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Existence, continuity and utility representation of strictly monotonic preferences on continuum of goods commodity spaces

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14. March 2009

Existence, continuity and utility representation of strictly monotonic preferences on continuum of goods commodity spaces[*]

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May 4, 2009

Abstract

It is an easy task for most commodity spaces, to find examples of strictly monotonic preference relations. For example in l^∞. However, it is not easy for spaces like $l^\infty([0,1])$. In this note we investigate the roots of this difficulty. We show that strictly monotonic preferences on $l^\infty(K)$ always exist. However, if K is uncountable no such preference is continuous and none of them have a utility representation.

Keywords: utility representation, strictly monotonic preferences
JEL Classification: D11; D50.

1   Introduction

This note is motivated by the lack of examples in the literature of strictly monotonic preferences in economies with a continuum of goods. For example,

how would we picture a strictly monotonic preference relation in $l^\infty([0,1])$? Are there strictly monotonic utility functions in this space? Are there continuous ones? Indeed, we will show that there are strictly monotonic preference relations on $l^\infty(K)$. Our example will be similar to the lexicographic order. Thus, our example has no utility representation and is discontinuous. Can we find an example that is continuous? Even better can we find a continuous utility function on $l^\infty(K)$ which is strictly monotonic? We will answer these questions negatively;

i) no strictly monotonic preference on $l^\infty(K)$ can be continuous in a linear topology if K is uncountable;

ii) no such preference has a utility representation.

Thus our results completely explain the lack of examples in the literature.

2 Notation and assumptions

The notation is quite standard. We use χ_A to denote the characteristic function of the set A. That is, $\chi_A(x) = 1$ if $x \in A$ and is 0 otherwise.

We denote by K the set of elementary commodities. The set of bounded function on K is denoted B. That is

$$B = l^\infty(K) = \{f : K \to \mathbb{R}; f \text{ is bounded}\}.$$

We use the standard partial order on B. Thus, if $f, g \in B$ we have that $f \geq g$ if $f(k) \geq g(k)$ for every $k \in K$. And we write $f > g$ if $f \geq g$ and $f \neq g$. If $f \geq g$ we define the interval $[g, f] = \{h \in B; g \leq h \leq f\}$.

The commodity space, X, is a convex subset of B. We suppose that $X \supset [0, \chi_K]$. A consumption plan $f \in X$ specifies an amount $f(k) \in \mathbb{R}$ of each commodity $k \in K$.

A preference relation $\succeq$ is a complete and transitive relation on X. Thus for every $f, g \in X$ either $f \succeq g$ or $g \succeq f$. Moreover if $f \succeq g \succeq h$ then $f \succeq h$. A preference relation has a utility representation (in short, is representable) if there is a function $U = U_\succeq : X \to \mathbb{R}$ such that $f \succeq g$ if and only if $U(f) \geq U(g)$. For $f, g \in X$ we write $f \succ g$ if $f \succeq g$ but $g \not\succeq f$.

It is quite easy to check that if $U : X \to \mathbb{R}$ is a function then $\succeq_U :=$ $\{(x, y) \in X^2; U(x) \geq U(y)\}$ define a complete and transitive preference relation on X. Also $x \succ_U y$ if and only if $U(x) > U(y)$.

The preference relation on X is strictly monotonic if $f, g \in X$ and $f > g$ implies that $f \succ g$. The preference relation on X is continuous if $\succeq$ is a closed subset of $X \times X$.

Let K be a set and $\leq$ an order on K. We say that $\leq$ is a well-order in K if for every non-empty subset $A \subset K$ there exists $\min A$. The following theorem will be quite useful.

Zermelo's theorem For every set K there is a well-ordering[1] of K.

3 Results

If K is countable it is quite easy to find strictly monotonic preferences on B. Let $a_n > 0$ be such that $\sum_{n=1}^{\infty} a_n < \infty$. The preference relation given by the function $U(x) = \sum_{n=1}^{\infty} a_n x_n$ is strictly monotone, continuous[2] and–obviously–representable by a utility function. However the next theorem shows that we cannot go much farther. In the proof we need to consider an order on K. For example if $K = [a, b]$ we take the usual order of the real numbers. More generally if $a, b \in \mathbb{R}^n$ and $K = \{(1 - t) a + tb; 0 \leq t \leq 1\}$ is the segment joining a and b we order K using the order on $[0, 1]$. Since Zermelo's theorem says that every set has a well-ordering, to suppose that K is ordered is without loss of generality.

Theorem 1 *Suppose $(K, \leq)$ is an uncountable ordered set. Then every strictly monotonic preference relation on X is non-representable.*

Proof. Suppose $\succeq$ is a strictly monotonic preference relation on X. Suppose $U : X \to \mathbb{R}$ represents $\succeq$. Define[3] for $t \in K$ the functions $x_t = \chi_{\{k \in K; k < t\}}$ and $y_t = \chi_{\{k \in K; k \leq t\}}$. Since $y_t > x_t$ we have that $U(y_t) > U(x_t)$. Now if $t < s$ we have that $x_s > y_t$ and therefore $U(x_s) > U(y_t)$. In particular we conclude that the set of intervals $\{I_t; t \in K\}$ where $I_t = (U(x_t), U(y_t))$ is pairwise disjoint and uncountable. An impossibility.   QED

Remark 1 *The same proof would work if instead of supposing that X contains the interval $[0, \chi_K]$ we suppose that X contains an interval $[u, v]$ where u and v are such that $v(k) - u(k) > 0$ for every $k \in K$.*

The theorem above will have little meaning if there are no strictly monotonic preference relations on X. We fill this gap in the next theorem.

Theorem 2 *There exists a strictly monotonic preference relation on X.*

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[1]This theorem was proved by E. Zermelo. For a proof see Kelley's book page 33.

[2]In the weak* topology and in the norm topology a fortiori.

[3]The reader may suppose $K = [a, b] \subset \mathbb{R}$ on a first reading.

Proof. Let $\leq_K$ be a well-ordering of K. Let $x \neq y \in X$. Let $k^* = \min\{k; x(k) \neq y(k)\}$. If $x(k^*) > y(k^*)$ we define $x \succ y$. Otherwise we define $y \succ x$. It is a simple task to verify that $\succeq$ so defined is complete, transitive and strictly monotonic.

The preference relation defined above is a generalization of the lexicographic ordering. It is discontinuous and non-representable (see Debreu (1954), footnote 1). Is it possible to find a strictly monotonic preference on B which is continuous in a suitable topology? For example if τ is such that (B, τ) is a Hausdorff topological vector space can we find such a preference? The following theorem shows that this is not possible.

Theorem 3 *Suppose (B, τ) is a topological vector space. If $\succeq$ is a continuous preference relation on X then $\succeq$ restricted to $[0, \chi_K]$ has a utility representation.*

Proof. Let $\succeq' := \succeq \cap ([0, \chi_K] \times [0, \chi_K])$. That is, $\succeq'$ is $\succeq$ restricted to $[0, \chi_K]$. It is therefore continuous, has a most preferred point, namely $\chi_K \equiv 1$ and a worst point, 0. Thus, since τ is a linear topology, and $[0, \chi_K]$ is a path connected space, the preference has a utility representation by Corollary 2, page 150 of Monteiro (1987).   QED

In non-separable metric spaces there is always a non-representable continuous preference relation (M. Estévez and C. Hervés-Beloso 1995, page 306). Our results complements this nicely: It is not possible to strengthen the result by requiring strict monotonicity.[4]

To make a counterpoint to the above results, let us consider the following example.

Example 1 *There is a continuous strictly monotonic utility function on the space $l^1_+([0,1])$. To see this define $U(f) = |f|_1$. This does not contradict theorem ?? since any $f \in l^1_+([0,1])$ is such that $\{t; f(t) \neq 0\}$ is countable. In this consumption space no agent ever consumes an uncountable set of commodities.*

4   Conclusion

Having considered a general scenario where monotonicity is meaningful, we have proved that strictly monotonic preferences always exist. However, quite surprisingly, strict monotonicity is incompatible with continuity and utility representation if there is a continuum of goods.

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[4] In spaces like B of course.

It is important to highlight that our incompatibility results do not apply, for example, to the case of the Banach spaces of the class of integrable functions. In fact,the argument that leads to the proof of our theorem **??**, requires that the characteristic functions of the intervals $[0, x)$ and $[0, x]$ be different. However this is not the case if we consider classes of integrable functions on a σ-finite measure space.

References

Debreu, G., 1954, Representation of a preference ordering by a numerical function, in: R.M. Thrall, C.H. Coombs and R.L. Davis, eds., Decision Processes (Wiley, New York) 159-165; also in Mathematical economics: twenty papers of Gerard Debreu (Cambridge University Press, Cambridge), 105-110.

Estévez, M. and Hervés, C., 1995, On the existence of continuous preference orderings without utility representations, Journal of Mathematical Economics 24, 305-309.

Kelley, J., General Topology, 1955, Graduate Texts in Mathematics 27

Monteiro, P. K., 1987, Some results on the existence of utility functions on path connected spaces, Journal of Mathematical Economics, 147-156.