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Abstract

We model pre-investment R&D decisions in the presence of spillover effects in an option pricing framework with analytic tractability. Two firms face two decisions that are solved for interdependently in a two-stage game. The first-stage decision is: what is the optimal level of coordination (optimal policy/technology choice)? The second-stage decision is: what is the optimal effort for a given level of the spillover effects and the cost of information acquisition? The framework is extended to a two-period stochastic game with (path-dependency inducing) switching costs that make strategy revisions harder. Strategy shifts are easier to observe in more volatile environments.

1. Introduction

We discuss in a game theoretic context managerial intervention directed towards value enhancement in the presence of uncertainty and spillover effects. Two firms face real investment options. Embedded in these are (optional) actions that allow firms to enhance value of their prospects directly (through R&D that improves product attributes or reduces costs, advertisement, etc.) or indirectly through information acquisition (exploratory drilling, marketing research, etc.) Due to spillovers, each firm´s action affects the other firm. In this framework both firms can act strategically and take advantage of the positive spillovers (or take pre-emptive action against the negative spillovers). In equilibrium, the degree of coordination can be higher or lower. The implementation of strategy by each firm can either be implicit, or explicit, i.e., by forming a research joint venture. Each firm must thus decide: (a) How much from this effort to share with its counterpart if this level can vary (the first-stage decision) and (b) How much to spend for such (R&D, etc.) actions, given the spillover effects (the second-stage decision). For the optimal effort (a tactical) decision we allow for a continuous set of alternatives, whereas for the optimal coordination (a strategic) one, we only allow a discrete set of choices. Let for example a pharmaceutical and a chemical firm trying to enhance the value of their investment prospect, by taking R&D actions to develop new technologies. Each one can learn at least part of the results of the effort of the other for free even if both act on their own. However the two firms will optimize the value of their investment options if they strategically determine the amount of R&D they share (in forming for example a research joint venture and deciding the degree of coordination that will take place and the type of research they will engage in). Another example would be two oil firms that own rights in adjacent oil fields. Knowledge resulting from exploratory drilling could be shared benefiting both.

In the paper we adopt the real options framework from the financial economics literature and connect it with the research joint ventures literature of industrial organization. The literature on real options or otherwise irreversible investments under uncertainty (see Dixit and Pindyck, 1994, and Trigeorgis, 1996) examines the value of flexibility in investment and operating decisions under uncertainty where the traditional net present value (NPV) approach fails. Although the importance of learning actions like exploration, experimentation, and R&D was recognized early on (e.g., Roberts and Weitzman, 1981), the real options literature has paid little attention to management´s ability to intervene in order to acquire more information and/or enhance value (see Paxson, 2003, for many useful references). Majd and Pindyck (1989), and Pennings and Lint (1997) examine real options with passive learning. Although many state-variables are usually treated as observable, it is often more realistic to assume that they are only estimates of quantities that will be actually observed or realized later. Childs, et al. (2001), and Epstein, et al. (1999) use a filtering approach towards learning. Beyond learning actions there can also exist control actions that directly enhance investment value as in Huchzermeier and Loch (2001) who study R&D problems as impulse-control problems. Martzoukos (2000) considers true value conditional on activation of costly control actions as a random variable with a known probability distribution. He examines real options with impulse-type controls of random size (random controls) to model intervention of management as intentional actions to affect the underlying value with uncertain outcome (similarly to Korn, 1997), and also to model learning/exploration actions to reduce or eliminate estimation error about the true value of the underlying asset. When the method is used to capture learning, it is designed to avoid the aforementioned bias and as we discuss later it is also consistent with a Bayesian approach. He also assumes that sequential R&D actions are independent of each other and of the exogenous uncertainty. We adopt this random controls methodology to examine in a game theoretic framework the behavior of two firms in the existence of spillovers at the pre-investment stage.

Real options papers in a game-theoretic context include for example Smit and Ankum (1993) with an exogenously determined set of alternative corporate strategies; Smit and Trigeorgis (2004) that include uncertainty under various types of competition in the product markets; Weeds (2002) who considers R&D investments with binary outcome (in the case of success an investment option is created, whereas in the case of failure there is no investment option available); and Lambrecht and Perraudin (2003), and Shackleton, Tsekrekos, and Wojakowski (2003) who consider strategic preemption.

In our paper firms own an investment option each (although not necessarily a profitable one) and each firm tries to either directly maximize its profit potential, or improve the relevant information set, with (optional) costly research actions before the final capital intensive decision takes place. The interaction between firms that results in a game theoretic framework comes from the existence of spillovers due to the pre-investment (R&D) decisions of each player. We allow firms to optimally choose the research/technology policy (a strategic choice between a discrete number of policies), and to also endogenously determine the optimal amount of effort (a tactical decision between a continuum of research budget allocations). The importance of intra-industry spillovers has been documented in the literature. Foster and Rosenzweig (1995) for example emphasize the importance of learning spillovers in agriculture, and Carey and Bolton (1996) argue that collusion in advertisement can be successful due to significant spillovers from generic advertising. Spillover effects are also significant even among different sectors, as discussed in Bernstein and Nadiri (1988).

Firms may coordinate their R&D efforts without necessarily colluding in the product markets (in a case like that we have a semi-collusion to use the term in Brod and Shivakumar, 1999, etc.) Coordination can take also either the form of research joint ventures (RJV) or non-equity co-development (COD) according to the access the firms have to the innovation (see for example Tao and Wu, 1997). The seminal theoretic model of R&D choice in the presence of spillovers under various structures in the product market is d´Aspremont and Jacquemin (1988), where the degree of spillover is the same. In Kamien et.al. (1992) the degree of spillovers varies. In most papers firms operate in the same product market, and authors search for symmetric equilibria.

In our paper, both the optimal effort to exert and the choice of the level of spillovers are endogenously determined. Since we adopt an option pricing framework, we could easily allow at the maturity of this investment option interactions in the product market (joint determination of equilibrium quantity and price) like for example in Smit and Trigeorgis (2004). For purposes though of analytic tractability, we focus on the case where the firms cannot affect each other in the product market (i.e., they either have monopoly power over their investment option, or prices are exogenously determined). This assumption will allow us the use of models isomorphic to the familiar Black and Scholes (1973). Since firms can operate in different product markets, we allow for asymmetric equilibria in the degree of spillovers between them. We first demonstrate the general model for costly controls at a pre-investment stage that allows for different types of actions for each agent, and we introduce the framework for the two-stage game. We then present in detail the real option game with pure *learning* controls (where value enhancing comes through information acquisition, i.e., exploration, pharmaceutical experimentation, but also marketing research, etc.), and the game with *impact* controls (that allow each player direct value enhancing outcomes, i.e., R&D for attribute improvements or cost reduction, but also advertisement, etc.). We show that unlike in the impact control case where in equilibrium both players exert a positive effort, in the pure learning case there exist equilibria where one player *delegates* all effort to the other (*free lunch*). For both cases we discuss applications with numerical results (parameter values for the mean and uncertainty of the control´s impact are very plausible as can be seen for example in the empirical R&D papers of Grabowski, and Vernon, 1990, and Kothari, Laguerre, and Leone, 2002). Although we originally restrict to single-period games, the framework is eventually extended to multiple periods with an application to a two-period game with switching costs that induce path-dependency. The final section concludes.

2. The real option game with noisy assets, multiplicative controls, and spillovers.

We consider costly R&D (control) actions that managers use to affect the value of an investment opportunity at the pre-investment stage. The outcome of these actions will be observed instantaneously, but the controls´ outcome is random. We classify them into two types: pure *learning* control actions with the sole purpose of information acquisition that reduce uncertainty about the estimated investment value; and *impact* control actions with a direct value enhancement (or similarly a cost reduction) purpose but with random outcome. Ex ante we simply know the probability distribution of the outcome, thus we call them random controls. The impact control is the most obvious one, since it is an impulse type (like in Huchzermeier and Loch, 2001) but with random outcome (similarly to Korn, 1997). Advertisement, process improvement, product attribute enhancement, etc., are actions that result directly in adding value, increasing sales volume and/or price per item, enhancing market share, or reduce production costs. In contrast, pure learning actions are intended to improve the information about (but not to directly affect) a quantity, potential sales price, etc. Exploratory drilling for example will improve information about the value of an oil field, and marketing research will help to better assess market share, etc. Learning actions are thus activated when a parameter significant for the decision making process is estimated with error. Management intends to eliminate or at least reduce this error in order to make optimal investment decisions. If uncertainty is fully resolved, exercise of an investment option on stochastic asset S^* with exercise cost X yields $S^* - X$. Has a learning action not been taken before the investment decision is made, resolution of uncertainty (learning) would occur ex post. Ex ante, the investment decision must be made based solely on expected (instead of actual) outcomes. In this case exercise of the real option is expected to provide $E[S^*] - X$. The real investment prospect is a claim contingent on $S = E[S^*]$, and we assume that $E[S^*]$ follows a geometric Brownian motion, just like S^*. Thus, S will follow the same process before and after learning. Consider for example the case where the underlying asset is a product of two variables, a stochastic but observable one, and a constant but unobservable one. We seek to learn about the unobservable entity, and in doing so we will not affect the stochastic process of the product of the two. At learning, we will simply revise our estimate of the product. *Fully revealing* learning actions are designed to resolve uncertainty completely (assuming this is economically or technically feasible), but in the most general case *partly revealing* actions will be employed either because complete resolution of uncertainty is infeasible, or it is too costly. Each firm faces an investment decision, and either $S = E[S^*]$ is common for both (or simply differs by a constant), or each firm´s claim is contingent on a different asset, simply necessitating separate notation for S_1 and S_2 which again follow geometric Brownian motions. In both cases (pure learning or impact), the effect is modeled as an impulse-type control with random outcome, activated at a cost. This is for methodological convenience only, since the learning action reduces parameter uncertainty (as in a Bayesian framework), while the impact control intentionally affects the underlying asset (with random outcome).

Formally, we assume that each underlying asset (project) value, S, subject to i optional (and often costly) controls, follows in the risk-neutral probability measure a stochastic process of the form:

$$\frac{dS}{S} = \left(r - \delta\right)dt + \sigma dZ + \sum_{i=1}^{N} k_i dq_i \,, \tag{1}$$

where σ is the instantaneous standard deviation, dZ is an increment of a standard Wiener process, and dq_i is a control counter for managerial activation of action i. This is a control (not a random) variable, not to be confused with the case jump-diffusion; the following notation is simply convenient due to the multiplicative nature of the control. Here r is the riskless rate of interest, while the parameter δ represents any form of a "dividend yield" (e.g., in McDonald and Siegel, 1984, δ is a deviation from the equilibrium required rate of return, while in Brennan, 1991, δ is a convenience yield). We assume that an intertemporal capital asset pricing model as in Merton (1973) holds. As in Merton (1976), we assume the control risk to be diversifiable (and hence not priced).

For each control i, we assume that the distribution of its size, $1 + k_i$, is log-normal, i.e., $\ln(1 + k_i) \sim \boldsymbol{N}(\gamma_i - .5\sigma_{C_i}^2,\ \sigma_{C_i}^2)$, with $\boldsymbol{N}(.,.)$ denoting the normal density function with mean $\gamma_i - .5\sigma_{C_i}^2$ and variance $\sigma_{C_i}^2$, and $E[k_i] \equiv \bar{k}_i = \exp(\gamma_i) - 1$. The control outcome is assumed independent of the Brownian motion -- although in a more general setting it can be dependent on time and/or the value of S. In general we can assume any plausible form, but the log-normality assumption will allow analytic tractability. Stochastic differential equation (1a) can alternatively be expressed in integral form as:

$$\ln\left[S\left(T\right)\right] - \ln\left[S\left(0\right)\right] = \int_0^T \left(r - \delta - 0.5\sigma^2\right)dt + \int_0^T \sigma dZ\left(t\right) + \sum_{i=1}^{N} dq_i \ln\left(1 + k_i\right). \tag{2}$$

Conditional on control activation

$$E[S^* \mid \text{activation of control } i] = E[S^*](1 + \bar{k}_i) = S(1 + \bar{k}_i)$$

and if the control is a pure learning (information acquisition) action ($\bar{k}_i = 0 = \gamma_i$)

$$E[S^* \mid \text{activation of control } i] = S\,.$$

For the case of learning actions, the fact that we model the percent change in the estimate guarantees that the resulting learning has no bias. The approach is consistent with Pindyck (1993), only we allow the final investment decision to be made without first eliminating all uncertainty (thus allowing an investment decision based on imperfect parameter estimates which is how real-life decisions are actually made). The method of learning is consistent with a Bayesian approach, since the (ex-ante) distribution of the learning actions´ outcome, can be estimated (depending on data availability) using the standard Bayesian techniques. The distribution of the outcome can be seen as the pre-posterior distribution in Bayesian learning (for such an analysis see Pratt, Raiffa and Schlaifer, 1995, and for the specific case of the lognormal distribution, see Kaufman, 1963). Recently, Davis and Samis (2006) use in a real options context the Martzoukos (2000) learning approach and demonstrate the Bayesian estimation of the parameters of the pre-posterior distribution following Kaufman (1963). Of course, instead of Bayesian methods, control parameters can be approximated with traditional econometrics from empirical R&D literature, as for example in Eberhart, Maxwell and Siddique (2004), Hall, and Oriani (2004), and Chan, Lakonishok, and Sougiannis (2001).

Each firm's management seeks to optimally activate controls that belong to an admissible set M, so that each firm´s claim F on the underlying asset must satisfy (subject to the actions of the other firm, and the exact definition of the claim which in this paper is a European call option on S with exercise price X and time to maturity T) the following optimization problem:

$$F^* = \max_{M}\left[F\left(S,T,M\right)\right] \tag{3}$$

subject to:

$$\frac{dS}{S} = \left(r-\delta\right)dt + \sigma dZ + \sum_{i=1}^{N} k_i dq_i$$

and

$\ln(1 + k_i)$ is normally distributed with mean: $\gamma_i - 0.5\,\sigma_{C_i}^2$, and variance: $\sigma_{C_i}^2$.

The next follows directly from the log-normality assumption of the multiplicative controls.

Property 1: *Assuming independence between the controls´ outcome and the increment dZ of the standard Wiener process, the conditional solution to the European call option (excluding the controls´ cost) is given by:*

$$F(S, X, T, \sigma, \delta, r; \gamma_i, \sigma^2_{C_i}) = e^{-rT} E[(S^*_T - X)^+ \mid \text{activation of } N \text{ controls}]. \tag{4}$$

The present value of the risk-neutral expectation E[.] conditional on activation of the controls at t = 0, is isomorphic to the Black-Scholes (1973) model:

$$E[(S^*_T - X)^+ \mid \text{activation of } N \text{ controls}] =$$

$$S\, e^{\left(rT - \delta T + \sum_{i=1}^{N} g_i(f_i \gamma_i)\right)} N(d_1) - X\, N(d_2) \tag{5}$$

where

$$d_1 \equiv \frac{\ln(S/X) + (r-\delta)T + \sum_{i=1}^{N} g_i(f_i\gamma_i) + .5\sigma^2 T + .5\sum_{i=1}^{N} g_i\left(f_i\sigma^2_{C_i}\right)}{\left(\sigma^2 T + \sum_{i=1}^{N} g_i\left(f_i\sigma^2_{C_i}\right)\right)^{1/2}}$$

and

$$d_2 \equiv d_1 - \left(\sigma^2 T + \sum_{i=1}^{N} g_i\left(f_i\sigma^2_{C_i}\right)\right)^{1/2}.$$

The $\boldsymbol{N}(d)$ denotes the cumulative standard normal density evaluated at d. The degree of spillovers (parameter) f may differ between firms. The functions $g(.)$ with arguments parameters of the controls´ distribution and the degree of spillovers will be shown more clearly in the next sections. The exact form depends on the type of control that is activated (impact or pure learning control). Controls´ outcome can be generated intentionally due to a firm´s own action, or unintentionally due to a spillover effect from the actions of the other firm. Finally the controls´ cost $\beta = \theta g(.)$ where θ is a cost parameter, must be subtracted from the firm´s *conditional* real option value. Given activation of learning controls, the (risk-neutral) probability $P(S_T > X)$ that the call option will be exercised at maturity T equals $\boldsymbol{N}(d_2)$. In pure learning actions intended just to resolve uncertainty about the true value of the unobservable variable S^* the impact parameters $\gamma_i = 0$. In the most general R&D case the impact parameters would differ from zero, and they can be positive if they affect revenues or negative if they affect fixed or variable costs. The spillover impact from the other firm can have either sign.

The Tactical Resource Allocation Decision.

The two firms must solve their optimization problem simultaneously, seeking thus an equilibrium in this (tactical) decision (see Figure 1). Let us denote with β_1 and β_2 the cost (effort) of the first and the second firm´s actions. The impact on option value is a function of that cost. Given the actions β_2 of the second firm, the impact on the net option value of firm one equals $F_1(\beta_1 \mid \beta_2) - \beta_1$, and given β_2 the first firm must maximize $F_1 - \beta_1$ through the first order condition

$$\frac{\partial\left(F_1\left(\beta_1 \mid \beta_2\right) - \beta_1\right)}{\partial \beta_1} = 0 . \tag{6}$$

Similarly, firm two conditional on the first firm´s action β_1 must maximize $F_2(\beta_2 \mid \beta_1) - \beta_2$ through the first order condition

$$\frac{\partial\left(F_2\left(\beta_2 \mid \beta_1\right) - \beta_2\right)}{\partial \beta_2} = 0 . \tag{6a}$$

The first order conditions are necessary for the existence of a maximum. Furthermore, if the second order conditions (that the second derivative is negative) are satisfied everywhere (or at least in some admissible range), this maximum is unique (in the admissible range).

The optimal cost effort functions $\beta_1^*(\beta_2)$ and $\beta_2^*(\beta_1)$ for each firm are depended on the other firm's actual effort. In this duopolistic game, both firms optimize their actions simultaneously and the equilibrium solution pair β_1^{**} and β_2^{**} is shown at the intersection of $\beta_1^*(\beta_2)$ and $\beta_2^*(\beta_1)$. Since the cost efforts β_1 and β_2 affect F_1 and F_2 through the *impact* (γ_i) and *learning* (σ_{C_i}) parameters and some cost parameter θ, we must define the mappings $\beta_1(\theta_1, \gamma_1, \sigma^2_{C_1})$ and $\beta_2(\theta_2, \gamma_2, \sigma^2_{C_2})$. As will be seen in the examples presented later, it is more intuitive to optimize directly with respect to the learning (or the impact) parameter. In order to solve numerically these two equations in the most general case, the 2x2 Jacobian matrix of the 2nd order analytic derivatives is needed and an iterative two-dimensional Newton-Raphson scheme is implemented. The following proposition holds for cost functions that are homothetic of degree one to the cost parameter θ.

Proposition 1: *The equilibrium efforts of the tactical decision are invariant to identically proportional changes in the price of the underlying asset S, the exercise price X, and the cost β (or equivalently a cost parameter θ) of the control. The constant of proportionality may differ between the two firms.*

Proof: We can verify through equations (6) – (6a) that the property of option prices to be homogenous of degree one in the underlying asset and the exercise price, can also be preserved in this game theoretic context, due to the multiplicative nature of the random control. With the proper choice of the cost function, the conditional option prices (for each firm) can be homogeneous of degree one in the underlying asset, the exercise price, and the control's cost θ, as clearly seen in equations (10a, b) and (12a, b).

[Enter Figures 1 and 2 about here]

The Strategic Coordination Decision.

We now must consider the strategic coordination choice. Firms must decide on the optimal degree of coordination of their R&D efforts. For example, in a 2x2 game, each firm can decide to exert high (*H*) or low (*L*) coordination effort (see Figure 2). The degree of coordination determines the extent of spillover effects (through the parameter *f*), and potentially the cost of R&D (through the cost parameter θ). We assume for ease of exposition two strategies available for each firm, but more than two (or even a continuous set of alternatives) could exist. The choice sets (*H*, *H*), (*H*, *L*), etc. uniquely determine the degree of spillover effects. The optimal choice for the two firms is provided by the pure *Nash* equilibrium(a), or alternatively the mixed strategies equilibrium. Equilibria off the diagonal can occur because of the asymmetry in the direct spillover effects and the cost reduction results of coordination. This is justifiable when the two firms operate in different product markets, and can be for example technology dependent. The solution to the tactical decision in Figure 1 is nested to the solution of the strategic one in Figure 2. The next follows directly from Proposition 1.

Corollary 1: *If the constant of proportionality (as discussed in proposition* 1*) is the same for both firms, then the Nash equilibrium strategy is invariant to the choice of this constant.*

3. The (one-period) real options game with costly information acquisition.

Let us consider the case of pure learning actions. Two companies face an investment opportunity each. Before they decide to invest they also have the option to invest in order to acquire information about the true value or at least a better estimate of the investment. Thus, both impact parameters equal zero since they do not pursue to directly enhance value but they do so indirectly by reducing the error of the estimate. We consider i (= 1, 2) the two pure learning actions ($\gamma_i = 0$) coming from firm one and firm two respectively: These actions may affect either firm so we denote the destination of the action as firm j (= 1, 2). The parameters $f_{i \to j}$ define the degree of spillovers. For the influence of own actions, most often we assume $f_{1 \to 1} = f_{2 \to 2} = 1$. To find equilibrium, given the action of the other firm, each firm j must maximize the conditional option value given below as an application of Property 1:

$$F_j - \beta_j = S_j e^{-\delta_j T} N(d_1) - X_j e^{-rT} N(d_2) - \beta_j \tag{7}$$

where

$$d_1 \equiv \frac{\ln(S_j / X_j) + (r - \delta_j)T + 0.5\sigma_j^2 T + 0.5\sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2}{\left(\sigma_j^2 T + \sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2\right)^{1/2}}$$

and

$$d_2 \equiv d_1 - \left(\sigma_j^2 T + \sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2\right)^{1/2}.$$

The summation captures effects from firms´ own action and the action of the other firm. The information revelation potential is bounded from above (as well as positive)

$$\sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2 \leq \sigma_{Cmax_j}^2$$

and affects (positively) option prices. After the action, the amount

$$\sigma_{R_j}^2 = \sigma_{Cmax_j}^2 - \sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2$$

defines exactly the unresolved (residual) uncertainty $\sigma_{R_j}^2$ (the orthogonality condition implying that the *experiment* produces information and no noise). The distribution of the learning control´s outcome captures the information revelation potential, and its parameters may be estimated via Bayesian methods. We can use the pre-posterior distribution as the one that captures the information revelation ex-ante (before actually observing the outcome of the learning action). As discussed in Davis and Samis (2006) (drawing on Kaufman, 1963) the prior distribution, the posterior, and the pre-posterior (that we will effectively use in real option pricing) are log-normal. The residual uncertainty does not affect the pricing of the option, only the uncertainty of the information revelation (captured by the pre-posterior) does. Note that the firm will make the final investment decision using its best estimate of the underling value. When the firm acts knowingly that this value is estimated with error, the firm makes a costly effort to reduce or eliminate this error. We assume that this firm has a well diversified project portfolio and thus any residual uncertainty about the true asset value is not priced and does not affect the investment option value. In addition, the information revelation uncertainty has no correlation with the market uncertainty and thus is also not priced, but is of course included in the option valuation model and affects its value. We must also observe that beyond the information revelation we also have the exogenous uncertainty σ_j^2 since we have made the assumption that the estimate of the underlying value follows a geometric Brownian motion before and after learning takes place. Only at the time of learning the estimate will change in a multiplicative fashion. We have thus assumed that there are several factors affecting the underlying value, like quantity, price, etc., so the geometric Brownian motion is due to one (say, price) and the learning due to error in the estimate of another (say, quantity).

Uncertainty is reduced or resolved at a cost, and the cost function is assumed for simplicity quadratic in the learning effort

$$\beta_j = \theta_j f_{j\to j} \sigma_{C_j}^2 . \tag{8}$$

Note that the cost parameter θ and the spillover parameters f are conditional on the strategic decision. In this application we assume that the optimal learning efforts are below the upper bound for the maximum feasible learning (non-binding constraint) without loss of generality. It is natural to assume that complete elimination of uncertainty would almost be infeasible, or, after some point, exceedingly costly. Else, if the constraint were binding, we would simply incorporate it explicitly in the numerical solution.

The two first order conditions

$$\frac{\partial\left(F_1 - \theta_1 f_{1\to 1} \sigma_{C_1}^2\right)}{\partial \sigma_{C_1}} = 0 , \tag{9a}$$

and

$$\frac{\partial\left(F_2-\theta_2 f_{2\to 2}\sigma_{C_2}^2\right)}{\partial\sigma_{C_2}}=0 \qquad (9b)$$

are conditional on the other firm's move, and must be solved simultaneously. Specifically we get,

$$\frac{\partial\left(F_1-\theta_1 f_{1\to 1}\sigma_{C_j}^2\right)}{\partial\sigma_{C_1}}=$$
$$X_1 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{1\to 1}\sigma_{C_1}}{\left(\sigma_1^2 T+f_{2\to 1}\sigma_{C_2}^2+f_{1\to 1}\sigma_{C_1}^2\right)^{0.5}}-2\theta_1 f_{1\to 1}\sigma_{C_1}=0 \qquad (10a)$$

and

$$\frac{\partial\left(F_2-\theta_2 f_{2\to 2}\sigma_{C_2}^2\right)}{\partial\sigma_{C_2}}=$$
$$X_2 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{2\to 2}\sigma_{C_2}}{\left(\sigma_2^2 T+f_{2\to 2}\sigma_{C_2}^2+f_{1\to 2}\sigma_{C_1}^2\right)^{0.5}}-2\theta_2 f_{2\to 2}\sigma_{C_2}=0 \qquad (10b)$$

After solving for the optimal learning efforts $\sigma_{C_1}^{2**}$ and $\sigma_{C_2}^{2**}$, we find the optimal cost efforts β_1^{**} and β_2^{**}. The second order conditions so that the solution is a maximum are

$$\frac{\partial^2\left(F_j-\theta_j f_{j\to j}\sigma_{C_j}^2\right)}{\partial\left(\sigma_{C_j}\right)^2}<0$$

The general form of the response functions is as in Figure 3a with the unique solution at *E*. Figure 3b presents the case with the unique equilibrium *E* on the horizontal axis when firm 2 gets a free lunch by exerting zero effort and benefiting from the spillovers from the actions of firm 1 (symmetrically when firm 1 gets the free lunch). We observe that the first order conditions are always satisfied at $\sigma_{C_j}^2=0$, so if also the second order conditions are satisfied, this is a solution. Notice that by the model construction this actually is an interior and not a corner solution.

These figures indicate the existence of a solution. The uniqueness of this solution depends on the slope and the curvature of the response functions. Figure 3c for example, presents a case with three equilibria (E_1, E_2 and E_3), in which case equilibrium E_2 is not a stable one (in the numerical examples presented in this paper, such a case has not been observed). In the stable equilibrium point E in Figure 3a, the slope of the second player´s response function is lower than that of the first player, unlike E_2 in Figure 3c where the opposite holds. We have the following stability condition (see also Seade, 1980) for each equilibrium point

$$\left|\frac{\partial \sigma_{C_2}^*}{\partial \sigma_{C_1}}\right| < \frac{1}{\left|\frac{\partial \sigma_{C_1}^*}{\partial \sigma_{C_2}}\right|}$$

In our numerical examples all solutions satisfy the SOC and the stability conditions. Still, we cannot exclude the possibility of the existence of an infinite number of equilibria when the response functions coincide (such a case is observed and identified with asterisk in Table 1).

[Enter Figures 3a, 3b, 3c about here]

In Table 1 we see the optimal decisions for a wide variety of asymmetric spillover effects and asymmetric costs. The degree of influence of these parameters on optimal effort is very significant. We see that the higher the R&D cost for a firm and the highest the spillover benefits the firm receives from the other, the more likely it is to reduce R&D spending, and at the extreme we encounter a *free-lunch* (zero effort, *delegation* to the other player), as the preferred choice.

[Enter Table 1 and Figure 4 about here]

As a result of Proposition 1, in the example discussed we can multiply the price of the underlying asset S_1, the exercise price X_1, and the cost θ_1 with a positive constant, then multiply S_2, X_2, and θ_2 with another positive constant, and the results regarding the equilibrium effort will not change. Optimal option values will of course change by the relevant constant. For example, the middle panel in Table 3 (for $\theta_2 = 75.00$) would also provide the optimal efforts for the case where $S_1 = X_1 = 100.00$, $S_2 = X_2 = 100.00$, and $\theta_2 = 100.00$.

Figure 4 presents the results for the strategic decision where the two had to choose the optimal degree of R&D coordination. In the top panel we see that the Nash equilibrium is when both decide on the maximum degree of coordination. The first one however, decides not to spend on learning at all (see the results in parenthesis), whereas the second exhibits a very high effort. The bottom panel presents a case where the second one concedes to a high degree and the first to a low degree of coordination. As a result of Corollary 1, if for the two firms the constants of multiplication are the same, then the equilibrium strategy for the games will be unaffected, since all payoffs will be multiplied by the same positive constant. Note that under no Nash equilibrium, mixed strategies could be considered as an alternative approach, providing the probabilities that each firm would play a High or a Low strategy.

4. The (one-period) real options game with costly impact controls.

In the previous section we focused on the pure learning (information acquisition) case. If the impact parameters are not zero (a direct effort to enhance value), this case of random control would be similarly solved through the use of the first order conditions (6) and (6a). Again we define the conditional option value by applying Property 1

$$F_j - \beta_j = S_j e^{\left(-\delta_j T + \sum_{i=1}^{2} f_{i\to j}\gamma_i\right)} N\left(d_1\right) - X_j e^{-rT} N\left(d_2\right) - \beta_j \tag{11}$$

where

$$d_1 \equiv \frac{\ln\left(S_j / X_j\right) + \left(r - \delta_j\right)T + \sum_{i=1}^{2} f_{i\to j}\gamma_i + 0.5\sigma_j^2 T + 0.5\sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2}{\left(\sigma_j^2 T + \sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2\right)^{1/2}},$$

$$d_2 \equiv d_1 - \left(\sigma_j^2 T + \sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2\right)^{1/2},$$

and a cost function quadratic in the impact effort $\beta_j = \theta_j\left(f_{j\to j}\gamma_j\right)^2$.

For the tactical decision we consider the two first order conditions

$$\begin{aligned}\frac{\partial\left(F_1 - \theta_1\left(f_{1\to1}\gamma_1\right)^2\right)}{\partial\gamma_1} &= S_1 e^{\left(-\delta_1 T + \sum_{i=1}^{2} f_{i\to1}\gamma_i\right)} f_{1\to1} N\left(d_1\right) \\ &+ X_1 e^{-rT}\frac{e^{-d_2^2/2}}{2\sqrt{2\pi}}\frac{s_{1\to1} f_{1\to1}\sigma_{C_1}^2}{\left(\sigma_1^2 T + s_{1\to1} f_{1\to1}\gamma_1\sigma_{C_1}^2 + s_{2\to1} f_{2\to1}\gamma_2\sigma_{C_2}^2\right)^{0.5}} - 2\theta_1 f_{1\to1}^2\gamma_1 = 0\end{aligned} \tag{12a}$$

and

$$\frac{\partial\left(F_2-\theta_2\left(f_{2\to 2}\gamma_2\right)^2\right)}{\partial\gamma_2}=S_2 e^{\left(-\delta_2 T+\sum_{i=1}^{2} f_{i\to 2}\gamma_i\right)} f_{2\to 2} N\left(d_1\right)$$

$$+X_2 e^{-rT}\frac{e^{-d_2^2/2}}{2\sqrt{2\pi}}\frac{s_{2\to 2} f_{2\to 2}\sigma_{C_2}^2}{\left(\sigma_2^2 T+s_{1\to 2} f_{1\to 2}\gamma_1\sigma_{C_1}^2+s_{2\to 2} f_{2\to 2}\gamma_2\sigma_{C_2}^2\right)^{0.5}}-2\theta_2 f_{2\to 2}^2\gamma_2=0 \qquad \text{(12b)}$$

where the constants $s_{i\to j}$ are introduced to simply guarantee the positivity of the variance components. Subsequently and similarly with the pure learning case we solve the two equations simultaneously and we get the optimal impact efforts γ_1^{**} and γ_2^{**} and through them the optimal cost efforts β_1^{**} and β_2^{**}. Since each player´s intension is to enhance value, the admissible impact effort is non-negative (for a put option it would be non-positive). It is also bounded from above by the minimum of γ_{max_j} (which defines an economically or technically feasible range) and the point where the second derivative becomes positive (which guarantees uniqueness).

In the impact control case, we expect most often both players to exert a positive effort (and avoid delegation or free lunch) with conditional solutions (γ_1^*| γ_2) and (γ_2^*| γ_1) positive (see also Figure 3d). Using the cost function $\beta_j=\theta_j(f_{j\to j}\gamma_j)^2$, the second order conditions are

$$\frac{\partial^2\left(F_j-\theta_j\left(f_{j\to j}\gamma_j\right)^2\right)}{\partial\left(\gamma_j\right)^2}<0$$

and the stability condition is

$$\left|\frac{\partial\gamma_2^*}{\partial\gamma_1}\right|<\frac{1}{\left|\frac{\partial\gamma_1^*}{\partial\gamma_2}\right|}.$$

In the discussion that follows, all solutions again satisfy the SOC and the stability conditions.

An example for the cost function $\beta_j=\theta_j(f_{j\to j}\gamma_j)^2$, with the range of admissible impact parameters positive and bounded below 100% is given in Figure 5 that presents two examples where the Nash equilibrium for the strategic decision is the highest (*H*/*H*) and the lowest (*L*/*L*) degree of coordination respectively.

[Enter figures 3d and 5 about here]

In the top panel the impact spillover is positive (like in the case of generic advertisement) whereas in the bottom panel the impact spillover can be negative (when advertisement is more competitive).

5. The two-period stochastic game with path-dependency inducing switching costs for strategy revision.

The single-period setting discussed for the two-stage game can be extended to a multi-period setting with switching costs that induce path-dependency. For simplicity and due to the high computational complexity arising because of path-dependency we will discuss two-period games. We assume that firms have successive investment opportunities, and can invest in learning/control before undertaking them. In the simplest setting investments would be independent among consecutive periods. One investment decision in the first period would not affect the investment decision in period two, etc. Furthermore, we assume no spillover in learning/control from period to period. A framework of complete independence among consecutive periods would allow such problems of sequential decisions to be solved through simple backwards induction. Here we want to solve more interesting problems where there is dependency from period to period. We assume the presence of switching costs among repeated strategy/technology choices for pre-investment actions in sequential investments. These costs can be explicit penalties due to changing strategy like in the case of *breach of contract* by a firm that participates in a research joint venture; or implicit costs (or benefits) in a *learning-by-doing* situation where sequential research/control actions benefit the player that does not switch strategy. The sequential strategy decisions are affected through switching penalties, or through differing parameter values for the cost functions and the spillovers. The resulting game is path-dependent (see Figure 6 for the two-period case).

[Enter Figures 6, 7 and 8, and Table 2 about here]

We implement a numerical (lattice-based) framework to represent uncertainty (the evolution of the states of *nature*) in the standard textbook fashion (see Hull, 2006). Then, in order to solve the two-stage game we use a backwards-forwards looking solution algorithm that captures path-dependency as described in Trigeorgis (1996). The computational burden now expands considerably due to the path-dependency inducing switching costs, and the solution method is effectively one of discrete optimization with exhaustive search. To make numerical solutions feasible (due to the considerable computational burden) we avoid many periods and for simplicity we solve problems in two-periods. The lattice captures the different states of nature in consecutive periods. In each period, the problem is solved using the analytic approach described in the earlier sections. In the second period the game is solved conditional on the previous actions that define the new set of parameter values. The solution values are discounted and added to each player´s value so that in period one the total value for the solution to the strategic coordination game is the discounted expectation from the optimal next period solution plus the value from the optimal solution in period one. In each period the tactical resource allocation game is solved using the analytic framework discussed in earlier sections (with parameter values as defined by previous actions).

As expected, the presence of switching costs makes it harder to deviate from previous strategies. These results are consistent with the insights in Lipman and Wang (2000) who demonstrate in the Prisoners´ Dilemma that switching costs make complex threats credible, and Beggs and Klemperer (1992) who studied the effect of switching costs for consumers on competition between firms. We focus on the effects of Learning-by-doing. There is ample empirical evidence that firms do achieve cost reduction and improved quality through accumulated experience in research and production (see for example Irwin, D. A., and P. J. Klenow, 1994, for a discussion of learning-by-doing in the semiconductor industry). We now present numerical results to demonstrate the impact of switching costs. First we see that in the absence of learning-by-doing effects (in Figure 7), the optimal strategy for the two players is *H*/*H*. It so happens that if we look at the first period independently, the optimal solution is again *H*/*H*. We then allow a cost reduction in research for the first player in case he/she retains a *L* strategy, and we see that beyond a certain level of cost reduction, the optimal strategy in the first period shifts to a *L*/*H* strategy. Due to path-dependency, the local optimum would not be part of the global optimum. The critical threshold where strategy shift occurs is at a cost reduction equal to 25.38% (see Figure 8). Comparing the two figures we see that the pay-offs differ, but the optimal effort in the first period is the same in both cases, without and with learning-by-doing. This is because of our assumption that the impact of research affects one period at a time only and the only dependence between periods is in the cost of research. Relaxing this assumption and allowing for multi-period spillovers would not permit the analytic-type solution of the single-period 2-stage game, but it can be overcome with a discretization of the amount of effort exerted. Finally Table 2 demonstrates the critical cost reduction threshold for different degree of asymmetry, volatility, riskless rate and dividend yield. Using plausible parameter values we observe that, as expected, the impact of reducing the dividend yield is opposite to that of reducing the riskless rate. In addition, in a more volatile environment (in terms of the exogenous volatility) it is easier to observe a strategy shift.

6. Conclusions

This paper presents and solves a real options game that jointly addresses at the pre-investment stage the strategic decision about the extent of coordination between two firms, and the decision about the optimal effort invested in R&D in the presence of uncertainty and spillover effects. We assume that the two firms can influence each other´s decision at the pre-investment stage, whereas at the investment decision each firm has monopoly power over its investment and there is no further interaction between the two. Firms want to enhance value and to resolve (or reduce) uncertainty of real (investment) opportunities, before they make a commitment. Managerial actions are treated as optional, costly impulse-type multiplicative controls with random size whose realization is a random variable with a known probability distribution. We used a contingent claims framework with noisy assets and costly control actions, and without loss of generality or any sacrifice in insights gained we made the assumption that the two firms face investment opportunities of the European type, allowing thus the use of analytic models isomorphic to Black and Scholes (1973). Alternatively, fully numerical methods like lattice or numerical solutions to partial differential equations could have been used, but the iterative solution to this continuous game would have been much more intensive computationally and less accurate. Within such a numerical solution framework it would be easy to incorporate further firm interactions in the product markets. Note that the model discussed above pertains to call options where players try to enhance the value of the underlying asset. Symmetrically we could have worked for a put option where the players could pursue a cost-reduction strategy.

Subsequently, the solution to the firms' optimal strategic and tactical R&D decision-making is found as the solution of a two-stage game. In contrast with other real option game-theoretic literature, our effort is not on finding the optimal timing decision of a single action, but on deciding the best action strategy, and the amount of effort. This decision, as expected, is heavily dependent on the effectiveness of R&D investments, their cost, and the degree of coordination that is optimal for the two firms. Some times *high* coordination and other times *low* coordination will be optimal. The degree of coordination affects both the degree of spillovers, and the parameters of the cost function. In the cases of pure learning actions, there are instances where a firm will *delegate* research by agreeing on a high degree of coordination (lowering thus the R&D cost of the other firm and increasing the degree of spillovers) and reap afterwards the rewards. In general we have shown in a single-period game that optimal coordination and optimal R&D efforts are essential for value enhancement and optimal investment decision-making.

The single-period approach can be extended to multiple-period (finitely repeated) stochastic games with switching costs for strategy revision. Switching costs can arise due to penalties for violating contracts in research joint ventures, or due to differing parameter values in successive games with learning-by-doing. As expected, switching costs make strategy revisions harder. A strategy shift is easier to observe in a more volatile environment.

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Table 1
Optimal R&D Learning Effort (firm 1 / firm 2)

Cost θ_2	Spillovers $2 \rightarrow 1$	Spillovers $1 \rightarrow 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.14938 0.24898	0.14938 0.23751	0.14938 0.22546	0.14938 0.21273	0.14938 0.19919
	0.25	0.08256 0.24898	0.08527 0.24530	0.08826 0.24103	0.09160 0.23600	0.09534 0.23000
	0.50	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
	0.75	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
	1.00	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
75.00	0.00	0.14938 0.14938	0.14938 0.12937	0.14938 0.10563	0.14938 0.07469	0.14938 0.00000
	0.25	0.12937 0.14938	0.13361 0.13361	0.13830 0.11292	0.14352 0.08286	0.14938 0.00000
	0.50	0.10563 0.14938	0.11292 0.13830	0.12197 0.12197	0.13361 0.09448	0.14938 0.00000
	0.75	0.07469 0.14938	0.08286 0.14352	0.09448 0.13361	0.11292 0.11292	0.14938 0.00000
	1.00	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	*0.10563 0.10563
100.00	0.00	0.14938 0.09078	0.14938 0.05160	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.25	0.14232 0.09078	0.14698 0.05329	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.50	0.13488 0.09078	0.14420 0.05516	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.75	0.12702 0.09078	0.14091 0.05724	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	1.00	0.11863 0.09078	0.13698 0.05958	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000

Notes. The numbers give the optimal learning efforts for each firm in the solution to the tactical game (first number for firm 1 and second for firm 2). Investment options' parameters are: underlying assets $S_1 = S_2 = 100.00$, exercise prices $X_1 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$, and costs of learning (per unit of variance) $\theta_1 = 100.00$ (with $\theta_2 = 50.00$, 75.00, or 100.00) with spillover of learning 0.00, 0.25, 0.50, 0.75 and 1.00 (for $1 \rightarrow 2$ and for $2 \rightarrow 1$).

* Due to the relative symmetry of the assumptions, the response functions coincide to provide an infinite number of equilibria, and this point is approximated at the limit from $f_{1\rightarrow 2} = f_{2\rightarrow 1} \rightarrow 1^-$ or $f_{1\rightarrow 2} = f_{2\rightarrow 1} \rightarrow 1^+$.

Table 2
Critical Strategy Switching Cost Reduction Level in a 2-period Stochastic Game

	$S_2 = X_2 = 75$	$S_2 = X_2 = 100$	$S_2 = X_2 = 125$
Base case	0.1959 (0.1950)	0.2538 (0.2438)	0.3014 (0.2905)
$\sigma = 0.20$	0.1379 (0.1430)	0.1829 (0.1896)	0.2276 (0.2353)
$r = 0.05$	0.2062 (0.2083)	0.2911 (0.2844)	0.3859 (0.3605)
$\delta = 0.05$	0.1564 (0.1568)	0.1981 (0.1970)	0.2356 (0.2324)

Notes. Investment options' parameters are: underlying assets $S_1 = S_2 = 100.00$, exercise prices $X_1 = X_2 = 100.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$. The impact induced volatility for both firms equals $\sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i\to j}\gamma_i | 0.05^2$. Admissible impact range is bounded below 100%. In the 2-period stochastic game the 2 periods are one year apart. We implement a lattice discretization of one step per year which is easy for other researchers to replicate (and the more accurate 25 steps per year in parenthesis). Calculations provide the critical reduction (in the cost parameter) for the 1st player if strategy L is played repeatedly which induces a switch in the equilibrium strategy from H/H to L/H.

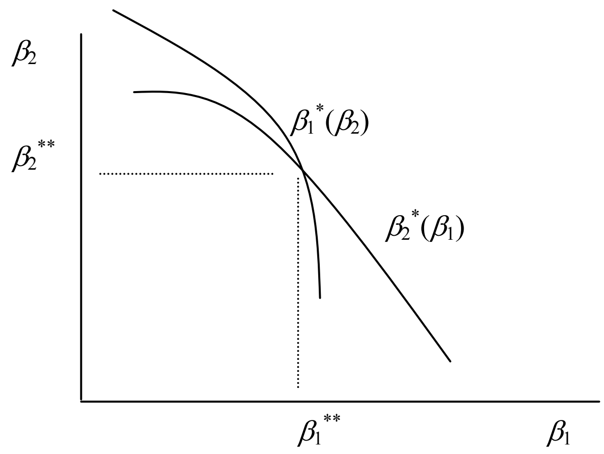

Figure 1
Game with Spillovers: Tactical Resource Allocation Decision

Notes: β_1, β_2 are the learning costs incurred by the two firms, $F_1(\beta_1 \mid \beta_2)$, $F_2(\beta_2 \mid \beta_1)$ are the investment option values before learning costs are subtracted, and $\beta_1^*(\beta_2)$, $\beta_2^*(\beta_1)$ are the optimal cost efforts (of each firm *conditional* on the effort of the other). The equilibrium solution pair (β_1^{**}, β_2^{**}) is given by the intersection of the two optimal *conditional* cost effort curves.

		FIRM 1	
		H	*L*
FIRM 2	*H*	$F_1(H, H)$, $F_2(H, H)$	$F_1(L, H)$, $F_2(L, H)$
	L	$F_1(H, L)$, $F_2(H, L)$	$F_1(L, L)$, $F_2(L, L)$

Figure 2
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. F_1 and F_2 are the investment option values (before the cost of investments in coordinated R&D are subtracted). $F_1(H, L) = F_1[\beta_1^{**}(H, L), \beta_2^{**}(H, L)]$, $F_2(H, L) = F_2[\beta_1^{**}(H, L), \beta_2^{**}(H, L)]$.

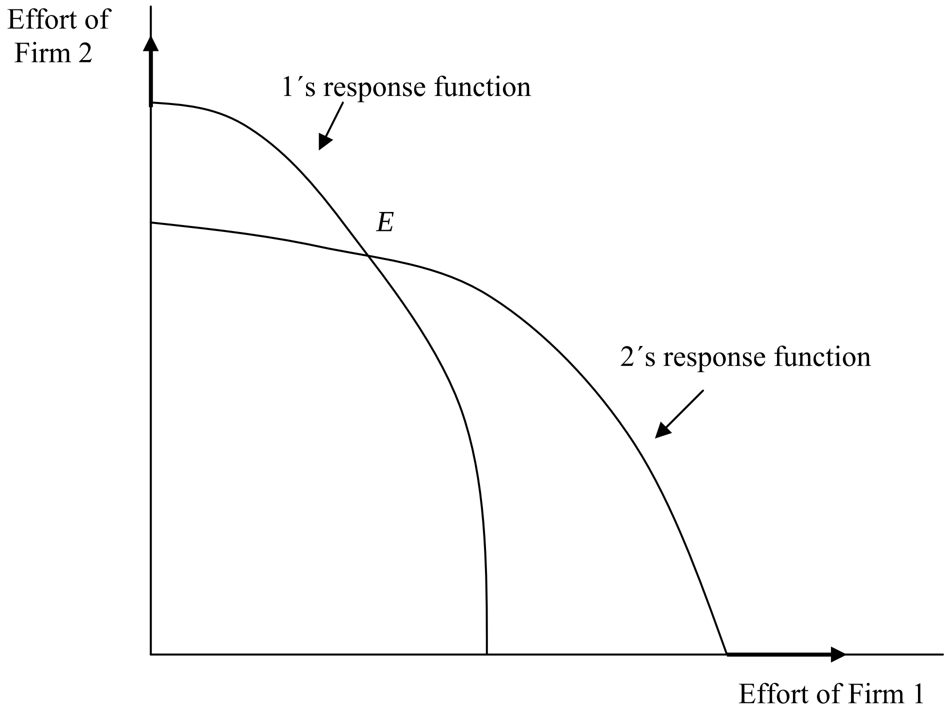

Figure 3a
Unique Optimal (Learning) R&D Solution – the general case

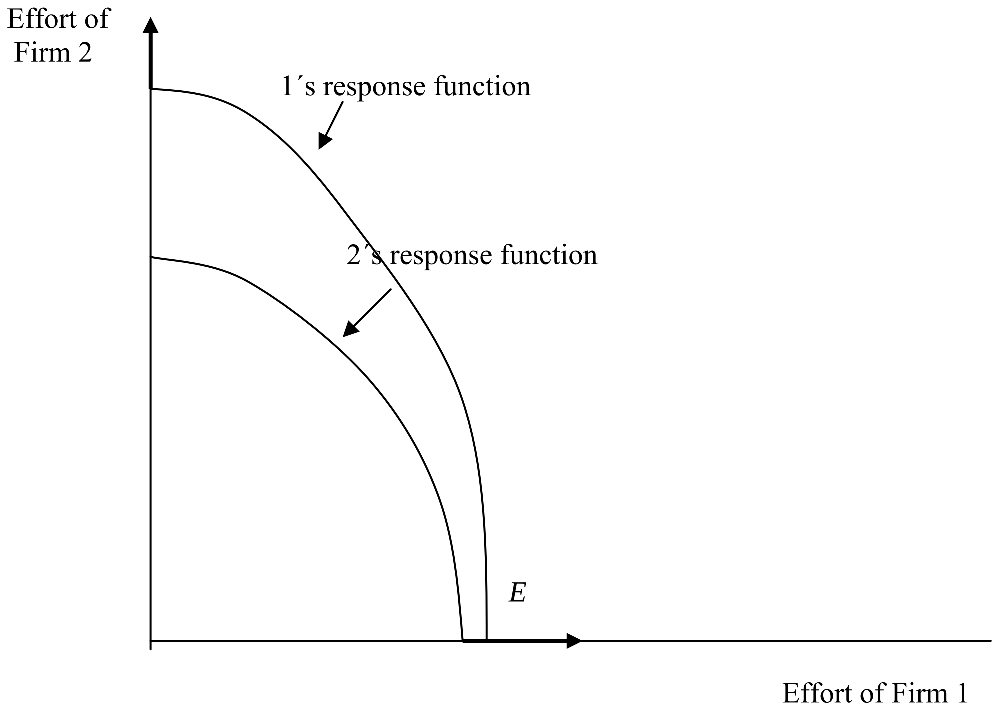

Figure 3b
Unique Optimal (Learning) R&D Solution – free lunch for firm 2

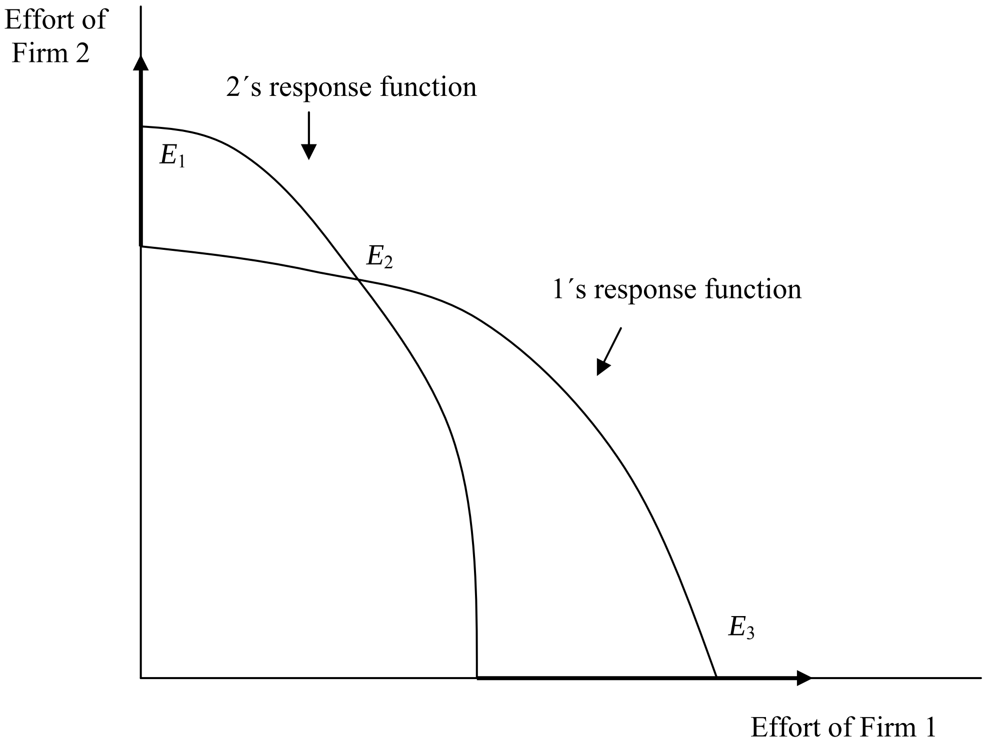

Figure 3c
Multiple Optimal (Learning) R&D Solutions

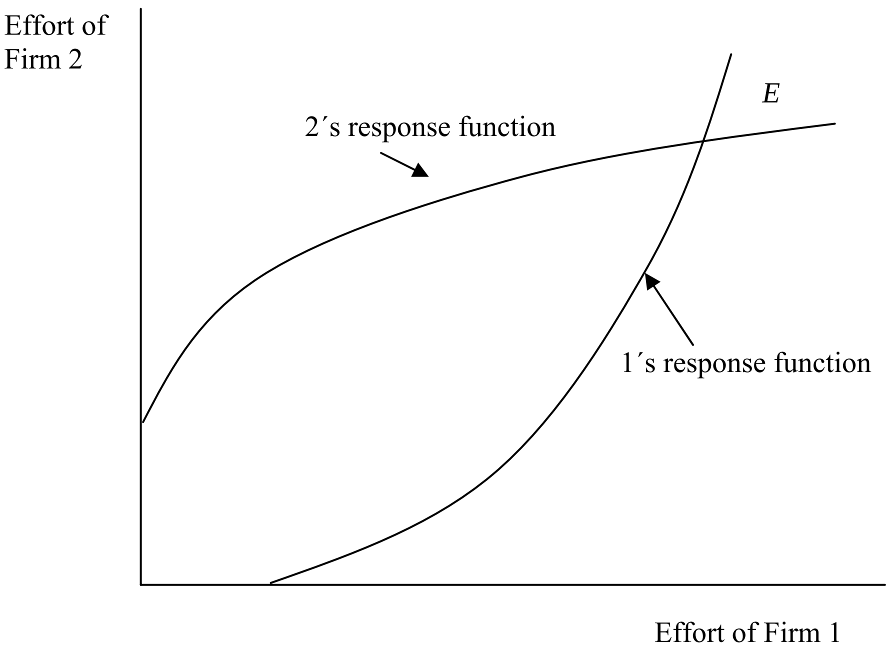

Figure 3d
Unique Optimal (Impact) R&D Solution – the most general case (it admits no free lunch)

		FIRM 1	
		H	*L*
FIRM 2	*H*	**16.3155, 7.4114** (0.0000, 0.5136) 80.00, 25.00 0.75, 0.75	5.7594, 3.7296 (0.1702, 0.1228) 80.00, 75.00 0.75, 0.25
	L	9.9166, 7.4114 (0.0000, 0.5136) 100.00, 25.00 0.25, 0.75	4.5915, 3.4628 (0.1374, 0.1171) 100.00, 80.00 0.25, 0.25

		FIRM 1	
		H	*L*
FIRM 2	*H*	4.5076, 4.8640 (0.1727, 0.0000) 90.00, 75.00 0.75, 0.75	**5.1513, 3.4829** (0.1506, 0.0977) 90.00, 85.00 0.75, 0.25
	L	4.4205, 4.3453 (0.1435, 0.0829) 100.00, 75.00 0.25, 0.75	4.5055, 3.4209 (0.1405, 0.1013) 100.00, 85.00 0.25, 0.25

Figure 4

Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' optimal learning effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, and for both options $T = 1.00$.

		FIRM 1 *H*	*FIRM* 1 *L*
FIRM 2	*H*	**21.3807, 16.4136** (0.2566, 0.1856) 250.00, 250.00 0.50, 0.50	14.2699, 7.2619 (0.2366, 0.0702) 250.00, 500.00 0.50, 0.25
	L	9.1118, 9.3363 (0.0955, 0.1652) 500.00, 250.00 0.25, 0.50	6.7082, 4.8011 (0.0854, 0.0592) 500.00, 500.00 0.25, 0.25

		FIRM 1 *H*	*FIRM* 1 *L*
FIRM 2	*H*	7.1494, 4.5636 (0.1153, 0.0578) 400.00, 500.00 0.15, 0.15	10.6827, 1.4637 (0.2251, 0.0271) 250.00, 500.00 0.15, -0.25
	L	3.0131, 6.1968 (0.0542, 0.1508) 500.00, 250.00 -0.25, 0.15	**9.1651, 1.6091** (0.2192, 0.0384) 250.00, 400.00 -0.25, -0.25

Figure 5

Game with Impact Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' optimal effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, and for both options $T = 1.00$. The impact induced volatility for both firms equals $\sum_{i=1}^{2} | f_{i\rightarrow j}\gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i\rightarrow j}\gamma_i | 0.05^2$. Admissible impact range is bounded below 100%.

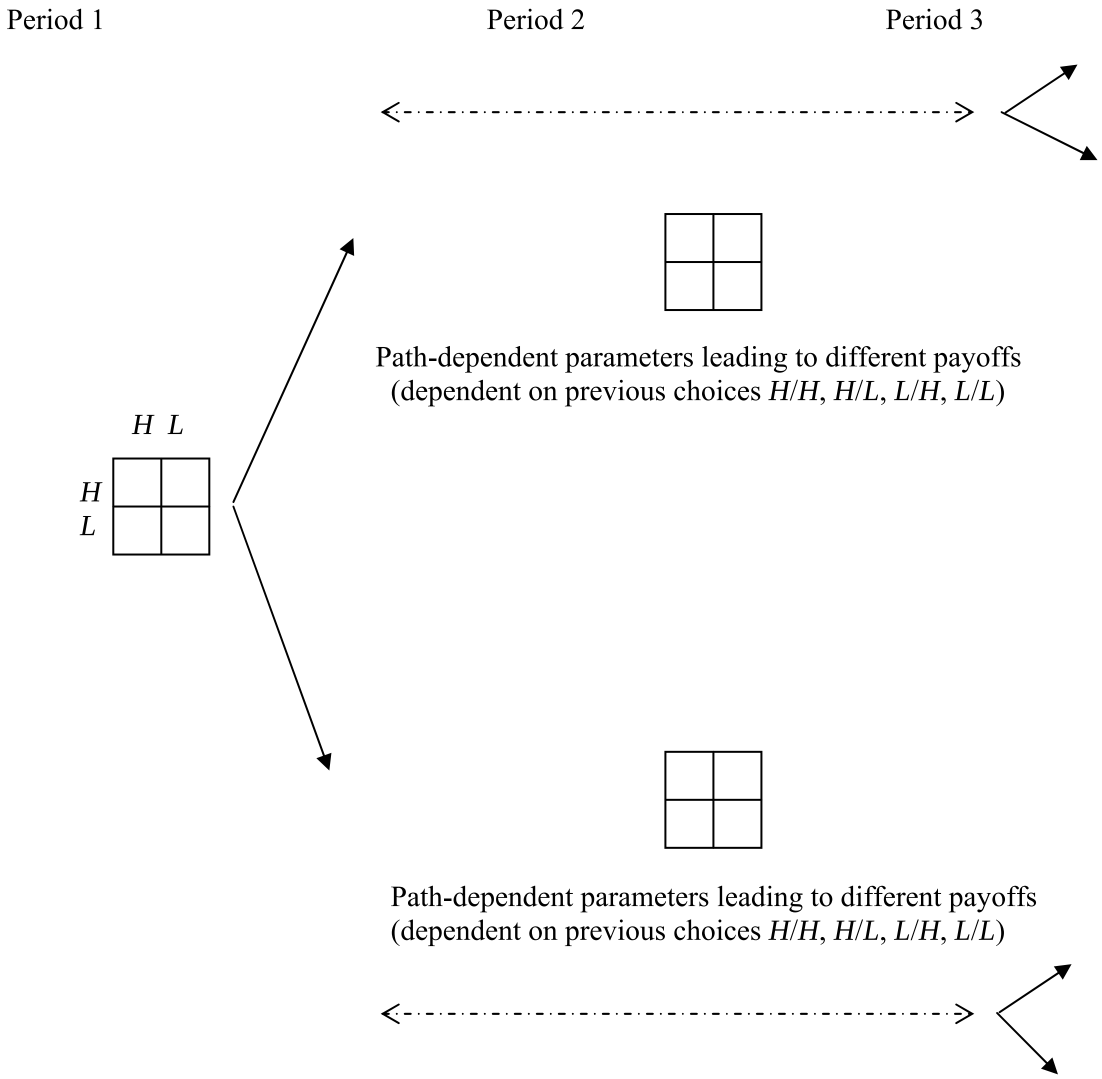

Figure 6
Multiperiod Stochastic Game with Path-Dependency

Note: We see a multiperiod stochastic game with (for simplicity of exposition) two states of nature (up and down as demonstrated by solid-line arrows). The 2nd-period parameter values depend on the 1st period moves (*H*/*H*, *H*/*L*, *L*/*H* or *L*/*L*) and the payoffs depend also on the results from all subsequent periods.

Path-dependency is induced from explicit or implicit switching costs. Explicit costs arise for example in case of breach of contracts (i.e. research joint ventures). Implicit costs can be due to differing spillovers or costs of research in case of a strategy switch (i.e., in a learning-by-doing case, sequential research will benefit the player that does not switch strategy, etc.).

FIRM 1

		H	*L*
FIRM 2	*H*	**16.8407, 16.8407** (0.0738, 0.0738) 600.00, 600.00 0.50, 0.50	16.2424, 15.0304 (0.0715, 0.0600) 600.00, 650.00 0.50, 0.25
	L	15.0304, 16.2424 (0.0600, 0.0715) 650.00, 600.00 0.25, 0.50	14.7864, 14.7864 (0.0589, 0.0589) 650.00, 650.00 0.25, 0.25

Figure 7
A 2-period Stochastic Game without Learning-by-doing

Note: The parameters for the two investment options are S_1 = 100, X_1 = 100, S_2 = 75 and X_2 = 75. Asset values S_1 and S_2 are perfectly correlated with a stochastic state-variable with σ = 0.10, and δ = 0.10. The riskless rate r = 0.10 and time for each investment option maturity T = 1. Figures in parenthesis provide the optimal effort for the first period only. Optimal effort for the second period is dependent on the level of the state-variable, etc. There is no learning-by-doing effect here.

FIRM 1

		H	*L*
FIRM 2	*H*	16.8407, 16.8407 (0.0738, 0.0738) 600.00, 600.00 0.50, 0.50	**16.8411, 14.1196** (0.0715, 0.0600) *600.00, 650.00 0.50, 0.25
	L	15.0304, 16.2424 (0.0600, 0.0715) 650.00, 600.00 0.25, 0.50	15.3851, 13.8756 (0.0589, 0.0589) *650.00, 650.00 0.25, 0.25

Figure 8
A 2-period Stochastic Game with Learning-by-doing

Note: The parameters for the two investment options are S_1 = 100, X_1 = 100, S_2 = 75 and X_2 = 75. Asset values S_1 and S_2 are perfectly correlated with a stochastic state-variable with σ = 0.10, and δ = 0.10. The riskless rate r = 0.10 and time for each investment option maturity T = 1. Figures in parenthesis provide the optimal effort for the first period only. Optimal effort for the second period is dependent on the level of the state-variable, etc.
* There is a learning-by-doing effect here for the 1st player: if strategy *L* is played repeatedly, in the 2nd period a cost reduction 25.38% applies. This is actually the critical cost reduction threshold for the 1st player to switch strategy from *H* to *L*.