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This paper presents hedging strategies for European and exotic options in a Lévy market. By applying Taylor's theorem, dynamic hedging portfolios are constructed under different market assumptions, such as the existence of power jump assets or moment swaps. In the case of European options or baskets of European options, static hedging is implemented. It is shown that perfect hedging can be achieved. Delta and gamma hedging strategies are extended to higher moment hedging by investing in other traded derivatives depending on the same underlying asset. This development is of practical importance as such other derivatives might be readily available. Moment swaps or power jump assets are not typically liquidly traded. It is shown how minimal variance portfolios can be used to hedge the higher order terms in a Taylor expansion of the pricing function, investing only in a risk-free bank account, the underlying asset and potentially variance swaps. The numerical algorithms and performance of the hedging strategies are presented, showing the practical utility of the derived results.

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KEY WORDS: Lévy process; Hedging; Exotic option; Variance swap, Power jump asset; Moment swap, Chaotic Representation Property.

1 Introduction

This paper provides perfect hedging strategies and minimal variance portfolios for European and exotic options in a Lévy market. It is well known, see Schoutens (2000, p.71), that Brownian motion has the Chaotic Representation Property (CRP), which states that every square integrable random variable adapted to the filtration generated by a Brownian motion can be represented as a sum of its mean and an infinite sum of iterated stochastic integrals with respect to the Brownian motion, with deterministic integrands. A consequence of this is the so-called Predictable Representation Property (PRP) for Brownian motion. The PRP states that every square integrable random variable adapted to the filtration generated by a Brownian motion can be represented as a sum of its mean and a stochastic integral with respect to the Brownian motion, where the integrand is a predictable process. The PRP implies the completeness of the Black-Scholes option pricing model and gives the admissible self-financing strategy of replicating a contingent claim whose price only depends on the time to maturity and the current stock price.

Unfortunately, this kind of PRP, where the stochastic integral is with respect to the underlying process only, is a property which is only possessed by a few martingales, including Brownian motion, the compensated Poisson process, and the Azéma martingale (see Schoutens (2003) or Dritschel & Protter (1999)). When the underlying asset is driven by a Lévy process, perfect hedging using only a risk-free bank account and the underlying asset is not in general possible. The market is therefore said to be incomplete. However, even in this case, further developments are possible. There are two different types of chaos expansions for Lévy processes: Itô (1956) proved a Chaotic Representation Property (CRP) for any square integrable functional of a general Lévy process. This CRP is written in terms of multiple integrals with respect to a two-parameter random measure associated with the Lévy process. Nualart & Schoutens (2000) proved the existence of a new version of the CRP for Lévy processes which satisfy some exponential moment conditions. This new version states that every square integrable random variable adapted to the filtration generated by a Lévy process can be represented as an infinite sum of iterated stochastic integrals with respect to the orthogonalised compensated power jump processes of the underlying Lévy process. The market can be completed by allowing trades in these processes while risks due to jumps and fat tails are considered. In light of the new version of the PRP, Corcuera *et al.* (2005) suggested that the market should be enlarged with power jump assets so that perfect hedging could still be implemented. Corcuera *et al.* (2006) used this completeness to solve the portfolio optimisation problem using the martingale method. Another form of commonly traded financial derivative is the variance swap which depends functionally on the volatility of the underlying asset. Since variance swaps are already traded commonly in the over-the-counter (OTC) markets, Schoutens (2005) suggested trading in moment swaps, which are a generalisation of variance swaps. Based on the CRP derived by Itô (1956), Benth *et al.* (2003) derived a minimal variance portfolio for hedging contingent claims in a Lévy market.

Inspired by these papers, we derive practical and implementable hedging strategies based on the PRP derived from Taylor approximations to the option pricing formulae. We apply Taylor's theorem directly to the option pricing formulae and derive perfect hedging strategies by investing in power jump assets, moment swaps or some traded derivatives depending on the same underlying asset. The hedging of the higher moments terms in the Taylor expansion of a contingent claim using other contingent claims in a Lévy market is a technique introduced by this paper. When these financial derivatives are not available, we demonstrate how to use the minimal variance portfolios derived by Benth *et al.* (2003) to hedge the higher order terms in the Taylor expansion. While we apply Taylor expansions to decompose the pricing formula into an infinite sum of higher moment terms, Corcuera *et al.* (2005) applied the Itô's formula to obtain the PRP of a contingent claim. Note that the Itô's formula is derived as a result of the elementary Taylor expansion, see Kijima (2002). In practice, when implementing a hedging strategy numerically, we have to discretise the time variable. Hence, it is more natural to work directly from Taylor's theorem as this discretisation can be acknowledged explicitly. In fact, the delta and gamma hedges commonly used by traders in the market, given in Section 3.2.4, are derived using a Taylor expansion. We construct dynamic hedging strategies for European and exotic options in a Lévy market. Although static hedging is only applied to European options, exotic options can be decomposed into a basket of European options so that static hedging can be achieved; in this case see for example Derman *et al.* (1995). It is practically important to be able to statically hedge since static hedging has several advantages over dynamic hedging. Static hedging is less sensitive to the assumption of zero transaction costs (both commissions and the cost of paying individuals to monitor the positions). Moreover, dynamic hedging tends to fail when liquidity dries up or when the market makes large moves, but especially in these situations effective hedging is needed.

We discuss how hedging can be implemented by applying Taylor's theorem to a pricing formula. We investigate the approximation of the derivatives of the pricing formula and present the numerical procedures used to construct the hedging strategies. Performance of the hedging is assessed and the difficulties encountered are discussed. Thus, this paper constitutes a practical development for the hedging of contingent claims in a Lévy market.

The rest of the paper is arranged as follows: Section 2 gives the background about Lévy processes and the CRPs in terms of power jump processes and Poisson random measures. Section 3 gives hedging strategies by investing in variance swaps, moment swaps or power jump assets and extend the delta and gamma hedging strategies to higher moment hedging. Section 4 gives the minimal variance portfolios in the case where perfect hedging is not possible. Section 5 gives the approximation procedures of the hedging strategies and Section 6 gives the performance of the hedging strategies implemented on a set of different types of options as illustration of the performance of the proposed method. An example of static hedge of a one year European option on real life data is given. In Section 7, some concluding remarks are provided. Proofs and tables are included in the appendices at the end.

2 Background

In this section, we give a brief introduction to Lévy processes and the two versions of the CRP discussed in the introduction. We discuss Taylor expansions which will later be used to derive the hedging portfolios of exotic options.

Let $X = \{X_t, t \geq 0\}$ be a Lévy process in a complete probability space $(\Omega, \mathcal{F}, P)$ on $\mathbb{R}^d$, where $\mathcal{F}$ is the filtration generated by $X : \mathcal{F}_t = \sigma\{X_t, 0 \leq s \leq t\}$, where σ denotes the sigma-algebra generated by X. A detailed account of Lévy process can be found in Sato (1999). Denote the *left limit process* by $X_{t-} = \lim_{s \to t, s<t} X_s,\ t > 0$, and the *jump size* at time t by $\Delta X_t = X_t - X_{t-}$. Let ν be the Lévy measure of X. In the rest of the paper, we assume that all Lévy measures concerned satisfy, for some $\varepsilon > 0$ and $\lambda > 0$, $\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda |x|)\, \nu(\mathrm{d}x) < \infty$. This condition implies that for $i \geq 2$, $\int_{-\infty}^{+\infty} |x|^i \nu(\mathrm{d}x) < \infty$, and that the characteristic function $E[\exp(\mathrm{i}uX_t)]$ is analytic in a neighborhood of 0. Denote the i-th *power jump process* by $X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i,\ i \geq 2$, and for completeness let $X_t^{(1)} = X_t$. Clearly $E[X_t] = E[X_t^{(1)}] = m_1 t$, where $m_1 < \infty$ is a constant and by Protter (2004, p.32), we have

$$E[X_t^{(i)}] = E[\sum_{0<s\leq t} (\Delta X_s)^i] = t \int_{-\infty}^{\infty} x^i \nu(\mathrm{d}x) = m_i t < \infty, \quad \text{for } i \geq 2, \tag{2.1}$$

thus defining the moments m_i. Nualart & Schoutens (2000) introduced the *compensated power jump process* (or *Teugels martingale*) of order i, $Y^{(i)} = \left\{ Y_t^{(i)}, t \geq 0 \right\}$, defined by

$$Y_t^{(i)} = X_t^{(i)} - E[X_t^{(i)}] = X_t^{(i)} - m_i t \quad \text{for } i = 1, 2, 3, ... \tag{2.2}$$

$Y^{(i)}$ is constructed to have a zero mean. It was shown by Nualart & Schoutens (2000, Section 2) that there exist constants $a_{i,1}, a_{i,2}, ..., a_{i,i-1}$ such that the processes defined by

$$H_t^{(i)} = Y_t^{(i)} + a_{i,i-1} Y_t^{(i-1)} + \cdots + a_{i,1} Y_t^{(1)}, \tag{2.3}$$

for $i \geq 1$ are a set of pairwise strongly orthogonal martingales. Nualart & Schoutens (2000) proved the CRP and PRP in terms of these orthogonalised compensated power jump processes, $H^{(i)}$'s.

Theorem 1 (Chaotic Representation Property (CRP)) *Every random variable F in $L^2(\Omega, \mathcal{F})$ has a representation of the form*

$$F = E(F) + \sum_{j=1}^{\infty} \sum_{i_1,...,i_j \geq 1} \int_0^{\infty} \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,...,i_j)}(t_1, ..., t_j) \mathrm{d}H_{t_j}^{(i_j)} ... \mathrm{d}H_{t_2}^{(i_2)} \mathrm{d}H_{t_1}^{(i_1)}, \tag{2.4}$$

where the $f_{(i_1,...,i_j)}$'s are functions in $L^2(\mathbb{R}_+^j)$ and H's are defined in equation (2.3).

Theorem 2 (Predictable Representation Property (PRP)) *The CRP stated above implies that every random variable F in $L^2(\Omega, \mathcal{F})$ has a representation of the form*

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^{\infty} \phi_s^{(i)} \mathrm{d}H_s^{(i)}, \tag{2.5}$$

where $\phi_s^{(i)}$'s are predictable, that is, they are $\mathcal{F}_{s-}$-measurable.

In contrasts, Itô (1956) proved a chaos expansion for general Lévy processes in terms of multiple integrals with respect to the compensated Poisson random measure. One may convert the representation to one involving iterated integrals (see Løkka (2004)). The stochastic integrals are in terms of both Brownian motion, W, and the compensated Poisson measure $\tilde{N}$,

$$\tilde{N}\left(\mathrm{d}t, \mathrm{d}x\right) = N\left(\mathrm{d}t, \mathrm{d}x\right) - \nu\left(\mathrm{d}x\right)\mathrm{d}t, \tag{2.6}$$

where $\nu\left(\mathrm{d}x\right)$ is the Lévy measure of the underlying Lévy process, and

$$N\left(B\right) = \#\left\{t : \left(t, \Delta X_t\right) \in B\right\}, B \in \mathcal{B}\left(\left[0, T\right] \times \mathbb{R}_0\right),$$

is the Poisson random measure of the Lévy process where $\mathcal{B}\left(\left[0, T\right] \times \mathbb{R}_0\right)$ is the *Borel σ-algebra* of $[0, T] \times \mathbb{R}_0$ and $\mathbb{R}_0 = \mathbb{R} - \{0\}$. The compensator of $N\left(\mathrm{d}t, \mathrm{d}x\right)$ is known to be $E\left[N\left(\mathrm{d}t, \mathrm{d}x\right)\right] = \nu\left(\mathrm{d}x\right)\mathrm{d}t$. Benth *et al.* (2003) derived relations between the two chaos expansions. When the underlying Lévy process is a pure jump process, the compensated power jump process defined in (2.2) satisfies

$$Y_t^{(i)} = \int_0^t \int_{\mathbb{R}} x^i \tilde{N}\left(\mathrm{d}s, \mathrm{d}x\right), \quad 0 \leq t \leq T, \ i = 1, 2, ... \tag{2.7}$$

This relationship is very important in the development of the chaotic representation of Lévy processes. Since the introduction of the chaos expansion by Itô (1956), the development of representations in the literature has been focused on expansions with respect to the Poisson random measure. Unfortunately, we cannot trade in the Poisson random measure. Note that trading in a finite set of power jump assets is possible because the i-th power jump asset contains information of the i-th moment of the Lévy process, given that i is finite. Therefore, it is theoretically possible to construct a financial product which contains information of the i-th moment of the underlying process. For example, if we want to hedge the risk introduced by the variance of the underlying process, we can trade in the variance swaps or the second power jump asset. However, the Poisson random measure contains **all** the information of the moments up to infinity and hence it is not clear how to construct such a financial product unless information of all the higher moments are obtained. This limits the application of the CRP in terms of Poisson random measures and also the application of Lévy processes in finance. The equation (2.7) links the two important expansions together and hence the results derived for expansions in terms of Poisson random measures can be applied to the expansions in terms of power jump processes.

To unify notation, Benth *et al.* (2003) defined the following notation:

$$U_1 = \left[0, T\right], U_2 = \left[0, T\right] \times \mathbb{R}, \mathrm{d}Q_1\left(\cdot\right) = \mathrm{d}W\left(\cdot\right), Q_2\left(\cdot\right) = \tilde{N}\left(\cdot, \cdot\right),$$

$$\int_{U_1} g\left(u^{(1)}\right) Q_1\left(\mathrm{d}u^{(1)}\right) = \int_0^t g\left(s\right) W\left(\mathrm{d}s\right), \int_{U_2} g\left(u^{(2)}\right) Q_2\left(\mathrm{d}u^{(2)}\right) = \int_0^t \int_{\mathbb{R}} g\left(s, x\right) \tilde{N}\left(\mathrm{d}s, \mathrm{d}x\right).$$

The CRP in terms of Brownian motion and Poisson random measures is given by:

Theorem 3 (Chaos expansion for general Lévy process by Itô (1956)) *Let F be a square integrable random variable adapted to the underlying Lévy process, X. We have*

$$F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,...,j_n=1,2} \tilde{J}_n \left(g_n^{(j_1,...,j_n)} \right), \tag{2.8}$$

for a unique sequence $g_n^{(j_1,...,j_n)}$ $(j_1, ..., j_n = 1, 2;\ n = 1, 2, ...)$ of deterministic functions in the corresponding L_2-space, $L_2(G_n)$, where

$$G_n = \left\{ \left(u_1^{(j_1)}, ..., u_n^{(j_n)} \right) \in \Pi_{i=1}^n U_{j_i} : 0 \le t_1 \le \cdots \le t_n \le T \right\} \tag{2.9}$$

with $u^{(j_i)} = t$ if $j_i = 1$, and $u^{(j_i)} = (t, x)$ if $j_i = 2$, and

$$\begin{aligned} &\tilde{J}_n \left(g_n^{(j_1,...,j_n)} \right) \\ &= \int_{\Pi_{i=1}^n U_{j_i}} g_n^{(j_1,...,j_n)} \left(u_1^{(j_1)}, ..., u_n^{(j_n)} \right) 1_{G_n} \left(u_1^{(j_1)}, ..., u_n^{(j_n)} \right) Q_{j_1} \left(\mathrm{d}u_1^{(j_1)} \right) \cdots Q_{j_n} \left(\mathrm{d}u_n^{(j_n)} \right). \end{aligned}$$

So far we have given the theoretical results on the chaotic representations. We now discuss their financial applications. Under the Black-Scholes model, the PRP of Brownian motions allows perfect hedging of European options. Unfortunately, the derivation of hedging strategies of options in an incomplete market is not as simple and has been the focus of considerable study in the literature, see for example Carr *et al.* (2001), He *et al.* (2005) and Cont *et al.* (2005). In this paper, by extending the ideas of Corcuera *et al.* (2005), Schoutens (2005) and Benth *et al.* (2003), we derive and implement some hedging strategies for European and exotic options. Numerical procedures are provided and performance of the hedging strategies is discussed.

As mentioned above, the PRP is useful in option hedging. For option pricing functions which are infinitely differentiable in the stock price, we can simply apply the Itô's formula to obtain such a predictable representation. Assuming power jump assets are traded in the market, Corcuera *et al.* (2005) derived a self-financing replicating portfolio for a contingent claim whose payoff function only depends on the stock price at maturity. Their hedging formula is derived from the Itô's formula and given in terms of an infinite sum of stochastic integrals. In this paper, we use a different approach to determine a self-financing replicating portfolio, which, in some cases, can be used in both static and dynamic hedging with a flexible Δt, where Δt denotes the time change during the hedging period. We discuss this in more detail in Section 3. We apply Taylor's theorem directly to the option pricing formulae to obtain the hedging portfolios. In the literature, the results on option hedging using CRP of Lévy processes, has previously focused on the theoretical aspects of the problem, see, for example, Corcuera *et al.* (2005) and Løkka (2004). We aim to investigate the problem from a practical point of view by providing methods to obtain the hedging portfolios explicitly using numerical methods and shall discuss the difficulties encountered. Our approach can be applied to a range of exotic options in the case of dynamic hedging, for example, barrier options, lookback options and Asian options.

3 The perfect hedging strategies

In this section, we derive hedging strategies using Taylor's theorem. Firstly, we specify the model of the underlying asset, S_t. Following Corcuera *et al.* (2005, Theorem 3), we assume

$$\mathrm{d}S_t = bS_{t-}\mathrm{d}t + S_{t-}\mathrm{d}X_t, \tag{3.1}$$

where $X = \{X_t, t \geq 0\}$ is a general Lévy process. Let the risk-free bank account be $B_t = \exp(rt)$, where r is the continuously compounded risk-free rate. Let $F(t,x)$ be the option pricing function at time $t < T$ and stock price equal to x, where T is the maturity of the option. Let $D_1^i F(t,x)$ be the i-th derivative of $F(t,x)$ with respect to the first variable (time), and $D_2^i F(t,x)$ be the i-th derivative of $F(t,x)$ with respect to the second variable (stock price). Suppose $F(t,x)$ is continuous and infinitely differentiable in the second variable and satisfies

$$\sup_{x<K, t\leq t_0} \sum_{n=2}^{\infty} \left| D_2^n F(t,x) \right| R^n < \infty \qquad \text{for all } K, R > 0, t_0 > 0.$$

Let Δt be the time change during the hedging period and $\Delta S_t = S_{t+\Delta t} - S_t$. Applying Taylor's theorem twice to the option pricing formula, $F(t, S_t)$, we obtain

$$F(t+\Delta t, S_t + \Delta S_t) - F(t, S_t) = \sum_{i=1}^{\infty} \frac{D_2^i F(t+\Delta t, S_t)}{i!} (\Delta S_t)^i + \sum_{i=1}^{\infty} \frac{D_1^i F(t, S_t)}{i!} (\Delta t)^i, \tag{3.2}$$

which is true as long as $D_2^i F(t+\Delta t, S_t)$ and $D_1^i F(t, S_t)$ exist for $i = 1, 2, 3, ...$ Note that it is not necessary to apply the multivariate Taylor's theorem since the value of Δt is known at time t. Let $M^{(q)}(t,x)$ be the price of a financial derivative such that $M^{(q)}(0, S_0) = F(0, S_0)$ and

$$M^{(q)}(t+\Delta t, S_t + \Delta S_t) - M^{(q)}(t, S_t) = \sum_{i=1}^{q} \frac{D_2^i F(t+\Delta t, S_t)}{i!} (\Delta S_t)^i + \sum_{i=1}^{\infty} \frac{D_1^i F(t, S_t)}{i!} (\Delta t)^i, \tag{3.3}$$

where q is a positive integer. Therefore, we have $\lim_{q\to\infty} M^{(q)}(T, S_T) = F(T, S_T)$, that is, the value of the financial derivative $M^{(q)}$ converges to F as q goes to infinity. Our aim is to construct a self-financing hedging portfolio for $M^{(q)}$. Note that the hedging error at time Δt,

$$\begin{aligned}\left[F(t+\Delta t, S_t + \Delta S_t) - F(t, S_t)\right] - \left[M^{(q)}(t+\Delta t, S_t + \Delta S_t) - M^{(q)}(t, S_t)\right] \\ = \sum_{i=q+1}^{\infty} \frac{D_2^i F(t+\Delta t, S_t)}{i!} (\Delta S_t)^i,\end{aligned}$$

can be approximated using standard techniques in calculating the remainder terms in a Taylor expansion. Let $\mathcal{P}_t^{(i)}$ be the value of a basket of financial derivatives such as the risk-free bank account, the underlying stock, variance swaps, moment swaps, power jump assets or other financial derivatives depending on the same underlying stock such that $(\Delta S_t)^i = \Delta\mathcal{P}_t^{(i)} = \mathcal{P}_{t+\Delta t}^{(i)} - \mathcal{P}_t^{(i)}$. Note that $\mathcal{P}_t^{(i)}$ is a basket of assets that would not lead to arbitrage opportunities. We will show later how to construct such a basket of tradable assets. Therefore, we have

$$\begin{aligned} M^{(q)}(t+\Delta t, S_t+\Delta S_t) - M^{(q)}(t, S_t) &= \sum_{i=1}^{\infty} \frac{D_1^i F(t, S_t)}{i!} (\Delta t)^i + D_2^1 F(t+\Delta t, S_t)\Delta S_t \\ &+ \sum_{i=2}^{q} \frac{D_2^i F(t+\Delta t, S_t)}{i!} \Delta\mathcal{P}_t^{(i)}. \end{aligned} \tag{3.4}$$

The self-financing portfolio to hedge $M^{(q)}(t+\Delta t, S_t+\Delta S_t) - M^{(q)}(t, S_t)$ is then

(i) Invest

$$\frac{1}{(\exp(r\Delta t) - 1)} \sum_{i=1}^{\infty} D_1^i F(t, S_t)(\Delta t)^i / i! \tag{3.5}$$

in a riskless bank account such that at time $t+\Delta t$, the deposit is worth

$$\frac{1}{(\exp(r\Delta t) - 1)} \sum_{i=1}^{\infty} D_1^i F(t, S_t)(\Delta t)^i \exp(r\Delta t) / i!$$

and the change of value of the investment is $\sum_{i=1}^{\infty} \frac{D_1^i F(t,S_t)}{i!} (\Delta t)^i$.

(ii) Invest $D_2^1 F(t+\Delta t, S_t)$ in the underlying stock;

(iii) Invest $\frac{D_2^i F(t+\Delta t, S_t)}{i!}$ in $\mathcal{P}_t^{(i)}$ for $i = 2, 3, ..., q$.

In real life application, we have to find a reasonable value for q and we discuss methods of choosing q in Section 6. Note that the approximation in (3.3) requires the existence of $D_1^i F(t, S_t)$ for $i = 1, 2, 3, ...$ and $D_2^i F(t+\Delta t, S_t)$ only for $i = 1, 2, 3, ..., q$. The value of q determines how many different financial derivatives needed to hedge the option up to a pre-specified level of accuracy. If $q = 1$, we only need to hedge the deterministic term that appears as the first term in equation (3.4) by investing in a risk-free bank account, and the term $D_2^1 F(t+\Delta t, S_t)\Delta S_t$ by investing in the underlying stock, which is a simple extension to the delta hedging discussed in Section 3.2.4. If $q = 2$, we can hedge by investing in a risk-free bank account, the underlying stock and the variance swaps currently traded in the market, which is discussed in Section 3.2.1. If $q \geq 3$, we can consider perfect hedging in three cases: (a) trading in moment swaps, discussed in Section 3.2.2, (b) trading in power jump assets, discussed in Section 3.2.3 and (c) trading in some financial derivatives depending on the same underlying assets, discussed in Section 3.2.4. Note that (a) and (b) are not liquidly traded in the market while (c) might be more readily available. If all of these financial derivatives are not available for trading, we can employ the minimal variance portfolios derived in Section 4.

The approximation in (3.3) can be used in both static and dynamic hedging for European options by just changing Δt. The reason why static hedging may not be applicable to exotic options is because if during the hedging period, Δt, the value of the $S_{t+\Delta s}$, where $\Delta s < \Delta t$ is explicitly occurring in the formulae, then this must be used in the calculation of the option price. In this case, we have to apply Taylor's theorem with respect to both $\Delta S_t = (S_{t+\Delta t} - S_t)$ and $(S_{t+\Delta s} - S_t)$. In the case of dynamic hedging, we can assume that the minimum time period for a change of value of S to take place is equal to Δt , the hedging period. Although static hedging can only be applied to European options, some exotic options can be decomposed into a basket of European options such that static hedging can still be achieved, see for example Derman *et al.* (1995). In Section 6, we show the approximation results for both static hedging (Δt equals to 3 months) and dynamic hedging (Δt equals to 5 minutes) for European options and dynamic hedging for barrier options. The advantage of static hedging over dynamic hedging is that in real life, transaction costs and bid-ask spreads of option prices are not negligible. The replicating portfolio is not truly self-financing since extra investment must be made to pay for these additional costs. Hence, it is preferable to hedge statically rather than dynamically as the costs involved will be less and constant rebalancing is not required. In the literature and in practice, it is common to assume that ΔS_t is very small such that the approximation in (3.3) can be truncated without loss of accuracy; this is the main assumption behind the delta and gamma hedges commonly used by traders in the market. However, in real life, the price of every traded asset in the market moves by a tick size, such as 0.5 or 1. After a very short period of time, the price of the traded asset either stays unchanged or moves by a multiple of the tick size. Hence, the assumption of ΔS_t being very small in hedging is not sufficiently accurate. It would not in general be reasonable to assume that ΔS_t is small when modelling S as a process with jumps. Thus, we consider $\Delta S_t \geq 1$ for both static and dynamic hedging in our simulation analysis in Section 6.

3.1 Hedging instruments

In this section, we consider the use of moment swaps (including variance swaps) and power jump assets in our hedging strategies. Recall the PRP for Lévy processes involves stochastic integrals with respect to power jump processes, which are related to the higher moments of the underlying Lévy process. In equation (3.3), they are represented through the terms $\frac{D_2^i F(t+\Delta t, S_t)}{i!} (\Delta S_t)^i$. To hedge these terms, we need to invest in some financial derivatives related to these higher moments. We show how moment swaps introduced by Schoutens (2005) and power jump assets by Corcuera *et al.* (2005) can be used to construct $\mathcal{P}_t^{(i)}$ used in the hedging portfolio given in (3.4). Variance swaps, introduced by Demeterfi *et al.* (1999), are commonly traded over-the-counter (OTC) derivatives. Schoutens (2005) generalised variance swaps to moment swaps, which are not liquidly traded in the market. Windcliff *et al.* (2006) gave a detailed discussion on volatility swaps. There are two common contractual definitions of returns of stock price. Let $\{s_1, s_2, ..., s_n\}$ be the sampling points of the contract, where the s's are equally spaced with length Δs. The *actual return* and the *log return* are defined to be

$$R_{\text{actual},i} = \left(S_{s_{i+1}} - S_{s_i}\right) / S_{s_i} \quad \text{and} \quad R_{\log,i} = \log\left(S_{s_{i+1}} / S_{s_i}\right). \tag{3.6}$$

The annualised realised variance, $\sigma^2_{\text{realised}}$, is defined by $\sigma^2_{\text{realised}} = \frac{1}{\Delta s(n-2)} \sum_{i=1}^{n-1} R_i^2$ where R_i is either the actual return or log return of the stock price. In the case of log return, $R_i = R_{\log,i}$, Schoutens (2005) generalised variance swaps to moment swaps. The annualised realised k-th moment is defined by $M^{(k)}_{\text{realised}} = \frac{1}{\Delta s(n-2)} \sum_{i=1}^{n-1} R_i^k$. This definition can be easily extended to the case where $R_i = R_{\text{actual},i}$. We can now give the definition of the k-th moment swap.

Definition 1 *A k-th moment swap is a forward contract on annualised realised k-th moment, $M^{(k)}_{realised}$. Its payoff to the holder at expiration is equal to $\left(M^{(k)}_{realised} - M^{(k)}_{strike}\right) N$, where $M^{(k)}_{realised}$ is the realised k-th moment (quoted in annual terms) over the life of the contract, $M^{(k)}_{strike}$ is the pre-defined delivery price for the k-th moment, and N is the notional amount of the swap. The annualised realised k-th moment is calculated based on the pre-specified set of sampling points over the period, $\{s_1, s_2, ..., s_n\}$.*

Corcuera *et al.* (2005) suggested enlarging the Lévy market with power jump assets, where the i-th power jump asset is defined by

$$T_t^{(i)} = \exp(rt)\, Y_t^{(i)}, \quad i \geq 2, \tag{3.7}$$

and $Y_t^{(i)}$ is defined in (2.2). The authors derived the dynamic hedging portfolio trading in these assets using the Itô's formula and noted that the 2nd power jump process is related to the realised variance. However, the 2nd power jump asset is not the same as a variance swap and we consider their usages separately in Section 3.2.

3.2 Hedging strategies

In the last section, we introduce two different kinds of financial derivatives involving higher moments, namely, the moment swaps and the power jump assets. In this section, we explain how to use them to construct the basket of financial derivatives, $\mathcal{P}_t^{(i)}$, in order to hedge the terms in equation (3.3). We also discuss the delta and gamma hedges in the literature and we extend them in order to obtain perfect hedging by trading in certain financial derivatives depending on the same underlying asset, which may be available in the market.

3.2.1 Hedging with variance swaps

To hedge the term $(\Delta S_t)^2$ in equation (3.3), we construct $\mathcal{P}_t^{(2)}$ which invest in a risk-free bank account and variance swaps. If Δt is negligible compared to ΔS_t, from (3.1), we have

$$(\Delta S_t)^2 = S_t^2 (\Delta X_t)^2 . \tag{3.8}$$

We cannot use the variance swaps using log return, $R_{\log,i}$ defined in (3.6) to hedge. We have

$$\left[\log\left(\frac{S_{t+\Delta t}}{S_t}\right)\right]^2 \backsimeq [\log(1+\Delta X_t)]^2$$

since we assume Δt to be negligible. From (3.8), we need $(\Delta X_t)^2$ rather than $[\log(1+\Delta X_t)]^2$ to hedge, therefore the variance swaps using log returns are not useful in this case. Even if we use the model $S_{t+\Delta t} = S_t \exp(\Delta X_t)$ such that $\log(S_{t+\Delta t}/S_t) = \Delta X_t$, we then have $(\Delta S_t)^2 = (S_{t+\Delta t} - S_t)^2 = S_t^2 [\exp(\Delta X_t) - 1]^2$, which still can not be hedged by the variance swaps using log returns. Therefore, in our case where we apply Taylor's theorem with respect to ΔS_t, we should invest in the variance swaps using absolute returns, $R_{\text{actual},i}$, as defined in (3.6).

Recall in Section 3.1 that there is a set of sampling points, $\{s_1, s_2, ..., s_n\}$, for each contract. We invest in the variance swap at time t where the last two sampling points are equal to t and $t+\Delta t$: $s_{n-1} = t$ and $s_n = t+\Delta t$ and maturity equal to $t+\Delta t$. At maturity, we receive the payoff $\sigma^2_{\text{realised}} - \sigma^2_{\text{strike}}$, where

$$\sigma^2_{\text{realised}} = \frac{1}{\Delta s\,(n-2)} \sum_{i=1}^{n-1} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2 = \frac{1}{\Delta s\,(n-2)} \left[\left(\frac{\Delta S_t}{S_t}\right)^2 + \sum_{i=1}^{n-2} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2\right]$$

and the value of $\sum_{i=1}^{n-2} \left[(S_{t_{i+1}} - S_{t_i})/S_{t_i}\right]^2$ is known at time t. In the following, we give the hedging strategy to hedge the term

$$Q_2 = \frac{D_2^2 F(t+\Delta t, S_t)}{2} (\Delta S_t)^2 = C_2 (\Delta S_t)^2 \tag{3.9}$$

in equation (3.3) by constructing $\mathcal{P}_t^{(2)}$.

Proposition 1 *To hedge the term Q_2 in equation (3.9) we invest in C_2 units of $\mathcal{P}_t^{(2)}$ at time t, consisting of $\Delta s\,(n-2)\, S_t^2$ units of the variance swap with sampling points $\{..., s_{n-1} = t, s_n = t+\Delta t\}$, maturity $t+\Delta t$, strike σ^2_{strike} and*

$$\frac{S_t^2 \Delta s\,(n-2)}{[\exp(r\Delta t) - 1]} \left[\sigma^2_{strike} - \frac{1}{\Delta s\,(n-2)} \sum_{i=1}^{n-2} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2\right] + \frac{P_V \Delta s\,(n-2)\, S_t^2}{[\exp(r\Delta t) - 1]}$$

units of cash in a risk-free bank account, where P_V is the price of one unit of the variance swap.

Proof. Let

$$\overline{S}_{n,2} = \frac{1}{\Delta s\,(n-2)} \sum_{i=1}^{n-2} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2 = \frac{1}{\Delta s\,(n-2)} \widetilde{S}_{n,2}. \tag{3.10}$$

The initial investment at time t equals the price of the variance swap plus the deposit into the risk-free bank account, which is equal to

$$C_2 \Delta s\,(n-2)\, S_t^2 P_V \left[1 + \frac{1}{e^{r\Delta t} - 1} \right] + \frac{C_2 S_t^2 \Delta s\,(n-2)}{[\exp(r\Delta t) - 1]} \left[\sigma^2_{\text{strike}} - \overline{S}_{n,2} \right].$$

At maturity, the portfolio is worth

$$C_2 (\Delta S_t)^2 + C_2 S_t^2 \Delta s\,(n-2) \left[\sigma^2_{\text{strike}} - \overline{S}_{n,2} \right] / \left[\left(e^{r\Delta t} - 1 \right) \right] + C_2 P_V \frac{\Delta s\,(n-2)\, S_t^2 e^{r\Delta t}}{e^{r\Delta t} - 1}.$$

Hence, the change of value of the hedging portfolio is equal to

$$C_2 (\Delta S_t)^2 + C_2 \Delta s\,(n-2)\, S_t^2 P_V \left[\frac{e^{r\Delta t}}{e^{r\Delta t} - 1} - 1 - \frac{1}{e^{r\Delta t} - 1} \right] = C_2 (\Delta S_t)^2,$$

as desired. □

3.2.2 Hedging with moment swaps

In the last section, we explained how to hedge the term Q_2 in equation (3.9) using variance swaps. The idea can be extended easily to moment swaps to hedge the term

$$Q_i = \frac{D_2^i F(t + \Delta t, S_t)}{i!} (\Delta S_t)^i = C_i (\Delta S_t)^i$$

for $i = 3, 4, 5, \ldots$, which can be done by investing in the i-th moment swap at time t with sampling points $s_{n-1} = t$ and $s_n = t + \Delta t$ and maturity equal to $t + \Delta t$. At maturity, we receive the payoff $M^{(i)}_{\text{realised}} - M^{(i)}_{\text{strike}}$, where

$$M^{(i)}_{\text{realised}} = \frac{1}{\Delta s\,(n-2)} \left[\left(\frac{\Delta S_t}{S_t} \right)^i + \sum_{i=1}^{n-2} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^i \right] = \frac{1}{\Delta s\,(n-2)} \left[\left(\frac{\Delta S_t}{S_t} \right)^i + \widetilde{S}_{n,i} \right],$$

and the value of $\widetilde{S}_{n,i}$ is known at time t. In the following, we give the hedging strategy to hedge the term Q_i by constructing $\mathcal{P}_t^{(i)}$.

Proposition 2 *To hedge the terms Q_i we invest in C_i units of $\mathcal{P}_t^{(i)}$ at time t, consisting of $\Delta s\,(n-2)\, S_t^i$ units of the i-th moment swap with sampling points $\{\ldots, s_{n-1} = t, s_n = t + \Delta t\}$,* *maturity* $t+\Delta t$ *and strike* $M^{(i)}_{strike}$*, and*

$$\frac{S_t^i \Delta s\,(n-2)}{[\exp(r\Delta t)-1]}\left[M^{(i)}_{strike}-\frac{1}{\Delta s\,(n-2)}\widetilde{S}_{n,i}\right]+\frac{\Delta s\,(n-2)\,S_t^i P_M}{[\exp(r\Delta t)-1]}$$

units of cash in a risk-free bank account where P_M *is the price of one unit of the moment swap.*

Proof. The proof follows in the same fashion as for Proposition 1.

□

3.2.3 Hedging with power jump processes of higher orders

In the last two sections, we discuss how to hedge the term $\sum_{i=1}^{q} Q_i$ for $q \geq 2$ using variance swaps and moment swaps. We can instead hedge using power jump assets, discussed in Section 3.1, if we allow trading of them. Using Itô's formula, see Corcuera *et al.* (2005, Section 2.3), equation (3.1) has the solution

$$S_t = S_0 \exp\left(X_t + \left(b-\sigma^2/2\right)t\right)\prod_{0<s\leq t}\left(1+\Delta L_s\right)\exp\left(-\Delta L_s\right), \tag{3.11}$$

where b is defined in (3.1), σ^2 is the Brownian variance parameter and L is the pure jump part of the Lévy process X, see Corcuera *et al.* (2005, Section 2) for details. In the following, we consider the simplified case where there is at most one jump of X between t and $t+\Delta t$, and the general case where there can be infinite number of jumps. Note that the latter case might not be realistic because in reality, we only observe a discrete series of the underlying stock S, while the power jump processes of the Lévy process with infinite activity are not observable. Therefore, it appears to be more practical to consider trading in moment swaps rather than power jump processes. We consider both assets for completeness and theoretical interest.

The simplified case If Δt is negligible compared to ΔS_t, from (3.1), (3.7) and assuming there is at most one jump of X between t and $t+\Delta t$, we have

$$\left(\Delta S_t\right)^i = S_t^i\left[\exp\left(-r\left(t+\Delta t\right)\right)T^{(i)}_{t+\Delta t}-\exp\left(-rt\right)T^{(i)}_t+m_i\Delta t\right]. \tag{3.12}$$

Therefore, we can hedge the term Q_i by constructing $\mathcal{P}^{(i)}_t$:

Proposition 3 *If* Δt *is negligible compared to* ΔS_t*, to hedge* Q_i*, we invest in* C_i *units of* $\mathcal{P}^{(i)}_t$*, consisting of* $S_t^i\exp\left(-r\left(t+\Delta t\right)\right)$ *units of* $T^{(i)}_t$ *and*

$$S_t^i\left\{e^{-r(t+\Delta t)}T^{(i)}_t - e^{-rt}T^{(i)}_t + m_i\Delta t\right\}/\left[e^{r\Delta t}-1\right]$$

units of cash in a risk-free bank account.

Proof. The proof is included in Appendix A.1. □

If Δt is not negligible compared to ΔS_t, assuming $\sigma = 0$ and there is only one jump of X between times t and $t + \Delta t$ as before, we have from (3.11)

$$\Delta S_t = S_t \left[\exp\left(b\Delta t\right)\left(1 + \Delta X_t\right) - 1\right]. \tag{3.13}$$

Note that if $\Delta t \to 0$, $\exp(b\Delta t) \to 1$, we have $\Delta S_t = S_t\left(\Delta X_t\right)$, as in the case above. Squaring both sides, we have

$$(\Delta S_t)^2 = S_t^2 \left\{ e^{2b\Delta t}\left(\Delta X_t\right)^2 + 2e^{b\Delta t}\left[e^{b\Delta t} - 1\right]\Delta X_t + \left[e^{b\Delta t} - 1\right]^2 \right\}.$$

Substituting ΔX_t by $\left[\frac{\Delta S_t}{S_t} + 1\right]\exp(-b\Delta t) - 1$, we have

$$(\Delta S_t)^2 = 2S_t\left[\exp\left(b\Delta t\right) - 1\right]\Delta S_t + S_t^2 \exp\left(2b\Delta t\right)\left(\Delta X_t\right)^2 - S_t^2\left[\exp\left(b\Delta t\right) - 1\right]^2.$$

Similarly to (3.12) above,

$$(\Delta S_t)^2 = -S_t^2\left[e^{b\Delta t} - 1\right]^2 + 2S_t\left[e^{b\Delta t} - 1\right]\Delta S_t + S_t^2 e^{2b\Delta t}\left[e^{-r(t+\Delta t)}T_{t+\Delta t}^{(2)} - e^{-rt}T_t^{(2)} + m_2\Delta t\right]. \tag{3.14}$$

We can then hedge the term Q_2 by constructing $\mathcal{P}_t^{(2)}$:

Proposition 4 *If Δt is not negligible compared to ΔS_t, to hedge the term Q_2, we invest in C_2 units of $\mathcal{P}_t^{(2)}$, consisting of $S_t^2 e^{2b\Delta t} e^{-r(t+\Delta t)}$ units of $T_t^{(2)}$ and*

$$\left\{ S_t^2 e^{2b\Delta t - r(t+\Delta t)} T_t^{(2)} - S_t^2\left[e^{b\Delta t} - 1\right]^2 + 2S_t\left[e^{b\Delta t} - 1\right]\Delta S_t + S_t^2 e^{2b\Delta t}\left[-e^{-rt}T_t^{(2)} + m_2\Delta t\right] \right\} / \left[e^{r\Delta t} - 1\right]$$

units of cash in a risk-free bank account.

Proof. The proof is similar to that of Proposition 3. □

To hedge C_i for $i > 2$ if Δt is not negligible compared to ΔS_t, we start from (3.13),

$$(\Delta S_t)^i = S_t^i \left\{ \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} \exp\left(jb\Delta t\right) \left[1 + j\Delta X_t + \sum_{k=2}^{j} \binom{j}{k} \left(\Delta X_t\right)^k \right] \right\}.$$

Substituting ΔX_t by $\left[\frac{\Delta S_t}{S_t}+1\right]\exp\left(-b\Delta t\right)-1$, we have

$$(\Delta S_t)^i = S_t^i \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} e^{jb\Delta t} \left\{ 1 + j\left(e^{-b\Delta t}-1\right) + je^{-b\Delta t}\frac{\Delta S_t}{S_t} + \sum_{k=2}^{j}\binom{j}{k}(\Delta X_t)^k \right\}.$$

Let

$$c_0^{(i,j)} = S_t^i \binom{i}{j} (-1)^{i-j} \exp\left(jb\Delta t\right)\left\{1 + j\left(\exp\left(-b\Delta t\right)-1\right)\right\} \tag{3.15}$$

$$c_1^{(i,j)} = S_t^{i-1} \binom{i}{j} (-1)^{i-j}\, j \exp\left(\left(j-1\right)b\Delta t\right) \tag{3.16}$$

$$c_k^{(i,j)} = S_t^i \binom{i}{j} (-1)^{i-j} \exp\left(jb\Delta t\right)\binom{j}{k} \qquad \text{for } k = 2,3,...,j\ , \tag{3.17}$$

we have

$$(\Delta S_t)^i = \sum_{j=0}^{i}\left[c_1^{(i,j)}\Delta S_t + \sum_{k=2}^{j} c_k^{(i,j)}(\Delta X_t)^k + c_0^{(i,j)}\right].$$

Similar to (3.12) above,

$$(\Delta S_t)^i = \sum_{j=0}^{i}\left[c_1^{(i,j)}\Delta S_t + \sum_{k=2}^{j} c_k^{(i,j)}\left[\exp\left(-r\left(t+\Delta t\right)\right)T_{t+\Delta t}^{(k)} - \exp\left(-rt\right)T_t^{(k)} + m_k\Delta t\right] + c_0^{(i,j)}\right].$$

Therefore, we can hedge the term Q_i by constructing $\mathcal{P}_t^{(i)}$:

Proposition 5 *To hedge Q_i for $i>2$ if Δt is not negligible compared to ΔS_t, we invest in C_i units of $\mathcal{P}_t^{(i)}$, consisting of $\sum_{j=k}^{i} c_k^{(i,j)}\exp\left(-r\left(t+\Delta t\right)\right)$ units of $T_t^{(k)}$ for $k=2,3,...i$, and*

$$\frac{1}{\left[\exp\left(r\Delta t\right)-1\right]}\sum_{j=0}^{i}\left\{\sum_{k=2}^{j} c_k^{(i,j)}\exp\left(-r\left(t+\Delta t\right)\right)T_t^{(k)} + c_1^{(i,j)}\Delta S_t + \sum_{k=2}^{j} c_k^{(i,j)}\left[-\exp\left(-rt\right)T_t^{(k)} + m_k\Delta t\right] + c_0^{(i,j)}\right\}$$

units of cash in a risk-free bank account, where $c_0^{(i,j)}$, $c_1^{(i,j)}$ and $c_k^{(i,j)}$ are defined in (3.15)-(3.17).

Proof. The proof is similar to that of Proposition 3. □

The general case In the case where there are infinite number of jumps from t to $t+\Delta t$, we need the following results on explicit formulae of CRP proved by Yip *et al.* (2007). Let

$$\mathcal{I}_k = \left\{ (i_1, i_2, ..., i_l) \mid l \in \{1,2,...,k\}, i_q \in \{1,2,...,k\} \text{ and } \sum_{q=1}^{l} i_q \le k \right\} \tag{3.18}$$

and

$$\mathcal{L}_k = \left\{ (i_1, i_2, ..., i_l) \,|\, l \in \{1,2,...,k\}, i_q \in \{1,2,...,k\}, i_1 \geq i_2 \geq \cdots \geq i_l, \sum_{q=1}^{l} i_q = k \right\}. \tag{3.19}$$

The number of distinct values in a tuple $\phi_k = \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)$ in $\mathcal{L}_k$ is less than or equal to l. When it is less than l, it means some of the value(s) in the tuple are repeated. Let the number of times $r \in \{1, 2, 3, .., k\}$ appears in the tuple $\phi_k = \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)$ be $p_r^{\phi_k}$. Denote the terms which do not contain any stochastic integral in $(X_{t+\Delta t} - X_t)^k$ by $C^{(k)}_{\Delta t,\sigma}$.

Proposition 6

$$C^{(k)}_{\Delta t,\sigma} = \sum_{\phi_k = \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right) \in \mathcal{L}_k} \frac{1}{l!} \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)! \left(p_1^{\phi_k}, p_2^{\phi_k}, ..., p_k^{\phi_k}\right)! \left[\prod_{q \in \phi_k} m'_q\right] t^l, \tag{3.20}$$

where $i_1^{(k)}, ..., i_l^{(k)}$ *are the elements of* ϕ_k, $p_j^{\phi_k}$*'s are defined above and* $\left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)!$ *is the multinomial coefficient:* $\left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)! = \frac{\left(\sum_{j=1}^{l} i_j^{(k)}\right)!}{i_1^{(k)}! i_2^{(k)}! \cdots i_l^{(k)}!}$, $m'_q = m_q$ *for* $q \neq 2$ *and* $m'_2 = m_2 + \sigma^2$.

Denote the coefficient of the stochastic integral $\int_t^{t+\Delta t} \int_t^{t_1 -} \cdots \int_t^{t_{j-1}-} \mathrm{d}Y_{t_j}^{(i_1)} \cdots \mathrm{d}Y_{t_2}^{(i_{j-1})} \mathrm{d}Y_{t_1}^{(i_j)}$ in $(X_{t+\Delta t} - X_t)^k$ by $\Pi^{(k)}_{(i_1,i_2,...,i_j),\Delta t,\sigma}$. We then have the following result.

Proposition 7

$$\Pi^{(k)}_{(i_1,i_2,...,i_j),\Delta t,\sigma} = (i_1, i_2, ..., i_j, n)! C^{(n)}_{\Delta t,\sigma} \text{ where } n = k - \sum_{p=1}^{j} i_p. \tag{3.21}$$

Theorem 4 *For any Lévy process X, the representation of $(X_{t+\Delta t} - X_t)^n$ is given by*

$$(X_{t+\Delta t} - X_t)^n = \sum_{\theta_n \in \mathcal{I}_n} \Pi^{(n)}_{\theta_n,\Delta t,\sigma} \mathcal{S}'_{\theta_n,\Delta t,t} + C^{(n)}_{\Delta t,\sigma},$$

where $\mathcal{I}_n$ is defined in (3.18), $\Pi^{(n)}_{\theta_n,\Delta t,\sigma}$ *and* $C^{(n)}_{\Delta t,\sigma}$ *are defined above and* $\mathcal{S}'_{(i_1,i_2,...,i_j),\Delta t,t}$ *is defined to be the integral*

$$\mathcal{S}'_{(i_1,i_2,...,i_j),\Delta t,t} = \int_t^{t+\Delta t} \int_t^{t_1 -} \cdots \int_t^{t_{j-1}-} \mathrm{d}Y_{t_j}^{(i_1)} \cdots \mathrm{d}Y_{t_2}^{(i_{j-1})} \mathrm{d}Y_{t_1}^{(i_j)}.$$

If Δt is negligible compared to ΔS_t, from (3.1) and Theorem 4, we have

$$(\Delta S_t)^n = S_t^n (\Delta X_t)^n = S_t^n (X_{t+\Delta t} - X_t)^n = S_t^n \left[\sum_{\theta_n \in \mathcal{I}_n} \Pi^{(n)}_{\theta_n,\Delta t,\sigma} \mathcal{S}'_{\theta_n,\Delta t,t} + C^{(n)}_{\Delta t,\sigma}\right]. \tag{3.22}$$

In order to hedge $(\Delta S_t)^n$, we can invest in the *power jump integral process*:

$$\mathcal{U}_{(i_1,i_2,\ldots,i_j),\Delta t,t} = \exp(r\Delta t)\,\mathcal{S}'_{(i_1,i_2,\ldots,i_j),\Delta t,t}.$$

Note that since $Y^{(i)}$'s are martingales, $\left\{\mathcal{S}'_{(i_1,i_2,\ldots,i_j),\Delta t,t}, t \geq 0\right\}$'s are also martingales. Therefore, the discounted versions of the $\mathcal{U}_{(i_1,i_2,\ldots,i_j),\Delta t,t}$ are Q-martingales:

$$E_Q\left[\exp(-r\Delta t)\,\mathcal{U}_{(i_1,i_2,\ldots,i_j),\Delta t,t}|\mathcal{F}_s\right] = E_Q\left[\mathcal{S}'_{(i_1,i_2,\ldots,i_j),\Delta t,t}|\mathcal{F}_s\right] = \mathcal{S}'_{(i_1,i_2,\ldots,i_j),s-t,t}, \quad t \leq s \leq t+\Delta t.$$

Hence the market allowing trade in the bond, the stock and the power jump integral assets remains arbitrage-free. From (3.22), we have $(\Delta S_t)^n = S_t^n\left[\sum_{\theta_n\in\mathcal{I}_n}\Pi^{(n)}_{\theta_n,\Delta t,\sigma}\exp(-r\Delta t)\,\mathcal{U}_{\theta_n,\Delta t,t} + C^{(n)}_{\Delta t,\sigma}\right]$.

Proposition 8 *If Δt is negligible compared to ΔS_t, to hedge Q_i, we invest in C_i units of $\mathcal{P}_t^{(i)}$, consisting of $S_t^i\Pi^{(i)}_{\theta_i,\Delta t,\sigma}\exp(-r\Delta t)$ units of $\mathcal{U}_{\theta_i,\Delta t,t}$ for $\theta_i \in \mathcal{I}_i$ and $\frac{S_t^iC^{(i)}_{\Delta t,\sigma}}{(\exp(r\Delta t)-1)}$ units of cash in a risk-free bank account.*

Remark 1 *In this general case, we can only derive simple hedging strategy when Δt is negligible. Note that both power jump assets introduced by Corcuera* et al. *(2005) and power jump integral assets introduced here are imaginary assets. In reality, we only observe a discrete series of stock price, S, while there are an infinite number of jumps between any finite time interval if the underlying Lévy process has infinite activity. In other words, the values of these assets cannot be observed in the market and hence cannot be traded. The moment swaps introduced by Schoutens (2005) depend on the increment of the underlying stock, ΔS, and can hence be observed and traded in reality. We include the discussion on power jump assets for theoretical interest.*

Alternatively, note that in $\mathcal{S}'_{(i_1,i_2,\ldots,i_j),\Delta t,t}$, the integrand $\int_t^{t_1-}\cdots\int_t^{t_{j-1}-}\mathrm{d}Y^{(i_1)}_{t_j}\cdots\mathrm{d}Y^{(i_{j-1})}_{t_2}$ is a predictable function. Since we assume Δt to be very small, we can hedge $(\Delta S_t)^n$ by investing in the power jump assets. Let $\phi^{(n)}_{j,s}$ be the predictable function such that

$$(\Delta S_t)^n = S_t^n\left[\sum_{\theta_n\in\mathcal{I}_n}\Pi^{(n)}_{\theta_n,\Delta t,\sigma}\mathcal{S}'_{\theta_n,\Delta t,t} + C^{(n)}_{\Delta t,\sigma}\right] = \sum_{j=1}^n\int_t^{t+\Delta t}\phi^{(n)}_{j,s}\mathrm{d}Y^{(j)}_s + S_t^nC^{(n)}_{\Delta t,\sigma}, \tag{3.23}$$

where $\phi^{(n)}_{j,s}$'s can be calculated by rearranging the terms in $S_t^n\sum_{\theta_n\in\mathcal{I}_n}\Pi^{(n)}_{\theta_n,\Delta t,\sigma}\mathcal{S}'_{\theta_n,\Delta t,t}$'s. We then have

$$(\Delta S_t)^n = \int_t^{t+\Delta t}\sum_{j=1}^n -e^{-2rs}T_s^{(j)}\phi^{(n)}_{j,s}\mathrm{d}e^{rs} + S_t^nC^{(n)}_{\Delta t,\sigma} + \sum_{j=1}^n\int_t^{t+\Delta t}\phi^{(n)}_{j,s}e^{-rs}\mathrm{d}T_s^{(j)}.$$

Hence, to hedge $(\Delta S_t)^n$, we invest $\sum_{j=1}^n -e^{-2r\Delta t}T_t^{(j)}\phi^{(n)}_{j,t} + \frac{S_t^nC^{(n)}_{\Delta t,\sigma}}{\exp(r\Delta t)-1}$ in a riskless bank account and invest $\phi^{(n)}_{j,t}e^{-r\Delta t}$ units of $T_t^{(i)}$ for $j = 1, 2, \ldots, n$.

3.2.4 Extension of delta and gamma hedges

So far we have discussed the hedging strategies using moment swaps and power jump assets. In this section, we give a brief introduction to delta and gamma hedging strategies and extend it to obtain perfect hedging in a Lévy market. Let Π be the value of the portfolio under consideration. The delta and gamma dynamic hedging strategies are constructed using a Taylor expansion:

$$\delta\Pi = \frac{\partial\Pi}{\partial S}\delta S + \frac{\partial\Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\delta S^2 + \frac{1}{2}\frac{\partial^2\Pi}{\partial t^2}\delta t^2 + \frac{\partial^2\Pi}{\partial S\partial t}\delta S\delta t + ..., \tag{3.24}$$

where $\delta\Pi$ and δS are the changes in Π and S in a small time interval δt. Hull (2003, Chapter 14) gave detailed descriptions of the strategies in finance. The *delta* of a portfolio is defined as $\frac{\partial\Pi}{\partial S}$. Delta hedging eliminates the first term on the right-hand side of (3.24). The *gamma* of a portfolio is defined as $\frac{\partial^2\Pi}{\partial S^2}$. Gamma hedging eliminates the third term on the right-hand side of (3.24).

Below we extend the gamma hedge in order to obtain a perfect hedging strategy in a Lévy market. Note that equation (3.24) is a multivariate Taylor expansion and it is assumed that all the cross derivative terms are negligible. In equation (3.3), we applied Taylor expansions twice to avoid the cross derivative terms, since the value of Δt is deterministic and known at time t. Hence, for fixed n, the approximation by:

$$F\left(t+\Delta t, S_t+\Delta S_t\right) - F\left(t, S_t\right) = \sum_{i=1}^{\infty}\frac{D_1^i F\left(t, S_t\right)}{i!}\left(\Delta t\right)^i + \sum_{i=1}^{n}\frac{D_2^i F\left(t+\Delta t, S_t\right)}{i!}\left(\Delta S_t\right)^i \tag{3.25}$$

is more accurate than

$$F\left(t+\Delta t, S_t+\Delta S_t\right) - F\left(t, S_t\right) = \sum_{i=1}^{\infty}\frac{D_1^i F\left(t, S_t\right)}{i!}\left(\Delta t\right)^i + \sum_{i=1}^{n}\frac{D_2^i F\left(t, S_t\right)}{i!}\left(\Delta S_t\right)^i.$$

Moreover, in the literature, Δt and ΔS are assumed to be very small (such that the cross terms and higher terms are negligible). We provide the flexibility of specifying the values of Δt and ΔS_t such that static hedging is possible in some cases.

It is natural to extend the delta and gamma hedging strategies to the n-th derivative of the portfolio with respect to the underlying asset using the approximation of equation (3.25). Let F be the value of our portfolio to be hedged and there are $n-1$ traded financial derivatives, F_i, $i = 2, ..., n$, which are linearly independent of each other. Suppose we add w_i number of F_i into our portfolio, $i = 2, ..., n$ and add w_1 number of the underlying asset, which is denoted by F_1. We assume that $D_2^j F_i\left(t+\Delta t, S_t\right)$ are nonzero for $j = i$ and can be zero, or not, for $j = 1, 2, ..., i-1, i+1, ..., n$. In general, to make the portfolio k-th moment neutral for $k = 1, ..., n$, we need $D_2^k F\left(t+\Delta t, S_t\right) + \sum_{i=1}^{n} w_i D_2^k F_i\left(t+\Delta t, S_t\right) = 0$ for $k = 1, 2, ..., n$. Therefore, we have n equations for n unknown, w_i's. Note that whether the system of equations is solvable depends on the values of $D_2^k F_i\left(t+\Delta t, S_t\right)$, $i, k = 1, 2, ..., n$. Therefore, the traded financial derivatives have to be chosen such that the system of equations are solvable.

4 Minimal variance portfolios in a Lévy market

In Section 3, we gave the perfect hedging portfolios, given that the moment swaps, power jump assets and certain financial derivatives that depend on the same underlying asset, are available in the market. In this section, we demonstrate how to use the minimal variance portfolios derived by Benth *et al.* (2003) to hedge the higher order terms in the Taylor expansion, investing only in a risk-free bank account, the underlying asset and, if possible, variance swaps.

Benth *et al.* (2003) derived the minimal variance hedging portfolio of a contingent claim in a market such that the stock prices are independent Lévy martingales in terms of Malliavin derivatives. We demonstrate how to use their results to hedge the terms Q_i. Following Benth *et al.* (2003), to derive the minimal variance portfolio, we need to confine ourselves to the case of Lévy processes, $\eta = \{\eta(t), 0 \leq t \leq T\}$, which are martingales on the filtered probability space under consideration. That is, $E[\eta(t)] = 0$ and $E\left[\eta^2(t)\right] = \left(\sigma^2 + \int_{\mathbb{R}} x^2 \nu(\mathrm{d}x)\right) t$. Benth *et al.* (2003) called such processes *Lévy martingales of the second order.* From Benth *et al.* (2003, equation (2.1)), $\eta(t)$ has the following representation formula:

$$\eta(t) = \sigma W(t) + \int_0^t \int_{\mathbb{R}} x \tilde{N}(\mathrm{d}s, \mathrm{d}x), \quad \text{for } 0 \leq t \leq T, \tag{4.1}$$

where $\sigma \in \mathbb{R}^+$, $W(t)$ is the standard Brownian motion and $\tilde{N}(\mathrm{d}t, \mathrm{d}x)$ is defined in (2.6).

Based on the methodology developed by Benth *et al.* (2003), we modify their results to express the minimal variance portfolio for independent securities without referring to Malliavin calculus. Benth *et al.* (2003) assumed the underlying asset is directly represented by the Lévy martingale, that is, $S_t = \eta(t)$. We find it more natural to employ an exponential model and allow a drift term in the model of the underlying asset since the mean of $\eta(t)$ is zero. By extending (3.1), we suppose there are k independent securities prices $S_1, ..., S_k$, modeled as follows:

$$\mathrm{d}S_j(t) = b_j S_j(t_-)\,\mathrm{d}t + S_j(t_-)\,\mathrm{d}\eta_j(t), \quad j = 1, ..., k, \tag{4.2}$$

where $b_j \in \mathbb{R}$. Let $L_2(\Omega) = L_2(\Omega, \mathcal{F}, P)$ and $\xi \in L^2(\Omega)$ be a random variable to be hedged. Let $\mathcal{A}$ be the set of all admissible portfolios. The minimal variance portfolio is an admissible portfolio, $\varphi \in \mathcal{A}$ such that

$$E\left[\left(\xi - E[\xi] - \sum_{j=1}^{k} \int_0^T \varphi_j(s)\,\mathrm{d}S_j(s)\right)^2\right] = \inf_{\psi \in \mathcal{A}} E\left[\left(\xi - E[\xi] - \sum_{j=1}^{k} \int_0^T \psi_j(s)\,\mathrm{d}S_j(s)\right)^2\right]. \tag{4.3}$$

This is known as the *minimal variance hedging* for incomplete markets. Define a measure of the length of ξ by $\|\xi\| = \left(\int_\Omega |\xi(\omega)|^2 P(\mathrm{d}\omega)\right)^{1/2} = \left(E\left[|\xi|^2\right]\right)^{1/2}$. Following Benth *et al.* (2003, Definition 3.10 (a)), let $\mathbb{D}_{1,2}$ be the set of all $\xi \in L_2(\Omega)$ such that the chaos expansion defined in (2.8) satisfies the condition

$$\|\xi\|_{\mathbb{D}_{1,2}}^2 = E\left[\xi^2\right] + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=1,2} \int_{U_{j_n}} \left\| g_n^{(j_1,\ldots,j_n)}\left(\cdot, u_n^{(j_n)}\right) \right\|_{L_2(G_{n-1})}^2 \mathrm{d}\left\langle Q_{j_n}\right\rangle \left(u_n^{(j_n)}\right) < \infty,$$

where G_n is defined in (2.9). The chaotic representation derived by Benth *et al.* (2003) implies that every ξ satisfying some moment conditions can be expressed in the form

$$\xi = E[\xi] + \sum_{j=1}^{k} \int_0^T f_1(\xi; s, j)\,\mathrm{d}W_j(s) + \sum_{j=1}^{k} \int_0^T \int_{\mathbb{R}} f_2(\xi; s, x, j)\,\tilde{N}_j(\mathrm{d}s, \mathrm{d}x), \tag{4.4}$$

where $f_1(\xi; s, j)$ and $f_2(\xi; s, x, j)$ are predictable functions. Yip *et al.* (2007) derived the computationally explicit representation formula for $f_1(\xi; s, j)$ and $f_2(\xi; s, x, j)$ when ξ is the power of increments of a Lévy process, see Theorem 4. The minimal variance portfolio consisting of independent securities driven by (4.2), can be obtained by modifying Theorem 4.1 in Benth *et al.* (2003):

Proposition 9 *For any $\xi \in \mathbb{D}_{1,2}$, the minimal variance portfolio $\varphi = (\varphi_1, ..., \varphi_k)$ in (4.3),*

$$\hat{\xi} = E[\xi] + \sum_{j=1}^{k} \int_0^T \varphi_j(s)\,\mathrm{d}S_j(s),$$

admits the following representation:

$$\varphi_j(s) = \frac{f_1(\xi; s, j)\,\sigma_j + \int_{\mathbb{R}} x f_2(\xi; s, x, j)\,\nu_j(\mathrm{d}x)}{\left\{\sigma_j^2 + \int_{\mathbb{R}} x^2 \nu_j(\mathrm{d}x)\right\} S_j(s)},$$

where $f_1(\xi; s, j)$ and $f_2(\xi; s, x, j)$ are predictable functions defined in (4.4).

Proof. The proof is included in Appendix A.2. □

Although variance swaps are traded in OTC markets, there might be times that the appropriate variance swaps needed are not available. Hence, we firstly discuss how to use a minimal variance portfolio to hedge $\sum_{i=2}^{q} Q_i$ using only a risk-free bank account and the underlying stock. As in Section 3.2.3, we consider the simplified case where there is at most one jump of X between t and $t + \Delta t$, and the general case where there can be infinite number of jumps.

4.1 The simplified case

If Δt is negligible compared to ΔS_t, from (3.12),

$$\sum_{i=2}^{q} Q_i = \sum_{i=2}^{q} C_i S_t^i \left[\int_t^{t+\Delta t} \mathrm{d}Y_s^{(i)} + m_i \Delta t\right]. \tag{4.5}$$

Proposition 10 *If Δt is negligible compared to ΔS_t, the minimal variance portfolio to hedge $\sum_{i=2}^{q} Q_i$ using only a risk-free bank account and the underlying asset is to*
1) invest $\sum_{i=2}^{q} \frac{C_i}{(\exp(r\Delta t)-1)} S_t^i m_i \Delta t$ in a risk-free bank account, and
2) buy $\frac{1}{[\sigma^2+m_2]} \sum_{i=2}^{q} C_i S_t^{i-1} m_{i+1}$ units of the underlying stock, S_t, where m_i is defined in (2.1).

Proof. The proof is included in Appendix A.3. □

In the following, we discuss how to hedge the terms $\sum_{i=3}^{q} Q_i$ using a risk-free bank account, the underlying stock and variance swaps. If Δt is negligible compared to ΔS_t, from (3.12),

$$\sum_{i=3}^{q} Q_i = \sum_{i=3}^{q} C_i S_t^i \left[\int_t^{t+\Delta t} \mathrm{d}Y_s^{(i)} + m_i \Delta t \right]. \tag{4.6}$$

Therefore, we have the following hedging portfolio.

Proposition 11 *If Δt is negligible compared to ΔS_t, the minimal variance portfolio to hedge $\sum_{i=3}^{q} Q_i$ by investing in a risk-free bank account, the underlying asset and variance swaps is given by:*
1) buy $\phi \Delta s \,(n-2)\, S_t^2$ units of the variance swap at time t with sampling points $\{..., s_{n-1} = t, s_n = t + \Delta t\}$, maturity $t + \Delta t$ and strike σ^2_{strike}, where

$$\phi = \frac{\sum_{i=3}^{q} C_i S_t^{i-2} \int_{\mathbb{R}} x^i \nu\,(\mathrm{d}x)}{\int_{\mathbb{R}} x^2 \nu\,(\mathrm{d}x)} = \frac{\sum_{i=3}^{q} C_i S_t^{i-2} m_i}{m_2},$$

m_i are defined in (2.1) and P_V is the price of one unit of the variance swap.
2) invest nothing in the underlying asset, S_t,
3) invest

$$\frac{1}{e^{r\Delta t}-1} \left\{ \sum_{i=3}^{q} C_i S_t^i m_i \Delta t + \phi S_t^2 \left\{ \Delta s\,(n-2) \left[\sigma^2_{strike} - \overline{S}_{n,2} \right] + P_V \Delta s\,(n-2) - m_2 \Delta t \right\} \right\}$$

in a risk-free bank account, where $\overline{S}_{n,2}$ is defined in (3.10).

Proof. The proof is similar to those of Propositions 9 and 10. □

4.2 The general case

If Δt is negligible compared to ΔS_t, from (3.22),

$$(\Delta S_t)^n = S_t^n \sum_{\theta_n \in \mathcal{I}_n} \Pi^{(n)}_{\theta_n,\Delta t,\sigma} \mathcal{S}'_{\theta_n,\Delta t,t} + S_t^n C^{(n)}_{\Delta t,\sigma},$$

where the expression can be calculated explicitly using Theorem 4. Let

$$\sum_{i=2}^{q} Q_i = \sum_{j=1}^{q} C_i \int_t^{t+\Delta t} \phi_{j,s}^{(q)} \mathrm{d}Y_s^{(j)} + \sum_{i=2}^{q} C_i S_t^i C_{\Delta t,\sigma}^{(i)},$$

where $\phi_{j,s}^{(q)}$ is defined in (3.23).

Proposition 12 *If Δt is negligible compared to ΔS_t, the minimal variance portfolio to hedge $\sum_{i=2}^{q} Q_i$ using only a risk-free bank account and the underlying asset is to*
1) invest $\sum_{i=2}^{q} \frac{C_i}{\exp(r\Delta t)-1} S_t^i C_{\Delta t,\sigma}^{(i)}$ in a risk-free bank account, and
2) buy $\frac{1}{[\sigma^2+m_2]} \sum_{j=1}^{q} C_i \phi_{j,s}^{(q)} S_t^{-1} m_{i+1}$ units of the underlying stock, S_t, where m_i is defined in (2.1).

Proof. The proof is similar to that of Proposition 10. □

In the following, we discuss how to hedge the terms $\sum_{i=3}^{q} Q_i$ using a risk-free bank account, the underlying stock and variance swaps.

Proposition 13 *If Δt is negligible compared to ΔS_t, the minimal variance portfolio to hedge $\sum_{i=3}^{q} Q_i$ by investing in a risk-free bank account, the underlying asset and variance swaps is given by:*
1) buy $\phi \Delta s\,(n-2)\, S_t^2$ units of the variance swap at time t with sampling points $\{..., s_{n-1} = t, s_n = t + \Delta t\}$, maturity $t + \Delta t$ and strike σ_{strike}^2, where

$$\phi = \frac{\sum_{i=1}^{q} C_i \phi_{j,s}^{(q)} S_t^{-2} \int_{\mathbb{R}} x^i \nu\,(\mathrm{d}x)}{\int_{\mathbb{R}} x^2 \nu\,(\mathrm{d}x)} = \frac{\sum_{i=1}^{q} C_i \phi_{j,s}^{(q)} S_t^{-2} m_i}{m_2},$$

m_i are defined in (2.1) and P_V is the price of one unit of the variance swap.
2) invest nothing in the underlying asset, S_t,
3) invest

$$\frac{1}{e^{r\Delta t}-1} \left\{ \sum_{i=2}^{q} C_i S_t^i C_{\Delta t,\sigma}^{(i)} + \phi S_t^2 \left\{ \Delta s\,(n-2) \left[\sigma_{strike}^2 - \overline{S}_{n,2}\right] + P_V \Delta s\,(n-2) - m_2 \Delta t \right\} \right\}$$

in a risk-free bank account, where $\overline{S}_{n,2}$ is defined in (3.10).

Proof. The proof is similar to that of Proposition 10. □

5 Simulation algorithm

In this section, we discuss the approximation of the derivatives, $D_2^i F\,(t + \Delta t, S_t)$, and computational implementation of the hedging strategies. Assuming that the terms $\sum_{i=2}^{\infty} \frac{D_1^i F(t,S_t)}{i!} (\Delta t)^i$ do not contribute to the approximation significantly and can be ignored (which is found to be true in our simulation study), we have

$$F(t+\Delta t, S_t+\Delta S_t) - F(t, S_t) = D_1^1 F(t, S_t)\,\Delta t + \sum_{i=1}^{q} \frac{D_2^i F(t+\Delta t, S_t)}{i!} (\Delta S_t)^i,$$

which is true as long as $D_1^1 F(t, S_t)$ and $D_2^i F(t+\Delta t, S_t)$ exist for $i = 1, 2, 3, \ldots$. Note that the assumption $\sum_{i=2}^{\infty} \frac{D_1^i F(t,S_t)}{i!} (\Delta t)^i \approx 0$ is only for simplicity here since we are more interested in finding ways to hedge $\sum_{i=1}^{q} \frac{D_2^i F(t+\Delta t,S_t)}{i!} (\Delta S_t)^i$. The deterministic terms $\sum_{i=2}^{\infty} \frac{D_1^i F(t,S_t)}{i!} (\Delta t)^i$ can be hedged by investing in a risk-free bank account, as in equation (3.5). Since the pricing formulae for options with underlying driven by Lévy processes are in general not analytic, we need to approximate the derivatives of the pricing formulae, $D_2^i F(t+\Delta t, S_t)$, for $i = 1, 2, 3, \ldots$. We employ the Taylor's series based central difference approximation of arbitrary p-th degree derivatives introduced by Khan & Ohba (2003, Section 1), which is quoted in Appendix B.

In the following, we discuss how to calculate the derivatives of the option prices. We note that the most time consuming step in the approximation procedures is the calculation of $\sum_i \frac{1}{X(i)^2}$ in finding $d_k^{(p)}$ in equation (B.2) in the central difference approximation of derivatives. It is because the vector X contains the product of all the possible combinations of length c in Y, where Y contains all integers from 1 to N except $|k|$. For example, if we want to approximate the 31st derivative and set $N = 33$ (the accuracy of the approximation increases with the value of N), $c = 15$ and $k = 1$, the number of values in Y is 32 and the number of possible combinations of length c in Y is $C_{15}^{32} = \frac{32!}{15!(32-15)!} = 565,722,720$, which takes quite a while to calculate. Nevertheless, this calculation is the same for all functions $f(t)$. Therefore, we can build up a look-up table to store values of $C_{N,k} \sum_i \frac{1}{X(i)^2}$ for different N, c and k and use it for all options. Although the calculation for large N can take a very long time, we only need to do this once.

Algorithm	
1.	Construct the look-up table of $C_{N,k} \sum_i \frac{1}{X(i)^2}$ defined in equation (B.2).
2.	Calculate sample paths of S with different values of the current stock price, S_t.
3.	Use Monte Carlo simulation to calculate the option prices with respect to different values of the current stock price.
4.	Calculate the derivatives with respect to the underlying, $D_2^i F(t+\Delta t, S_t)$.
5.	Calculate the first derivative with respect to time, $D_1^1 F(t, S_t)$.

Table 5.1: The simulation algorithm to calculate the derivatives in Taylor expansions.

Step 1 For a fixed N, construct the look-up table of $C_{N,k} \sum_i \frac{1}{X(i)^2}$, where $k = 0,1,2,\ldots,N$ and $c = 3, 4, \ldots, c_{\max}$, where $c_{\max} = N-1$ (since $2N > p$ and c is the largest integer less than or equal to $\frac{p-1}{2}$). Therefore, the maximum derivative obtainable is $(2N-1)$-th.

Note that we should loop through c and then k. For each value of c, we use a vector to save the intermediate values of $\sum_i \frac{1}{X(i)^2}$ for each k. Therefore, we only need to calculate the combination of choosing c from Y once for each c.

Step 2 Calculate sample paths of S with different values of the current stock price, S_t.

Step 3 Use Monte Carlo simulation to calculate the option prices with respect to different values of the current stock price, using the sample paths of S generated in Step 2.

Step 4 Using the finite different method given in Appendix B, calculate the derivatives with respect to the underlying, $D_2^i F\left(t+\Delta t, S_t\right)$, using the look-up table produced in Step 1.

Step 5 Similar to Step 4, calculate the first derivative with respect to time, $D_1^1 F\left(t, S_t\right)$.

After calculating the derivatives, we show the performance of the proposed hedging strategies in the next section.

6 Performance of the hedging strategies

In this section, we investigate the performance of the hedging strategies given in Section 3 on European options and barrier options. We also give an example of static hedging of an one year European option on real life data. We truncate the infinite sum in (3.2) and calculate $\sum_{i=1}^p \frac{D_2^i F(t+\Delta t, S_t)}{i!}\left(\Delta S_t\right)^i + D_1^1 F\left(t, S_t\right)\Delta t$ for some fixed p. By comparing the values on the L.H.S. and R.H.S. of (3.2), it may be noted that for some $q \in \mathbb{N}$, the terms $\frac{D_2^i F(t+\Delta t, S_t)}{i!}\left(\Delta S_t\right)^i \simeq 0$ for $i > q$. This approximation is very useful, since in practice it is ideal to hedge by investing in as few kinds of products as possible, due to cost of transaction and administration. By fixing a tolerance level, α_{tol}, we can find the smallest value of p such that

(6.1)
$$\left|\left[F\left(t+\Delta t, S_t+\Delta S_t\right) - F\left(t, S_t\right)\right] - \left[D_1^1 F\left(t, S_t\right)\Delta t + \sum_{i=1}^p \frac{D_2^i F\left(t+\Delta t, S_t\right)}{i!}\left(\Delta S_t\right)^i\right]\right| \leq \alpha_{\text{tol}}$$

and we call it q. For a given tolerance level, α_{tol}, the following approximation is then assumed satisfactory:

$$F\left(t+\Delta t, S_t+\Delta S_t\right) - F\left(t, S_t\right) = D_1^1 F\left(t, S_t\right)\Delta t + \sum_{i=1}^q \frac{D_2^i F\left(t+\Delta t, S_t\right)}{i!}\left(\Delta S_t\right)^i. \tag{6.2}$$

Thus the magnitude of α_{tol} determines the number of terms required for a Taylor expansion to obtain a satisfactory approximation. In option hedging, we want the number of terms to be as small as possible since we have to invest in an additional financial derivative to hedge each term. In practice as we noted before, transaction costs, bid-ask spreads and the cost of administration make the trades of a large number of different financial derivatives not preferable. Therefore, there is a trade-off between the accuracy of the hedging and the additional costs involved.

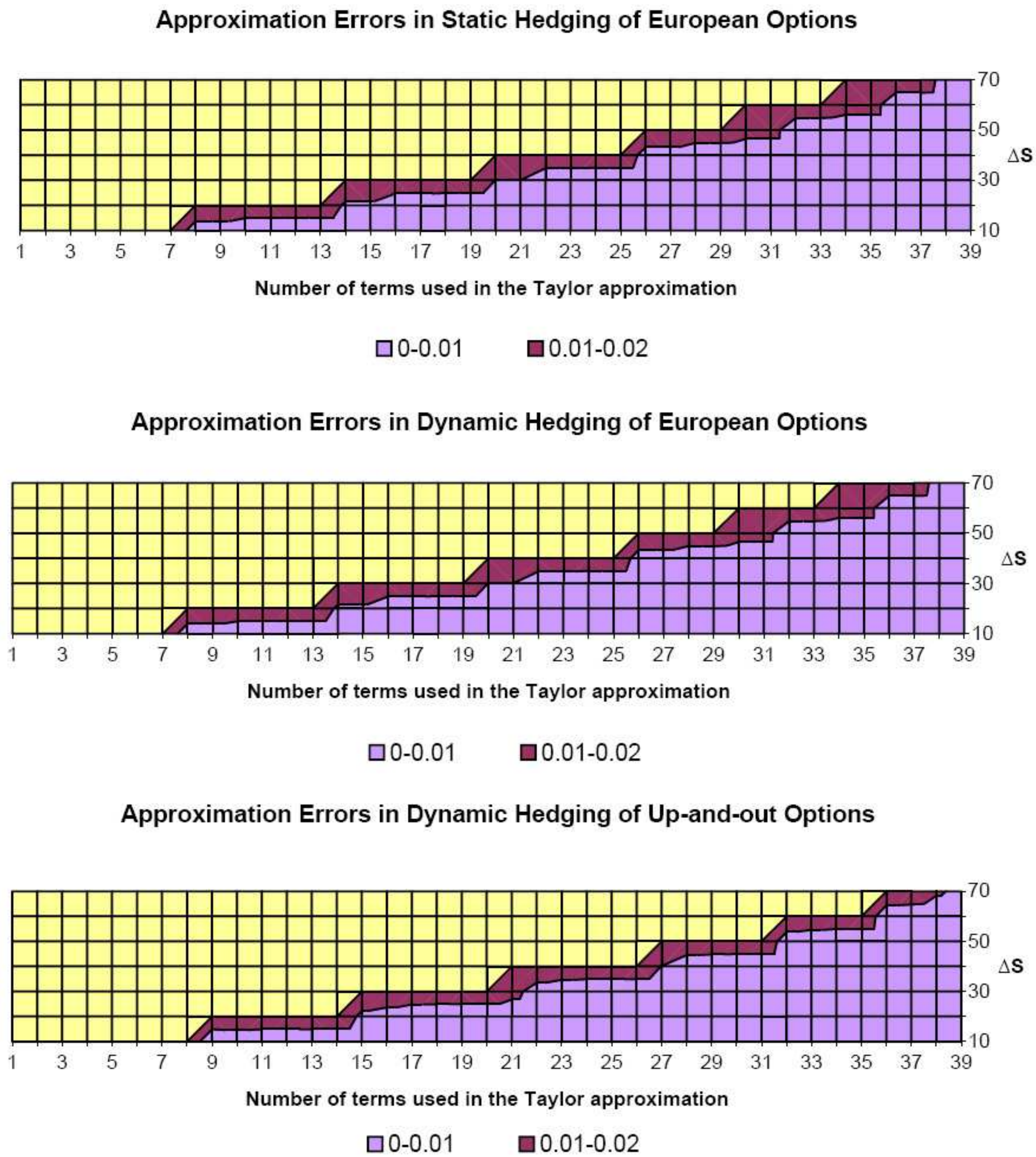

Figure 6.1: The approximation errors in static hedging of European options, dynamic hedging of European options and dynamic hedging of up-and-out options. The x-axis gives the value of q and the y-axis gives ΔS. The area of the graph is coloured in light purple when the approximation error ≤ 0.01 and in deep purple when the approximation error is between 0.01 and 0.02.

In the following, we give the performance of the static and dynamic hedging strategies on European, up-and-out, up-and-in, down-and-out and down-and-in options. We investigate how many terms in the Taylor expansions are needed to obtain a satisfactory approximation, that is, we determine the value of q for a given α_{tol}, defined in (6.1). In our simulations, we set $\alpha_{\text{tol}} = 0.01$. It is because in practice, we are hedging the prices of the options, the lowest price change is 0.01. We assume the current stock price, S_0, is 5000 and the strike price of the options, K, are 5000. Note that our strategies work for all values of K. We consider the cases where the change in the price of the stock price ΔS_t is equal to 10, 20, ..., 70. For static hedging, we assume $\Delta t = 1$, and the options are expiring in 1 year as well, that is, $T = 1$. For dynamic hedging, we set $\Delta t = 9.5129 \times 10^{-6}$, approximately 5 minutes, and $T = 1.1416 \times 10^{-4}$, approximately 1 hour.

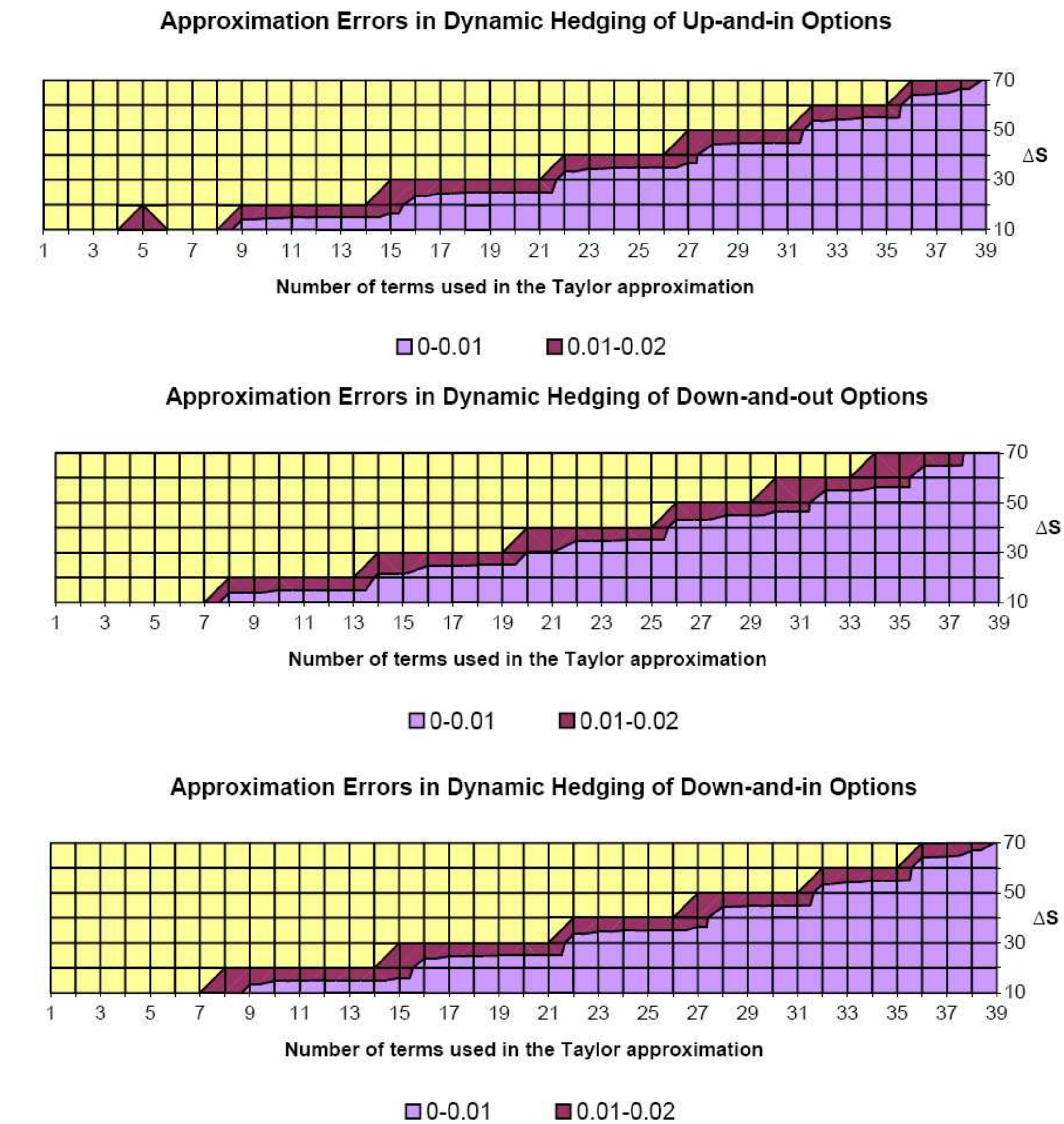

Figure 6.2: The approximation error in dynamic hedging of up-and-in options, down-and-out options and down-and-in options. The x-axis gives the value of q and the y-axis gives ΔS.

The performance of static and dynamic hedging of European options is given in Figure 6.1. We can see that the values of q required are the same in the cases of static and dynamic hedging. The value of q, that is, the number of terms required in the Taylor approximation, such that the error $\leq \alpha_{\text{tol}}$ increases gradually as the value of ΔS_t increases. This verifies the discussion given in the beginning of this section, that is, for a given tolerance level, the number of terms required in the Taylor expansions is finite. The values of q for different values of ΔS_t is also given in Table 6.1.

The performance of dynamically hedging of up-and-out options is given in Figure 6.1. We assume the barrier is given by $H = 5050$. The values of q required are bigger than the ones for European options due to the more complicated payoff function. The values of q for different values of ΔS_t is also given in Table 6.1. Similarly, the hedging performance of up-and-in options, down-and-out and down-and-in options are given in Figure 6.2 and Table 6.1.

In static hedging of European options in Figure 6.1,								In dynamic hedging of up-and-in options in Figure 6.2,							
ΔS_t	10	20	30	40	50	60	70	ΔS_t	10	20	30	40	50	60	70
q	8	14	20	26	32	36	38	q	9	16	22	28	32	36	39
In dynamic hedging of European options in Figure 6.1,								In dynamic hedging of down-and-out options in Figure 6.2,							
ΔS_t	10	20	30	40	50	60	70	ΔS_t	10	20	30	40	50	60	70
q	8	14	20	26	32	36	38	q	8	14	20	26	32	36	38
In dynamic hedging of up-and-out options in Figure 6.1,								In dynamic hedging of down-and-in options in Figure 6.2,							
ΔS_t	10	20	30	40	50	60	70	ΔS_t	10	20	30	40	50	60	70
q	9	15	22	27	32	36	39	q	9	16	22	28	32	36	39

Table 6.1: The values of q for given ΔS_t in static hedging of European options, dynamic hedging of European, up-and-out, up-and-in, down-and-out and down-and-in options.

The performance of hedging some other exotic options, such as lookback options and Asian options, can be obtained similarly since we employ Monte Carlo simulation in calculating the option prices. Recall in Section 5, as N increases, the number of derivatives that can be calculated increases. The results show that q increases rapidly with increasing ΔS_t. Note that the bigger the value of ΔS_t, the slower the convergence rate of Taylor expansion and this is why dynamic hedging is more popular in the literature. From our simulation results, we note that $\frac{D_2^i F(t+\Delta t, S_t)}{i}$ become very small as i increases, but the value of $(\Delta S_t)^i$ increases very rapidly. Therefore, we cannot ignore the terms $\frac{D_2^i F(t+\Delta t, S_t)}{i!}(\Delta S_t)^i$. To enable perfect hedging using moment swaps, power jump assets or some other traded derivatives depending on the same underlying asset, the market has to allow trading in these financial derivatives in a unit as small as $\frac{D_2^i F(t+\Delta t, S_t)}{i}$.

In summary, as long as we can find the q such that the Taylor approximations are accurate for all possible values of ΔS_t under consideration, the perfect hedging using moment swaps, power jump assets or other traded derivatives depending on the same underlying asset works very well.

To show the trading strategy is applicable to real life data, we fit the VG model to European option price on FTSE index and derive a static hedging strategy on a one year European option. On 4th January 2007 and 4th January 2008, the spot FTSE 100 index are 6287 and 6348.5, respectively. The change in value of the underlying, ΔS, is therefore 61.5. We apply our hedging strategy to the one year European option on 4th January 2007 and show how hedging can be achieved. The option with strike 6287 is worth 410.3 on 4th January 2007, where the risk-free interest rate is 5.43%, the dividend on the FTSE 100 index is 3.51% and the implied volatility is 14.65%. We fit these data using the VG model and obtain the parameter values: $\theta = -0.2721$, $\nu = 0.3032$ and $\sigma = 3.02\%$. The Monte Carlo (MC) simulated option price using these parameters is 410.914. The pricing error due to calibration and simulation is then 0.614. At maturity, the option is in the money and the payoff is $(6348.5 - 6287) = 61.5$. Therefore, the change of value of the option is $61.5 - 410.914 = -349.414$ according to the MC calculation. The hedging performance is given in Table 6.2. The first column shows the number of terms used in the Taylor expansion, the second column shows the value of the derivative, $D_2^{(i)}(t + \Delta t, \Delta S)$ and the third column shows the approximated price. We see that 12 terms are needed to obtain the change in option price, -349.414. Perfect hedging is achieved according to the MC price. It shows that the hedging strategy works very well in replicating the MC price and the hedging error in this example is entirely due to calibration and simulation of the VG model, which is out of the scope of this paper. We note that the contributions of the odd number terms except the first term are almost negligible and can be ignored. Therefore we can reduce the number of instruments invested in this case.

1	0.5	-380.164	6	4.80557e-011	-349.141	11	-1.39873e-032	-349.413
2	0.01107	-338.294	7	-4.05377e-023	-349.141	12	-3.40129e-025	-349.414
3	-8.97421e-015	-338.294	8	-1.4317e-015	-349.434	13	1.36317e-037	-349.414
4	-9.39954e-007	-351.741	9	9.58199e-028	-349.434	14	3.01623e-030	-349.414
5	9.30204e-019	-351.741	10	2.6928e-020	-349.413	15	-9.35744e-043	-349.414

Table 6.2: The performance of the hedging strategy on a one year European option price on FTSE 100 index on 4th January 2007.

7 Conclusion

In this paper, we provided some perfect hedging strategies and minimal variance portfolios in a Lévy market. Many financial institutions hold derivative securities in their portfolios, and frequently these securities need to be hedged for extended periods of time. Failure to hedge properly can expose an institution to sudden swings in the values of derivatives, such as options, resulting from large, unanticipated changes in the levels or volatilities of the underlying asset. Research in the techniques employed for hedging derivative securities is therefore of crucial importance. Under the assumption of the famous Black-Scholes model, the market is complete and an European option can be hedged perfectly by investing in a risk-free bank account and the underlying stock. However, there is statistical evidence, such as the volatility smile, that the Black-Scholes model is not sufficiently flexible to model the price process. As a result, the study of Lévy process, which is a generalisation of Brownian motion with jumps, has become increasingly important in mathematical finance. If the underlying asset is driven by a Lévy process, the market is not complete, that is, a contingent claim cannot be hedged using only a risk-free bank account and the underlying asset. By applying a Taylor expansion to the pricing formulae, we derived dynamic perfect hedging strategies of European and some exotic options by trading in moment swaps, power jump assets or certain traded derivatives depending on the same underlying asset. In the case of European options, static hedging can also be achieved. We extended the delta and gamma hedging strategies to higher moment hedging by investing in other traded derivatives depending on the same underlying asset. We demonstrated how to use the minimal variance portfolios derived by Benth *et al.* (2003) to hedge the higher order terms in the Taylor expansion, investing only in a risk-free bank account, the underlying asset and, potentially, variance swaps. We explicitly addressed numerical issues in the procedures, such as the approximation of the derivatives in the Taylor expansion, as well as investigated the performance of the hedging strategies. If as many derivatives as the Taylor expansion needed for accuracy can be determined and the financial derivatives required to hedge are available in the specified amounts, perfect hedging is possible.

APPENDICES

A Proof of Propositions and Lemma

A.1 Proof of Proposition 3

The initial investment at time t is

$$C_i \left\{ S_t^i e^{-r(t+\Delta t)} T_t^{(i)} + \frac{S_t^i e^{-r(t+\Delta t)} T_t^{(i)}}{e^{r\Delta t} - 1} + \frac{S_t^i \left[-e^{-rt} T_t^{(i)} + m_i \Delta t \right]}{e^{r\Delta t} - 1} \right\}.$$

At maturity, the value of the portfolio is equal to

$$C_i S_t^i \left\{ e^{-r(t+\Delta t)} T_{t+\Delta t}^{(i)} + \frac{e^{r\Delta t}}{e^{r\Delta t} - 1} \left\{ e^{-r(t+\Delta t)} T_t^{(i)} - e^{-rt} T_t^{(i)} + m_i \Delta t \right\} \right\}.$$

Hence, by equation (3.12), the change of value of the portfolio equals

$$C_i \left\{ S_t^i e^{-r(t+\Delta t)} T_{t+\Delta t}^{(i)} + S_t^i \left[-e^{-rt} T_t^{(i)} + m_i \Delta t \right] \right\}.$$

A.2 Proof of Proposition 9

Let

$$\xi = \xi^0 + \sum_{j=1}^{k} \int_0^T \varphi_j(s)\, \mathrm{d}S_j(s). \tag{A.1}$$

where ξ^0 denotes the difference of value between ξ and $\sum_{j=1}^{k} \int_0^T \varphi_j(s)\, \mathrm{d}S_j(s)$ for the portfolio $\varphi = (\varphi_1, ..., \varphi_k)$. By the results of Monat & Stricker (1995, Section 4.2), the Hilbert space argument in Benth *et al.* (2003, Theorem 2.3) and equation (4.1), the following orthogonality condition is satisfied:

$E\left[\left(\xi-\hat{\xi}\right)\Theta\right]=E\left[\left\{\xi^0-E\left[\xi\right]\right\}\Theta\right]=E\left[\xi^0\Theta\right]-E\left[\xi\right]E\left[\Theta\right]=0$, where

$$\Theta=\sum_{j=1}^{k}\int_0^T\theta_j\left(s\right)\sigma_jS_j\left(s_-\right)\mathrm{d}W_j\left(s\right)+\sum_{j=1}^{k}\int_0^T\int_{\mathbb{R}}x\theta_j\left(s\right)S_j\left(s_-\right)\tilde{N}_j\left(\mathrm{d}s,\mathrm{d}x\right) \tag{A.2}$$

for all $\theta=\left(\theta_1,...,\theta_k\right)\in\mathcal{A}$. Since $E\left[\Theta\right]=0$, we have $E\left[\xi^0\Theta\right]=0$. From (4.1) and (4.2),

$$\sum_{j=1}^{k}\int_0^T\varphi_j\left(s\right)\mathrm{d}S_j\left(s\right)=\sum_{j=1}^{k}\int_0^T\varphi_j\left(s\right)S_j\left(s_-\right)b_j\mathrm{d}s+\sum_{j=1}^{k}\int_0^T\varphi_j\left(s\right)\sigma_jS_j\left(s_-\right)\mathrm{d}W_j\left(s\right)$$
$$+\sum_{j=1}^{k}\int_0^T\int_{\mathbb{R}}x\varphi_j\left(s\right)S_j\left(s_-\right)\tilde{N}_j\left(\mathrm{d}s,\mathrm{d}x\right).$$

Hence, from (4.4) and (A.1),

$$\xi^0=E\left[\xi\right]-\sum_{j=1}^{k}\int_0^T\varphi_j\left(s\right)S_j\left(s_-\right)b_j\mathrm{d}s+\sum_{j=1}^{k}\int_0^T\left(\frac{1}{\sigma_j}f_1\left(\xi;s,j\right)-\varphi_j\left(s\right)S_j\left(s_-\right)\right)\sigma_j\mathrm{d}W_j\left(s\right)$$
$$+\sum_{j=1}^{k}\int_0^T\int_{\mathbb{R}}\left(f_2\left(\xi;s,x,j\right)-x\varphi_j\left(s\right)S_j\left(s_-\right)\right)\tilde{N}_j\left(\mathrm{d}s,\mathrm{d}x\right).$$

Hence, from (A.2) and the well-known isometry, see Ikeda & Watanabe (1989), we have

$$E\left[\xi^0\Theta\right]=\sum_{j=1}^{k}E\left[\int_0^T\theta_j\left(s\right)S_j\left(s_-\right)\left\{\left(f_1\left(\xi;s,j\right)-\sigma_j\varphi_j\left(s\right)S_j\left(s_-\right)\right)\sigma_j\right.\right.$$
$$\left.\left.+\int_{\mathbb{R}}x\left(f_2\left(\xi;s,x,j\right)-x\varphi_j\left(s\right)S_j\left(s_-\right)\right)\nu_j\left(\mathrm{d}x\right)\right\}\mathrm{d}s\right]=0.$$

$$\Rightarrow f_1\left(\xi;s,j\right)\sigma_j+\int_{\mathbb{R}}xf_2\left(\xi;s,x,j\right)\nu_j\left(\mathrm{d}x\right)=\varphi_j\left(s\right)S_j\left(s\right)\left\{\sigma_j^2+\int_{\mathbb{R}}x^2\nu_j\left(\mathrm{d}x\right)\right\}$$
$$\varphi_j\left(s\right)=\left[f_1\left(\xi;s,j\right)\sigma_j+\int_{\mathbb{R}}xf_2\left(\xi;s,x,j\right)\nu_j\left(\mathrm{d}x\right)\right]/\left[\left\{\sigma_j^2+\int_{\mathbb{R}}x^2\nu_j\left(\mathrm{d}x\right)\right\}S_j\left(s\right)\right].$$

A.3 Proof of Proposition 10

From equation (4.5), the term $\sum_{i=2}^{q}C_iS_t^im_i\Delta t$ can be hedged by investing

$$\sum_{i=2}^{q}\frac{C_iS_t^im_i\Delta t}{\exp\left(r\Delta t\right)-1}$$

in a risk-free bank account. To hedge the term $\sum_{i=2}^{q}C_iS_t^i\int_t^{t+\Delta t}\mathrm{d}Y_s^{(i)}$, we let

$$\xi=\sum_{i=2}^{q}\int_t^{t+\Delta t}C_iS_t^i\mathrm{d}Y_s^{(i)}=\sum_{i=2}^{q}\int_t^{t+\Delta t}\int_{\mathbb{R}}C_iS_t^ix^i\tilde{N}\left(\mathrm{d}s,\mathrm{d}x\right)$$

by (2.7) and let the minimal variance portfolio to hedge ξ be $\hat{\xi}=E\left[\xi\right]+\int_t^{t+\Delta t}\varphi_s\mathrm{d}S_s=\int_t^{t+\Delta t}\varphi_s\mathrm{d}S_s$ since $E\left[\xi\right]=0$. Hence, using Proposition 9 and equation (4.4) by putting $f_1\left(\xi;s,j\right)=0$ and $f_2\left(\xi;s,x,j\right)=$

$\sum_{i=2}^{q} C_i S_t^i x^i$, we have

$$\varphi_s = \frac{\int_{\mathbb{R}} \sum_{i=2}^{q} C_i S_t^i x^{i+1} \nu(\mathrm{d}x)}{\left[\sigma^2 + \int_{\mathbb{R}} x^2 \nu(\mathrm{d}x)\right] S_s}.$$

Hence, to hedge the terms $\sum_{i=2}^{q} Q_i$ by minimal variance portfolio, we need to invest the amount

$$\sum_{i=2}^{q} \frac{C_i S_t^i m_i \Delta t}{\exp(r\Delta t) - 1}$$

in a risk-free bank account and buy

$$\frac{\int_{\mathbb{R}} \sum_{i=2}^{q} C_i S_t^i x^{i+1} \nu(\mathrm{d}x)}{\left[\sigma^2 + \int_{\mathbb{R}} x^2 \nu(\mathrm{d}x)\right] S_t} = \frac{\sum_{i=2}^{q} C_i S_t^{i-1} m_{i+1}}{\left[\sigma^2 + m_2\right]}$$

amount of the underlying stock, S_t, where m_i are defined in (2.1).

B Central difference approximation of arbitrary degree

Khan & Ohba (2003, Section 1) showed that Taylor's series based central difference approximation of arbitrary p-th degree derivative of a function $f(t)$ at $t = t_0$ can be written for an order $2N$ as

$$f_0^{(p)} = \frac{1}{T^p} \sum_{k=-N}^{N} d_k^{(p)} f_k, \tag{B.1}$$

where T is the sampling period, $2N + 1$ is the number of nodes used in the approximation, f_k denotes the value of function $f(t)$ at $t = t_0 + kT$, $2N$ is an integer bigger than p and $d_0^{(p)} = 0$ if p is odd, otherwise $d_0^{(p)} = -2\sum_{k=1}^{N} d_k^{(p)}$, and

$$d_k^{(p)} = (-1)^{k+c_1} \frac{p!}{k^{1+c_2}} C_{N,k} \sum_i \frac{1}{X(i)^2}, \quad \text{for } k = -N, -N+1, ..., -1, 1, ..., N-1, N, \tag{B.2}$$

$C_{N,k} = \frac{N!^2}{(N-k)!(N+k)!}$, c = largest integer less than or equal to $(p-1)/2$, $c_1 = 1$ if c is even, otherwise $c_1 = 0$, $c_2 = 1$ if p is even, otherwise $c_2 = 0$, and the vector X is generated in the following way:

1. Take a vector Y containing all integers from 1 to N except $|k|$ (in Khan & Ohba (2003, p. 121), it was except k, but from the derivation of the formula, it should be $|k|$).

2. The vector X contains the product of all the possible combinations of length c in Y.

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