Bertrand-Edgeworth games under oligopoly with a complete characterization for the triopoly

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Abstract

The paper extends the analysis of price competition among capacity-constrained sellers beyond the cases of duopoly and symmetric oligopoly. We first provide some general results for the oligopoly and then focus on the triopoly, providing a complete characterization of the mixed strategy equilibrium of the price game. The region of the capacity space where the equilibrium is mixed is partitioned according to the features of the mixed strategy equilibrium arising in each subregion. Then computing the mixed strategy equilibrium becomes a quite simple task. The analysis reveals features of the mixed strategy equilibrium which do not arise in the duopoly (some of them have also been discovered by Hirata (2008)).

1 Introduction

The issue of price competition among capacity-constrained sellers has attracted considerable interest since Levitan and Shubik's (1972) modern reappraisal of Bertrand and Edgeworth. Assume there are a given number of firms producing at the same constant unit cost up to some fixed capacity. Assume, also, a non-increasing and concave demand and that any rationing takes place according to the surplus maximizing rule. Then there are a few well-established facts about equilibrium of the price game. First, at any pure strategy equilibrium the firms are charging the competitive price. However, a pure strategy equilibrium need not exist, unless the capacity of the largest firm is small enough compared to total capacity (see, for instance, Vives, 1986). When a pure strategy equilibrium does not exist, existence of a mixed strategy equilibrium is guaranteed by Theorem 5 of Dasgupta and Maskin (1986) for discontinuous games.

A characterization of mixed strategy equilibrium has been provided by Kreps and Scheinkman (1983) for the duopoly within a two-stage capacity and price game, assuming concavity of demand and identical unit costs. The model was subsequently extended by Osborne and Pitchik (1986) to allow for non-concavity of demand and by Deneckere and Kovenock (1996) to allow for differences in unit cost among the duopolists. Both changes lead to the emergence of new phenomena, such as the possibility of the supports of the equilibrium strategies being disconnected and not identical for the duopolists.

Yet, there is still much to be learned about mixed strategy equilibria under oligopoly, even for the case of identical (and constant) unit cost and concave demand: to the best of our knowledge, a complete characterization of the mixed strategy equilibrium is only available for the case of equal capacities (see, among others, Vives, 1986). This paper purports to make progress in this direction by allowing for differences in size among the firms while retaining equality in unit cost, concavity of demand, and the efficient rationing rule. In this connection, we first point out a number of general properties of mixed strategy equilibrium under oligopoly, which are subsequently used to provide a comprehensive analysis of the price game for the triopoly. Although far more complex than in duopoly, characterizing mixed strategy equilibria turns out to be a quite tractable task in a triopoly. Most important, the task proves to be worth pursuing: several new phenomena do indeed appear as soon as one departs from duopoly, suggesting that the latter is a rather special case. First of all, the supports of the equilibrium distributions need no longer coincide for all the firms. Second, the supports need no longer be connected for all the firms: we identify circumstances where there is a gap in the support for the smallest firm. Third, the equilibrium need no longer be unique. There are an infinity of mixed strategy equilibria when the capacity of the largest firm is high enough - a result which extends straightforwardly to oligopoly: in such circumstances, the equilibrium distributions of the other $n-1$ firms are determined up to $n-2$ degrees of freedom.

The paper is organized as follows. Section 2 contains definitions and the basic assumptions of the model along with a few basic results about equilibrium payoffs under oligopoly. Section 3 shows that several characteristic of mixed strategy equilibrium extend from duopoly to the oligopoly. Most notably, as far as the largest firm is concerned: the minimum element in the support of its equilibrium strategy is determined like in duopoly; the highest element - also determined like in duopoly - is charged with positive probability if its capacity is strictly higher than for any other firm.

The remainder of the paper is devoted to the triopoly. Section 4 characterizes equilibrium profits and bounds the supports of the equilibrium strategies for all the firms, in any point of the region of mixed strategy equilibria. It is found that, whenever maxima cannot be the same for all the firms, it is the support of the smallest firm the one with the lowest maximum. In contrast, when the minima of the supports cannot be the same for all the firms, there are regions of the capacity space where the support with the highest minimum pertains to the smallest firm as well as one region where it pertains to the intermediate-size firm: in either case, we show how that minimum is determined. The section also addresses the - up to that point, hypothetical - event of the support being disconnected for some firm. More precisely, it is shown how that event can possibly be detected and how the gap in the support is to be determined. Section 5 applies all the above findings and proves that the event of a disconnected support is a concrete one. More specifically, we compute the mixed strategy equilibrium for capacities all different from each other and lying in one of the two regions where the supports of the equilibrium strategies have the same bounds for all the firms. That region is partitioned into two subsets according to the nature of the equilibrium: in one the supports are identical and connected for all the firms, in the other, the support for the smallest firm has a gap. Section 6 briefly concludes.

POSTSCRIPT. The results contained in this paper have been achieved by the authors over the last couple of years, a few of them being reached at the end of last year. At that point we were stopped since we were not yet able to exclude the possibility of a gap in the support of the largest firm in the subset E_1 introduced in Section 4. Two weeks ago, when coming back to our paper, we noticed a working paper by D. Hirata (2008, posted in March 30 in the Munich Personal RePEc Archive), also dealing with mixed strategy equilibria in the triopoly. The present paper includes all of the results presented in Hirata's paper, which we arrived at independently in the past, along with a number of others, so it is worth to clarify the differences between Hirata's contribution and our own.

Hirata proves that there are regularities of mixed strategy equilibria under duopoly which need not hold in the triopoly (and therefore presumably need not hold in an oligopoly with more than three firms). We introduce a number of procedures which allow us to provide a complete characterization of the mixed strategy equilibria for the triopoly case. As a consequence we find all the facts which can happen, including those found by Hirata. In this connection we are able to identify the entire region where there is an indeterminacy of equilibrium over part of range $[p_m, p_M]$ and the entire region where $p_m^{(3)} > p_m^{(1)} = p_m^{(2)} = p_m$ (i.e., where the minimum of the support is higher for firm 3 - the smallest firm - than it is for firms 1 and 2 (respectively, the largest and the medium-size firms). Further, we recognize that there is a non-degenerate region (our region F) where $p_m^{(2)} > p_m^{(1)} = p_m^{(3)} = p_m$. (Hirata only discovers this feature for a degenerate region (see his footnote 4).)[1] Further, the fact that under certain circumstances there is a gap in the support of the equilibrium strategy of the smallest firm (see Section 5 below) is a novel feature of our paper. Finally, while recognizing in his Claim 3 that there are circumstances where equilibrium profit per unit of capacity is larger for firm 3 than it is for firm 2, how firm 3's equilibrium profit is determined in those circumstances is not addressed by Hirata. This is done in Proposition 5 below; further, the Theorem in Section 4 below fully specifies the subsets of the capacity space where equilibrium profit per unit of capacity is higher for firm 3 than firm 2.

2 Preliminaries

There are n firms, $1, 2, ..., n$, supplying a homogeneous good. The firms are assumed to produce at the same constant unit cost, normalized to zero, up to capacity. The demand is denoted as $D(p)$ and its inverse as $P(x)$. When positive, $D(p)$ is assumed to be decreasing and concave. Without loss of generality, we consider the subset of the capacity space $(K_1, K_2, ...K_n)$ such that $K_1 \geq K_2 \geq ... \geq K_n$, and we define $K = K_1 + ... + K_n$. As already said, the firms are charging the competitive price, $p^c = \max\{0, P(K)\}$ at any pure strategy equilibrium of the price game. Thus such an equilibrium fails to exist when $\arg\max p(D(p) - \Sigma_{j\neq 1} K_j) > p^c$, or, to put it more throroughly, when either

$$\Sigma_{j\neq 1} K_j < D(0),\ \ p^c = 0; \tag{1}$$

or

$$K_1 > -p^c \left[D'(p)\right]_{p=p^c},\ \ p^c > 0. \tag{2}$$

[1] The fact that a full characterization is not provided by Hirata is also clear by looking at his Claim 4 (p. 13): in the circumstances of that claim, the equilibrium can exhibit different features (only some of them identified by him), but Hirata does not clarify which obtains when. By the way, statements of Claims 4 and 5 of Hirata include an obvious misprint. It should be $K_1 < D(a^*) < K_1 + K_3$ instead of $K_1 < D(a^*) < K_3$.

It is assumed throughout that either (1) or (2) holds, so that we are in the region of mixed strategy equilibria. Note that, in the subset of the capacity space here considered, any point with $\Sigma_{j\neq 1}K_j < D(0)$ belongs to such a region provided K_1 is not less than or close enough to $D(0) - \Sigma_{j\neq 1}K_j$.[2] We henceforth denote by Π_i^* firm i's equilibrium payoff (expected profit), by $\Pi_i(p)$ firm i's expected profit when charging p and the rivals are playing their equilibrium profile of distributions, $\phi_{-i}(p)$, by $\phi_i(p) = \Pr(p_i < p)$ firm i's equilibrium (cumulative) distribution, where $\Pr(p_i < p)$ is the probability of i charging less than p, by S_i the support of ϕ_i, and by $p_M^{(i)}$ and $p_m^{(i)}$ the maximum and the minimum of S_i, respectively. More specifically, we say that $p \in S_j$ when $\phi_j(\cdot)$ is increasing at p, that is, when $\phi_j(p+h) > \phi_j(p-h)$ for any $h > 0$, whereas $p \notin S_j$ if $\phi_j(p+h) = \phi_j(p-h)$ for some $h > 0$. Of course, any $\phi_i(p)$ is non-decreasing and everywhere continuous except at $p^\circ : \Pr(p_i = p^\circ) > 0$, where it is left-continuous $(\lim_{p\to p^{\circ -}} \phi_i(p) = \phi_i(p^\circ))$, but not continuous.

Obviously, $\Pi_i^* \geq \Pi_i(p)$ everywhere and $\Pi_i^* = \Pi_i(p)$ almost everywhere in S_i. Some more notation is needed to go deeper through the properties of $\Pi_i(p)$. Let $N = \{1, ..., n\}$ be the set of firms, $N_{-i} = N - \{i\}$, and $\mathcal{P}(N_{-i}) = \{\psi\}$ be the power set of N_{-i}. Further, let

$$Z_i(p; \phi_{-i}) := p \sum_{\psi\in\mathcal{P}(N_{-i})} q_{i,\psi} \Pi_{r\in\psi}\phi_r \Pi_{s\in N_{-i}-\psi}(1 - \phi_s), \tag{3}$$

where $q_{i,\psi} = \max\{0, \min\{D(p) - \sum_{r\in\psi} K_r, K_i\}\}$ is firm i's output when any firm $r \in \psi$ charges less than p and any firm $s \in N_{-i} - \psi$ charges more than p.[3] $Z_i(p; \phi_{-i})$ is continuous in p and concave almost everywhere (for every p there is ϵ and ϵ' such that $Z_i(p; \phi_{-i})$ is concave in the intervals $[p, p+\epsilon]$ and $[p-\epsilon', p]$; as a consequence it is locally concave whenever it is differentiable.[4] $Z_i(p; \phi_{-i})$ is continuous and differentiable in ϕ_j (each $j \neq i$). Differentiation of $Z_i(p; \phi_{-i})$ with respect to ϕ_j yields, after rearrangement,[5]

$$\frac{\partial Z_i}{\partial \phi_j} = p \sum_{\psi\in\mathcal{P}(N_{-i})} (q_{i,\psi} - q_{i,\psi'})\Pi_{r\in\psi'}\phi_r \Pi_{s\in N_{-i}-\psi}(1 - \phi_s) \tag{4}$$

[2] The right-hand side of (2) converges to 0 as K_1 converges to $D(0) - \Sigma_{j\neq 1}K_j$, hence inequality (2) is met with K_1 in a left neighbourhood of $D(0) - \Sigma_{j\neq 1}K_j$.

[3] Note that $\Pi_{r\in\psi}\phi_r$ is the empty product, hence equal to 1, when $\psi = \emptyset$; and it is similarly $\Pi_{s\in N_{-i}-\psi}(1 - \phi_s) = 1$ when $\psi = N_{-i}$.

[4] $Z_i(p; \phi_{-i})$ is kinked at any $p = P(\sum_{r\in\psi} K_r)$, where it is locally convex so long as $K_r \neq K_i$ for all $r \in \psi$.

[5] Note that $q_{i,\psi} - q_{i,\psi'} = 0$ whenever $j \notin \psi$. This allows to simplify the notation.

where $\psi' = \psi - \{j\}$. Since $-K_j \leq q_{i,\psi} - q_{i,\psi'} \leq 0$, $\partial Z_i/\partial\phi_j \leq 0$. More precisely $\partial Z_i/\partial\phi_j < 0$ if there exists some ψ containing j such that

$$\Pi_{r\in\psi'}\phi_r\Pi_{s\in N_{-i}-\psi\}}(1-\phi_s) > 0. \tag{5}$$

and

$$0 < D(p) - \sum_{h\in\psi'} K_h < K_i + K_j. \tag{6}$$

It is immediately recognized that for each p in which all functions $\phi_j(p)$ $(j \neq i)$ are continuous, then

$$\Pi_i(p) = Z_i(p; \phi_{-i}(p)),$$

whereas

$$Z_i(p^\circ; \phi_{-i}(p^\circ)) \geq \Pi_i(p^\circ) \geq \lim_{p\to p^\circ+} Z_i(p; \phi_{-i}(p))$$

if $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$. This is enough to state the following

Lemma 1. *For any $i \in N$, $\Pi_i^* \geq \Pi_i(p)$ with $\Pi_i^* = \Pi_i(p)$ for p in the interior of S_i.*

Proof. Suppose contrariwise that $\Pi_i^* > \Pi_i(p^\circ)$ for some p° internal to S_i. This reveals that p° is not charged by i: it is $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$ and $Z_i(p^\circ; \phi_{-i}(p^\circ)) > \Pi_i(p^\circ) > \lim_{p\to p^\circ+} Z_i(p; \phi_{-i}(p))$. As a consequence there is a right neighbourhood of p° in which $\Pi_i^* > \Pi_i(p)$: a contradiction. ■

Let $p_M = \max_i p_M^{(i)}$ and $p_m = \min_i p_m^{(i)}$, $M = \{i : p_M^{(i)} = p_M\}$ and $L = \{i : p_m^{(i)} = p_m\}$. Moreover, if $\#M < n$, then we define $\widehat{p}_M = \max_{i\notin M} p_M^{(i)}$ and, with an abuse of language, if $\#M = n$, then we say that $\widehat{p}_M = p_M$. Similarly, if $\#L < n$, then we define $\widehat{p}_m = \min_{i\notin L} p_m^{(i)}$ whereas $\widehat{p}_m = p_m$ if $\#L = n$. When evaluated over some range α, $\phi_i(p)$ and $\Pi_i(p)$ are denoted as $\phi_{i\alpha}(p)$ and $\Pi_{i\alpha}(p)$, respectively. Finally, $\lim_{p\to h+} \Pi_{i\alpha}(p)$ and $\lim_{p\to h-} \Pi_{i\alpha}(p)$ are denoted as $\Pi_{i\alpha}(h^+)$ and $\Pi_{i\alpha}(h^-)$, respectively.

Since Kreps and Scheinkman it is known that $p_M = p_M^{(1)} = p_M^{(2)} = \arg\max p(D(p) - K_2)$ in a duopoly with $K_1 \geq K_2$; also, $\phi_1(p_M) < \phi_2(p_M) = 1$ if $K_1 > K_2$, while $\phi_1(p_M) = \phi_2(p_M) = 1$ if $K_1 = K_2$. Therefore $\Pi_i^* = p_M(D(p_M) - K_2)$ for any i such that $K_i = K_1$. The next proposition summarizes some generalizations of these results to oligopoly that have been made recently.

Proposition 1 *$p_M = \arg\max p(D(p) - \Sigma_{j\neq 1}K_j)$ and, for any $i : K_i = K_1$, $\Pi_i^* = \max p(D(p) - \Sigma_{j\neq 1}K_j)$; furthermore, $p_M^{(i)} = p_M$ for any $i : K_i = K_1$ and $\phi_j(p_M) = 1$ for any $j : K_j < K_1$.*

Proof. This statement is an obvious consequence of the statement that $p_M^{(i)} = p_M$ for *some* $i : K_i = K_1$ and that $\Pi_i^* = \max p(D(p) - \Sigma_{j\neq 1}K_j)$ for *any* $i : K_i = K_1$.(A complete proof of this statement is in De Francesco (2003); see also Boccard and Wauthy (2001) and, for a more recent proof, Loertscher (2008).) ■

According to this result, in the region of mixed strategy equilibria, the equilibrium payoff of the largest firm is decreasing in the capacity of any rival and is independent on its own capacity. The fact that $\Pi_i^* = \max p(D(p) - \Sigma_{j\neq 1}K_j)$ for any $i : K_i = K_1$ has a nice interpretation. Note that, in the region of the capacity space where the equilibrium is in mixed strategies, $\max p(D(p) - \Sigma_{j\neq 1}K_j)$ is nothing but the minimax payoff for any $i : K_i = K_1$.[6] Thus, what Proposition 1 is actually saying is that the equilibrium payoff of (any of) the largest firm(s) equals its minimax payoff.

Since Kreps and Scheinkman it is also known that, in a duopoly, $\#L = 2$ and $\Pr(p_i = p_m) = 0$ for $i = 1, 2$, so that $\Pi_1^* = p_m \min\{D(p_m), K_1\}$. This implies that $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$, where $\widehat{p} \equiv \Pi_1^*/K_1$ and $\widehat{\widehat{p}}$ is the smallest solution of the equation in p

$$pD(p) = \Pi_1^*.$$

Finally, firm 2's equilibrium payoff is $\Pi_2^* = p_m K_2$. Since $S_1 = S_2 = [p_m, p_M]$, then $\phi_1(p)$ and $\phi_2(p)$ are found straightforwardly by solving the two-equation system $\Pi_i^* = Z_i(p; \phi_{-i}(p))$. It will be seen below to which extent these results generalize beyond duopoly.

3 Some properties of equilibrium for the oligopoly

In this section we establish a number of general properties of mixed strategy equilibria under oligopoly. The following proposition presents a number of basic properties, which represent generalizations of analogous results holding for duopoly.

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[6]Let σ_{-i} denote any mixed strategy profile on the part of firm i's rivals, where $i : K_i = K_1$ and let $p(\sigma_{-i})$ denote any of firm i's best response to σ_{-i}. It is immediately understood that $\Pi_i(p(\sigma_{-i})) \geq p_M(D(p_M) - \Sigma_{j\neq 1}K_j)$ with strict equality holding for some σ_{-i}.

Proposition 2 *(i)* $\#M \geq 2$ *and* $\#L \geq 2$.

(ii) At any $p^\circ \in (p_m, p_M)$*, it cannot be* $\#\{i : p^\circ \in S_i\} = 1$.

(iii) For any $p^\circ \in (p_m, p_M)$, $p^\circ > P(\Sigma_{i:p_m^{(i)}<p^\circ} K_i)$.

(iv) $i \in L$ *for any* $i : K_i = K_1$.

(v) Let $i \in N_{-1}$ *and* $j \in N_{-1} - \{i\}$*. At any* $p \in (p_m, p_M)$*:*

(v.a) $\partial Z_1/\partial\phi_i < 0$ *and* $\partial Z_i/\partial\phi_1 < 0$ *for any i;*

(v.b) if $p \geq P(K_1)$, $\partial Z_i/\partial\phi_j = 0$*;*

(v.c) if $p < P(K_1)$ *and n=3, then* $\partial Z_i/\partial\phi_j < 0$*; if* $p < P(K_1)$ *and* $n > 3$, *then, for each* $i \in R(p)$ *(for each* $j \in R(p)$*), there is some* $j \in R(p)$ *(resp., some* $i \in R(p)$*) such that* $\partial Z_i/\partial\phi_j < 0$*, where* $R(p) = \{r : p_m^{(r)} \leq p\}$.

(vi) $p_m > P(\Sigma_{j\in L} K_j)$.

(vii) For any $p^\circ \in (p_m, p_M)$, $\Pr(p_j = p^\circ) = 0$ *for any j.*

(viii) $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$.

Proof. (i) This is so because $Z_i(\cdot)$ is concave in p on a right neighbourhood of p_m and on a left neighbourhood of p_M. Suppose contrariwise that $\#L = 1$ and let $L = \{i\}$. Then $\phi_j' = 0$ for all $j \neq i$ in a neighbourhood of p_m. Hence $d\Pi_i(p)/dp = \partial Z_i/\partial p$, contrary to the fact that $\Pi_i(p) = \Pi_i^*$ in a right neighbourhood of p_m. A similar argument rules out the event of $\#M = 1$.

(ii) The proof is similar to the previous one, given the fact that $Z_i(\cdot)$ is concave on a right neighbourhood of any p and a left neighbourhood of any p.

(iii) Otherwise for $i : p_m^{(i)} < p$ it would be $\Pi_i(p) = pK_i$ for $p \in S_i \cap [p_m, p^\circ]$: a contradiction.

(iv) Since $D(p_M) > \sum_{j\neq 1} K_j$, if $p_m < p_m^{(i)}$ for some $i : K_i = K_1$, then *a fortiori* $D(p) > \sum_{j\in L} K_j$ for $p \leq p_M$: as a consequence, for any $j \in L$, $\Pi_j(p)$ is increasing for $p \in [p_m, p_m^{(i)})$: a contradiction.

(v.a) A crucial role is played here by statements (i) to (iv) above and the fact that $D(p) > \sum_{j\neq 1} K_j$. To see that $\partial Z_1(p)/\partial\phi_i < 0$ one must check that at least one product on the right-hand side of (4) is strictly negative. This is so for $\psi = R(p) - \{1\}$ if $j \in R(p)$ and $\psi = R(p) \cup \{j\} - \{1\}$ if $j \notin R(p)$: in fact, $q_{1,\psi} - q_{1,\psi'} < 0$ since $0 < q_{1,\psi} < K_1$ and, at the same time, $\Pi_{r\in\psi'}\phi_r \Pi_{s\in N_{-i}-\psi}(1 - \phi_s) > 0$. One can similarly see that $\partial Z_i(p)/\partial\phi_1 < 0$: in fact, we take $\psi = R(p)$ if $i \notin R(p)$ and $\psi = R(p) - \{i\}$ if $i \in R(p)$, and see that $q_{1,\psi} - q_{1,\psi'} < 0$.

(v.b) Now $q_{i,\psi} - q_{i,\psi'} = K_i - K_i = 0$ for any $\psi \in N_{-i}$ such that $1 \notin \psi$, while $q_{i,\psi} - q_{i,\psi'} = 0 - 0 = 0$ for any $\psi \in N_{-i}$ such that $1 \in \psi$.

(v.c) Note that the first inequality (6) holds for $\psi = \{1, j\}$ since $p < P(K_1)$, whereas the second inequality (6) holds for $\psi = N_{-i}$ since $D(p) < K$: therefore $\partial Z_i/\partial \phi_j < 0$ for $n = 3$, since then $\{1, j\} = N_{-i}$. Turning to the oligopoly, note that, by statement (iii), the second inequality (6) also holds for $\psi = R(p) - \{i\}$. Thus $\partial Z_i/\partial \phi_j < 0$ if $\#R(p) = 3$, since then $\{1, j\} = R(p) - \{i\}$. Finally, with $\#R(p) > 3$, let Ψ_1 (Ψ_2) be the set of the subsets ψ of N_{-i} which satisfy the first (resp., the second) inequality (6): neither Ψ_1 nor Ψ_2 are empty. If $\Psi_1 \cap \Psi_2 \neq \emptyset$, then $\partial Z_i/\partial \phi_j < 0$. If instead $\Psi_1 \cap \Psi_2 = \emptyset$, then for any $\psi \in \Psi_1$,

$$D(p) - \sum_{h \in \psi} K_h \geq K_i > -K_j,$$

while, for any $\psi \in \Psi_2$,

$$D(p) - \sum_{h \in \psi} K_h \leq -K_j < K_i.$$

Of course, there is some $\psi_l \in \Psi_1$ such that $\psi_l \cup \{l\} \in \Psi_2$ where $l \in R(p) - \{i, j\}$ and therefore

$$-K_j \geq D(p) - \sum_{h \in \psi_l} K_h - K_l \geq K_i - K_l.$$

Thus $K_l \geq K_i + K_j$. But this cannot hold if either i or j is the largest firm in $R(p)$ apart for firm 1. This completes the proof of the claim.

(vi) If $\#L = n$, then inequality $p_m \leq P(\Sigma_{j \in L} K_j)$ implies that each firm earns no more than its competitive profit, contrary to Proposition 1. Suppose next $\#L < n$. If $p_m < P(\Sigma_{j \in L} K_j)$, then $\Pi_j(p)$ would be increasing over the range $[p_m, \min\{\widehat{p}_m, P(\Sigma_{j \in L} K_j)\}]$ for any $j \in L$. To rule out the event of $p_m = P(\Sigma_{j \in L} K_j)$ when $\#L < n$, it will be shown that otherwise it would be $\lim_{p \to p_m^+} \phi_i'(p) < 0$ for each $i \in L$. Note that $\Pi_i^* = p_m K_i$ and

$$\begin{aligned} \Pi_i^* = \Pi_i(p) &= p[D(p) - \Sigma_{j \in L - \{i\}} K_j] \Pi_{j \in L - \{i\}} \phi_j + p K_i (1 - \Pi_{j \in L - \{i\}} \phi_j) \\ &= p[D(p) - D(p_m)] \Pi_{j \in L - \{i\}} \phi_j + p K_i \end{aligned}$$

in a neighborhood of p_m. Therefore

$$\Pi_{j \in L - \{i\}} \phi_j = \frac{(p_m - p) K_i}{p[D(p) - D(p_m)]}.$$

Then

$$\frac{d \Pi_{j \in L - \{i\}} \phi_j}{dp} = K_i \frac{-p_m [D(p) - D(p_m) + p D'(p)] + p^2 D'(p)}{p^2 [D(p) - D(p_m)]^2},$$

and

$$\lim_{p \to p_m^+} \frac{d\Pi_{j \in L-\{i\}} \phi_j}{dp} = K_i \frac{p_m D''(p) + 2D'(p)}{2p_m^2 [D'(p)]^2} < 0.$$

This in its turn implies that $\lim_{p \to p_m^+} \phi_i'(p) < 0$ for each $i \in L$ since, in the present case, $\phi_i(p) K_i = \phi_j(p) K_j$ for each $i, j \in L$.

(vii) A distinction is drawn according as to whether $p^\circ \in S_1$ or $p^\circ \notin S_1$. In the former case, if contrariwise $\phi_j(p^\circ) < \lim_{p \to p^{\circ +}} \phi_j(p)$ for some $j \neq 1$, then $\lim_{p \to p^{\circ +}} \Pi_1(p) < \lim_{p \to p^{\circ -}} \Pi_i(p)$ since $\partial Z_1 / \partial \phi_j < 0$ because of statement (v): a contradiction. In a similar way it is also proved that $\phi_1(p^\circ) = \lim_{p \to p^{\circ +}} \phi_1(p)$. Assume now that $p^\circ \notin S_1$. It must preliminarily be noted that such an event might only arise (if ever) when $p^\circ < P(K_1)$. Indeed, if $p^\circ \geq P(K_1)$ and $p^\circ \notin S_1$, then, as a consequence of statement (v) above, $d\Pi_i(p)/dp = \partial Z_i / \partial p$ in a neighbourhood of p°, contrary to the fact that $\Pi_i(p)$ is constant in a neighbourhood of p° for any i such that $p^\circ \in S_i$. If, on the other hand, $p^\circ < P(K_1)$, then, according to statement (v.c), $\partial Z_i / \partial \phi_j < 0$ for some i such that $p_m^{(i)} \leq p$. Therefore, if $\phi_j(p^\circ) < \lim_{p \to p^{\circ +}} \phi_j(p)$ for some j, it would be $\lim_{p \to p^{\circ +}} \Pi_i(p) < \lim_{p \to p^{\circ -}} \Pi_i(p)$: a contradiction.

(viii) Since $\Pi_1(p) \leq p \min\{D(p), K_1\}$, then $\Pi_1(p) < \Pi_1^*$ for $p < \max\{\widehat{p}, \widehat{\widehat{p}}\}$. At the same it cannot be that $p_m > \max\{\widehat{p}, \widehat{\widehat{p}}\}$: if it were, then it would be $\Pi_1(p_m^-) = p_m \min\{D(p_m), K_1\} > \Pi_1^*$. ■

Note that, since $\widehat{p}$ is decreasing in K_1, the event of $\widehat{\widehat{p}} \geq \widehat{p}$ arises at relatively large levels of K_1. An immediate consequence of statement (viii) is

Corollary 1. $p_m \geq P(K_1)$ if and only if $\widehat{\widehat{p}} \geq \widehat{p}$.

Note that if $\Pi_j^* = p_m K_j$ for all $j \neq 1$ and $S_i = [p_m, p_M]$ for all i, then the equilibrium distributions would be found, as in duopoly, by solving the n-equation system $\Pi_i^* = Z_i(p, \phi_{-i}(p))$ throughout $[p_m, p_M]$. But there is no guarantee that the above features hold, hence we are not yet in a position to the determine the equilibrium. Yet, we can make some remarks regarding p_M.

Proposition 3 *(i) Let $K_1 > K_2$. Then $\phi_1(p_M) < 1$. (ii) If $K_r = K_1$, then $\phi_r(p) = \phi_1(p)$ for $p \in [p_m, p_M]$, and $\phi_r(p_M) = \phi_1(p_M) = 1$. Furthermore, if at the same time $K_j < K_1$ for some j, then $p_M^{(j)} < p_M$.*

Proof. (i) Suppose contrariwise that $\phi_1(p_M) = 1$. As a consequence, $\Pi_i^* = \Pi_i(p_M^-) = p_M \max\{D(p_M) - \sum_{j \neq i} K_j, 0\}$ for $i \in M - \{1\}$. If $D(p_M) \leq$

$\sum_{j\neq i} K_j$, then $\Pi_i^* = 0$ while $\Pi_i(p_m^-) = p_m K_i > 0$: a contradiction. If, instead, $D(p_M) - \sum_{j\neq i} K_j > 0$, then, since $\arg\max p[D(p) - \sum_{j\neq i} K_j] \in (0, p_M)$ for $i \in M - \{1\}$, it would be $\Pi_i(p) > \Pi_i(p_M^-)$ for some p: a contradiction. Thus it must be $\Pr(p_1 = p_M) > 0$ and $\Pi_i(p_M^-) > \Pi_i(p_M)$.

(ii) Since $D(p_M) > \sum_{j\neq 1} K_j$, we can write $\Pi_r^* = \Pi_r(p) = p\phi_1 E(x_r \mid p_1 < p) + p(1-\phi_1)K_r = p\phi_1[E(x_r \mid p_1 < p) - K_r] + pK_r$, where p is internal to S_r and $E(x_r \mid p_1 < p)$ denotes r's expected output at p conditional on firm 1 charging less than p. Similarly, we can write $\Pi_1^* = \Pi_1(p) = p\phi_r[E(x_1 \mid p_r < p) - K_1] + pK_1$ for p internal to S_1. Obviously, $E(x_r \mid p_1 < p) = E(x_1 \mid p_r < p)$, so that $\Pi_r(p) = \Pi_1(p)$ - as required by Proposition 1 - if and only if $\phi_r = \phi_1$. Further, it cannot be $\phi_r(p_M) = \phi_1(p_M) < 1$, otherwise $\Pi_r(p_M^-) > \Pi_r(p_M)$ contrary to the presumption that p_M is quoted with positive probability by firm r. Nor can it be $p_M^{(j)} = p_M$ for any $j : K_j < K_1$. By arguing as in the proof of the previous statement we would obtain that $\Pi_j(p) > \Pi_j(p_M)$ at some $p < p_M$. ■

4 The triopoly: a complete characterization

In the preceding sections we have seen how there are a number of properties which generalize from the duopoly to oligopoly. Equipped with these results and in order to get further insights for the oligopoly, in the remainder of the paper we provide a comprehensive study of mixed strategy equilibria in the triopoly. Compared to the duopoly, the triopoly will be seen to allow for much wider diversity throughout the region of mixed equilibria, the equilibrium being affected on several grounds by the ranking of p_m and p_M relative to the demand prices of different aggregate capacities, namely, $P(K_1 + K_2)$, $P(K_1 + K_3)$, and $P(K_1)$.

Without loss of generality, in the region of mixed strategy equilibria of the (K_1, K_2, K_3)-space we restrict ourselves to the subset where $K_1 \geq K_2 \geq K_3$. In light of what will be found in this section, that subset can be partitioned in the following way. (Note that, because of Proposition 1 and statement (viii) of Proposition 2, p_M and p_m are known once K_1, K_2, and K_3 are given.)

$$A = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, p_m \leq P(K_1 + K_2), p_M \leq P(K_1 + K_3)\}$$
$$B_1 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3,$$
$$p_m \leq P(K_1 + K_2), P(K_1 + K_3) < p_M \leq P(K_1)\}$$
$$E_1 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, p_m \leq P(K_1 + K_2), p_M > P(K_1)\}$$
$$C_1 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, P(K_1 + K_2) < p_m < P(K_1 + K_3)\}$$
$$C_2 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3, P(K_1 + K_3) \leq p_m, p_M \leq P(K_1)\}$$
$$C_3 = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3,$$
$$P(K_1 + K_3) \leq p_m < \frac{K_1 - K_3}{K_1} P(K_1), p_M > P(K_1)\}$$
$$F = \{(K_1, K_2, K_3) : K_1 \geq K_2 > K_3,$$
$$\max\{P(K_1 + K_3), \frac{K_1 - K_3}{K_1} P(K_1)\} \leq p_m < P(K_1), p_M > P(K_1)\}$$
$$D = \{(K_1, K_2, K_3) : K_1 \geq K_2 \geq K_3, p_m \geq P(K_1)\}$$
$$B_2 = \{(K_1, K_2, K_3) : K_1 \geq K_2 = K_3, p_m < P(K_1), p_M \leq P(K_1)\}$$
$$E_2 = \{(K_1, K_2, K_3) : K_1 \geq K_2 = K_3, p_m < P(K_1), p_M > P(K_1)\}$$

It is easily checked that it is actually $K_1 > K_2 + K_3$ whenever $p_M \geq P(K_1)$, hence at any $(K_1, K_2, K_3) \in C_3 \cup D \cup E_1 \cup E_2 \cup F$, and $K_1 > K_2$ whenever $p_M \geq P(K_1 + K_3)$, hence at any $(K_1, K_2, K_3) \in B_1 \cup C_2$.

The following theorem collects most of the results to be achieved in this section.

Theorem. (a) *In A, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 2, 3\}$ and $M = \{1, 2\}$.*

(b) *In $B_1 \cup B_2$, $\Pi_i^* = p_m K_i$ for all i and $L = M = \{1, 2, 3\}$.*

(c) *In $C_1 \cup C_2 \cup C_3$, $\Pi_i^* = p_m K_i$ for $i \neq 3$ and $\Pi_3^* > p_m K_3$; $L = M = \{1, 2\}$; $p_M^{(3)} < P(K_1)$.*

(d) *In D, $\Pi_1^* = p_m D(p_m)$ and $\Pi_j^* = p_m K_j$ for $j \neq 1$; $\phi_1(p) = 1 - p_m/p$, while $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions such that $pK_2\phi_2 + pK_3\phi_3 = pD(p) - \Pi_1^*$, $\phi_j(p_m) = 0$ and $\phi_j(p_M) = 1$ for $j \neq 1$.*

(e) *In $E_1 \cup E_2$, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 2, 3\}$ and $\#M \geq 2$ with $\widehat{p}_M \geq P(K_1)$. Over $[P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$, and $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions such that $pK_2\phi_2 + pK_3\phi_3 = pD(p) - \Pi_1^*$, $\phi_j(P(K_1)^+) = \phi_j(P(K_1)^-)$ and $\phi_j(p_M) = 1$ for $j \neq 1$.*

(f) *In F, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 3\}$ and $p_m^{(2)} \geq P(K_1)$. Over the range $[P(K_1), p_M]$ distributions are determined like in $E_1 \cup E_2$.*

(g) *In A, $B_1 \cup B_2$, and $C_1 \cup C_2 \cup C_3$, the equilibrium is unique throughout $[p_m, p_M]$. In F, all equilibria share the same ϕ_i over range $[p_m, P(K_1)]$.*

In addition, we will see how to determine $p_m^{(3)}$ and Π_3^* when $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$. We will also deal with the event of a disconnected support, the fact that such an event may hold being established in the following section.

The route leading to the results listed in the Theorem begins with the determination of $\#L$ in the various subsets making up the partition of the region of mixed strategy equilibria. Then we will address the determination of L and the $\Pi_i' s$. Finally, we will determine M in each subset of the partition. In connection to the first task an intermediate step is made by the following Lemma.

Lemma 2. *If* $\#L = 2$, *then* $\Pr(p_j = p_m) = 0$ *for each* $j \in L$*; if* $\#L = 3$ *and* $\Pr(p_i = p_m) > 0$ *for some* i, *then* $\Pr(p_j = p_m) = 0$ *for each* $j \neq i$.

Proof. Let $L = \{i, j\}$. If $\Pr(p_j = p_m) > 0$, then, taking account of statement (vi) of Proposition 2, $\Pi_i^* = \Pi_i(p_m^+) < p_m \min\{D(p_m), K_i\}$ while $\Pi_i(p_m^-) = p_m \min\{D(p_m), K_i\}$: a contradiction. A similar argument establishes the second part of the statement, relating to the event of $L = \{i, j, k\}$.

■

We are now ready to address the determination of $\#L$. First of all note that if $\#L = 3$ then equilibrium distributions constitute a solution of system

$$\Pi_i^* = Z_i(p, \phi_{-i}(p)), \phi_i > 0, \phi_i' \geq 0 \text{ for each } i, \tag{7}$$

in an open to the left right neighbourhood of p_m, where Π_2^* and Π_3^* are constants to be determined. Note, furthermore, that $\Pr(p_i = p_m) = \phi_i(p_m^+)$. The following result addresses the determination of $\#L$ in the whole region of mixed strategy equilibria except set D along with the determination of $\Pr(p_i = p_m)$ throughout the partition. In this connection, it must be noted that subset $B_2 \cup E_2$ can be partitioned into two subsets, one in which $p_m \leq P(K_1 + K_2) = P(K_1 + K_3)$ and one in which $P(K_1 + K_2) = P(K_1 + K_3) < p_m < P(K_1)$. It is shown that whether $\#L = 2$ or $\#L = 3$ depends on the size of p_m relative to $P(K_1 + K_2)$ and $P(K_1)$, as well as on whether $K_2 > K_3$ or $K_2 = K_3$.

Proposition 4 *(i) Let* $p_m \leq P(K_1 + K_2)$ *or, equivalently, let* $(K_1, K_2, K_3) \in A \cup B_1 \cup E_1$ *or* (K_1, K_2, K_3) *fall in the subset of* $B_2 \cup E_2$ *where* $p_m \leq P(K_1 + K_2)$*. Then* $\#L = 3$ *and* $\Pr(p_i = p_m) = 0$ *for each* i.

(ii) Let (K_1, K_2, K_3) *fall in the subset of* $B_2 \cup E_2$ *where* $P(K_1 + K_2) = P(K_1 + K_3) < p_m < P(K_1)$*. Then* $\#L = 3$ *and* $\Pr(p_i = p_m) = 0$ *for each* i.

(iii) Let $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3 \cup F$*, or, equivalently,* $P(K_1 + K_2) < p_m < P(K_1)$ *and* $K_2 > K_3$*. Then* $\#L = 2$.

(iv) Let $(K_1, K_2, K_3) \in D$*, that is,* $p_m \geq P(K_1)$*. Then* $\Pr(p_i = p_m) = 0$ *for each* i*.*

(v) $\Pr(p_i = p_m) = 0$ *for each* $i \in L$*.*

(vi) $\Pi_i^* = p_m K_i$ *for each* $i \in L$*, except that* $\Pi_1^* = p_m D(p_m)$ *in set* D*.*

Proof. (i) The first part is an obvious consequence of statement (vi) of Proposition 2. The second part of the statement is proved by showing that $\phi_i(p_m^+) = 0$ for each i at any solution of system (7). Suppose first that $p_m < P(K_1 + K_2)$. Then the equations in system (7) read

$$\begin{aligned} \Pi_1^* &= p\phi_2\phi_3[D(p) - K] + pK_1, \\ \Pi_2^* &= p\phi_1\phi_3[D(p) - K] + pK_2, \\ \Pi_3^* &= p\phi_1\phi_2[D(p) - K] + pK_3. \end{aligned}$$

Hence $\left[dZ_i(p, \phi_{-i}(p))/dp\right]_{p=p_m^+} = 0$ for each i if and only if

$$\begin{aligned} (D-K)[\phi_2\phi_3 + p_m(\phi_2'\phi_3 + \phi_2\phi_3')] + D'p_m\phi_2\phi_3 + K_1 &= 0, \\ (D-K)[(\phi_1\phi_3 + p_m(\phi_1'\phi_3 + \phi_1\phi_3')] + D'p_m\phi_1\phi_3 + K_2 &= 0, \\ (D-K)[(\phi_1\phi_2 + p_m(\phi_1'\phi_2 + \phi_1\phi_2')] + D'p_m\phi_1\phi_2 + K_3 &= 0, \end{aligned}$$

where $D, D', \phi_1, \phi_2, \phi_3, \phi_1', \phi_2'$, and ϕ_3' are all to be undertood as limits for $p \to p_m^+$. Now, suppose contrariwise that, say, $\phi_1(p_m^+) > 0$ (one might as well suppose either $\phi_2(p_m^+) > 0$ or $\phi_3(p_m^+) > 0$). Then, according to Lemma 2, $\phi_2(p_m^+) = \phi_3(p_m^+) = 0$, and the system above becomes

$$\begin{aligned} p_m(D-K)(\phi_2'\phi_3 + \phi_2\phi_3') &= -K_1, \\ p_m(D-K)(\phi_1'\phi_3 + \phi_1\phi_3') &= -K_2, \\ p_m(D-K)(\phi_1'\phi_2 + \phi_1\phi_2') &= -K_3. \end{aligned}$$

But this system cannot hold. Indeed, in order for the first equation to hold it must be either $\phi_2' = \infty$ or $\phi_3' = \infty$ (or both): then, either the third equation or the second equation (or both) cannot hold. The same logic applies when $p_m = P(K_1 + K_2)$, regardless of whether $K_2 > K_3$ or $K_2 = K_3$. For example, in the former case, the equations in system (7) read

$$\begin{aligned} \Pi_1^* &= p\phi_2[D(p) - K_1 - K_2] - p\phi_2\phi_3K_3 + pK_1, \\ \Pi_2^* &= p\phi_1[D(p) - K_1 - K_2] - p\phi_1\phi_3K_3 + pK_2 \\ \Pi_3^* &= p(1 - \phi_1\phi_2)K_3, \end{aligned}$$

in a right neighbourhood of p_m and the same procedure proves the statement in this case too.

(ii) Assume contrariwise that $p_m^{(1)} = p_m^{(2)} < p_m^{(3)}$. Then

$$\Pi_2^* = Z_2(p, \phi_{-2}(p)) = p\phi_1(D(p) - K_1) + p(1 - \phi_1)K_2$$

$$\Pi_3(p) = Z_3(p, \phi_{-3}(p)) = p\phi_1(1 - \phi_2)(D(p) - K_1) + p(1 - \phi_1)K_2$$

for $p \in (p_m, p_m^{(3)}]$. It is immediately seen that $Z_3(\cdot) < Z_2(\cdot)$ for any $\phi_1, \phi_2 > 0$. Consequently, $\Pi_3^* = Z_3(p_m^{(3)+}, \phi_{-3}(p_m^{(3)})) < \Pi_2^*$: firm 3 has not made a best response since it can guarantee itself Π_2^* by charging p_m. To establish the second part of the statement, assume contrariwise that $\phi_i(p_m^+) > 0$ for some i. Then $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq i$: a contradiction.

(iii) The statement is proved by showing that, if $\#L = 3$, then either $Z_i(p_m^+, \phi_{-i}(p_m^+)) < Z_i(p_m, \phi_{-i}(p_m))$ for some i - a clear contradiction - or system (7) has no solution. The proof runs somewhat differently according as to whether $P(K_1 + K_2) < p_m < P(K_1 + K_3)$ or $P(K_1 + K_3) \leq p_m < P(K_1)$.

(iii.a) $P(K_1 + K_2) < p_m < P(K_1 + K_3)$.

There are three cases to consider: either $\phi_i(p_m^+) > 0$ for some $i \in \{1, 2\}$, or $\phi_3(p_m^+) > 0$, or $\phi_i(p_m^+) = 0$ for each i. In the first case $\Pi_j^* = \Pi_j(p_m^+) < \Pi_j(p_m^-) = p_m K_j$ for $j \in \{1, 2\}$ and $j \neq i$. In both the second and third case the equations in system (7) read

$$\begin{aligned}
\Pi_1^* &= p\phi_2[D(p) - K_1 - K_2] - p\phi_2\phi_3 K_3 + pK_1, \\
\Pi_2^* &= p\phi_1[D(p) - K_1 - K_2] - p\phi_1\phi_3 K_3 + pK_2, \\
\Pi_3^* &= p(1 - \phi_1\phi_2)K_3,
\end{aligned}$$

over range $(p_m, P(K_1 + K_3))$. Then $\left[dZ_i(p, \phi_{-i}(p))/dp\right]_{p=p_m^+} = 0$ if and only if

$$\begin{aligned}
p_m[\phi_2'\phi_3 K_3 + \phi_2\phi_3' K_3 - \phi_2'(D - K_1 - K_2)] &= K_1, \\
p_m[\phi_1'\phi_3 K_3 + \phi_1\phi_3' K_3 - \phi_1'(D - K_1 - K_2)] &= K_2, \\
p_m(\phi_1'\phi_2 + \phi_1\phi_2') &= 1.
\end{aligned}$$

Since $\phi_i' \geq 0$, the first two equations cannot hold unless ϕ_2' and ϕ_1' are both finite, whereas the third equation requires that at least one of them is not.

(iii.b) $P(K_1 + K_3) \leq p_m < P(K_1)$.

Then the equations in system (7) read

$$\begin{aligned}
\Pi_1^* = p[&\phi_2(D(p) - K_1 - K_2) - \phi_2\phi_3(D(p) - K_1) \\
&+ \phi_3(D(p) - K_1 - K_3) + K_1], \\
\Pi_2^* = p[&\phi_1(D(p) - K_1 - K_2) - \phi_1\phi_3(D(p) - K_1) + K_2], \\
\Pi_3^* = p[&\phi_1(D(p) - K_1 - K_3) - \phi_1\phi_2(D(p) - K_1) + K_3],
\end{aligned}$$

for $p \in (p_m, \min\{\widehat{p}_M, P(K_1)\}]$. We consider a partition of four cases. In the first case, $P(K_1 + K_3) < p_m$ and $\phi_i(p_m^+) > 0$ for some i. If $i = 1$, then $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq i$; if $i \in \{2,3\}$, then $\Pi_1^* = Z_1(p_m^+, \phi_{-1}(p_m^+)) < \Pi_1(p_m^-) = p_m K_1$. A similar contradiction is obtained in the second case, in which $P(K_1 + K_3) = p_m$ and $\phi_i(p_m^+) > 0$ for some $i \in \{1,2\}$. Then, $\Pi_j^* = Z_j(p_m^+, \phi_{-1}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq i$ and $j \in \{1,2\}$. As third case, assume that $P(K_1 + K_3) = p_m$ and $\phi_1(p_m^+) = \phi_2(p_m^+) = 0$. Then the proof follows as in the last two cases inspected in (iii.a). The partition is completed by the case where $P(K_1 + K_3) < p_m$ and $\phi_i(p_m^+) = 0$ for each i. Arguing as before it is now obtained

$$\begin{gathered} p_m\left[\phi_2'\phi_3(D-K_1) + \phi_2\phi_3'(D-K_1) - \phi_2'(D-K_1-K_2) + \right. \\ \left. -\phi_3'(D-K_1-K_3)\right] = K_1, \\ p_m[\phi_1'\phi_3(D-K_1) + \phi_1\phi_3'(D-K_1) - \phi_1'(D-K_1-K_2)] = K_2, \\ p_m[\phi_1'\phi_2(D-K_1) + \phi_1\phi_2'(D-K_1) - \phi_1'(D-K_1-K_3)] = K_3. \end{gathered}$$

On close scrutiny, a *necessary* condition for such equations to hold is that $0 < \phi_i' < \infty$ for each i. Granted this, the last two equations become

$$\begin{aligned} -p_m\phi_1'(D-K_1-K_2) &= K_2, \\ -p_m\phi_1'(D-K_1-K_3) &= K_3. \end{aligned}$$

These two equalities cannot simultaneously hold since $K_2 > K_3$ and $D(p_m) > K_1$.

(iv) Under the present circustances, equation $\Pi_1^* = Z_1(p, \phi_{-1})$ reads

$$\Pi_1^* = p_m[D(p_m) - \phi_2 K_2 - \phi_3 K_3].$$

If either $\phi_2(p_m^+) > 0$ or $\phi_3(p_m^+) > 0$, then $\Pi_1^* = Z_1(p_m^+, \phi_{-1}(p_m^+)) < \Pi_1(p_m^-) = p_m D(p_m)$: a contradiction. To dispose of the event of $\phi_1(p_m^+) > 0$, note that $Z_j(p, \phi_{-j}(p)) = p(1-\phi_1)K_j$ for $j \neq 1$: then, if $\phi_1(p_m^+) > 0$, it would be $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \in L - \{1\}$.

(v) It is a consequence of statements (i)-(iv) and Lemma 2.

(vi) It is a consequence of previous statement and Corollary 1. ■

We know from Sections 2 and 3 that p_m and p_M are determined just as in the duopoly. Unlike in duopoly, however, the supports S_i need not be the same for all i, as is immediately revealed by the fact that $\#L = 2$ may hold. One group of related questions is then whether $L = \{1,2\}$ or $L = \{1,3\}$ and how $\widehat{p}_m$ is determined under the circumstances of statement (iii) of Proposition 4. According to the following proposition, $L = \{1,2\}$ in $C_1 \cup C_2 \cup C_3$ and $L = \{1,3\}$ in F. Furthermore, the proposition points the indeterminacy affecting the equilibrium at $p > P(K_1)$ when $\widehat{p}_M > P(K_1)$.

Proposition 5 *(a) Let $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$. Then: (a.i) $L = \{1, 2\}$ and $\Pi_i^* = p_m K_i$ for $i \neq 3$; (a.ii) $(p_m, p_m^{(3)}] \subset S_1 \cap S_2$, $\Pi_3^* = \max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p) > p_m K_3$ and $p_m^{(3)} = \arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p)$, where $\widetilde{\alpha} = [p_m, p_M^*]$, p_M^* is such that $\phi_{2\alpha}(p_M^*) = 1$, $\phi_{1\alpha}$ and $\phi_{2\alpha}$ are such that $\Pi_1^* = Z_1(p, \phi_{-1\alpha})$ and $\Pi_2^* = Z_2(p, \phi_{-2\alpha})$ in the assumption that $\phi_{3\alpha} = 0$, and $\Pi_{3\alpha}(p) = Z_3(p, \phi_{-3\alpha})$.*

(b) If $(K_1, K_2, K_3) \in D$, then $\Pi_1^* = p_m D(p_m)$ and $\Pi_j^* = p_m K_j$ for $j \neq 1$; $\phi_1(p) = 1 - p_m/p$, while $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions such that

$$\phi_2 = \frac{pD(p) - \Pi_1^* - pK_3\phi_3}{pK_2}, \tag{8}$$

$\phi_j(p_m) = 0$ and $\phi_j(p_M) = 1$ for $j \neq 1$. Equation (8) is consistent with $L = \{1, 2, 3\}$, $L = \{1, 2\}$ and $L = \{1, 3\}$, as well as $M = \{1, 2, 3\}$, $M = \{1, 2\}$ and $M = \{1, 3\}$, and even with (non-overlapping) gaps in S_2 and/or S_3. Among the infinite solutions, there exists a simmetric one in ϕ_2 and ϕ_3.

(c) If $(K_1, K_2, K_3) \in F$, then $L = \{1, 3\}$, $p_m^{(2)} \geq P(K_1)$, $(p_m, p_m^{(2)}] \subset S_1 \cap S_3$ and $\Pi_i^* = p_m K_i$ for all i. Over the range $[P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$ while $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions meeting (8) and such that $\phi_3(P(K_1^+) = \phi_3(P(K_1^-)$ and $\phi_2(P(K_1^+) = 0$. It is $\phi_3(P(K_1)) < 1$ unless $\frac{K_1 - K_3}{K_1} P(K_1) = p_m$: in this special case, $S_2 \cap S_3 = \{P(K_1)\}$ and the equilibrium is completely determined.

Proof. (a.i) Given statement (iii) of Proposition 4, we just need to rule out the event of $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$. Consider first $(K_1, K_2, K_3) \in C_1$. Under that event $\Pi_3^* = Z_3(p, \phi_{-3}(p)) = pK_3$ for $p \in (p_m, \min\{p_m^{(2)}, P(K_1 + K_3)\}]$: an obvious contradiction. Next let $(K_1, K_2, K_3) \in C_2$. If it were $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$, then, for $i \in \{1, 3\}$ and $j \notin \{i, 2\}$ it would be $\Pi_i^* = p\phi_j(D(p) - K_j) + p(1 - \phi_j)K_i$ over the range $[p_m, p_m^{(2)}]$, and, as a consequence,

$$\phi_j = \frac{(p_m - p)K_i}{p[D(p) - K_i - K_j]} \tag{9}$$

over that range. By charging a price there firm 2 would get

$$\Pi_2(p) = Z_2(p, \phi_{-2}(p)) = p\phi_1(1 - \phi_3)[D(p) - K_1] + p(1 - \phi_1)K_2,$$

which is lower than $p_m K_2$ at any $p < P(K_1)$. As a consequence, if $p_M < P(K_1)$, $\Pi_2^* = \Pi_2(p_m^{(2)}) < \Pi_2(p_m)$: a contradiction. If instead $p_M = P(K_1)$, then one can avoid this contradiction only by taking $p_m^{(2)} = p_M^{(2)} = p_M$, that is, $\Pr(p_2 = p_M) = 1 > 0$, contrary to Proposition 1. Finally, let

$(K_1, K_2, K_3) \in C_3$. Now, with $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$ it should be $p_m^{(2)} \geq P(K_1)$, to avoid the previous contradiction; but then, according to (9), $\phi_3(P(K_1)) > 1$ since $p_m K_1 < (K_1 - K_3)P(K_1)$.

(a.ii) We first prove that $(p_m, p_m^{(3)}] \subset S_1 \cap S_2$ (namely, S_1 and S_2 exhibit no gaps over that range). If not, then either statement (ii) of Proposition 2 is contradicted or there exists some range $(p^\circ, p^{\circ\circ}) \subset [p_m, p_m^{(3)}]$ which is off S_1 as well as S_2 while $p^\circ \in S_1 \cap S_2$. Now, since $p^\circ \in S_1$, $\left[\frac{d\Pi_1(p)}{dp}\right]_{p=p^\circ -} = \left[\frac{dZ_1(p;\phi_{-1}(p))}{dp}\right]_{p=p^\circ -} = 0$; by statement (v) of Proposition 2, this in its turn implies that $\left[\frac{\partial Z_1(p;\phi_{-1}(p))}{\partial p}\right]_{p=p^\circ -} \geq 0$. This leads to a contradiction: if $\left[\frac{\partial Z_1(p;\phi_{-1}(p))}{\partial p}\right]_{p=p^\circ -} > 0$, then 1 would earn more than Π_1^* by charging slightly more than p°; if $\left[\frac{\partial Z_1(p;\phi_{-1}(p))}{\partial p}\right]_{p=p^\circ -} = 0$ then it could not be $\left[\frac{d\Pi_1(p)}{dp}\right]_{p=p^\circ -} = 0$ on the right of p°, contrary to the fact that $p_M^{(1)} = p_M$.

Let $\alpha = [p_m, p_m^{(3)}]$. Note that, for all i, $\phi_{i\alpha}$ is firm i's equilibrium distribution over α. Furthermore, $\Pi_{3\alpha}(p) = p\phi_{1\alpha}(1-\phi_{2\alpha}) \max\{0, \min\{K_3, \min\{D(p) - K_1\}\} + p(1-\phi_{1\alpha})K_3$ on a neighbouhood of p_m. Of course, $\Pi_{3\alpha}(p) \leq \Pi_3^*$ for $p \in \alpha$. It is easily checked that $\Pi_{3\alpha}(p_m) = p_m K_3 = \Pi_{3\alpha}(P(K_1))$ and, if $P(K_1) > p_M^*$, $\Pi_{3\alpha}(p_M^*) < p_m K_3$; furthermore, $\Pi'_{3\alpha}(p)_{p=p_m} > 0$.[7] It follows immediately that $\Pi_3^* > p_m K_3$, since firm 3 will earn more than $p_m K_3$ at a price higher than and sufficiently close to p_m, and that $\Pi_{3\alpha}(p)$ has an internal maximum over the range $[p_m, \min\{P(K_1), p_M^*\}]$. Thus it cannot be $p_m^{(3)} > \arg\max \Pi_{3\alpha}(p)$, otherwise $\Pi_3^* = \Pi_{3\alpha}(p_m^{(3)}) < \max \Pi_{3\alpha}(p)$, while firm 3 can earn $\max \Pi_{3\alpha}(p)$ by charging $\arg\max \Pi_{3\alpha}(p)$. This being so, let $\beta = [p_m^{(3)}, \min\{P(K_1), \widehat{p}_M\}]$. To rule out the event of $p_m^{(3)} < \arg\max \Pi_{3\alpha}(p)$, note that, on a right neighbourhood of $p_m^{(3)}$, $\Pi_i^* = \Pi_{i\beta}(p) = Z_i(p, \phi_{-i\beta}(p))$ for all i and $\Pi_i^* = \Pi_{i\alpha}(p) = Z_i(p, \phi_{-i\alpha}(p))$ for $i \in \{1, 2\}$. Thus, taking account of statement (v) of Proposition 2, $\phi_{2\beta} < \phi_{2\alpha}$ and $\phi_{1\beta} < \phi_{1\alpha}$ since $\phi_{3\beta} > \phi_{3\alpha} = 0$), implying that $Z_3(p, \phi_{-3\beta}(p)) > Z_3(p, \phi_{-3\alpha}(p))$. But this inequality is contradicted from the fact that $p_m^{(3)} < \arg\max \Pi_{3\alpha}(p)$, implying that $Z_3(p, \phi_{-3\alpha}(p)) > Z_3(p_m^{(3)}, \phi_{-3\alpha}(p_m^{(3)})) = \Pi_3^* = Z_3(p, \phi_{-3\beta}(p))$ on a right neighbourhood of $p_m^{(3)}$.[8]

(b) It is immediately checked that $\Pi_j^* = p\,(1 - \phi_1)\,K_j$ for $j \neq 1$ and

[7] In C_1, $\Pi'_{3\alpha}(p)_{p=p_m} = K_3$, in $C_2 \cup C_3$, $\Pi'_{3\alpha}(p)_{p=p_m} = p_m\,[\phi'_{1\alpha}]_{p=p_m}\,[D(p_m) - K_1 - K_3] + K_3$, where $[\phi'_{1\alpha}]_{p=p_m} = -\frac{K_2}{p_m[D(p_m) - K_1 - K_2]}$.

[8] One might wish to account for the event of $\Pi_{3\alpha}(p)$ reaching its maximum more than once in $\widetilde{\alpha}$. Arguing as in the text, it is established that $\phi_{3\beta} = 0$ for any $p \leq$

$p \in S_j$. Since $\#L > 1$, $\Pi_j^* = p_m K_j$ for some $j \neq 1$, in its turn implying $\phi_1(p) = 1 - \frac{p_m}{p}$. Further, equation $\Pi_1^* = Z_1(p, \phi_{-1}(p))$ reads

$$\Pi_1^* = p\phi_2\phi_3[D(p) - K_2 - K_3] + p\phi_2(1-\phi_3)[D(p) - K_2] \\ + p(1-\phi_2)\phi_3[D(p) - K_3] + p\,(1-\phi_2)\,(1-\phi_3)D(p).$$

This leads to equation (8), leaving one conditional degree of freedom in the determination of ϕ_2 and ϕ_3, additional constraints being, of course, $\phi_j' \geq 0$ throughout $[p_m, p_M]$ for all $j \neq 1$, $\phi_j(p_m) = 0$, and $\phi_j(p_M) = 1$.[9]

As one can easily check, these constraints are met at the symmetric solution of (8), namely,

$$\phi_j(p) = \frac{pD(p) - \Pi_1^*}{p(K_2 + K_3)} \; for \; j \neq 1. \tag{8'}$$

(c) Again taking account of statement (iii) of Proposition 4, we just need to rule out the event of $p_m^{(1)} = p_m^{(2)} < p_m^{(3)}$. Under such an event, $\Pi_3(p) = Z_3(p, \phi_{-2}(p)) = p\phi_1(1-\phi_2)[D(p) - K_1] + p(1-\phi_1)K_3$ in a neighbourhood of p_m, where $\phi_1(p)$ and $\phi_2(p)$ are given by equations (9). It is easily checked that $\Pi_3(p) > p_m K_3$ in an open to the left neighbourhood of p_m. This implies, first, that $\Pi_3^* > p_m K_3$ and, second, that $p_M^{(3)} < P(K_1)$, otherwise $p\,(1-\phi_1)\,K_3 = \Pi_3^* > p_m K_3$ and firm 2 would get $\Pi_2(p) = p\,(1-\phi_1)\,K_2 > p_m K_2 = \Pi_2^*$ at $p \geq P(K_1)$. As a result, $\Pi_1^* = p\phi_2[D(p) - K_2 - K_3] + p(1-\phi_2)[D(p) - K_3]$ over the range $(p_M^{(3)}, P(K_1))$ and $\phi_2 = \frac{p[D(p)-K_3]-\Pi_1^*}{pK_2}$.[10] But then $\phi_2(P(K_1)) \leq 0$ since $p_m K_1 \geq (K_1 - K_3)P(K_1)$: an obvious contradiction. Thus it must be $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$. Further, it cannot be $p_m^{(2)} < P(K_1)$, otherwise - as shown in the proof of statement (a.i) - $\Pi_2^* = \Pi_2(p_m^{(2)}) < p_m K_2$.

By following a procedure used above we get that $(p_m, p_m^{(2)}] \subset S_1 \cap S_3$ (cf. proof of statement (a.ii)). Thus ϕ_1 and ϕ_3 are given by equations (10) over the range $[p_m, P(K_1)]$. Over the range $(P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$, whereas ϕ_2 and ϕ_3 are any pair of non-decreasing functions meeting equation (8) and such that $\phi_3(P(K_1)^+ = \phi_3(P(K_1)^-$. (Note that $\phi_2(P(K_1)^+) = 0$ whenever $\phi_3(P(K_1)^+ = \phi_3(P(K_1)^-$.) Quite interestingly, it can be $p_m^{(2)} > P(K_1)$ rather than $p_m^{(2)} = P(K_1)$. In the former case, $\phi_3 = \frac{pD(p)-\Pi_1^*}{pK_3}$

$\max\{\arg\max_{p \in \tilde{\alpha}} \Pi_{3\alpha}(p)\}$, hence $p_m^{(3)} = \max\{\arg\max_{p \in \tilde{\alpha}} \Pi_{3\alpha}(p)\}$.

[9] By the way, holding equation (9), $\phi_2(p_m) = 0$ if $\phi_3(p_m) = 0$ and $\phi_2(p_M) = 1$ if $\phi_3(p_M) = 1$.

[10] In the assumption that $(p_M^{(3)}, P(K_1)) \subset S_1 \cap S_2$. Assuming otherwise that this range belongs neither to S_1 nor to S_2 would lead to a contradiction.

over the range $[P(K_1), p_m^{(2)}]$ and it would still be $\phi_2(p_m^{(2)})^+) = 0$. Finally, $\phi_3(P(K_1)) = 1$ if and only if $\frac{K_1-K_3}{K_1}P(K_1) = p_m$; in this special case, $\phi_2 = \frac{pD(p)-\Pi_1^*-pK_3}{pK_2}$ over range $[P(K_1), p_M]$. ■

A few remarks are in order as regards the regions of indeterminacy of equilibrium. One can generate solutions with any of the qualitative features claimed in statement (b) of Proposition 5, by slightly perturbing $\phi_3(p)$ around $\phi_j(p)$ (the symmetric solution in ϕ_2 and ϕ_3) over some $[p^\circ, p^{\circ\circ}] \subseteq [p_m, p_M]$. For example, one can obtain infinitely many equilibria with $L = \{1, 2\}$ and $M = \{1, 2, 3\}$ by using a procedure like the following. Take $\phi_3(p) = 0$ for $p \in [p_m, p^\circ]$, next, let $\phi_3(p) = r + s(p - p_m)$ for $p \in [p^\circ, p^{\circ\circ}]$, with r and s chosen such that $\phi_3(p^{\circ\circ}) = \phi_j(p^{\circ\circ})$, and let $\phi_3(p) = \phi_j(p)$ for $p \in [p^{\circ\circ}, p_M]$. It is immediately seen that $\phi_3' > 0$ throughout (p°, p_M) and $\phi_2' > 0$ throughout $(p_m, p^\circ] \cup (p^{\circ\circ}, p_M)$. Straightforward calculus would also show that $\phi_2(p)$ (to be determined according to equation (8)) is such that $\phi_2' > 0$ throughout $(p^\circ, p^{\circ\circ}]$, provided $p^{\circ\circ}$ is chosen close enough to p_m. A similar procedure can be used to generate equilibria with, say, $L = \{1, 3\}$ and $\#M \geq 2$, and even equilibria such that $\phi_j' = 0$ for some $j \neq 1$, over a subset of $[p_m^{(j)}, p_M^{(j)}]$: in other words, S_j need not be connected.[11] Finally, it is worth looking at what underlies the indeterminacy of equilibrium when $(K_1, K_2, K_3) \in D$ or when $(K_1, K_2, K_3) \in F$. Except in a duopoly, this feature can arise when K_1 is sufficiently large. With $n \geq 3$, the output of any $i \neq 1$ when charging $p > P(K_1)$ does not depend on p_j $(j \notin \{i, 1\})$ - the demand forthcoming to i being zero whenever $p_1 < p$ and higher than K_i whenever $p_1 > p$. (Recall that $D(p) > \sum_{i \neq 1} K_i$ at any $p \leq p_M$.) Thus, in region D - where $p_m \geq P(K_1)$ - ϕ_i (each $i \neq 1$) only affects firm 1's payoff at any $p \in (p_m, p_M)$: consequently, there is one degree of freedom in the determination of ϕ_2 and ϕ_3 all over (p_m, p_M). A similar feature holds, in region F, throughout $(P(K_1), p_M)$.

Two remarks are in order about statement (a.ii). If $\arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p) \neq P(K_1 + K_3)$, then $[\phi_{3\beta}']_{p=p_m^{(3)}} = 0$ and $[\phi_{j\beta}']_{p=p_m^{(3)}+} = [\phi_{j\alpha}']_{p=p_m^{(3)}-}$ for $j = 1, 2$; whereas if $\arg\max_{p \in \widetilde{\alpha}} \Pi_{3\alpha}(p) = P(K_1 + K_3)$, then $[\phi_{3\beta}']_{p=p_m^{(3)}} > 0$ and $[\phi_{j\beta}']_{p=p_m^{(3)}+} < [\phi_{j\alpha}']_{p=p_m^{(3)}-}$ for $j = 1, 2$. (We omit the proof, which can be derived straightforwardly.)

[11] In a not dissimilar fashion, with $(K_1, K_2, K_3) \in F$ one can devise a simple method to completing the construction of the equilibrium on the right of $P(K_1)$. Take ϕ_3 as constant at $\phi_3(P(K_1))$ until $\frac{pD(p)-\Pi_1^*-pK_3\phi_3(P(K_1))}{pK_2} = \phi_3(P(K_1))$. Over the remaining range, let $\phi_2(p) = \phi_3(p)$ (the symmetric solution). It can easily be checked that $\phi_2' > 0$ over the range where ϕ_3 is constant.

We still have to determine M in all regions but D and F.

Proposition 6 *(i) Let $(K_1, K_2, K_3) \in A \cup C_1 \cup C_2 \cup C_3$. Then $M = \{1, 2\}$, $p_M^{(3)} < P(K_1)$ and $(p_M^{(3)} p_M] \subset S_1 \cap S_2$.*

(ii) Let $(K_1, K_2, K_3) \in B_1 \cup B_2$. Then $\#M = 3$. Furthermore, $\phi_2(p) = \phi_3(p)$ whenever $K_2 = K_3$.

(iii) Let $(K_1, K_2, K_3) \in E_1 \cup E_2$. Then $p_M^{(j)} \geq P(K_1)$ for $j \neq 1$. For $p \in [P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$ while $\phi_2(p)$ and $\phi_3(p)$ are any non-decreasing functions consistent with equation (8) and such that $\phi_j(P(K_1)^+) = \phi_j(P(K_1)^-)$ and $\phi_j(p_M) = 1$ for $j \neq 1$. This is consistent with $\#M = 3$, $p_M^{(2)} < p_M$, and $p_M^{(3)} < p_M$, and even with (non-overlapping) gaps in S_2 and/or S_3.

Proof. (i) In order to establish that $M = \{1, 2\}$, the set $A \cup C_1 \cup C_2 \cup C_3$ is partitioned into the following regions: region (a), where $p_M \leq P(K_1 + K_2)$; region (b), where $p_m \leq P(K_1 + K_2) < p_M < P(K_1 + K_3)$; region (c), where $p_m \leq P(K_1 + K_2) < P(K_1 + K_3) = p_M$; region (d), where $P(K_1 + K_2) < p_m < p_M < P(K_1 + K_3)$; region (e), $C_2 \cup C_3$; region (f), where $P(K_1 + K_2) < p_m < P(K_1 + K_3) \leq p_M$. This is a partition because regions (a), (b), and (c) make up set A, while regions (d) and (f) make up set C_1.

A constructive argument is provided for region (a). By statement (i) of Proposition 4, $p \in S_1 \cap S_2 \cap S_3$ in a neighbourhood of p_m. Hence, over that neighbourhood equilibrium distributions are the solution of the three-equation system $\Pi_i^* = p\phi_j\phi_r(D(p) - K_j - K_r) + p(1 - \phi_j\phi_r)K_i$, so that $\phi_i = (K_j/K_i)\phi_j$. Based on this, it can be neither $\#M = 3$ nor $p_M^{(2)} < p_M$. It is instead $p_M^{(3)} < p_M$ and $S_1 = S_2 = [p_m, p_M]$ and $S_3 = [p_m, p_M^{(3)}]$ at one equilibrium.[12]

As to regions (b) through (f), we first rule out the event of $\#M = 3$ and then the event of $p_M^{(2)} < p_M$. Recall that, by Proposition 3, with $\#M = 3$ it is $\phi_1(p_M) < 1 = \phi_2(p_M) = \phi_3(p_M)$. Further, in a left neighbourhood of p_M equilibrium distributions would be the solutions of the three-equation system (7), call them $\phi^\circ{}_i$. Let us consider region (c) first. As seen more thoroughly in the following section, solving this system yields $\phi^\circ{}_1 = \sqrt{\frac{K_2}{K_1}\frac{(p-p_m)}{p}}$, $\phi^\circ{}_2 = \frac{K_1}{K_2}\phi^\circ{}_1$, and $\phi^\circ{}_3 = \frac{D(p)-K_1-K_2}{K_3} + \frac{K_1}{K_3}\phi^\circ{}_1$ for $p \in \beta = [P(K_1 + K_2), P(K_1 + K_3)]$. Since $\phi^\circ{}_2(P(K_1 + K_3)) = 1$, then $\phi^\circ{}_1(P(K_1 + K_3)) = K_2/K_1$; upon differentiation of $\phi^\circ{}_3$ and recalling that

[12] By Lemma 4 below, the equilibrium is unique.

$D(p_M) - K_2 - K_3 + p_M \left[D'(p)\right]_{p=p_M} = 0$ and $\Pi_1^* = p_M[D(p_M) - K_2]$, it is found $\left[\phi^{\circ\prime}_3(p)\right]_{p=P(K_1+K_3)^-} = \frac{\left[D'(p)\right]_{p=p_M}}{2K_3} < 0$: a contradiction.

The event $\#M = 3$ in regions (b), (d), (e), and (f) can be dismissed more easily. Under that event, $\Pi_2(p_M^-) = Z_2(p_M, \phi_{-2}(p_M)) = \Pi_2^*$ and $\Pi_3(p_M^-) = Z_3(p_M, \phi_{-3}(p_M)) = \Pi_3^*$. These two equations contradict each other since $\phi_2(p_M) = \phi_3(p_M) = 1$. For example, if the former holds, then $\Pi_3(p_M^-) < \Pi_3^*$ and the latter cannot hold. Let us see how this works in each case. Note that, both in (e) and (f), $p_M \geq P(K_1 + K_3)$. Hence, in either case, under our working assumption it would be $\Pi_2^* = p_m K_2 = p_M[1 - \phi_1(p_M)]K_2$. This yields $\phi_1(p_M) = 1 - p_m/p_M$, in its turn implying $Z_3(p_M^-) = p_M[1 - \phi_1(p_M)]K_3 = p_m K_3$, contrary to statement (a.ii) of Proposition 5. In (d), $\Pi_2^* = p_m K_2 = Z_2(p_M^-) = p_M[\phi_1(p_M)(D(p_M) - K_1 - K_3) + (1 - \phi_1(p_M))K_2]$, yielding $\phi_1(p_M) = \frac{p_M - p_m}{p_M} \frac{K_2}{K - D(p_M)}$. By substituting this into $Z_3(p_M^-) = p_M[1 - \phi_1(p_M)]K_3$ it is obtained $Z_3(p_M^-) = \frac{p_M[K_1 + K_3 - D(p_M)] + p_m K_2}{K - D(p_M)} K_3$. Note that $\frac{p_M[K_1 + K_3 - D(p_M)] + p_m K_2}{K - D(p_M)} < p_m$ since $P(K_1 + K_3) > p_M$; hence $Z_3(p_M^-) < p_m K_3$, contrary to statement (a.ii) of Proposition 5. A similar argument applies to (b).

It remains to dismiss the event of $p_M^{(2)} < p_M$ in regions (b), (c), (d), (e), and (f). This is done by showing that it would otherwise be $\Pi_2(p) > \Pi_2^*$ in a left neighbourhood of p_M. If $p_M^{(2)} < p_M$ in regions (d), (e) and (f), then $\Pi_3(p_M^-) = p_M[1 - \phi_1(p_M)]K_3 = \Pi_3^* > p_m K_3$, implying $\phi_1(p_M) = 1 - \frac{\Pi_3^*}{p_M K_3} < 1 - \frac{p_m}{p_M}$ and hence $\Pi_2(p_M^-) = p_M \phi_1(p_M) \max\{0, D(p_M) - K_1 - K_3\} + p_M[1 - \phi_1(p_M)]K_2 \geq \frac{\Pi_3^*}{K_3} K_2 > \frac{p_m K_3}{K_3} K_2 = p_m K_2$. If $p_M^{(2)} < p_M$ under (b) or (c), then $\phi_1(p) = 1 - \frac{p_m}{p}$ in a neighbourhood of p_M. Consequently, by charging a price in that neighbourhood firm 2 would earn $\Pi_2(p) = p\phi_1\phi_3[D(p) - K_1 - K_3] + p\phi_1(1 - \phi_3)[D(p) - K_1] + p(1 - \phi_1)K_2 > p(1 - \phi_1)K_2 = p_m K_2 = \Pi_2^*$.

Next we prove that $p_M^{(3)} < P(K_1)$. This is trivial when $p_M \leq P(K_1)$, i.e. in $A \cup C_2$. In $C_1 \cup C_3$, $\Pi_3^* > p_m K_3$ and if $p_M^{(3)} \geq P(K_1)$, then $\Pi_3^* = P(K_1)[1 - \phi_1(P(K_1))]K_3 = p_m K_3$ and $p_m K_2 = P(K_1)[1 - \phi_1(P(K_1))]K_2$: an obvious contradiction.

The claim that $(p_M^{(3)}, p_M] \subset S_1 \cap S_2$ is a bit cumbersome to prove. The event of there being $p^\circ \geq p_M^{(3)}$ such that $p^\circ \in S_1 \cap S_2$ and $(p^\circ, p^{\circ\circ})$ belonging neither to S_1 nor to S_2 is dismissed by arguing similarly as in the proof of statement (a) of Proposition 5. Next we rule out the joint event of a gap $(p^\circ, p^{\circ\circ})$ in S_1 and a gap $(p_M^{(3)}, p^{\circ\circ})$ in S_2 when $p^\circ < p_M^{(3)}$ and $p^\circ \in S_1 \cap S_2$. Under such an event, $\left[\frac{d\Pi_2(p)}{dp}\right]_{p=p^{\circ-}} = \left[\frac{dZ_2(p;\phi_{-2}(p))}{dp}\right]_{p=p^{\circ-}} = 0$, implying

$\left[\frac{\partial Z_2(p;\phi_{-2}(p))}{\partial p}\right]_{p=p^{\circ -}} \geq 0$. Likewise, $\left[\frac{\partial Z_2(p;\phi_{-2}(p))}{\partial p}\right]_{p=p_M^{(3)-}} \geq 0$ as a consequence of the fact that $p_M^{(3)} \in S_2$. However, one can check that $Z_2(p;\phi_{-2}(p))$ is differentiable and concave everywhere in p when $\phi_3(p) = 1$. Therefore, a contradiction would at any rate be reached: if $\left[\frac{\partial Z_2(p;\phi_{-2}(p))}{\partial p}\right]_{p=p_M^{(3)-}} > 0$, then firm 2 would get more than Π_2^* by charging a price slightly higher than $p_M^{(3)}$; if $\left[\frac{\partial Z_2(p;\phi_{-2}(p))}{\partial p}\right]_{p=p_M^{(3)-}} = 0$, then it would be $\left[\frac{\partial Z_2(p;\phi_{-2}(p))}{\partial p}\right]_{p=p^{\circ\circ +}} < 0$, contrary to the fact that $p^{\circ\circ} \in S_2$. Finally, the joint event of a gap $(p^\circ, p^{\circ\circ})$ in S_2 and a gap $(p_M^{(3)}, p^{\circ\circ})$ in S_1 when $p^\circ < p_M^{(3)}$ and $p^\circ \in S_1 \cap S_2$ is ruled out (even more easily) by relying on the concavity of $Z_1(p;\phi_{-1}(p))$.

(ii) Note that, by statements (i) , (ii), and (vi) of Proposition 4, $\#L = 3$ and $\Pi_i^* = p_m K_i$. Consider first the case where $(K_1, K_2, K_3) \in B_1$. If $p_M^{(j)} < p_M$ for some $j \neq 1$, then one can easily check that $\Pi_j(p) > \Pi_j^*$ for $p \in [\max\{p_M^{(3)}, P(K_1 + K_3)\}, p_M]$. Turn next to the case where $(K_1, K_2, K_3) \in B_2$. Here $\Pi_2^* = \Pi_3^* = p_m K_2$ (recall that $K_2 = K_3$), hence $p_m K_2 = Z_2(p, \phi_{-2}(p)) = Z_2(p, \phi_{-3}(p))$ on a right neighbourhood of p_m. Therefore, $\phi_2(p) = \phi_3(p)$ and, of course, $\#M = 3$.

(iii) According to statements (i), (ii) and (vi) of Proposition 4, $\Pi_j^* = p_m K_j$ for any $j \neq 1$. Also, $\Pi_j(p) = p(1 - \phi_1) K_j$ for $p \in [P(K_1), p_M]$: this leads to $\phi_1(p) = 1 - p_m/p$ since $p_M^{(j)} = p_M$ for some $j \neq 1$. Now, if it were $p_M^{(r)} < P(K_1)$ then it would be $\Pi_r(p) > \Pi_r^*$ for $p \in [p_M^{(r)}, P(K_1)]$, as one can easily check. Also, the argument in the proof of statement (ii) of Proposition 5 leads to the stated relationship between ϕ_2 and ϕ_3 over $[P(K_1), p_M]$. Any ϕ_2 and ϕ_3 consistent with equation (8) constitutes a pair of equilibrium distributions so long as $\phi_j' \geq 0$, $\phi_j(p_M) = 1$ for $j \neq 1$, and $\phi_j(p)$ is continuous in $P(K_1)$. ■

It remains to establish statement (g) of the Theorem, concerning the uniqueness of the equilibrium. First, equilibrium distributions are uniquely determined over $[p_m, \widehat{p}_m]$ and over $[\widehat{p}_M, p_M]$.

Proposition 7 *In $C_1 \cup C_2 \cup C_3$, ϕ_1 and ϕ_2 are uniquely determined over the range $[p_m, p_m^{(3)}] \cup [p_M^{(3)}, p_M] \subset S_1 \cap S_2$ and the same holds, in A, over the range $[p_M^{(3)}, p_M]$. Similarly, in F, ϕ_1 and ϕ_3 are uniquely determined over the range $[p_m, P(K_1)]$.*

Proof. The proposition follows from the fact that $\partial Z_1/\partial \phi_j < 0$ and $\partial Z_j/\partial \phi_1 < 0$. ■

As to the range $[\widehat{p}_m, \widehat{p}_M]$, an obvious candidate as a profile of equilibrium distributions is a solution to the equations in system (7). Denoting any such solution as $({\phi_1}^\circ, {\phi_2}^\circ, {\phi_3}^\circ)$, we have the following uniqueness result.

Lemma 3. (i) *$({\phi_1}^\circ, {\phi_2}^\circ, {\phi_3}^\circ)$ is unique at any $p \leq P(K_1)$.*

(ii) In A, $B_1 \cup B_2$ and $C_1 \cup C_2 \cup C_3$, if ${\phi_1}^\circ, {\phi_2}^\circ$, and ${\phi_3}^\circ$ are increasing over the range $(\widehat{p}_m, \widehat{p}_M)$, then ${\phi_1}^\circ, {\phi_2}^\circ$, and ${\phi_3}^\circ$ are the equilibrium distributions throughout $(\widehat{p}_m, \widehat{p}_M)$.

Proof. (i) Let contrariwise $(\widehat{\phi}^\circ{}_1, \widehat{\phi}^\circ{}_2, \widehat{\phi}^\circ{}_3)$ be another solution and let, without loss of generality, $\widehat{\phi}^\circ{}_1(p) < \phi^\circ{}_1(p)$ at some p. Then, since $\partial Z_3/\partial \phi_2 < 0$ and $\partial Z_2/\partial \phi_3 < 0$, it should be $\widehat{\phi}^\circ{}_2(p) > \phi^\circ{}_2(p)$ in order for $Z_3(p, \widehat{\phi}^\circ{}_{-3}) = \Pi_3^*$ and it should be $\widehat{\phi}^\circ{}_3(p) > \phi^\circ{}_3(p)$ in order for $Z_2(p, \widehat{\phi}^\circ{}_{-2}) = \Pi_2^*$. Consequently, since $\partial Z_1/\partial \phi_j < 0$ for $j \neq 1$, it would be $Z_1(p, \widehat{\phi}^\circ{}_{-1}) < \Pi_1^*$: a contradiction.

(ii) It must preliminarily be noted that $p < P(K_1)$ at any $p \in (\widehat{p}_m, \widehat{p}_M)$ in A, $B_1 \cup B_2$, and $C_1 \cup C_2 \cup C_3$; hence we are in the circumstances of statement (i). The statement is violated if and only if there is a gap $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m, \widehat{p}_M]$ in S_j for some j, so that $\phi_j(\widetilde{\widetilde{p}}) = \phi_j(\widetilde{p}^+)$. On the other hand, ${\phi_j}^\circ(\widetilde{\widetilde{p}}^+) > \phi_j(\widetilde{p}) = {\phi_j}^\circ(\widetilde{p})$: consequently, either $\widetilde{p}$ or $\widetilde{\widetilde{p}}$ or both are charged with positive probability, contrary to statement (vii) of Proposition 2. ■

Up to now, the first claim of statement (g) of the Theorem is proved if ${\phi_1}^\circ, {\phi_2}^\circ$, and ${\phi_3}^\circ$ are increasing throughout $[\widehat{p}_m, \widehat{p}_M]$. Unfortunately, this need not be so. An example is provided in the following section where, among other things, we will identify a gap in S_3 in a specific region of the capacity space. The task to be addressed now is how equilibrium distributions are determined when ${\phi_i}^{\circ\prime} < 0$ for some i over some interval(s) in $[\widehat{p}_m, \widehat{p}_M]$, with $\widehat{p}_M \leq P(K_1)$. We establish, first, a general property about any gap in S_i.

Lemma 4. *Let $N = \{i, j, r\}$ and let $(\widetilde{p}, \widetilde{\widetilde{p}})$ be a gap in S_i, with $\widetilde{\widetilde{p}} \leq P(K_1)$. Then it must be $\phi^\circ{}_i(\widetilde{\widetilde{p}}) = \phi^\circ{}_i(\widetilde{p})$ and $\phi^\circ{}_i(p) \geq \phi^\circ{}_i(\widetilde{p}) = \phi_i(p) = \phi^\circ{}_i(\widetilde{\widetilde{p}})$ at any $p \in (\widetilde{p}, \widetilde{\widetilde{p}})$.*

Proof. Suppose contrariwise that $\phi^\circ{}_i(\widetilde{\widetilde{p}}) > \phi^\circ{}_i(\widetilde{p})$. Since $\widetilde{p} \in S_i \cap S_j \cap S_r$ and $\widetilde{\widetilde{p}} \in S_i \cap S_j \cap S_r$, $\phi_i(\widetilde{\widetilde{p}}^+) > \phi_i(\widetilde{p})$, so that either $\widetilde{p}$ or $\widetilde{\widetilde{p}}$ or both would be charged with positive probability, contrary to statement (vii) of Proposition 2. Suppose next that $\phi^\circ{}_i(p) < \phi^\circ{}_i(\widetilde{p}) = \phi_i(p)$ for some $p \in (\widetilde{p}, \widetilde{\widetilde{p}})$. Since $Z_j(p, {\phi_i}^\circ(\widetilde{p}), \phi_r(p)) = \Pi_j^* = Z_j(p, {\phi_i}^\circ(p), {\phi_r}^\circ(p))$, $\phi_r(p) < \phi^\circ{}_r(p)$. Similarly, since $Z_r(p, {\phi_i}^\circ(\widetilde{p}), \phi_j(p)) = \Pi_r^* = Z_r(p, {\phi_i}^\circ(p), {\phi_j}^\circ(p))$,

$\phi_j(p) < \phi^\circ{}_j(p)$. Consequently, firm i would get more than its equilibrium profit by charging p: $Z_i(p, \phi_j, \phi_r) > \Pi_i^* = Z_i(p, \phi_j{}^\circ, \phi_r{}^\circ)$. ■

We can now determine the equilibrium when $\phi^\circ{}'_i < 0$ for some i over some subset of $[\widehat{p}_m, \widehat{p}_M]$.

Proposition 8 *Let $N = \{i, j, r\}$ and $\phi^\circ{}'_i < 0$ over range $\lambda_i \subset [\widehat{p}_m, \widehat{p}_M]$, where $\widehat{p}_M \le P(K_1)$. Let $\phi^\circ{}_i$, $\phi^\circ{}_j$, and $\phi^\circ{}_r$ be non-decreasing in $\gamma = [\widetilde{\widetilde{p}}, \widehat{p}_M]$, where $\widetilde{\widetilde{p}} = \sup \lambda_i$. Let $\widetilde{p}$ be the largest solution of $\phi_i{}^\circ(p) = \phi_i{}^\circ(\widetilde{\widetilde{p}})$ in the range $(\widehat{p}_m, \widetilde{\widetilde{p}})$. Finally, let $\alpha = [\widehat{p}_m, \widetilde{p}]$ and $\beta = (\widetilde{p}, \widetilde{\widetilde{p}})$.*

(a) As to range $\beta \cup \gamma$:

(a.i) equilibrium distributions are $\phi^\circ{}_i, \phi^\circ{}_j$, and $\phi^\circ{}_r$ over γ;

(a.ii) S_j and S_r are connected throughout $\beta \cup \gamma$ and S_i is disconnected with a gap equal to β.

(b) Moving on the left of $\widetilde{p}$:

(b.i) If $\phi^\circ{}_i$, $\phi^\circ{}_j$, and $\phi^\circ{}_r$ are never decreasing in α, then they are the equilibrium distributions throughout α.

(b.ii) If $\phi^\circ{}'_s < 0$ over range $\lambda_s \subset \alpha$ ($s \in \{i, j, r\}$), $\phi^\circ{}'_i$, $\phi^\circ{}'_j$, and $\phi^\circ{}'_r$ are non-negative at $p = \widetilde{p}$, and, similarly as before, $\phi^\circ{}_i$, $\phi^\circ{}_j$, and $\phi^\circ{}_r$ are never decreasing over range $[\widetilde{\widetilde{q}}, \widetilde{p}]$, where $\widetilde{\widetilde{q}} = \sup \lambda_s$, then: $\phi^\circ{}_i$, $\phi^\circ{}_j$, and $\phi^\circ{}_r$ are the equilibrium distributions over $[\widetilde{\widetilde{q}}, \widetilde{p}]$; there is a gap $(\widetilde{q}, \widetilde{\widetilde{q}})$ in S_s, where $\widetilde{q}$ is the (largest) solution of $\phi_s{}^\circ(p) = \phi_s{}^\circ(\widetilde{\widetilde{q}})$ in the range $(\widehat{p}_m, \widetilde{\widetilde{q}})$; the other two supports are connected throughout $[\widetilde{q}, \widetilde{p}]$.

(b.iii) If $\phi^\circ{}'_j < 0$ at $p = \widetilde{p}$, then there is a gap $(\widetilde{q}, \widetilde{p})$ in S_j, where $\widetilde{q}$ is the (largest) solution of $\phi_j{}^\circ(p) = \phi_j{}^\circ(\widetilde{p})$ in the range $(\widehat{p}_m, \widetilde{p})$.

(c) In the circumstances of statement (b.ii) or (b.iii), equilibrium distributions on the left of $\widetilde{q}$ are found along the same lines as in (b), until the stage is reached where $\phi^\circ{}_i$, $\phi^\circ{}_j$, and $\phi^\circ{}_r$ are never decreasing over the right neighbourhood of $\widehat{p}_m$ still left to analyze: then they provide the equilibrium distribution over that neighbourhood.

Proof. By construction, each firm gets its equilibrium payoff at any $p \in \gamma$, and the same holds for j and r at any $p \in \beta$: $Z_j(p, \phi_i{}^\circ(\widetilde{p}), \phi_r(p)) = \Pi_j^*$ and $Z_r(p, \phi_i{}^\circ(\widetilde{p}), \phi_j(p)) = \Pi_r^*$. Further, it does not pay for firm i to charge any $p \in \beta$: $Z_i(p, \phi_j, \phi_r) < \Pi_i^* = Z_i(p, \phi_j{}^\circ, \phi_r{}^\circ)$ since $\phi_j > \phi^\circ{}_j$ and $\phi_r > \phi^\circ{}_r$ throughout β. One can argue likewise while moving on the left of $\widetilde{p}$ and up to $\widehat{p}_m$: thus the strategy profile under consideration constitutes an equilibrium.

To check uniqueness, we begin by noting that, by Lemma 4, there are not equilibria where ϕ_i (or ϕ_j or ϕ_r) is constant over any interval in γ. Lemma 4 also allows us to dismiss any strategy profile such that any interval different from β constitutes a gap in S_i in the range $\beta \cup \gamma$. Nor can there be equilibria where any subset of β constitutes a gap in, say, S_j even if ${\phi^\circ}'_j < 0$ over some interval $\lambda_j \subset \beta$. (The argument goes likewise in the circumstances of statement (b.iii).) In order not to immediately violate Lemma 4, let $(\widehat{q}, \sup \lambda_j)$ be a gap in S_j, with $\widehat{q}$ such that ${\phi^\circ}_j(\widehat{q}) = {\phi^\circ}_j(\sup \lambda_j)$. Then it would be $\phi_i(\sup \lambda_j) = {\phi^\circ}_i(\sup \lambda_j) > {\phi^\circ}_i(\widetilde{\widetilde{p}})$: as a result, one could not construct a gap for S_i consistent with Lemma 4. ■

5 On the event of a disconnected support

Based on the results above one should be able to compute the mixed strategy equilibrium once the demand function and the firm capacities are fixed. This will be shown in this section. More precisely we will determine the equilibrium for $(K_1, K_2, K_3) \in B_1$. This region is of special interest. We will show, in fact, that S_3 may be disconnected under well-specified circumstances.

Let us partition the range $[p_m, p_M]$ into three subsets: $\alpha = [p_m, P(K_1 + K_2)$, $\beta = [P(K_1 + K_2), P(K_1 + K_3))$, and $\gamma = [P(K_1 + K_3), p_M]$. In α the equations in system (7) read

$$\begin{cases} \Pi_1^* = p\left\{\phi_{2\alpha}\phi_{3\alpha}(D(p) - K_2 - K_3) + [\phi_{2\alpha}(1 - \phi_{3\alpha}) + (1 - \phi_{2\alpha})]K_1\right\} \\ \Pi_2^* = p\left\{\phi_{1\alpha}\phi_{3\alpha}(D(p) - K_1 - K_3) + [\phi_{1\alpha}(1 - \phi_{3\alpha}) + (1 - \phi_{1\alpha})]K_2\right\} \\ \Pi_3^* = p\left\{\phi_{1\alpha}\phi_{2\alpha}(D(p) - K_1 - K_2) + [\phi_{1\alpha}(1 - \phi_{2\alpha}) + (1 - \phi_{1\alpha})]K_3\right\}, \end{cases}$$

and the solution is

$$\phi^\circ{}_{1\alpha} = \sqrt{\frac{K_2}{K_1}\frac{(p_m - p)K_3}{p(D(p) - K)}}, \phi^\circ{}_{2\alpha} = \frac{K_1}{K_2}\phi^\circ{}_{1\alpha}, \phi^\circ{}_{3\alpha} = \frac{K_1}{K_3}\phi^\circ{}_{1\alpha}. \tag{10}$$

In β, the equations in system (7) read

$$\begin{cases} \Pi_1^* = p\left[\phi_{2\beta}\phi_{3\beta}(D(p) - K_2 - K_3) + \phi_{2\beta}\left(1 - \phi_{3\beta}\right)(D(p) - K_2) + \left(1 - \phi_{2\beta}\right)K_1\right], \\ \Pi_2^* = p[\phi_{1\beta}\phi_{3\beta}(D(p) - K_1 - K_3) + \phi_{1\beta}\left(1 - \phi_{3\beta}\right)(D(p) - K_1) + \left(1 - \phi_{1\beta}\right)K_2], \\ \Pi_3^* = p[\phi_{1\beta}\left(1 - \phi_{2\beta}\right) + \left(1 - \phi_{1\beta}\right)]K_3, \end{cases}$$

and the solution is

$$\phi^\circ{}_{1\beta} = \sqrt{\frac{K_2}{K_1}\frac{(p - p_m)}{p}}, \phi^\circ{}_{2\beta} = \frac{K_1}{K_2}\phi_{1\beta}, \phi^\circ{}_{3\beta} = \frac{D(p) - K_1 - K_2}{K_3} + \frac{K_1}{K_3}\phi^\circ{}_{1\beta}. \tag{11}$$

In γ, the equation in system (7) read

$$\begin{cases} \Pi_1^* = p\left[\phi_{2\gamma}\phi_{3\gamma}(D(p)-K_2-K_3)+p\phi_{2\gamma}\left(1-\phi_{3\gamma}\right)(D(p)-K_2)\right. \\ \qquad\qquad \left. +\left(1-\phi_{2\gamma}\right)\phi_{3\gamma}(D(p)-K_3)+\left(1-\phi_{2\gamma}\right)(1-\phi_{3\gamma})K_1\right] \\ \Pi_2^* = p\left[\phi_{1\gamma}\left(1-\phi_{3\gamma}\right)(D(p)-K_1)+\left(1-\phi_{1\gamma}\right)K_2\right] \\ \Pi_3^* = p\left[\phi_{1\gamma}\left(1-\phi_{2\gamma}\right)(D(p)-K_1)+\left(1-\phi_{1\gamma}\right)K_3\right], \end{cases}$$

and the solution is

$$\begin{aligned} \phi^\circ{}_{1\gamma} &= \sqrt{\frac{K_2K_3(p-p_m)^2}{p^2(D(p)-K_1-K_2)(D(p)-K_1-K_3)+(p-p_m)K_1p(D(p)-K_1)}}, \\ \phi^\circ{}_{2\gamma}(p) &= 1-\frac{K_3}{K_2}+\frac{K_3}{K_2}\phi^\circ{}_{3\gamma} \qquad (12) \\ \phi^\circ{}_{3\gamma} &= \frac{(p-p_m)K_2+p\phi^\circ{}_{1\gamma}(p)(D(p)-K_1-K_2)}{p\phi^\circ{}_{1\gamma}(D(p)-K_1)}. \end{aligned}$$

In range α, $\phi^{\circ\prime}_{i\alpha}>0$ and $\phi^\circ{}_{i\alpha}(P(K_1+K_2))<1$ for all i. (If $\phi^{\circ\prime}_{i\alpha}<0$ for some i, then $\phi^{\circ\prime}_{j\alpha}<0$ for all $j\neq i$, thereby violating the requirement that $\Pi_i'=0$.) In range γ, $\phi_{1\gamma}(p_M)<1=\phi_{2\gamma}(p_M)=\phi_{3\gamma}(p_M)$ and $\phi_{i\gamma}'>0$ in the interior of γ, with $\phi_{3\gamma}'=\phi_{2\gamma}'=0$ at $p=p_M$ (see the Appendix). The problem lies with range β, as it might be $\phi^{\circ\prime}_{3\beta}<0$ in a left neighbourhood of $P(K_1+K_3)$. Note that

$$\phi^{\circ\prime}_{3\beta}=\frac{D'(p)}{K_3}+\frac{K_1}{K_3}\phi^{\circ\prime}_{1\beta}=\frac{D'(p)}{K_3}+\frac{1}{2}\left(\frac{K_2}{K_1}\frac{(p-p_m)}{p}\right)^{-1/2}\frac{K_2}{K_3}\frac{p_m}{p^2}.$$

Since $\phi^{\circ\prime}_{3\beta}$ is decreasing, it will be $\phi^{\circ\prime}_{3\beta}>0$ throughout β if and only if $[\phi^{\circ\prime}_{3\beta}]_{p=P(K_1+K_3)}\geq 0$. This in its turn amounts to

$$K_2p_m\geq -2\left[D'(p)\right]_{p=P(K_1+K_3)}\times[P(K_1+K_3)]^2\sqrt{\frac{K_2}{K_1}\left(1-\frac{p_m}{P(K_1+K_3)}\right)}. \qquad (13)$$

If this inequality holds, then equilibrium distributions are actually the $\phi^\circ{}_{i\beta}$'s throughout β. If not, then, by Proposition 8, S_3 has a gap, equal to $[\widetilde{p},P(K_1+K_3)]$. Two cases are possibile according as to whether $\phi^\circ{}_{3\beta}(P(K_1+K_3))\geq\phi^\circ{}_{3\beta}(P(K_1+K_2))$ or $\phi^\circ{}_{3\beta}(P(K_1+K_3))<\phi^\circ{}_{3\beta}(P(K_1+K_2))$. In the former case $\widetilde{p}$ is such that $\phi^\circ{}_{3\beta}(\widetilde{p})=\phi^\circ{}_{3\beta}(P(K_1+K_3))$, in the latter it is such that $\phi^\circ{}_{3\alpha}(\widetilde{p})=\phi^\circ{}_{3\beta}(P(K_1+K_3))$. In the former case, the equilibrium distributions are provided by equations (10) throughout α and by equations (11) over subset $[P(K_1+K_2), \widetilde{p}]$ of β, the remaining subset $[\widetilde{p}, P(K_1+K_3)]$ being the gap in S_3: here $\phi_3 = \phi°_{3\beta}(P(K_1+K_3))$, $\phi_1 = \frac{\Pi_2^* - pK_2}{p[D(p)-K_1-K_2-\phi_3 K_3]}$ and $\phi_2 = \frac{\Pi_1^* - pK_1}{p[D(p)-K_1-K_2-\phi_3 K_3]}$. In the latter case, equations (10) provide the equilibrium distributions over subset $[p_m, \widetilde{p}]$ of α and $\phi_3 = \phi°_{3\beta}(P(K_1+K_3))$ throughout range $[\widetilde{p}, P(K_1+K_3)]$, the gap in S_3. Now $\phi_1 = \frac{\Pi_2^* - pK_2}{p\phi_3(D(p)-K)}$ and $\phi_2 = \frac{\Pi_1^* - pK_1}{p\phi_3(D(p)-K)}$ over subset $[\widetilde{p}, P(K_1+K_2)]$ of the gap and $\phi_1 = \frac{\Pi_2^* - pK_2}{p[D(p)-K_1-K_2-\phi_3 K_3]}$ and $\phi_2 = \frac{\Pi_1^* - pK_1}{p[D(p)-K_1-K_2-\phi_3 K_3]}$ over the remaining subset $[\widetilde{p}, P(K_1+K_3)]$.

Examples.

First example: $D(p) = 10 - p$, $K_1 = 5.98$, $K_2 = 1$, and $K_3 = 0.97$. Then $p_M = 4.015$ and $p_m = 4.015^2/5.98$, while $\Pi_i^* = p_m K_i$ for each i. Condition (13) is met, hence $S_i = [p_m, p_M]$ for all i.

Second example: $D(p) = 10 - p, K_1 = 23/4$, $K_2 = 3$, $K_3 = 2$. Then $p_M = 4.015$ and $p_m = 25/23$, while $\Pi_i^* = \Pi_i^* = p_m K_i$ for each i. Condition (13) is violated, hence ϕ_3 is constant over range $[\widetilde{p}, P(K_1+K_3)]$, where $\widetilde{p} \approx 1.57358$ and $P(K_1+K_3) = 9/4$.

6 Concluding remarks

In this paper we have extended the analysis of price competition among capacity-constrained sellers beyond the cases of duopoly and symmetric oligopoly. We have first provided some general results for the oligopoly. They include the fact that the minimum element of the support of the equilibrium strategy is determined for the largest firm like in duopoly (a similar result was recently provided as for the maximum element). Apart from these results it emerged that mixed strategy equilibria might look quite different depending on the firm capacities: a taxonomy is required. For this reason we turned to the analysis of the triopoly and have provided a complete characterization for this case. In particular we have partitioned the region of the capacity space where the equilibrium is mixed according to the features of the mixed strategy equilibrium found to arise in each subregion. Having done this, computing the mixed strategy equilibrium becomes a much easier task, as exemplified by Section 5. Our analysis has discovered interesting new features, which do not arise in the duopoly. Among them, there is the fact that the supports of the equilibrium strategies need not coincide across all the firms and the fact that, in some subregions, there is one degree of freedom in the equilibrium distributions of firms other than the largest one (features which have also been discovered by Hirata (2008)). Another very interesting feature - not yet emerged before in the context of concave demand, constancy and equality of unit cost and efficient rationing - is the possibility of the support of the equilibrium strategy being disconnected.

7 Appendix

Since $\Pi_1' = 0$ we obtain:

$$\phi_{3\gamma}' = \frac{\Pi_1^* + p^2 D'(p)[\phi_{2\gamma} + \phi_{3\gamma} - \phi_{2\gamma}\phi_{3\gamma}]}{2p^2 K_3 \left[1 - \frac{D(p)-K_1}{K_2}\left(1-\phi_{3\gamma}\right)\right]} = \frac{\Pi_1^* + p^2 D'(p)[1 - \frac{K_3}{K_2}(1-\phi_{3\gamma})^2]}{2p^2 K_3 \left[1 - \frac{D(p)-K_1}{K_2}\left(1-\phi_{3\gamma}\right)\right]}.$$

This fraction is positive because both the denominator and the numerator are positive throughout γ. The denominator is immediately ascertained as positive. As to the numerator, recall that $\Pi_1^* = p_M[D(p_M) - K_2 - K_3]$, where $p_M = \arg\max p[D(p) - K_2 - K_3]$. Thus the numerator becomes $p_M[D(p_M) - K_2 - K_3 + p_M\left[D'(p)\right]_{p=p_M}] = 0$ at $p = p_M$. At $p < p_M$ the numerator is positive since $D(p)$ is concave and $\phi_{3\gamma} < 1$.

The proof that $\phi_{1\gamma}' > 0$ is more involved. Since $\phi_{1\gamma}' = (1/2)\phi_{1\gamma}^{-1}(d\phi_{1\gamma}^2/dp)$ and $\phi_{1\gamma} > 0$ throughout γ, then $\phi_{1\gamma}' > 0$ if and only if $d\phi_{1\gamma}^2/dp > 0$. This leads to

$$\Pi_1^*(p-p_m)[D(p)-K_1] > -2p_m p(D(p)-K_1-K_2)(D(p)-K_1-K_3)+$$
$$+(p-p_m)p^2D'(p)[D(p)-K_2-K_3+D(p)-K_1] - (p-p_m)K_1 p_m p D'(p),$$

which can be rearranged as

$$\begin{aligned} p_m(D(p)-K_1)(\varphi(p)-\Pi_1^*)+ \\ p_m p[K_2(K_1+K_3-D(p))+K_3(K_1+K_2-D(p)] > \\ (p-p_m)D'(p)p(\varphi(p)-\Pi_1^*), \end{aligned} \tag{14}$$

where $\varphi(p) = p[D(p) - K_2 - K_3] + p[D(p) - K_1]$. Note that $\varphi(p)$ is concave; furthermore, $\varphi(p_M) > \Pi_1^* > \varphi(P(K_1))$, and $\varphi(P(K_1 + K_2)) > \Pi_1^*$. Thus $\varphi(p) > \Pi_1^*$ throughout γ. As a consequence, inequality (14) is obviously met since the left-hand side is positive while the right-hand side is negative.

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