"Rationality in the Joint Allocation of Private and Public Goods".

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Rationality in the Joint Allocation of Private and Public Goods

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Abstract

In this study we assume that household demand for private and public goods are the efficient outcomes of the household decission process. From the efficient assumption we derive testable properties of these demands and we identify some characteristics of the intrahousehold distribution of the household expenditure. These results extend Chiappori (1988) main results to the case of joint consumption of private and public goods.

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1. INTRODUCTION

Although people within the household exchange material and immaterial goods, we can only observe the material aggregate consumption that results from this exchange process. The difficulty, of course, is that aggregate household consumption equals individual consumption in only three cases: single-person households, public goods, and in the case of goods whose consumption is exclusive to a household member (exclusive goods).

Traditional models of household demand based on a representative consumer are called *unitary models*. Because empirical properties derived from them (aggregation, homogeneity, symmetry and negative semi-definiteness of the Slutsky matrix) have received little empirical support, we may conclude that individual rationality presents serious difficulties when attempting to explain household behavior.

Alternatively, household behavior can be modeled as a decision process in which several individuals are involved. These kinds of models, known as *collective models*, have two principal objectives. i) to derive empirical properties of household demand, and ii) to recover the intrahousehold allocations of goods and welfare.

Within collective models, there is a strand of the literature that is based on the idea that "*the household is modeled as a two-member collectivity taking Pareto-efficient decisions*" (Chiappori, 1988). According to the assumptions of the model, we distinguish between three groups of papers which, in chronological order, are:

1) Manser and Brown (1980) and McElroy and Horney (1981) model household decisions as the equilibrium from a Nash bargaining game where the threat point is a function of exogenous variables called *extrahousehold environmental parameters* (EEP). The Nash bargaining game results in efficient outcomes, so these models fit into Chiappori's efficiency idea.

2) Chiappori (1988, 1992) provides the most general framework for the study of the intrahousehold allocation of private goods under the sole assumption of efficiency. Within this framework, there are two types of important results: those involving empirical properties of labor supply functions for egoistic agents, and those referring to the recovery of the sharing rule between household members.

3) The third group of models also starts from the efficiency premise but adds the assumption of the existence of, at least, a *distribution factor* that affects the reserve utility of both agents but does not affect either their preferences for goods or the budget constraint. A distribution factor is, therefore, an EEP in McElroy's terminology. This methodology is applied, for example, in Bourguignon *et al.* (1993, 1995), Browning and Chiappori (1994), Browning *et al.* (1994), and Chiappori *et al.* (1997).

The aim of this paper is to derive empirical properties of the household demands, and to recover intrahousehold allocations in the presence of public goods. For this purpose, we extend Chiappori's (1988) parametric model allowing for the joint choice between public and private goods. The assumption of weak separability between private and public goods, implicit in Chiappori's model, does not lead to any new result. But under a different separability assumption between public goods and exclusive goods, we obtain the following results:

1) Efficiency necessary conditions on household demands for private and public goods. We prove that the conditions obtained in Chiappori (1988) are a particular case of ours when the consumption of public goods is zero.

2) We do not recover the intrahousehold allocation of private goods completely, but we do recover its variation with respect to prices. The corresponding results

in Chiappori's model without public goods are nested into ours.

The theoretical importance of the assumption S is based on the possibility of measuring the effect of the public goods on the sharing rule. Also, the empirical restrictions allows the new effect of the public goods. The main limitation of the assumption S is the applicability of the model to labor supply. Since the wages could affect the public good demands, our assumption is not applicable to this case. If we consider clothing for the man and the woman as the exclusive goods and the clean house as the public good, we can test the assumption S.

The paper is organized as follows. In section 2 we introduce the notation and Chiappori's main results under weak separable preferences between private and public goods. In section 3 we introduce a different separability assumption and derive our results. In section 4 we apply the above results for a particular specification of household demands. In section 5 we discuss the main restrictions and contributions of our model to the empirical study of consumer behavior.

2. THE INITIAL MODEL: NOTATION AND CHIAPPORI'S RESULTS

Assume that a household consists of two adults who decide on how to allocate the household endowment, X, among goods, which we classify into three groups. First, there are *private goods* like food, alcohol, tobacco, and goods linked to education or entertainment activities for which individual consumptions are unobservable. What we observe is the aggregate consumption of private goods $Z = Z^1 + Z^2$, where Z^i is the amount consumed by agent i for $i = 1, 2$. Second, we observe individual expenditures for some adult *exclusive goods*, like women's or men's clothing expenditures. Leisure is a special case of an exclusive good included in Chiappori (1988) but which is excluded from our model. We denote by q^i the exclusive good consumed by agent i. Finally, household *public goods* (Q) are goods that are characterized by non-rivalry and non-exclusion, for which individual consumption coincides with the aggregate amount. Expenditures on housing and furniture and their maintenance are household public goods, which we call "clean house" following Pollak and Wachter (1975).

Household preferences are characterized by a pair of utility functions, $U^i\left(q^i, Z^i, Q\right)$ for $i = 1, 2$, which are assumed to be strictly monotonic, strongly quasi-concave, and twice continuously differentiable.

To sum up, the household is an economy, ξ, characterized by two consumption sets, $\left\{q^i, Z^i, Q\right\} \subset \mathbb{R}^3_+$, two utility functions $U^i : \left\{q^i, Z^i, Q\right\} \to \mathbb{R}$, $i = 1, 2$, and the income household endowment, X.

2.1. Chiappori's (1988) Results: A Reinterpretation

Chiappori's original model studies the following question: what conditions characterize the leisure $\left(L^i\right)$ and private goods (Z) demands if they arise from an efficient allocation? The demand functions with this property are called **collectively rational for egoistic agents.**

Definition 1. *Let T be the time endowment for both agents. For any wages, w_1, w_2, and non-labor income, y, the household demand for leisure $(\overline{L}^1(w_1, w_2, y), \overline{L}^2(w_1, w_2, y))$ is said to be* ***collectively rational for egoistic agents*** *(CREA) if there exist two demand functions $\overline{Z}^1$ and $\overline{Z}^2$ from $\mathbb{R}^3_+$ to $\mathbb{R}_+$, and two utility functions $U^1\left(L^1, Z^1\right)$ and $U^2\left(L^2, Z^2\right)$ such that the functions $\overline{L}^1, \overline{L}^2, \overline{Z}^1, \overline{Z}^2$ solve the following problem:*

$$\underset{L^1,L^2,Z^1,Z^2}{Max} \quad U^1\left(L^1, Z^1\right) + \mu\left(w_1, w_2, y\right) U^2\left(L^2, Z^2\right)$$

$$s.t. \qquad Z^1 + Z^2 \leq y + w^1\left(T - L^1\right) + w^2\left(T - L^2\right)$$

The first order conditions for this problem are:

$$\begin{aligned} U^1_{q^1}\left(L^1, Z^1\right) &= w_1 U^1_{z^1}\left(L^1, Z^1\right), 1 &\quad (2.1)\\ U^2_{q^2}\left(L^2, Z^2\right) &= w_2 U^2_{z^2}\left(L^2, Z^2\right), 2 &\quad (2.2)\\ Z^1 + Z^2 &= y + w_1\left(T - L^1\right) + w_2\left(T - L^2\right).3 &\quad (2.3)\end{aligned}$$

One of the main reasons why adults decide to live together is to enjoy scale economies in the consumption of public goods. From this perspective, the above model can be reinterpreted as a model under the assumption of weakly separable preferences between private and exclusive goods, on the one hand, and public goods, on the other. If we let C be the price of a single public good and replace labor supply $\left(T - L^i\right)$ by $-q^i$, wages w_i by prices p_i, and non-labor income y by private and exclusive goods expenditures $x = X - CQ$, then the problem becomes:

$$\underset{q^1,q^2,Z^1,Z^2,Q}{Max} \quad U^1\left(v^1_1\left(q^1, Z^1\right), v^1_2\left(Q\right)\right) + \mu U^2\left(v^2_1\left(q^2, Z^2\right), v^2_1\left(Q\right)\right) \qquad \text{(P2)}$$

$$s.t. \qquad p_1 q^1 + p_2 q^2 + Z^1 + Z^2 + CQ \leq X.$$

The first order conditions for this new problem are:

$$\begin{aligned} v^1_{q^1}\left(q^1, Z^1\right) &= p_1 v^1_{z^1}\left(q^1, Z^1\right), 4 &\quad (2.4)\\ v^2_{q^2}\left(q^2, Z^2\right) &= p_2 v^2_{z^2}\left(q^2, Z^2\right), 5 &\quad (2.5)\end{aligned}$$

$$\frac{U_2^1\left(v_1^1\left(q^1,Z^1\right),v_2^1\left(Q\right)\right)v_{2Q}^1\left(Q\right)}{U_1^1\left(v_1^1\left(q^1,Z^1\right),v_2^1\left(Q\right)\right)v_{1Z^1}^1\left(q^1,Z^1\right)}+\frac{U_2^2\left(v_1^2\left(q^2,Z^2\right),v_2^2\left(Q\right)\right)v_{2Q}^2\left(Q\right)}{U_1^2\left(v_1^2\left(q^2,Z^2\right),v_2^2\left(Q\right)\right)v_{1Z^2}^2\left(q^2,Z^2\right)}=c, \tag{6}$$

$$Z^2+p_2q^2=(X-CQ)-p_1q^1-Z^1. \tag{7}$$

This new problem can be interpreted as a two stage problem. At the first stage, the household allocates total expenditure between private, exclusive and public goods, and at the second stage, expenditures on private and exclusive goods are allocated among household members. Conditions (4), (5) and (7) are the same as conditions (1) to (3), respectively, in the problem (P1) without public goods. Therefore, Chiappori's results obtained from that problem apply to the study of the allocation of the private and exclusive goods expenditures in the second stage of problem (P2).

Under our assumptions, the demand functions q^1,q^2 from $\mathbb{R}_+^3$ to $\mathbb{R}_+$ are twice differentiable. We use the following notation from Chiappori (1988):

$M_k=\frac{\partial M}{\partial K}$, where $M=q^1,q^2,Z^1,Z^2$, etc., and $K=p_1,p_2,x$

$$A=\frac{q_{p_2}^1}{q_x^1},\qquad B=\frac{q_{p_1}^2}{q_x^2},\qquad and$$

$$\alpha=\begin{cases}\left[1-\frac{BA_x-A_{p_1}}{AB_x-B_{p_2}}\right]^{-1} & if\ AB_x-B_{p_2}\neq 0\\ 0 & \text{otherwise,}\end{cases}$$

$$\beta=1-\alpha.$$

Chiappori (1988) arrives to the following result:

Proposition 1. *Let the demand functions $q^1$and q^2satisfy the two following regularity conditions for each (p_1,p_2,x) :*

$$q_x^1.q_x^2\neq 0 \tag{R1}$$

$$AB_x-B_{p_2}\neq BA_x-B_{p_2}. \tag{R2}$$

For q^1,q^2 to be ***collectively rational for egoistic agents*** *in the sense of Definition 1, the following conditions are necessary:*

$$\alpha_xA+\alpha A_x-\alpha_{p_2}=0 \tag{CREA a}$$

$$\beta_xB+\beta B_x-\beta_{p_1}=0. \tag{CREA b}$$

If these conditions are satisfied, $Z^1$and Z^2 are unique up to an additive constant, and Z^i depends only on $q^i\left(p_1,p_2,x\right)$ and p_i $(i=1,2)$.

The conditions (CREA a, b) are empirical properties on first and second demand derivatives. In view of Proposition 1, if these properties are not satisfied, then we know that household outcomes are not efficient.

We are also interested in recovering the unobserved intrahousehold allocation of private goods. We have interpreted that the private goods expenditures in the second stage of problem (P2) are allocated between the two household members. Then, conditional on that allocation, each one solves his/her own utility maximization problem. From the Second Welfare Theorem, we know that if $\left(q^{1*}, q^{2*}, Z^{1*}, Z^{2*}\right)$ is an efficient allocation and $x > 0$ is the household expenditure on private and exclusive goods, there exists a price vector (p_1^*, p_2^*) in $\mathbb{R}_+^2$, such that $\left(q^{1*}, q^{2*}, Z^{1*}, Z^{2*}; p_1^*, p_2^*\right)$ is the competitive equilibrium that solves the pair of individual problems:

$$\begin{cases} \underset{q^i, Z^i}{Max} & v^i\left(q^i, Z^i\right) \\ s.t. & p_i q^i + Z^i = \phi^{i*} \qquad i = 1, 2, \end{cases}$$

where $\phi^{i*} = p_i^* q^{i*} + Z^{i*}$. Based on this Theorem we define the existence of a sharing rule function.

Definition 2. *Let $q^1\left(p_1, p_2, x\right)$and $Z^1\left(p_1, p_2, x\right)$ be two demand functions that solve the problem*

$$\begin{array}{ll} \underset{q^1, Z^1}{Max} & v^1\left(q^1, Z^1\right) \\ s.t. & p_1 q^1 + Z^1 = \phi\left(p_1, p_2, x\right). \end{array}$$

Then the function $\phi : \mathbb{R}_+^3 \rightarrow]0, X[$ is the ***sharing rule****, where*

$$\phi\left(p_1, p_2, x\right) = p_1 q^1\left(p_1, p_2, x\right) + Z^1\left(p_1, p_2, x\right).$$

The conditions (CREA a, b) of Proposition 1 imply the existence of such a sharing rule.

Proposition 2. *(Chiappori, 1988 and 1992). Given two demand functions q^1 and q^2 satisfying conditions (CREA a, b) of Proposition 1, a sharing rule is defined up to an additive constant; specifically, its partial derivatives are given by*

$$\begin{aligned} \phi_x &= \alpha \\ \phi_{p_1} &= -\beta B \\ \phi_{p_2} &= \alpha A \end{aligned}$$

The derivative of the sharing rule with respect to x, α, is the share of marginal expenditure received by member 1. $\beta = 1-\alpha$, is the share of marginal expenditure received by member 2. ϕ_{p_i} is the marginal change in expenditures when there is a change in the price p_i. We are interested in the sign of these derivatives for specific demand functions (see below the example in section 4).

Because of the weak separability assumption, this model does not allow us to analyze the effect of public goods on the intrahousehold allocation of private goods. In the following section we introduce another separability assumption between public and exclusive goods in order to obtain an expression for the relationship between the amount of public goods and the sharing rule which guides the allocation of the private good expenditures.

3. THE EXTENDED MODEL: PUBLIC GOOD EFFECT

In general, we expect that the price of a public good affects the demand for both private and exclusive goods. In this paper, however, we only allow for the public good effect on the private good demand but not on the exclusive good demand. For this purpose, we make the following separability assumption between household public goods and exclusive goods:

Assumption S. The public good price does not enter the exclusive good demand and the exclusive good price does not enter the public good demand. Mathematically:

$$q_c^i = 0, \qquad Q_{p_i} = 0 \quad i = 1, 2. \tag{S}$$

The advantage of this assumption is that it is testable. For example, if we estimate three demand functions for women's clothing, men's clothing and the clean house, we can test if clothing prices affect the clean house demand, and if the clean house price affects clothing demands. Intuitively, the wage rate may very well affect the demand for public goods. This is why we exclude leisure from the list of exclusive goods in what follows.

In this framework, we define a collectively rational behavior.

Definition 3. *Household demands for the exclusive goods* $\left(\overline{q}^1 (p_1, p_2, X), \overline{q}^2 (p_1, p_2, X)\right)$ *and for the public good* $\overline{Q} (c, X)$ *are said to be* ***collectively rational*** *(CR*) if there exist two demand functions* $\overline{Z}^1$ *and* $\overline{Z}^2$ *from* $\mathbb{R}_+^4$ *to* $\mathbb{R}_+$*, and two utility functions* $U^1 \left(L^1, Z^1, Q\right)$ *and* $U^2 \left(L^2, Z^2, Q\right)$ *such that, for all* (p_1, p_2, c, X) *in* $\mathbb{R}_+^4$ *and* $\mu \in \mathbb{R}_+$*, the functions* $\overline{q}^1, \overline{q}^2, \overline{Z}^1, \overline{Z}^2, \overline{Q}$*, solve the problem*

$$\underset{q^1, q^2, Z^1, Z^2, Q}{Max} \quad U^1 \left(q^1, Z^1, Q\right) + \mu U^2 \left(q^2, Z^2, Q\right) \tag{P3}$$

$$s.t. \qquad p_1q^1 + p_2q^2 + Z^1 + Z^2 + cQ \leq X.$$

The first order conditions are:

$$U^1_{q^1}\left(q^1, Z^1, Q\right) = p_1 U^1_{z^1}\left(q^1, Z^1, Q\right), \tag{8}$$

$$U^2_{q^2}\left(q^2, Z^2, Q\right) = p_2 U^2_{z^2}\left(q^2, Z^2, Q\right), \tag{9}$$

$$\frac{U^1_Q\left(q^1, Z^1, Q\right)}{U^1_z\left(q^1, Z^1, Q\right)} + \frac{U^2_Q\left(q^2, Z^2, Q\right)}{U^2_z\left(q^2, Z^2, Q\right)} = c, \tag{10}$$

$$p_1q^1 + p_2q^2 + Z^1 + Z^2 + cQ = X. \tag{11}$$

In what follows, we keep Chiappori's notation (A, B, α, β), but introduce new notations to include the public good demand:

$M_k = \frac{\partial M}{\partial K}$, $M = q^1,\ q^2,\ Q,\ Z^1,\ Z^2$, etc. and $K = p_1, p_2, C, X$.

$$A = \frac{q^1_{p_2}}{q^1_x},\ B = \frac{q^2_{p_1}}{q^2_x},\ D = \frac{Q_c}{Q_x},\ \delta = \frac{D+Q}{D},$$

$$\alpha = \begin{cases} \left[1 - \frac{BA_x - A_{p_1}}{AB_x - B_{p_2}}\right]^{-1} & \text{if } AB_x - B_{p_2} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma = \begin{cases} \frac{AB\left(\delta_x - \frac{\delta_c}{D}\right)}{A_{p_1} - BA_x - \left(B_{p_2} - AB_x\right)} & \text{if } A_{p_1} - BA_x \neq B_{p_2} - AB_x \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta = 1 - \alpha.$$

When a term is multiplied by δ, we denote it with $'$; for example, $\alpha' = \alpha\delta, \quad \alpha'_x = (\alpha\delta)_x = \alpha_x\delta + \delta_x\alpha$.

The following result is based on conditions (8), (9) and (11).

Proposition 1. *Let the demand functions q^1, q^2 and Q satisfy the two following regularity conditions. For each (p_1, p_2, c, X) in $\mathbb{R}^4_+$,*

$$q^1_x \neq 0,\ q^2_x \neq 0 \text{ and } \quad Q_c \neq 0, \tag{R*1}$$

$$AB_x - B_{p_2} \neq BA_x - B_{p_2}. \tag{R*2}$$

For q^1, q^2, Q to be ***collectively rational*** *in the sense of Definition 3, the following conditions are necessary:*

$$\left(\alpha' + \gamma\right) A_x + A\left(\alpha'_x + \gamma_x\right) = \frac{A}{D}\left(\alpha'_c + \gamma_c\right) + \left(\alpha'_{p_2} + \gamma_{p_2}\right), \tag{CR* a}$$

$$\left(\beta' - \gamma\right) B_x + B\left(\beta'_x - \gamma_x\right) = \frac{B}{D}\left(\beta'_c - \gamma_c\right) + \left(\beta'_{p_1} - \gamma_{p_1}\right). \qquad (\text{ CR* b})$$

If these conditions are satisfied, then we can recover the functions $Z^1_{p_i}$ and $Z^2_{p_i}$, and Z^i only depends on $q^i(p_1, p_2, X)$, $Q(c, X)$ and p_i $(i = 1, 2)$.

See the proof in Appendix 1.

Corollary 2. *The conditions (CREA a) and (CREA b) from the model without public goods are nested in conditions (CR* a) and (CR* b) for the case* $Q = 0$.

P roof. If $Q = 0$, then $\delta = \left(\frac{D+Q}{D}\right) = \frac{Q_c + QQ_x}{Q_c} = 1$. This implies that $\gamma = 0$, $\alpha' = \alpha\delta = \alpha$, $\beta' = \beta\delta = \beta$, and $\alpha_c = 0$. Therefore, replacing these in the expression (CR* a):
$\left\{\left(\alpha' + \gamma\right) A_x + A\left(\alpha'_x + \gamma_x\right) - \frac{A}{D}\left(\alpha'_c + \gamma_c\right) - \left(\alpha'_{p_2} + \gamma_{p_2}\right) = 0\right\}$, becomes expression (CREA a): $\{\alpha A_x + A\alpha_x - \alpha_{p_2} = 0\}$. Analogously, expression (CR* b) becomes (CREA b) when $Q = 0$. ■

The conditions (CR* a) and (CR* b) are empirical properties of the demand functions formulated in terms of their first and second derivatives with respect to prices and household expenditures. Thus, for a particular system of demand functions, these conditions will appear as parameter restrictions.

We are interested in the effect of public goods on the sharing rule which gives us the individual expenditure on private goods.

Definition 4. *Let* $q^i(p_1, p_2, X)$, $Z^i(p_1, p_2, c, X)$ *and* $Q(c, X)$, *for* $i = 1, 2$, *be the demand functions that solve the problem*

$$\underset{q^1, q^2, Z^1, Z^2, Q}{Max} \quad U^1\left(q^1, Z^1, Q\right) + \mu U^2\left(q^2, Z^2, Q\right)$$

$$s.t. \qquad p_1 q^1 + p_2 q^2 + Z^1 + Z^2 + CQ \leq X.$$

Then the function $\Phi : \mathbb{R}^4_+ \rightarrow]0, X[$ *is the* ***sharing rule for private goods****, where*

$$\Phi(p_1, p_2, c, X) = p_1 q^1(p_1, p_2, X) + Z^1(p_1, p_2, c, X).$$

As before, the efficiency conditions of Proposition 3 are sufficient for the existence of the sharing rule for private goods.

Proposition 3. *Given the demand functions q^1, q^2 and Q, satisfying conditions (CR* a) and (CR* b) of Proposition 3, the derivatives of the sharing rule for private goods with respect to prices are given by*

$$\begin{aligned}\Phi_{p_1} &= B(-\beta\delta+\gamma) = \phi_{p_1}\delta + B\gamma,\\ \Phi_{p_2} &= A(\alpha\delta+\gamma) = \phi_{p_2}\delta + A\gamma,\end{aligned}$$

where ϕ is the sharing rule in the model without public goods.

The proof is in Appendix 2.

Corollary 4. *In the case of zero consumption of public goods, the derivatives of the sharing rule with respect to prices in the model without public goods, ϕ_{p_i}, coincides with the derivatives of the sharing rule for private goods in the model with public goods, Φ_{p_i}.*

P roof. *If $Q = 0$, $\delta = 1$ and $\gamma = 0$, then*

$$\begin{aligned}\Phi_{p_1} &= \phi_{p_1}\delta + B\gamma = \phi_{p_1},\\ \Phi_{p_2} &= \phi_{p_2}\delta + A\gamma = \phi_{p_2}.\end{aligned}$$

■

Remark 1. *We observe two components in the derivatives of the* ***sharing rule*** *with respect to prices. If public goods are normal goods, then the first component, $\phi_{p_i}\delta$, is smaller in absolute value than ϕ_{p_i} because $\delta \in [0,1]$.*

We have $\delta = \frac{Q_c + QQ_x}{Q_c}$. In the optimal allocation, the numerator is the Slutsky equation, i.e., the substitution effect calculated from the Hicksian demand for public good. The denominator is the total effect calculated from the Marshallian demand. If public goods are normal goods, then the substitution effect has an absolute value smaller than the total effect, and we therefore have $\delta \in [0,1]$ and $\left|\phi_{p_i}\delta\right| \leq \left|\phi_{p_i}\right|$.

4. AN ILLUSTRATIVE EXAMPLE

We take the functional form of Chiappori's example (1988 and 1992) for exclusive goods demand, we add a public good demand, and we assume that such demands satisfy assumption (S) :

$$\begin{aligned}q^1(X,p_1,p_2) &= a_1 + b_1X + c_1X\log X + d_1^1p_1 + d_1^2p_2, && (4.1)\\ q^2(X,p_1,p_2) &= a_2 + b_2X + c_2X\log X + d_2^1p_1 + d_2^2p_2, && (4.2)\\ Q(X,C) &= a_3 + b_3X + c_3X\log X + d_3C. && (4.3)\end{aligned}$$

We calculate the terms used in Proposition 3:

$$q^i_{p_j} = d^j_i,\ Q_c = d_3,\ q^i_x = b_i + c_i + c_i \log X,\ q^i_{xx} = c_i/X,$$

$$Q_x = b_3 + c_3 + c_3 \log X, \quad Q_{xx} = c_3/X,$$

$$A = \frac{d^2_1}{q^1_x}, \quad B = \frac{d^1_2}{q^2_x}, \quad D = \frac{d_3}{Q_x},$$

$$\alpha = \frac{c_2}{c_2 b_1 - c_1 b_2} q^1_x, \quad \beta = \frac{c_1}{c_1 b_2 - c_2 b_1} q^2_x,$$

$$\delta = 1 + \frac{Q Q_x}{d_3},\ \gamma = \frac{c_3}{d_3 (c_1 b_2 - c_2 b_1)} q^1_x q^2_x Q,$$

$$\alpha_x = \frac{c_2}{c_2 b_1 - c_1 b_2} q^1_{xx} = \alpha \frac{q^1_{xx}}{q^1_x}, \quad \alpha_{p_2} = \alpha_c = 0,$$

$$\gamma_x = \gamma \left(\frac{q^1_{xx}}{q^1_x} + \frac{q^2_{xx}}{q^2_x} + \frac{Q_x}{Q} \right), \quad \gamma_c = \gamma \frac{Q_c}{Q}, \quad \gamma_{p_i} = 0,$$

$$\delta_x = \frac{Q_x Q_x}{Q_c} + \frac{Q Q_{xx}}{Q_c}, \quad \delta_c = Q_x.$$

4.1. The Initial Model without Public Goods

In this example, the regularity conditions *(R1)* $q^1_x q^2_x \neq 0$ and *(R2)* $(AB_x - B_{p_2} \neq BA_x - B_{p_2})$ are satisfied for all x when $c_1 b_2 \neq c_2 b_1$.

The demand functions are linear in prices, so that $\alpha_{p_2} = \beta_{p_1} = 0$. The necessary conditions imposed by colllective rationality for egoistic agents are:

$$\alpha_x A + \alpha A_x = 0, \tag{CRAE a}$$

$$\beta_x B + \beta B_x = 0. \tag{CRAE b}$$

Since $\alpha_x A + \alpha A_x = \frac{\partial}{\partial X} \alpha A = \frac{\partial}{\partial X} \left(\frac{c_2}{c_2 b_1 - c_1 b_2} q^1_x \frac{d^2_1}{q^1_x} \right) = 0$, these conditions are always satisfied for all values of the parameters $a_i, b_i, c_i, d^i_i, d^j_i,\ i, j = 1, 2$. Consequently, we cannot test efficiency for this functional form. We can, however, recover the derivatives of the sharing rule:

$$\begin{aligned}
\phi_x &= \alpha = \frac{c_2 (b_1 + c_1 + c_1 \log X)}{H} \\
\phi_{p_1} &= -\beta B = \frac{c_1 d^1_2}{H} \\
\phi_{p_2} &= \alpha A = \frac{c_2 d^2_1}{H}
\end{aligned}$$

where $H = c_2b_1 - c_1b_2$.

We want to know the sign of these derivatives in order to predict whether a change in private and exclusive goods expenditures, x, or a change in exclusive goods prices, will produce either an increase or a decrease in individual expenditures. If we assume that all goods are normal, then $q^i_x > 0$, $Q_x > 0$ for all X, ($b_i > 0$ and $c_i > 0$) and $q^i_{p_i} < 0$, $Q_c < 0$ ($d^i_i < 0, d_3 < 0$). The cross-price effect, $q^i_{p_j}$, is negative if the good q^i is a gross complement of good q^j $\left(d^j_i < 0\right)$ and it is positive if q^i is a gross substitute for q^j $\left(d^j_i > 0\right)$. The denominator, H, can be positive or negative.If $H > 0$ then, $sign\left(\phi_{p_1}\right) = sign\left(d^1_2\right)$. Thus, if p_1 increases and q^2 is a gross substitute for q^1, we then predict an increase in the share of expenditure received by member 1. In Chiappori's example, q^1 and q^2 are leisure demands, p_1 and p_2 are the wages, and the sharing rule is defined for non-labor income. In this case, we observe an increase in the man's wage and leisure is a normal good, and the man's labor supply will therefore increase. If the woman's leisure is a gross substitute (gross complement) of the man's leisure, $\phi_{p_1} > 0$ $\left(\phi_{p_1} < 0\right)$, then we predict an increase (decrease) of the man's share of non-labor income.

4.2. The Extended Model with Public Goods

The regularity conditions *(R*1)* and *(R*2)* are satisfied if $d_3 \neq 0$ and $c_1b_2 \neq c_2b_1$. Since, for this example, $\alpha'_{p_2} = \gamma_{p_2} = \beta_{p_1} = \gamma_{p_1} = 0$, the conditions (CR*a) and (CR*b) of Proposition 3 are:

$$\left(\alpha' + \gamma\right) A_x + A\left(\alpha'_x + \gamma_x\right) = \frac{A}{D}\left(\alpha'_c + \gamma_c\right), \tag{CR* a}$$

$$\left(\beta' - \gamma\right) B_x + B\left(\beta'_x - \gamma_x\right) = \frac{B}{D}\left(\beta'_c - \gamma_c\right). \tag{CR* b}$$

By replacing the equality terms in $(CR^{*}a)$ and simplifying, we have:

$$\left(\alpha\delta + \gamma\right) A_x + A\left(\left(\alpha\delta\right)_x + \gamma_x\right) = A\left(\alpha\left(\frac{Q_xQ_x}{Q_c} + \frac{QQ_{xx}}{Q_c}\right) + \gamma\frac{q^2_{xx}}{q^2_x} + \gamma\frac{Q_x}{Q}\right), \tag{4.4}$$

$$\frac{A}{D}\left(\left(\alpha\delta\right)_c + \gamma_c\right) = A\frac{Q_x}{Q_c}\left(\alpha Q_x + \gamma\frac{Q_c}{Q}\right). \tag{4.5}$$

So, the expression $(CR^{*}a)$ that results equating the right hand sides of (4.4) and (4.5) is:

$$\frac{\alpha QQ_{xx}}{Q_c} + \gamma\frac{q^2_{xx}}{q^2_x} = 0. \tag{4.6}$$

And on replacing (4.6) from the demand functions, we obtain:

$$\frac{c_2}{c_2 b_1 - c_1 b_2} \frac{c_3}{X d_3} q_x^1 Q - \frac{c_3}{d_3 \left(c_2 b_1 - c_1 b_2 \right)} \frac{c_2}{X} q_x^1 Q = 0. \tag{4.7}$$

Since condition (4.7) is always satisfied by these demand functions, we can not test efficiency in this example.

We calculate the derivatives of the sharing rule for private goods and obtain:

$$\Phi_{p_1} = \phi_{p_1} \delta + B\gamma = \frac{c_1 d_2^1}{H} \left(\frac{d_3 + Q Q_x}{d_3} \right) - \frac{c_3 d_2^1}{H d_3} Q q_x^1, \tag{4.8}$$

$$\Phi_{p_2} = \phi_{p_2} \delta + A\gamma = \frac{c_2 d_1^2}{H} \left(\frac{d_3 + Q Q_x}{d_3} \right) - \frac{c_3 d_1^2}{H d_3} Q q_x^2. \tag{4.9}$$

Since $\left| \phi_{p_i} \delta \right| \leq \left| \phi_{p_i} \right|$ (see Remark 1), the first effect of public good consumption is the reduction of the amount of the sharing rule change with respect to price variation. The second component is $\left(-\frac{c_3 d_2^1}{H d_3} Q q_x^1 \right)$, whose sign depends on those of H and d_2^1 . If $H > 0$, $sign \left(-\frac{c_3 d_2^1}{H d_3} Q q_x^1 \right) = sign \left(d_2^1 \right)$.

When we consider the two components, we conclude that, if $H > 0$, $sign \left(\Phi_{p_1} \right) = sign \left(d_2^1 \right)$ and the effect in the sharing rule produced by a change in p_1 has the same sign as in the model without public goods: the sign is positive if q^2 is a gross substitute of q^1 and it is negative if q^2 is a gross complement of q^1.

Summing up, the sign of a sharing rule change caused by a price variation does not depend on public goods consumption, but the presence of the public good does, in fact, modify the amount of this sharing rule change.

5. CONCLUSIONS

The main restriction of the model is our separability assumption which does not allow any effect of the exclusive good's prices on the demand for the public good. Since the public good demand depends on wages, our model cannot be used for the study of the empirical properties of labor supply, considered as an exclusive good. In contrast to the weak separability, however, our separability assumption allows the prices of public goods to enter the private good demand. Hence, our collective model allows the study of the effect of public goods on the allocation of private expenditures.

With regard to the contributions of this paper, we have derived new testable restrictions on observable behavior (the demand for public and exclusive goods). These restrictions are necesary conditions for Pareto efficiency. We show that the parametric restrictions derived by Chiappori (1988) for the case without public goods are nested in our conditions for the case with public goods. On the other hand, the collective setting developed here allows us to learn about the allocation of household expenditures in private goods between the two agents: the man and the woman. In particular, we can predict how the sharing of household expenditures between the man an the woman changes when the exclusive goods' prices change. This change depends on the amount of the public good. In the context of our model, we can then study the effect of any tax policy that changes the exclusive goods' prices on the distribution of household expenditures in private goods between the husband and the wife, and we can measure how the extent of this effect depends on the amount of public goods that exists in the home.

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6. APPENDIX 1

Let us consider Chiappori's Lemma, generalized for $m+1$ functions.

Lemma 1. *Let $\rho, X^1, X^2, ..., X^m$ be any $m+1$ C^∞ functions from some open, non-empty subset B of $\mathbb{R}^n$ to $\mathbb{R}$, with $n \geq m$ and such that $\overline{gradX^1}$, $\overline{gradX^2}$, .., $\overline{gradX^m}$ are noncolinear. For a function θ from $\mathbb{R}^m$ to $\mathbb{R}$, such that*

$$\begin{aligned} \text{for all } (x_1, .., x_n) &\in B, \\ \rho(x_1, .., x_n) &= \theta\left[X^1(x_1, .., x_n), X^2(x_1, .., x_n), .., X^m(x_1, .., x_n)\right] \end{aligned}$$

to exist in a neighborhood of any point of B, it is necessary and sufficient that the vectors $\overline{grad\rho}$, $\overline{gradX^1}$, $\overline{gradX^2}$, ..., $\overline{gradX^m}$ are always colinear.

Applying the above lemma to the first order conditions of the efficiency problem, we obtain the conditions for collective rationality and the expressions for the derivatives of the sharing rule. The first order conditions for the efficiency problem are:

$$\begin{array}{ll} (1) & U^1_{q^1}\left(q^1, Z^1, Q\right) = p_1 U^1_{z^1}\left(q^1, Z^1, Q\right), \\ (2) & U^2_{q^2}\left(q^2, Z^2, Q\right) = p_2 U^2_{z^2}\left(q^2, Z^2, Q\right), \\ (3) & \frac{U^1_Q\left(q^1, Z^1, Q\right)}{U^1_z\left(q^1, Z^1, Q\right)} + \frac{U^2_Q\left(q^2, Z^2, Q\right)}{U^2_z\left(q^2, Z^2, Q\right)} = c, \\ (4) & p_1 q^1 + p_2 q^2 + Z^1 + Z^2 + CQ = X. \end{array}$$

Consider any three functions q^1, q^2, Q. For these functions to be CR^* demand functions, it is necessary that there exist two functions Z^1 and Z^2 such that the first order conditions are satisfied.

Let (p_1, p_2, C, X) be a point in $\mathbb{R}^4$ such that q^1, q^2 and Q are not corner solutions and such that $q^1_x \neq 0$, $q^2_x \neq 0$ and $Q_c \neq 0$. Relation (1) can be written (and symmetrically relation (2)):

$$\left(U^1_{qq} - p_1 U^1_{Zq}\right)\overline{gradq^1} + \left(U^1_{qz} - p_1 U^1_{ZZ}\right)\overline{gradZ^1} + + \left(U^1_{qQ} - U^1_{zQ}\right)\overline{grad\, Q} - U^1_z\,\overline{grad\, p_1} = 0$$

Therefore, $\overline{gradZ^i}$, $\overline{grad\, q^i}$, $\overline{gradQ}$, $\overline{grad\, p_1}$, are colinear. The lemma applies directly here, with $n = 4$ and $X^1(.) = q^1(p_1, p_2, c, X)$, $X^2(.) = Q(p_1, p_2, C, X)$, $X^3(.) = p_1$, $\rho(.) = Z^1(p_1, p_2, C, X)$, since $q^1_x \neq 0$, $q^2_x \neq 0$ and $Q_c \neq 0$. Thus, locally, $Z^i = \theta\left(q^i, Q, p_i\right)$ for $i = 1, 2$.

The application of the lemma requires that $rank\left[\overline{grad\, q^1},\ \overline{gradQ},\ \overline{grad\, p_1}\right] = 3$. The separability assumption implies: $q^i_c = 0$, $Q_{p_i} = 0$, for $i = 1, 2$, so that the above matrix expression gives:

$$rank \begin{bmatrix} q^1_{p_1} & 0 & 1 \\ q^1_{p_2} & 0 & 0 \\ 0 & Q_c & 0 \\ q^1_x & Q_x & 0 \end{bmatrix} = 3, \Longleftrightarrow \begin{vmatrix} q^1_{p_1} & 0 & 1 \\ q^1_{p_2} & 0 & 0 \\ q^1_x & Q_x & 0 \end{vmatrix} \neq 0\ ,\ or \quad \begin{vmatrix} q^1_{p_1} & 0 & 1 \\ q^1_{p_2} & 0 & 0 \\ 0 & Q_c & 0 \end{vmatrix} \neq 0, or \begin{vmatrix} q^1_{p_1} & 0 & 1 \\ 0 & Q_c & 0 \\ q^1_x & Q_x & 0 \end{vmatrix} \neq 0\,.$$

Therefore, if we assume

$$q^1_x \neq 0, \quad q^2_x \neq 0, \quad Q_c \neq 0, \tag{R1}$$

the above rank condition holds. We call these conditions *regularity conditions* because these are fulfilled in most cases.

From the budget constraint (4), one obtains that

$$Z^2 + p_2 q^2 = (X - CQ) + (-p_1 q^1 - Z^1). \tag{A2}$$

Define $\varphi(q^1, Q, p^1) = -Z^1(q^1, Q, p^1) - p^1 q^1$. Note that $\varphi(q^1, Q, p^1) = -\Phi\,(p_1, p_2, c, X)$. Our objetive is to find the derivatives of the sharing rule for private goods (Φ), so we will develop the gradient expression for φ.

Since the left hand side of (A2) only depends on Z^2, q^2 and p_2, the right hand side, $(X - CQ) + \varphi(q^1, Q, p^1)$, depends on these same variables, and its gradient is colinear with the gradients of q^2, Q and p_2. Applying the lemma to this expression, one gets:

$$\left| \overline{grad\,(X - CQ) + \varphi\,(q^1, Q, p^1)} \quad \overline{grad\,q^2} \quad \overline{grad Q} \quad \overline{grad\,p_2} \right| = 0.\text{ Note that } \overline{grad\,p_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The above determinant is equal to:

$$\begin{vmatrix} \varphi_{q^1} q^1_{p_1} + \varphi_{p_1} & q^2_{p_1} & 0 \\ -Q + Q_c(\varphi_Q - C) & 0 & Q_c \\ 1 + \varphi_{q^1} q^1_x + Q_x(\varphi_Q - C) & q^2_x & Q_x \end{vmatrix} = 0.$$

Let $B = \frac{q^2_{p_1}}{q^2_x}$, and $D = \frac{Q_c}{Q_x}$, then, from this determinant we obtain:

$$BD\left(1 + \varphi_{q^1} q^1_x + Q_x(\varphi_Q - C)\right) - B\left(-Q + Q_c(\varphi_Q - C)\right) - D\left(\varphi_{q^1} q^1_{p_1} + \varphi_{p_1}\right) = 0.$$

Dividing by D, we eliminate the term $BQ_x(\varphi_Q - C)$ and get:

$$\varphi_{p_1} = B\left(\frac{D+Q}{D}\right) - \varphi_{q^1}\left(q^1_{p_1} - Bq^1_x\right),$$

and calling $\delta = \left(\frac{D+Q}{D}\right)$, the above expression gives:

$$\boxed{\varphi_{p_1}\text{=B}\delta\text{-}\varphi_{q^1}\left(q^1_{p_1} - Bq^1_x\right)} \tag{A3}$$

Since the term φ_{q^1} is not observable, we continue developing the above expression. The right hand side Bδ-$\varphi_{q^1}\left(q^1_{p_1} - Bq^1_x\right)$ in (A3) depends only on q^1, Q and p_1, that enter φ. Again, we can apply the lemma because of colinearity among gradients of this expression and of q^1, Q and p_1. Taking into account that the separability assumption makes $\delta_{pi} = 0$, $B_c = 0$ and $q^i_{p_i c} = q^i_{xc} = 0$, the determinant for the gradients is equal to:

$$\begin{vmatrix} B_{p_2}\delta - \varphi_{qq}q^1_{p_2}(q^1_{p_1} - Bq^1_x) - \varphi_q(q^1_{p_1 p_2} - B_{p_2}q^1_x - Bq^1_{p_2 x}) & q^1_{p_2} & 0 \\ B\delta_c - \varphi_{qQ}Q_c(q^1_{p_1} - Bq^1_x) & 0 & Q_c \\ \delta B_x + B\delta_x - (\varphi_{qq}q^1_x + \varphi_{qQ}Q_x)(q^1_{p_1} - Bq^1_x) - \varphi_q(q^1_{p_1 x} - B_x q^1_x - Bq^1_{xx}) & q^1_x & Q_x \end{vmatrix} = 0$$

If $A = \frac{q^1_{p_2}}{q^1_x}$, then the above determinant gives:

$$\begin{aligned} 0 \;=\; & AD\Big(\delta B_x + B\delta_x - (\varphi_{qq}q^1_x + \varphi_{qQ}Q_x)(q^1_{p_1} - Bq^1_x) - \varphi_q(q^1_{p_1 x} - B_x q^1_x - Bq^1_{xx})\Big) - \\ & -A\left(B\delta_c - \varphi_{qQ}Q_c(q^1_{p_1} - Bq^1_x)\right) - D\Big(B_{p_2}\delta - \varphi_{qq}q^1_{p_2}(q^1_{p_1} - Bq^1_x) - \varphi_q(q^1_{p_1 p_2} - B_{p_2}q^1_x - Bq^1_{p_2 x})\Big) \end{aligned}$$

Dividing by D, and since $Aq^1_x = q^1_{p_2}$, the terms multiplying φ_{qq} are eliminated. Since $\frac{Q_c}{D} = Q_x$, the term in φ_{qQ} is also eliminated . Consider also that $q^1_{p2x} - Aq^1_{xx} = A_x q^1_x$ and $q^1_{p1p2} - Aq^1_{p1x} = A_{p_1}q^1_x$. Carrying together in the left side the terms multiplied by φ_{q^1} gives:

$$\varphi_{q^1}q^1_x\left[A_{p_1} - BA_x - (B_{p_2} - AB_x)\right] = \delta B_{p_2} - A\left(\delta B_x + B\delta_x\right) + \frac{A}{D}B\delta_c.$$

We denote $B' = B\delta$, and the above expression gives:

$$\boxed{\varphi_{q^1}\text{q}^1_x\left[A_{p_1} - BA_x - (B_{p_2} - AB_x)\right]\text{=B}'_{p_2}\text{-AB}'_x\text{+}\tfrac{A}{D}\text{B}'_c} \tag{A4}$$

In (A4) the only unobservable term is φ_{q^1}, so we can obtain φ_{q^1} in terms of the demand parameters of public and exclusive goods. And replacing this expression in (A3), we also obtain an observable expression for φ_{p_1}. In order to calculate φ_{q^1}, we assume that the following regularity condition is satisfied:

$$(A_{p_1} - BA_x) \neq (B_{p_2} - AB_x) \neq 0. \tag{R2}$$

We can calculate φ_{q^1} from (A4) and get:

$$\varphi_{q^1} = \frac{-\alpha'}{q^1_x} + \frac{-\gamma}{q^1_x}, \tag{1}$$

where

$$\alpha' = \frac{\delta}{1 - \frac{A_{p_1} - BA_x}{B_{p_2} - AB_x}}, \quad \gamma = \frac{AB\left(\delta_x - \frac{\delta_c}{D}\right)}{A_{p_1} - BA_x - (B_{p_2} - AB_x)}.$$

Replacing φ_{q^1} in (A3), the expression for φ_{p_1} is:

$$\varphi_{p_1} = B' + \left(\alpha' + \gamma\right)\left(\frac{q^1_{p_1}}{q^1_x} - B\right). \tag{2}$$

Summing up, we have obtained expressions (1) and (2) for φ_{q^1} and φ_{p_1}. We can obtain the empirical restrictions implied by collective rationality (CR*1) and (CR*2) from these expressions.

Firstly, since φ_{q^1} depends on q^1, Q and p_1, again from the lemma:

$$\begin{vmatrix} (\alpha' + \gamma)\frac{q^1_{p_2x}}{(q^1_x)^2} - \frac{1}{q^1_x}(\alpha'_{p_2} + \gamma_{p_2}) & A & 0 \\ -\frac{1}{q^1_x}(\alpha_c + \gamma_c) & 0 & D \\ (\alpha' + \gamma)\frac{q^1_{xx}}{(q^1_x)^2} - \frac{1}{q^1_x}(\alpha'_x + \gamma_x) & 1 & 1 \end{vmatrix} = 0,$$

and this determinant gives:

$$A\left((\alpha' + \gamma)\frac{q^1_{xx}}{(q^1_x)^2} - \frac{1}{q^1_x}(\alpha'_x + \gamma_x)\right) + \frac{A}{D}\left(\frac{1}{q^1_x}(\alpha_c + \gamma_c)\right) - \left((\alpha' + \gamma)\frac{q^1_{p_2x}}{(q^1_x)^2} - \frac{1}{q^1_x}(\alpha'_{p_2} + \gamma_{p_2})\right) = 0$$

Since $\frac{q^1_{p_2x}}{q^1_x} - A\frac{q^1_{xx}}{q^1_x} = A_x$, we obtain the first empirical restriction:

$$\boxed{\left(\alpha' + \gamma\right)\mathrm{A}_x + \mathrm{A}\left(\alpha'_x + \gamma_x\right) = \frac{A}{D}\left(\alpha'_c + \gamma_c\right) + \left(\alpha'_{p_2} + \gamma_{p_2}\right)} \tag{CR* 1}$$

Secondly, for φ_{q^1} and $\varphi_{p_1}$to be compatible and for the integrability of φ, it is necessary that $\varphi_{q^1 p_1} = \varphi_{p_1 q^1}$. To compute this, consider

$$\begin{aligned}
\varphi\left(p_1, p_2, C, X\right) &= \varphi\left(q^1\left(p_1, p_2, X\right), Q\left(C, X\right), p_1\right) \Rightarrow \\
\varphi_{p_1 q^1} &= \frac{\partial}{\partial q^1}\varphi_{p_1} = \frac{\partial}{\partial p_1}\varphi_{p_1}\frac{\partial p_1}{\partial q^1} + \frac{\partial}{\partial p_2}\varphi_{p_1}\frac{\partial p_2}{\partial q^1} + \frac{\partial}{\partial C}\varphi_{p_1}\frac{\partial C}{\partial q^1} + \frac{\partial}{\partial X}\varphi_{p_1}\frac{\partial X}{\partial q^1} \\
\varphi_{q^1 p_1} &= \frac{\partial}{\partial p_1}\varphi_{q^1} = \frac{\partial}{\partial p_1}\varphi_{q^1}\frac{\partial p_1}{\partial p_1} + \frac{\partial}{\partial p_2}\varphi_{q^1}\frac{\partial p_2}{\partial p_1} + \frac{\partial}{\partial C}\varphi_{q^1}\frac{\partial C}{\partial p_1} + \frac{\partial}{\partial X}\varphi_{q^1}\frac{\partial X}{\partial p_1}
\end{aligned}$$

To calculate these derivatives we need the following Jacobian matrix $\frac{\partial(p_1,p_2,C,X)}{\partial(p_1,p_2,Q,q^1)}$. If we define the mapping $\Pi : (p_1, p_2, C, X) \rightarrow (p_1, p_2, Q, q^1)$, the Jacobian matrix of Π is:

$$J(\Pi) = \frac{\partial\left(p_1, p_2, Q, q^1\right)}{\partial\left(p_1, p_2, C, X\right)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Q_c & Q_x \\ q^1_{p_1} & q^1_{p_2} & 0 & q^1_x \end{bmatrix} \quad |J(\Pi)| = q^1_x Q_c.$$

And the Jacobian matrix of Π^{-1} is:

$$(J(\Pi))^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{q^1_{p_1}}{q^1_x}\frac{1}{D} & \frac{q^1_{p_2}}{q^1_x}\frac{1}{D} & \frac{1}{Q_c} & -\frac{1}{q^1_x}\frac{1}{D} \\ -\frac{q^1_{p_1}}{q^1_x} & -\frac{q^1_{p2}}{q^1_x} & 0 & \frac{1}{q^1_x} \end{bmatrix} = {}^1 J(\Pi^{-1}) \equiv \frac{\partial\left(p_1, p_2, C, X\right)}{\partial\left(p_1, p_2, Q, q^1\right)}.$$

Then, the expression for second derivatives of φ are:

$$\begin{aligned}
\varphi_{p_1 q^1} &= \frac{\partial}{\partial q^1}\varphi_{p_1} = \frac{\partial}{\partial q^1}\left(\left(\alpha'_x + \gamma_x\right)\left(\frac{q^1_{p_1}}{q^1_x} - B\right)\right) = -\frac{1}{q^1_x}\frac{1}{D}\frac{\partial}{\partial C}\left[B' + \left(\alpha' + \gamma\right)\left(\frac{q^1_{p_1}}{q^1_x} - B\right)\right] + \\
&+ \frac{1}{q^1_x}\frac{\partial}{\partial X}\left[B' + \left(\alpha' + \gamma\right)\left(\frac{q^1_{p_1}}{q^1_x} - B\right)\right], \\
\varphi_{q^1 p_1} &= \frac{\partial}{\partial p_1}\varphi_{q^1} = \frac{\partial}{\partial p_1}\left(\frac{-\alpha'}{q^1_x} + \frac{-\gamma}{q^1_x}\right) + \frac{q^1_{p_1}}{q^1_x}\frac{1}{D}\frac{\partial}{\partial C}\left(\frac{-\alpha'}{q^1_x} + \frac{-\gamma}{q^1_x}\right) - \frac{q^1_{p_1}}{q^1_x}\frac{\partial}{\partial X}\left(\frac{-\alpha'}{q^1_x} + \frac{-\gamma}{q^1_x}\right).
\end{aligned}$$

[1] By the Theorem of the Inverse Function

Equating the two above expressions we obtain:

$$B_x^{'} - (\alpha_x^{'} + \gamma_x)B - (\alpha^{'} + \gamma)B_x = -(\alpha_{p_1}^{'} + \gamma_{p_1}) + \frac{B_c^{'}}{D} - \frac{B}{D}\left(\alpha_c^{'} + \gamma_c\right),$$

and since $\alpha^{'} = \alpha\delta$, and $\beta^{'} = (1-\alpha)\,\delta = \beta\delta \Rightarrow \beta_x^{'} = \delta\beta_x + \beta\delta_x,\ \ \beta_c^{'} = \beta\delta_c,\ \beta_{p_1}^{'} = \delta\beta_{p_1}$, we express the above equality as the second empirical restriction:

$$\boxed{\left(\beta^{'} - \gamma\right)\mathrm{B}_x + \mathrm{B}\left(\beta_x^{'} - \gamma_x\right) = \left(\beta_{p_1}^{'} - \gamma_{p_1}\right) + \tfrac{B}{D}\left(\beta_c^{'} - \gamma_c\right)}. \tag{CR* 2}$$

7. APPENDIX 2

Once the compatibility condition $\varphi_{q^1 p_1} = \varphi_{p_1 q^1}$ is fulfilled, we can integrate and obtain φ, and since we have defined Φ as the sharing rule, then

$$\begin{aligned}\varphi(q^1, Q, p^1) &= -Z^1(q^1, Q, p^1) - p^1 q^1, \\ \Phi &= p_1 q^1 + Z^1 = -\varphi(q^1, Q, p_1).\end{aligned}$$

In Appendix 1 we obtained the following expressions:

$$\begin{aligned}\varphi_{q^1} &= \frac{-\alpha^{'}}{q_x^1} + \frac{-\gamma}{q_x^1} \\ \varphi_{p_1} &= B^{'} + \left(\alpha^{'} + \gamma\right)\left(\frac{q_{p_1}^1}{q_x^1} - B\right).\end{aligned}$$

Thus, the relationship between Φ and φ gives us the derivatives of the sharing rule with respect to prices:

$$\Phi_{p_1} = -\left(\varphi_{q^1} q_{p_1}^1 + \varphi_{p_1}\right) = B\left(-\beta\delta + \gamma\right),$$

$$\Phi_{p_2} = -\varphi_{q^1} q_{p_2}^1 = A\left(\alpha\delta + \gamma\right).$$

But we can not recover Φ_c and Φ_x

$$\Phi_c = -\varphi_Q Q_c,$$

$$\Phi_x = -\varphi_{q^1} q_x^1 - \varphi_Q Q_x,$$

since we do not know φ_Q.