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October 2008

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Abstract

We construct an index theorem for smooth infinite economies with separable utilities that shows that generically the number of equilbria is odd. As a corollary, this gives a new proof of existence and gives conditions that guarantee global uniqueness of equilibria.

JEL classification: D5, D50, D51

Keywords: Uniqueness, determinacy, equilibria, infinite economy, Fredholm map, equilibrium manifold, Banach manifold, index theorem, vector field, Rothe

1 Introduction

Models of competitive markets have a consumption space which may be infinite dimensional. Many authors have addressed the problem of studying if equilibrium prices are locally unique in infinite dimensions, including the earlier work of Araujo [2] and also the papers of Kehoe, Levine, Mas-Colell and Zame [10], Chichilnisky and Zhou [5], Shannon [12], Shannon and Zame [13] and Covarrubias [6]. In all these cases it has become clear that there is a trade-off between the generality of the consumption space, the generality of utility functions and the existence and differentiability of the individual demand functions.

However, an area that still remains largely unexplored in any such case is that of counting the numer of equilibria. When the consumption space is finite dimensional, Dierker [8] gave the first solution to this problem, and constructed an index theorem that showed that the number of equilibria is generically odd. He does this by interpreting the excess demand function as a vector field on the space of prices, and noticing that equilibria are the zeros of this vector field. He defines the notion of index of an equilibrium price system and shows that the sum of these indices is constant and equal to 1. Since the number of equilibria is generically odd, in particular it can never be zero and so Dierker's index theorem gives a new proof of existence of equilibria. Additionally, if the index at each equilibrium price is > 0 then the index theorem also gives conditions for *global* uniqueness of equilibria.

In infinite dimensions, one of the few results on uniqueness has been provided by Dana [7] taking into consideration a model of a pure exchange economy where the agents' consumption space is $L^p_+(\mu)$ and agents have additively separable utilities which fulfil the (RA) assumption that agents' relative risk aversion coefficients are smaller than one. In this case, Dana shows that one can work with the space of utility weights to avoid using the demand approach that may not be well defined. Dana finally shows that if utilities fulfil the (RA) assumption then the excess utility map is gross substitute which in turn implies existence and uniqueness of equilibrium.

In this paper, it is our aim to also consider separable utilities as in [7] but to construct an infinite dimensional analogue of Dierker's result: that the number of equilibria of smooth infinite economies is odd and hence to study conditions that guarantee global uniqueness of equilibria. We present an analytical notion that has not appeared in the economic literature which is that of a Z-Rothe vector field. When the aggregate excess demand function defines a Z-Rothe vector field, it allows us to construct an index theorem on the normalized infinite dimensional price space.

In section 2 we set the market and define aggregate excess demand functions in our setting; as usual, we will interpret them as vector fields on the space of prices. In section 3 we review the basic definitions of Fredholm theory, which is needed to extend differential topology to infinite dimensions. In section 4 we review the determinacy results obtained in [6] showing that most excess demand functions have isolated zeros; that is, that equilibria are locally unique. This guarantees that it makes sense to actually count the number of equilibria.

In section 5 we review the notion of Z-Rothe vector fields as developed by Tromba [15]. When an excess demand function is Z-Rothe, we can define a suitable index of equilibrium prices, that is, an index of zeros of a vector field. Then, in section 6, we construct an index theorem for smooth infinite economies. We show that the sum of indices of equilibrium prices is constant and equal to 1. Finally in section 7, we give a corollary to the index theorem analogous to [8], giving a new proof of existence of equilibrium and analyzing what condition an excess demand function needs to fulfill to give rise to a globally unique equilibrium.

2 The Market

We assume, following [5], that the commodity space is a subset of $C(M, \mathbb{R}^n)$, where M is any compact (Riemannian) manifold. For more general commodity spaces we refer to [4].

Example 1: In growth models a utility function on $C(M, \mathbb{R}^n)$ is a continuous-time version of a discounted sum of time-dependent utilities. Here M represents time.

Example 2: In finance, when the underlying parameters follow a diffusion process, a utility function on $C(M, \mathbb{R}^n)$ is the expectation of state-dependent utilities where M is the state space.

This commodity space is also mathematically convenient because in order to use differential techniques, we would like it to be a separable topological vector space for which the interior of its positive cone (the consumption space) is non-empty.

The **consumption space** is then $X = C^{++}(M, \mathbb{R}^n)$, the positive cone of $C(M, \mathbb{R}^n)$. Strictly speaking, prices are in the positive cone of the dual of $C(M, \mathbb{R}^n)$. However, it is shown in [5] that with separable utilities only a small subset of this space can support equilibria and we can actually consider the **price space** to be $S = \{P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1\}$ where $\|P\| = \sup_{t\in M} \|P(t)\|_{\mathbb{R}^n}$ with the standard metric $\|\cdot\|_{\mathbb{R}^n}$ on $\mathbb{R}^n$.

We denote by $\langle \cdot, \cdot \rangle$ the inner product on $C(M, \mathbb{R}^n)$ so that if $f, g \in C(M, \mathbb{R}^n)$ then

$$\langle f, g \rangle = \int_M \langle f(t), g(t) \rangle_{\mathbb{R}^n} dt$$

with the standard inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ in $\mathbb{R}^n$.

We consider a finite number I of agents. An **exchange economy** is parametrized for each agent $i = 1, \ldots, I$ by their initial endowments $\omega_i \in X$ and their individual demand functions $f_i : S \times (0, \infty) \to X$. The maps $f_i(P(t), w)$ are solutions to the optimization problem

$$\max_{\langle P(t), y \rangle = w} W_i(y)$$

where $W_i(x)$ is a separable utility function, i.e., it can be written as

$$W_i(x) = \int_M u^i(x(t), t)dt$$

We assume $u^i(x(t), t) : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in R^n_{++} : u^i(y, t) \geq u^i(x, t)\}$ is closed. In [5] is shown that this implies that $W_i(x)$ is strictly monotonic, concave and twice Fréchet differentiable.

In this paper we assume that the individual demand functions are fixed, so that the only parameters defining an economy are the initial endowments. Denote $\omega = (\omega_1, \ldots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \to C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(P) = \sum_{i=1}^{I} (f_i(P, \langle P, \omega_i \rangle) - \omega_i)$$

We also define $Z : \Omega \times S \to C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, P) = Z_\omega(P)$$

It satisfies $\langle P, Z_\omega(P) \rangle = 0$ for all $P \in S$.

Definition 1. *We say that $P \in S$ is an* ***equilibrium*** *of the economy $\omega \in \Omega$ if $Z_\omega(P) = 0$. We denote the* ***equilibrium set***

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, P) = 0\}$$

3 Fredholm Index Theory

We wish to explore the structure of aggregate excess demand functions and since we will be using tools of differential topology in infinite dimensions, we would like our maps to be Fredholm as introduced by Smale [14].

A (linear) **Fredholm operator** is a continuous linear map $L : E_1 \to E_2$ from one Banach space to another with the properties:

1. dim ker $L < \infty$
2. range L is closed
3. coker $L = E_2/\text{range}L$ has finite dimension

If L is a Fredholm operator, then its **index** is dim ker$L -$ dim coker$L$, so that the index of L is an integer.

A **Fredholm map** is a C' map $f : M \to V$ between differentiable manifolds locally like Banach spaces such that for each $x \in M$ the derivative $Df(x) : T_xM \to T_{f(x)}V$ is a Fredholm operator. The **index** of f is defined to be the index of $Df(x)$ for some x. If M is connected, this definition does not depend on x.

In our previous work [6] we have shown that the excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a Fredholm map of index zero.

4 Determinacy of equilibria

Since we would like to count the number of price equilibria of an economy, the first result that we need to establish is that generically equilibria will be isolated. Below we remind the reader the notion of a regular economy and of a regular price system.

Definition 2. *We say that an economy is* ***regular*** *(resp.* ***critical****) if and only if* ω *is a regular (resp. critical) value of the projection* $pr : \Gamma \to \Omega$ *.*

Definition 3. *Let* Z_ω *be the excess demand of economy* ω*. A price system* $P \in S$ *is a* ***regular equilibrium price*** *system if and only if* $Z_\omega(P) = 0$ *and* $DZ_\omega(P)$ *is surjective.*

In our previous work [6] we showed the relation between regular economies and regular equilibrium prices.

Proposition 1. *[6] The economy* $\omega \in \Omega$ *is regular if and only if all equilibrium prices of* Z_ω *are regular.*

Proposition 2 showed that for most economies, its aggregate excess demand function will have isolated zeros. Hence, it makes sense to try to count them.

Proposition 2. *[6] Almost all economies are regular. That is, the set of economies* $\omega \in \Omega$ *that give rise to an excess demand function* Z_ω *with only regular equilibrium prices, are residual in* Ω*.*

Since we haven shown that for most excess demand functions Z_ω will have isolated zeros, we will drop the explicit dependence on ω and will simply write Z.

5 Z-Rothe vector fields

Knowing that the excess demand function is a vector field on the price space, and that is a Fredholm map for which we know its index, we would like to give it the structure of a Z-Rothe vector field as developed by Tromba [15]. In section 6 it will become clear that we need a vector field that is outward pointing, so we insist $-Z_\omega$ to be Z-Rothe.

Let E be any Banach space and $\mathcal{L}(E)$ be the set of linear continuous maps from E to itself. Denote by $G\mathcal{L}(E)$ the general linear group of E; that is, the set of invertible linear maps in $\mathcal{L}(E)$. Let $C(E)$ be the linear space of compact linear maps from E to itself.

We write $\mathcal{S}(E) \subset G\mathcal{L}(E)$ to denote the maximal starred neighborhood of the identity in $G\mathcal{L}(E)$. Formally,

$$\mathcal{S}(E) = \{T \in G\mathcal{L}(E) : (\alpha T + (1-\alpha)I) \in G\mathcal{L}(E), \forall \alpha \in [0,1]\}$$

The **Rothe set** of E is defined as

$$\mathcal{R}(E) = \{A : A = T + C, T \in \mathcal{S}(E), C \in C(E)\}$$

and its invertible members by $G\mathcal{R}(E) = \mathcal{R}(E) \cap G\mathcal{L}(E)$.

A C^1 vector field X on a Banach manifold M is **Z-Rothe** if whenever $X(p) = 0$, $DX(P) \in \mathcal{R}(T_P M)$

Proposition 3. *The negative of the excess demand function, $-Z : S \to TS$ is a Z-Rothe vector field.*

Proof. If we want $-Z$ to be a Z-Rothe vector field, we need to check that whenever $Z(P) = 0$, $DZ(P) \in \mathcal{R}(T_P S)$. That is, we need to write $DZ(P)$ as the sum of an element of $\mathcal{S}(T_P S)$ and an element of $C(T_P S)$.

Chichilnisky and Zhou [5] have shown that for each agent i, his individual demand function Df_i can be written as the sum of the finite rank operator

$$-\frac{\lambda\langle P(t), (u^i_{xx})^{-1} DP(t)\rangle + \langle DP(t), f_i\rangle}{\langle P(t), (u^i_{xx})^{-1} P(t)\rangle} (u^i_{xx})^{-1} P(t) \tag{1}$$

and the invertible operator

$$\frac{(u^i_{xx})^{-1} P(t)}{\langle P(t), (u^i_{xx})^{-1} P(t)\rangle} Dw + \lambda (u^i_{xx})^{-1} DP(t) \tag{2}$$

Adding over agents in equation (1), define Z_C be the sum of the finite rank operators, that is,

$$Z_C = \sum_{i=1}^{I} -\frac{\lambda\langle P(t), (u^i_{xx})^{-1} DP(t)\rangle + \langle DP(t), f_i\rangle}{\langle P(t), (u^i_{xx})^{-1} P(t)\rangle} (u^i_{xx})^{-1} P(t) \tag{3}$$

Then Z_C has finite rank, and hence $Z_C \in C(T_P S)$. Now add over agents in equation (2) and define Z_R to be

$$Z_R = \sum_{i=1}^{I} \frac{(u^i_{xx})^{-1} P(t)}{\langle P(t), (u^i_{xx})^{-1} P(t) \rangle} Dw + \lambda (u^i_{xx})^{-1} DP(t) \tag{4}$$

The matrix (u^i_{xx}) is negative definite, and every negative definite matrix is invertible and its inverse is also negative definite. So Z_R has to be invertible and hence $Z_R \in G\mathcal{L}(T_P S)$

All we need to show then is that $Z_R \in \mathcal{S}(T_P S)$, that is, that

$$\alpha \left[-\sum_{i=1}^{I} \left[\frac{(u^i_{xx})^{-1} P(t)}{\langle P(t), (u^i_{xx})^{-1} P(t) \rangle} Dw - \lambda (u^i_{xx})^{-1} DP(t) \right] \right] + (1-\alpha) I$$

is invertible for all $\alpha \in [0,1]$. But this sum is just a homotopy of positive-definite operators.

□

6 The Index Theorem of Smooth Infinite Economies

Knowing that most economies are regular we need to find a right way of counting the number of equilibria. With an excess demand function that is a Fredholm map, we may use tools from infinite-dimensional differential topology that resembles the finite dimensional case.

Below we review the notion of index of a zero of a Z-Rothe vector field.

We also review the Euler characteristic, which is the topological invariant that we would like our index theorem to be equal to.

6.1 Euler Characteristic

A zero P of a vector field X is **nondegenerate** if $DX(P) : T_P M \to T_P M$ is an isomorphism.

Suppose that a Z-Rothe vector field X has only nondegenerate zeros, and let P be one of them. Then, $DX(P) \in G\mathcal{R}(T_P M)$. Tromba [15] shows that $G\mathcal{R}(T_P M)$ has two components; $G\mathcal{R}^+(E)$ denotes the component of the identity. Define

$$sgnDX(P) = \begin{cases} +1, & \text{if } DX(P) \in G\mathcal{R}^+(T_P M) \\ -1, & \text{if } DX(P) \in G\mathcal{R}^-(T_P M) \end{cases}$$

The **Euler characteristic** is then given by the formula

$$\chi(X) = \sum_{P \in Zeros(X)} sgnDX(P)$$

Tromba also shows that this Euler characteristic is invariant under homotopy of vector fields. All we have to do is to construct a vector field on S that has only one singularity and that is homotopic to the aggregate excess demand Z.

6.2 The Index Theorem of Smooth Infinite Economies

Suppose that the excess demand satisfies the 'boundary assumption' of Dierker [8], namely that if $P_n \in S$ and $P_n \to P \in \partial S$, then $\|Z(P_n)\| \to \infty$. Suppose also that Z is bounded below. Then $-Z$ is an outward-pointing vector field. Finally, assume that there are only finitely many zeros.

We are now ready to introduce our main result.

Proposition 4. *Suppose that an aggregate excess demand function Z is bounded from below and that it satisfies the boundary assumption. Suppose also that Z has only finitely many singularities and that they are all nondegenerate. Then,*

$$\sum_{P \in Zeros Z} sgn\,[-DZ(P)] = 1$$

Proof. For any fixed $Q \in C^{++}(M, \mathbb{R}^n)$ define the vector field $Z^Q : \bar{S} \to TS$ given by

$$Z^Q(P) = \left[\frac{Q(t)}{\langle P(t), Q(t)\rangle}\right] - P(t)$$

By construction, $Z^Q(P)$ has only one zero and is inward-pointing. Its derivative $DZ^Q_{(P)} : T\bar{S} \to T(TS)$ is given by

$$DZ^Q_{(P)}(h) = -\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2} - h$$

where $h \to -\frac{Q\langle h,Q\rangle}{\langle P,Q\rangle^2}$ is compact and $h \to -h$ is invertible; then $DZ^Q \in$

$\mathcal{R}(T_PS)$. Now let

$$-\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2} - h = h' \tag{5}$$

We need to solve for h. Then,

$$Q\langle h, Q\rangle + h\langle P, Q\rangle^2 = -h'\langle P, Q\rangle^2$$

Acting Q on both sides we get,

$$\langle Q, Q\rangle\langle h, Q\rangle + \langle h, Q\rangle\langle P, Q\rangle^2 = -\langle h', Q\rangle\langle P, Q\rangle^2$$

Solving for $\langle h, Q\rangle$ we get

$$\langle h, Q\rangle = \frac{-\langle h', Q\rangle\langle P, Q\rangle^2}{\langle Q, Q\rangle + \langle P, Q\rangle^2}$$

where the denominator never vanishes since $Q \in C^{++}(M, \mathbb{R}^n)$. Substituting $\langle h, Q\rangle$ in 5 we then get

$$h = h' + \frac{Q}{\langle P, Q\rangle^2}\left[\frac{\langle h', Q\rangle\langle P, Q\rangle^2}{\langle Q, Q\rangle + \langle P, Q\rangle^2}\right]$$

This shows that DZ^Q is invertible and therefore $DZ^Q \in G\mathcal{R}(T_PS)$. Furthermore, since it is not in the same component of the identity it has to be in $G\mathcal{R}^-(T_PS)$ and its only zero has index -1. The vector field Z^Q is inward pointing so reversing orientation will make outward poiting with index of +1.

$\square$

7 Concluding Remarks

We conclude from Proposition 4 that the number of equilibria of smooth infinite economies generically is odd. In particular, it can never be zero so this gives a new proof of existence.

Also, as a corollary of Proposition 4, we can also provide an infinite dimensional analogue of [8]; Dierker shows

Proposition 5. *[8] If the Jacobian of the excess supply function is positive at all Walras equilibria, then there is exactly one equilibrium.*

We show that:

Proposition 6. *If the sign of the derivative of the excess supply function is positive at all Walras equilibria, then there is exactly one equilibrium.*

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