on contestants $i_1, i_2, ..., i_K$ being respectively ranked from top

1 to top K, the probability that a contestant $n \in \tilde{\Omega}_{\tilde{K}+1}$ is the $(\tilde{K}+1)$-th ranked is

$$q(n|\mathbf{N}_K, \mathbf{x}) = z_n(x_n) \diagup \sum_{j \in \Omega_{K+1}} z_j(x_j), n \in \tilde{\Omega}_{\tilde{K}+1}, K = 1, ..., I-2. \tag{19}$$

We thus see from (18) and (19) that the resulted prize allocation plan is stochastically equivalent to that of a multiple-winner nested contest with output functions $z_i(x_i)$.

Q.E.D. ■

Why are they equivalent?

Theorem 4 states the strategic equivalence between our multi-winner race model and a nested lottery contest model where more than one prize is available. It remains to lay a micro foundation for this strategic equivalence. We now show that the argument proposed in Section 2.4 continues to apply and a selection mechanism that honors "the most favorable shock" is also hidden underneath the race model.

Theorem 5 *A multiple-winner race $R(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ is equivalent to a descending-order noisy-ranking contest (1) with the set of output functions $\mathbf{z}(\cdot)$ and the noises $\boldsymbol{\varepsilon}$ that are individually and independently distributed following an extreme value type I (maximum) distribution.*

One may not find this result very surprising, as we have established the equivalence between the multi-winner race model and a multiple-winner nested contest model. We will not lay out a dedicated technical proof, but present the reasoning by the following discussion. The hidden tie that connects all these models would surface as we set out to establish the result.

It is worth noting that t_i (the time the contestant i takes to finish a given task) can be modelled as the product of two multiplicatively separable components as follows

$$t_i = h_i(x_i) q_i, \ \forall i \in \mathbf{I}, \tag{20}$$

where $t_i, q_i \in (0, \infty)$ and $h_i(x_i) \triangleq z_i^{-1}(x_i)$. In other words, t_i is jointly determined by the deterministic component $h_i(x_i)$, which depends on only one's effort entry, and a stochastic term q_i. Obviously, the function $h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ strictly decreases with one's effort. As suggested by simple statistical facts, t_i follows a Weibull (minimum) distribution of (16) if and only if q_i follows a Weibull (minimum) distribution with c.d.f. $1 - e^{-q_i}$. Under this assumption, model (16) is equivalent to model (20).[14]

Model (20) can be further equivalently written as

$$\log t_i = \log h_i(x_i) + \eta_i,\ \forall i \in \mathbf{I}, \tag{21}$$

where i.i.d idiosyncratic noises $\eta_i \equiv \log q_i$ follow an extreme value type I extreme-value (minimum) distribution.[15] The c.d.f. and p.d.f. of η_i, respectively written as follows,

$$\Phi(\eta_i) = 1 - e^{-e^{\eta_i}}, \eta_i \in (-\infty, +\infty),\quad \forall i \in \mathbf{I}, \text{ and} \tag{22}$$

$$\varphi(\eta_i) = e^{\eta_i - e^{\eta_i}},\quad \forall i \in \mathbf{I}. \tag{23}$$

A closer look would reveal that an ascending-order model (21) is in fact equivalent to the framework we set up in Section 2.1. Note that model (21) can be equivalently written as

$$\log \widetilde{y}_i = \log z_i(x_i) + \xi_i,\ \forall i \in \mathbf{I}, \tag{24}$$

where $\widetilde{y}_i = t_i^{-1}$ and $\xi_i = \log q_i^{-1}$. Note that when $\eta_i = \log q_i$ follows an extreme value type I extreme-value (minimum) distribution, ξ_i must follow an extreme value type I extreme-value (maximum) distribution as given by (3). By the simple statistical fact, the extreme value type I (maximum) distribution is simply the inverse of its "minimum" counterpart (the extreme value type I (minimum) distribution)! Consequently, ranking (t_i) of model (21) in ascending order is equivalent to ranking (t_i^{-1}) of model (22) in descending order. In short, the race model (21) is the "dual" of the model we proposed in Section 2.1. With this observation, Theorem 5 immediately results, which reveals why our multi-winner race model is strategically equivalent to a sequential lottery contest (Theorem 4).

Model (21) thus provides a micro foundation for model (16). The multi-winner race model, as well as its "tweak" model (21), is underpinned by the same "preference relation" or "ordering mechanism" as the model we presented in Section 2.1. Both of them represent an evaluation mechanism that honors "the most favorable shock", which lays a common foundation for all the equivalence results we have established.

To see that, note that the random term η_i in (21) follows an extreme-value (minimum) type I distribution, which is also known as "log-Weibull (minimum)" distribution. A Weibull (minimum)

[14] We derive this setup by manipulating a race model. However, it is worth nothing that this model could also be applied to other competitive events, where contestants win by reducing the amounts of their "disvalued" outputs, e.g., pollution.

[15] The extreme value type I (minimum) distribution is also known as a "log-Weibull" distribution.

distribution describes the timing of "the minimum" from a collection of random sample from an arbitrary distribution with a support on $[0, \infty)$.[16] This fact, together with the ascending-order ranking rule, naturally corresponds to a selection mechanism that honors "the best luck", when the output is a "bad" to the decision maker, and contestants get ahead of others by contributing lesser amounts of their perceivable outputs: Under such a circumstance, "the most favorable shock" is seen by the realized minimum. A race directly exhibits such characteristics: One secures a more favorable rank by accomplishing his/her task as quick as possible, i.e., making t_i as "small" as possible!

3.2 The "Antithesis": An Example of Non-Lottery Contests

So far we have proposed a micro foundation, i.e., a hidden preference relation and an ordering rule, which underpins a wide range of contests. This discovery permits us to connect varieties of seeming disparate models on one hand, while it imposes a limit on this unity on the other: This family of contests may not include competitive events that do not honor "the most favorable shocks" when picking the winners.

To illustrate this point, we provide a contest model that hosts a different performance evaluation rule. One salient example is the noisy ranking contest model suggested by Hirshleifer and Riley (1992). Two contestants simultaneously submit their effort entries x_1 and x_2, and they are ranked by their composite output $q_i x_i$, where q_i is a random variable that follows a Weibull (minimum) distribution with c.d.f. $F(q_i) = 1 - e^{-aq_i}$. The contestant with higher output wins, thus outputs need to be ranked in a descending order. It can be easily verified that given the set of effort entries, the ex ante winning odd of a contestant is exactly identical to a standard Tullock success function $\frac{x_i}{x_1+x_2}, i = 1, 2$.

However, the equivalence between this model and a lottery contest does not hold when more than two contestants are in presence. We consider a more generalized variation of this model. We assume the deterministic component in the composite output takes the form $q_i g_i(x_i)$, where $g_i(x_i)$ is a strictly increasing function of the effort outlay x_i. Obviously, linear technology $g_i(x_i) = x_i$ is a special case of this setting. When $I = 3$, and when only one prize is available, contestant 1 wins

[16] This is the reason that it is the inverse of the extreme value type I maximum distribution.

with a probability

$$P_1 = 1 - \frac{g_2(x_2)}{g_1(x_1)+g_2(x_2)} - \frac{g_3(x_3)}{g_1(x_1)+g_3(x_3)} + \frac{g_2(x_2)g_3(x_3)}{g_1(x_1)g_3(x_3)+g_2(x_2)g_3(x_3)+g_1(x_1)g_2(x_2)}. \quad (25)$$

The proof is provided in the Appendix.

The source of this dichotomy is not difficult to detect as we look at the distribution of the noise term. The Weibull (minimum) distribution indicates the distribution of the incidence of the "minimum" among a collection of shocks. Referring to (20), readers would immediately realize that this model is no different from our race model except for the winning rule. One wins in a race by a smaller output $q_i x_i$. By way of contrast, a contestant in the model of Hirshleifer and Riley (1992) wins by a larger output. Namely, the decision maker does prefer a higher output. However, he/she does not honor the best shot, but rank their performances by their "weakest links".

This mechanism contradicts with the one underneath our model (21). It could represent those contests when the worst performance matters (the most) for one's win and contestants compete by improving their own blind sides. A close analogy is high-profile board game competitions such as Chess Olympic Championships. It is often perceived that one loses because of his/her most harmful misplay despite his/her marvelous moves. This dichotomy in underlying performance evaluation mechanism thus drives this observed disparity when the number of participants exceeds two, and excludes this type of contests from the family of models that can be represented as standard lottery contests.

4 Concluding Remarks

In this paper, we set forth a multi-winner contest model that links its prize allocation plan to a noisy ranking of contestants by their performances. The performance of a contestant is modelled as the sum of a deterministic output out of his/her spontaneous effort, and a random component. Contestants exert their one-shot effort simultaneously, and the ordered prizes are awarded to best performers by their ranks.

We find that if the contestants are evaluated and ranked by their "most favorable shocks" in a collection of attempts, our noisy-ranking model delivers exactly the same success functions as a lottery contest. We therefore establish the strategic equivalence between our noisy-ranking contest model and the family of lottery contests. The implications of this result are multi-fold. Firstly, this result provides an alternative interpretation of lottery contests, in particular, the multiple-winner nested contest model (Clark and Riis, 1996 and 1998a): a noisy ranking system can be recovered underneath its literally sequential lottery process. Secondly, this result illuminates a hidden common thread that connects a wide variety of seemingly disparate contests in the nutshell of ratio-form contest success functions: underlying all these contests is a common winning mechanism that honors contestants' most favorable shocks! Thus, this result provides a behavioral foundation that underpins the family of commonly adopted lottery contest models. Finally, our result nevertheless imposes a limit on the boundary of this broad class of models: the family of contests that can be united in the nutshell of lottery contests may not include competition schemes that do not honor "the most favorable shocks" on contestants' performance.

Appendix: The Proof of the "Antithesis"

In this appendix, we prove that when are three contestants ($I = 3$), the noisy-ranking contest model we present in 3.2 does not deliver a standard lottery contest.

Following Hirshleifer and Riley (1992), we use a formulation with multiplicative noise term:

$$y_i = q_i g_i(x_i), \tag{A.1}$$

where the q_i follows a Weibull minimum distribution with c.d.f. $1 - e^{-q_i}$. (A.1) can be equivalently written as

$$\log y_i = \log g_i(x_i) + \log q_i. \tag{A.2}$$

This distribution of $\varepsilon_i \triangleq \log q_i$ is a type I extreme-value (minimum) distribution. The c.d.f. and p.d.f. of ε_i are thus

$$\begin{aligned} F(\varepsilon_i) &= 1 - \exp(-e^{\varepsilon_i}), \text{ and} \quad &(A.3)\\ f(\varepsilon_i) &= e^{\varepsilon_i - e^{\varepsilon_i}}. \quad &(A.4)\end{aligned}$$

Consider the case of three contestants ($I = 3$). Given effort x_i, contestant 1 wins with the following probability

$$\begin{aligned}
&\int_{-\infty}^{+\infty} [\Pi_{j=2,3} F(\varepsilon_1 + \log g_1(x_1) - \log g_j(x_j))] f(\varepsilon_1) d\varepsilon_1 \\
= &\int_{-\infty}^{+\infty} [(1 - \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_2(x_2)}))(1 - \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_3(x_3)})] e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \\
= &\ 1 - \int_{-\infty}^{+\infty} \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_2(x_2)}) \cdot e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \\
&- \int_{-\infty}^{+\infty} \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_3(x_3)}) \cdot e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \\
&+ \int_{-\infty}^{+\infty} \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_2(x_2)}) \cdot \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_3(x_3)}) \cdot e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \int_{-\infty}^{+\infty} \exp(\varepsilon_1 - e^{\varepsilon_1}(1 + \frac{g_1(x_1)}{g_2(x_2)}))d\varepsilon_1 - \int_{-\infty}^{+\infty} \exp(\varepsilon_1 - e^{\varepsilon_1}(1 + \frac{g_1(x_1)}{g_3(x_3)}))d\varepsilon_1 \\
&\quad + \int_{-\infty}^{+\infty} \exp(\varepsilon_1 - e^{\varepsilon_1}(1 + \frac{g_1(x_1)}{g_2(x_2)} + \frac{g_1(x_1)}{g_3(x_3)}))d\varepsilon_1 \\
&= 1 - \int_{-\infty}^{+\infty} \exp(\varepsilon_1 - e^{\varepsilon_1 + \log(1 + \frac{g_1(x_1)}{g_2(x_2)})})d\varepsilon_1 - \int_{-\infty}^{+\infty} \exp(\varepsilon_1 - e^{\varepsilon_1 + \log(1 + \frac{g_1(x_1)}{g_3(x_3)})})d\varepsilon_1 \\
&\quad + \int_{-\infty}^{+\infty} \exp(\varepsilon_1 - e^{\varepsilon_1 + \log(1 + \frac{g_1(x_1)}{g_2(x_2)} + \frac{g_1(x_1)}{g_3(x_3)})})d\varepsilon_1 \\
&= 1 - \frac{g_2(x_2)}{g_1(x_1) + g_2(x_2)} - \frac{g_3(x_3)}{g_1(x_1) + g_3(x_3)} \\
&\quad + \frac{g_2(x_2)g_3(x_3)}{g_1(x_1)g_3(x_3) + g_2(x_2)g_3(x_3) + g_1(x_1)g_2(x_2)}.
\end{aligned}
$$

References

[1] Amegashie, J.A., Some Results on Rent-Seeking Contests with Shortlisting, *Public Choice*, 2000, 105, 245-253.

[2] Baye and Hoppe, The Strategic Equivalence of Rent-Seeking, Innovation, and Patent-Race Games, *Games and Economic Behavior*, 2003, 44, 217-226

[3] Barut, Y., and Kovenock, D., The Symmetric Multiple-Prize All-Pay Auction with Complete Information, *European Journal of Political Economy*, 1998, 14, 627-644.

[4] Clark, D.J., and Riis, C., A multi-winner Nested Rent-Seeking Contest, *Public Choice*, 1996, 87, 177-184.

[5] Clark, D.J., and Riis, C., Influence and the Discretionary Allocation of Several Prizes, *European Journal of Political Economy*, 1998a, 14, 605-615.

[6] Clark, D.J., and Riis, C., Competition Over More Than One Prize, *American Economic Review*, 1998b, 276-289.

[7] Dasgupta, Partha & Stiglitz, Joseph, Industrial Structure and the Nature of Innovative Activity, *Economic Journal*, 1980, 90, 266-93..

[8] Fu, Q., and Lu, J., The Optimal Multiple-Stage Contest, NAJ Economics, 2007, 14.

[9] Fullerton and McAfee, 1999, Auctioning Entry into Tournaments, *Journal of Political Economy*, 1999, 107, 573-605.

[10] Glazer, A., and Hassin, R., Optimal Contests, *Economic Inquiry*, 1988, 26, 133-143.

[11] Hirshlefer and Riley, *The Analysis of Uncertainty and Information*, 1992, Cambridge University Press, Cambridge UK.

[12] Konrad, K., Strategy in Contests-an Introduction, WZB Discussion Papers, 2007.

[13] Lazear and Rosen, Rank-order tournaments as optimum labor contracts, Journal of Political Economy, 1981, 89, 841-864.

[14] McFadden, D., Conditional Logit Analysis of Qualitative Choice Behavior, In P. Zarembka (ed.), *Frontier in Econometrics*, 1973, New York: Academic.

[15] McFadden, D., The measurement of Urban Travel Demand, *Journal of Public Economics*, 1974, 3, 303-28.

[16] Moldovanu, B., and Sela, A., The Optimal Allocation of Prizes in Contests, American Economic Reviews, 2001, 91, 542-558.

[17] Moldovanu, B., Sela, A., and Shi, X., Contests for Status, Forthcoming at *Journal of Political Economy,* 2007.

[18] Szymanski, S., and Valletti, T.M., Incentive effects of second prizes, *European Journal of Political Economy*, 2005, 21, 467-481.

[19] Skaperdas, S., Contest Success Function, *Economic Theory*, 1996, 7, 283-290.

[20] Yates, A.J., and Heckelman, J.C., Rent-Setting in Multiple Winner Rent-Seeking Contests, *European Journal of Political Economy*, 2001, 17, 835-852.