Una rassegna su alcuni modelli di crescita economica tipo Solow con dinamica caotica

Palmisani, Cesare

University of Naples "Federico II"

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A Survey on Chaotic Dynamics in Solow-type Growth Models

Cesare Palmisani

Dept. Matematica e Statistica, Federico II University, Naples (Italy)
e-mail: cpalmisani@unina.it

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Abstract

In this paper we review some Solow-type growth models, framed is discrete time, which are able to generate complex dynamic behaviour. For these models - put forward by Day (1982, 1983); Böhm and Kaas (2000); and Commendatore (2005) - we show that crucial features which could determine the emergence of regular or irregular growth cycles are (i) if the average saving ratio is constant or not; and (ii) the curvature of production function, representing the degree of substitutability between labour and capital. The lower the degree of substitutability, the higher the likelihood of complex behaviour.

Keywords Logistic Map, Li-York Chaos, Growth Models, Local Stability, Triangle Stability.

1 Introduction

The analysis of the fundamental issues in dynamical macroeconomics usually begins with the study of two (one-sector and one-dimensional) growth models: the Ramsey model (Ramsey, 1928) and the Solow model (Solow, 1956). In the Ramsey model a representative consumer has an infinite horizon of life and optimizes his/her utility. A basic Ramsey model in discrete time requires to find

$max\ W = \sum_{t=0}^{t=\infty} (\frac{1}{1+\varrho})^t u(c_t),$

subject to the constraints $y_t = f(k_t)$, $y_t = y_t + i_t$, $k_{t+1} = k_t + i_t$, where $f(k_t)$

is the production function, k_t is the capital-labor ratio at time t, y_t the income over labor at time t, $u(c_t)$ an utility function on the consumption per capita c_t at time t, i_t the investment over labor at time t, ϱ the discount rate, with the following properties $u(c_t) \geq 0$, $u'(c_t) > 0$, $u''(c_t) < 0$, $f(0) = 0$, $f'(0) = 0$, $f'(\infty) = 0$, $f'(k) > 0$, $f''(k) < 0$.

In the Solow model consumption is not optimal the representative agent saves a constant fraction of his income. In the next sections we will describe only the Solow model and the most relevant models for our paper. We note here that researches in several direction have spanned from the Solow model. For example, the Solow model inspired the works of Shinkay (1960), Meade (1961), Uzawa (1961,1963), Kurz (1963), Srinivasan (1962-1964), on two-sector growth models. Following this line of research, works about two-sector models appeared on the *Review of Economic Studies* in the 1960s (Drandakis (1963), Takajama (1963,1965), Oniki-Uzawa (1965), Hahn (1965), Stiglitz (1967), among others). This line of research has been further developed in the 80s with the introduction of chaos and Overlapping Generations (OLG) into the two-sector model (Galor and Ryder (1989), Galor (1992), Azariadis (1993), Galor and Lin (1994). Recently Karl Farmer and Ronald Wendner (2003) developed two-sector models including overlapping generation (OLG), instead Schmitz (2006) presented a two-sector model in discrete time that exhibits complex dynamics (topological chaos and strange attractors). Another line of research was opened by P. Diamond (1965) which was the first to extend the Solow model including OLG developing a one-sector and one-dimensional model with public debt. R.Farmer (1968) extended the Diamond model to the two-dimension case. Many authors developed model Farmer-type with chaos (Grandmont (1985), B. Jullien (1988), B. Reichlin (1986), A. Medio (1992), C. Azariadis (1993), V. Bohm (1993), A. Medio and G. Negroni (1996), de Vilder (1996), M. Yokoo (2000)). Moreover, the seminal ideas of Kaldor (1956, 1957), Pasinetti (1962), Samuelson and Modigliani (1966), Chiang (1973) about the influence on the growth path by different savings behaviour of two income group (labor and capital) originated two-class one-dimensional (Böhm and Kaas (2000)) and two-dimensional (Commendatore (2005)) discrete time models. We note that in the two-class extensions of the Solow model, the neoclassical features of the production function, the Inada conditions, are weakened or disappear, and both models present complex dynamics.

2 The Solow Growth Model in Discrete Time

Following Hans-Walter Lorenz (1989) and Costas Aziariadis (1993), we will develop a discrete time variant of the growth model due to Solow (1956). We consider *a single good economy, i.e. an economy in which only one good is produced and consumed.* We assume that the time t is *discrete*, that is $t = 0, 1, 2, \ldots$. The symbols Y_t, K_t, C_t, I_t, L_t, S_t indicate economywide aggregates respectively equal to *income, capital stock, consume, investment, labor force, saving at time* t. The capital stock K_0 and labor L_0 at time 0 are given. The constant s denotes the *marginal savings rate* and the constant n indicates the *growth rate of population.* We consider s and n as given exogenously. The map $F : (K_t, L_t) \to F(K_t, L_t)$ is *the production function.* We assume that:

1. $Y_t = C_t + I_t$: *for all time* $t = 0, 1, \ldots$, *the economy is in equilibrium, i.e. the supply of income* Y_t *is equal to the demand composed of the quantity* C_t *of good to consume plus the stock* I_t *of capital to invest (closed economy like a Robinson Crusoe economy)*;
2. $I_t = K_{t+1}$: *investment at time* t *corresponds to all capital available to produce at time* $t + 1$ *(working capital hypothesis)*;
3. $S_t = Y_t - C_t = sY_t$ $(0 < s < 1)$: *saving is a share of income*;
4. $Y_t = F(K_t, L_t)$, i.e. *at time* t *all income is equal to the output obtained by the inputs capital and labor*;
5. $L_t = (1 + n)^t L_0$ $(n > 0)$: *the labor force grows as a geometric progression at the rate* $(1 + n)$.

   From the first (3.) we deduce that in a short run equilibrium $Y_t = C_t + S_t$, which, after a comparison with (1.), gives $I_t = S_t$. Thus, applying (2.) and (3.), we have $K_{t+1} = sY_t$. Finally, from (4.) we obtain $K_{t+1} = sF(K_t, L_t)$.

   From the later expression, $\frac{K_{t+1}}{L_{t+1}} = \frac{sF(K_t,L_t)}{L_{t+1}}$.

   If F is *linear-homogeneous* (or it tells that F *exhibits constant returns to scale)*, i.e.

6. $F(\lambda K, \lambda L) = \lambda F(K, L)$ (for all $\lambda > 0$),

then we have

$\frac{K_{t+1}}{L_{t+1}} = \frac{sL_t F(\frac{K_t}{L_t},1)}{L_t(1+n)}$.

We set $k_t = \frac{K_t}{L_t}$ (*capital-labor ratio or capital per worker*) and $f(k_t) = f(\frac{K_t}{L_t}, 1)$. We call output *per worker the ratio* $y_t = \frac{Y_t}{L_t}$.

Therefore we get the equation of accumulation *for the Solow model in discrete time with the working capital hypothesis*:

$k_{t+1}(1+n) = sf(k_t)$ (2.1)

If we assume that *capital depreciates at the rate* $0 \leq \delta \leq 1$*(fixed capital hypothesis)*, the capital available at time $t+1$ corresponds to $K_{t+1} = K_t - \delta K_t + I_t$, from which $K_{t+1} = sF(K_t, L_t) + (1-\delta)K_t$.

As before we get the following time-map for capital accumulation

$k_{t+1}(1+n) = sf(k_t) + (1-\delta)k_t$ (2.2)

or

$k_{t+1} = h(k_t)$,

where $h(k_t) = \frac{1}{1+n}[sf(k_t) + (1-\delta)k_t]$.

We notice that I_t is the *gross investment* while $K_{t+1} - K_t = I_t - \delta K_t$ is *the net investment.*

Costas Azariadis (1993, *p*.4) tells us that *this model captures explicitly a simple idea that is missing in static formulations: there is a tradeoff between consumption and investment or between current and future consumption. The implications of this ever-present competition for resources between today and tomorrow are central to macroeconomics and can be explored only in a dynamic framework. Time is clearly of the essence.*

If $f(k_t)$ is a concave production function, for example, a Cobb-Douglas function $f(k_t) = Bk_t^{\beta}$ $(B > 0,\ 0 < \beta < 1,\ k \leq 0)$, then the equation (2.1) becomes $k_{t+1} = \frac{sBk_t^{\beta}}{1+n}$. Setting $h(k_t) = \frac{sBk_t^{\beta}}{1+n}$, we notice that $h(k_t)$ is monotonically increasing and concave for all $k < 0$:

$\frac{df(k)}{dk} = \frac{s}{1+n}\beta Bk^{\beta-1} > 0$ and $\frac{d^2f(k)}{dk^2} = \frac{s}{1+n}B\beta(\beta-1)k^{\beta-2} < 0$.

Remark 2.1 About the Cobb-Douglas, we observe that the assumption $0 < \beta < 1$ implies the concavity of $f(k)$. Moreover in the plane (k_t, k_{t+1}) *the graph of the Cobb-Douglas is below the* $45°$*-line if* $f(k_t) < k_t$*, from which* $k_t < (1/B)^{\frac{1}{B-1}}$.

Remark 2.2 About the Cobb-Douglas, we have also

$f^{'}(k) < 1$ if $k > (B\beta)^{\frac{1}{1-\beta}}$. As a matter of fact

$f^{'}(k) < 1 \Leftrightarrow B\beta k^{\beta-1} < 1 \Leftrightarrow k^{\beta-1} < \frac{1}{B\beta} \Leftrightarrow (k^{-1})^{1-\beta} < (B\beta)^{-1}$

$\Leftrightarrow k^{-1} < (B\beta)^{-\frac{1}{1-\beta}}$.Q.E.D.

For example, let $B = 0.2$ be and let $\beta = 0.7$ be, it needs that $k > 0.001425$.

Moreover the dynamical system $k_{t+1} = h(k_t)$ has two steady-states: the first, at $k = 0$, is *a trivial and repelling (or instable) fixed point*, while the second, at $k^* = [\frac{Bs}{1+n}]^{\frac{1}{\beta-1}}$, is *interior* and *asymptotically stable.*

3 Complex dynamics in the Solow Discrete Time Growth Model

R.H. Day (1982,1983) first has noticed that *complex dynamics can emerge from simple economic strutures* as, for example, the neoclassical theory of capital accumulation. In particulary Day argues that the nonlinearity of the $h(k_t)$ map and the lag present in (1.1) are not sufficient to lead to chaos. Instead making changes in (1.1) in the production function or thinking the saving propensity s as a function of k_t, i.e. $s = s(k_t)$, he obtains a *robust* result (Michele Boldrin and Michael Woodford, 1990).

In the former case he defines

$$f(k_t) = \begin{cases} Bk_t^{\beta}(m - k_t)^{\gamma}, & \text{if } k_t < m; \\ 0, & \text{otherwise,} \end{cases}$$

where m is a positive constant, $0 < \beta < 1$, $0 < \gamma < 1$ and $B > 0$.

In the latter case he sets $f(k_t) = Bk_t^{\beta}$ ($B \geq 2$, $0 < \beta < 1$) and he replaces the constant s with the saving function

$s(k_t) = a(1 - \frac{b}{r})\frac{k_t}{y_t}$,

where $r = f^{'}(k_t) = \beta\frac{y_t}{k_t}$, $a > 0, b > 0$.

Thus from the equation (2.1) we deduce respectively the equations

$k_{t+1} = \frac{1}{1+n}sBk_t^{\beta}(m - k_t)^{\gamma}$ (3.1)

and

$k_{t+1} = \frac{a}{1+n}k_t[1 - (\frac{b}{\beta B})k_t^{1-\beta}]$ (3.2).

It is very simple to solve the equation (4.1) when $m = \gamma = \beta = 1$. As a matter of fact we can rewrite it like this

$k_{t+1} = \frac{1}{1+n}sBk_t(1 - k_t)$ (3.3).

If we set $\mu = \frac{sB}{1+n}$ then the (3.3) becomes the well-known logistic equation (see **Appendix 1**)

$k_{t+1} = \mu k_t(1 - k_t)$.

We can use the Li-Yorke Theorem (see **Appendix 2**). Following Day (1982, 1983), first we observe that the right-hand side $h(k_t) = \frac{1}{1+n}sBk_t^{\beta}(m - k_t)^{\gamma}$ of equation (3.1) is a map concave, one-humped shaped, has a range equal to the interval $[0, h(k^c)]$, where k^c is the unique value of k_t which maximizes the map $h(k_t)$. Moreover fixing the parameters β, γ and m, the graph of $h(k_t)$ stretches upwards as B is increased and at same time the position of k^c doesn't changes because in the expression of k^c the parameter B don't appear while the maximum $h(k^c)$ depends linearly on B (See **Figure 1** and **Figure 2**).

As a matter of fact, from the equation

$$\frac{dk_{t+1}}{dk_t} = \frac{sB}{1+n}(\beta k_t^{\beta-1}(m-k_t)^{\gamma} - k_t^{\beta}\gamma(m-k_t)^{\gamma-1}) = 0,$$

we get $k^c = \frac{\beta m}{\gamma+\beta}$ and $h(k_t^c) = \frac{Bs}{1+n}\beta^{\beta}\gamma^{\gamma}(\frac{m}{\beta+\gamma})^{\beta+\gamma}$.

Moreover we assume that k^b is the backward iteration of k^c, i.e. $k^b = h^{-1}(k^c)$, k^m is the forward of k^c, i.e. $h(k^c) = k^m$ and k^m is the maximum k such that $h(k) = 0$. Thus $h(k^m) = 0$, $k^c = h(k^b)$, $k^m = h(k^c) = h(h(k^b))$, $h(k^m) = h(h(h(k^b))) = 0$. If B is large enough, k^c lies to left of the fixed point k^*, from which it follows that $k^b < k^c$.

The previous conditions

$$0 < k^b < k^c < k^m,$$

imply that

$$h(k^m) < k^b < h(k^b) < h(k^c),$$

which are equivalent to the inequalities

$$h^3(k^b) < k^b < h(k^b) < h^2(k^b).$$

Therefore the hypotheses of Li-Yorke theorem are satisfied.

From (3.2) we get

$$\frac{dk_{t+1}}{dk_t} = \frac{a}{1+n}\{[1 - \frac{b}{\beta B}k_t^{1-\beta}] + k_t[(-\frac{b}{\beta B})(1-\beta)k_t^{-\beta}]\}$$

$= \frac{a}{1+n}[1 - (2-\beta)\frac{b}{\beta B}k_t^{1-\beta}] = 0$

if and only if $k^* = [\frac{\beta B}{b(2-\beta)}]^{\frac{1}{1-\beta}}$.

If we call $\psi(k_t)$ the right-hand side of (3.2) we have

$\psi(k^*) = \frac{a}{1+n}[\frac{\beta B}{b(2-\beta)}]^{\frac{1}{1-\beta}}\frac{1-\beta}{2-\beta}$.

Let k_c the smaller root of the equation

$\psi(k_t) = x^*$ (3.4),

that is $\frac{a}{1+n}k_t[1 - (\frac{b}{\beta B})k_t^{1-\beta}] = [\frac{\beta B}{b(2-\beta)}]^{\frac{1}{1-\beta}}$ (4.5).

As above conditions of the of Li-Yorke Theorem are satisfied.

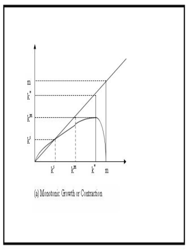

Figure 1: *In the expression of k^c the parameter B don't appear.*

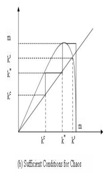

Figure 2: *The maximum $h(k^c)$ depends linearly on B.*

4 A Two Class Growth Model: A Model of Böhm and Kaas

4.1 Introduction

In the model of Böhm and Kaas (1999) there are two types of agents (*two class model*), called workers and shareholders, and only one good (or commodity) is produced which is consumed or invested (*one sector model*). Like Kaldor (1956,1957) and Pasinetti (1962), the workers and shareholders have constant savings propensities, denoted respectively with s_w and s_r ($0 \leq s \leq 1$ and $0 \leq s \leq 1$). The output is produced with two factors: labor and capital. We consider that the capital depreciates at a rate $0 < \delta \leq 1$ and the labor grows at rate $n \geq 0$. We write the production function $f : \Re \rightarrow \Re$ in intensive form (i.e. it is maps capital per worker k into output per worker y), and suppose that f satisfies the following conditions :

- f is C^2;
- $f(\lambda k) = \lambda f(k)$ (*constant returns to capital*);
- f is monotonically increasing and strictly concave (i.e. $f^{'}(k) > 0$ and $f^{''}(k) < 0$ for all $k > 0$);
- $\lim_{k \rightarrow \infty} f(k) = \infty$;

- (a) $\lim_{k\to 0} \frac{f(k)}{k} = \infty$ and (b) $\lim_{k\to\infty} \frac{f(k)}{k} = 0$ (*weak Inada conditions (WIC)*)

Remark 4.1.1 Following Böhm et al. (2007), we now introduce two families of production functions that violate the WIC: the *linear production functions* and the *Leontief production functions* given by $f(k) = a + bk$, $(a, b > 0)$ and $g(k) = min\{a, bk\}$ $(a > 0, b > 0)$ respectively.

Since

$\lim_{k\to 0} \frac{f(k)}{k} = \infty$ and $\lim_{k\to\infty} \frac{f(k)}{k} = b$,

f violates property (b) of WIC. Instead since

$\lim_{k\to 0} \frac{g(k)}{k} = b$ and $\lim_{k\to\infty} \frac{f(k)}{k} = 0$,

g does not satisfy property (a) of WIC. We conclude this remark offering an example of production functions that satisfy WIC: the *isoelastic production functions of the form*

$h(k) = Ak^{\alpha}$, $A > 0$, $0 < \alpha < 1$.

It easy verify that $h(k)$ satisfies WIC.

Remark 4.1.2 We observe that, *for any differentiable function* $f : \Re_+ \to \Re_+$, *the Inada conditions*

(α) $\lim_{k\to 0} f'(k) = \infty$ *and* (β) $\lim_{k\to\infty} f'(k) = 0$,

imply WIC . As a matter of fact, since

$\lim_{k\to 0} f(k) = 0$ and $\lim_{k\to\infty} f(k) = \infty$,

by l'Hôpital's rule,

$\lim_{k\to 0} f'(k) = \lim_{k\to 0} \frac{f(k)}{k}$ and $\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} \frac{f(k)}{k}$.

If we assume that the market is competitive then the wage rate $w(k)$ is coincident with the marginal product of labor, i.e. $w(k) = f(k) - kf'(k)$, and the interest rate (or investment rate) r is equal to the marginal product of capital, i.e. $r = f'(k)$. We suppose that $f(0)$ generally is not equal to 0. We observe that the total capital income per worker is $kf'(k)$. Moreover from WIC we deduce that:

- $w(k) \geq 0$;
- $w^{'}(k) = -kf^{''}(k) > 0$ ($w(k)$ is strictly monotonically increasing);
- $0 \leq kf^{'}(k) \leq f(k) - f(0)$;
- $\lim_{k \to 0} kf^{'}(k) = 0$.

Remark 4.1.3 There are several ways to obtain the inequality $0 \leq kf^{'}(k) \leq f(k) - f(0)$. The first way is the following. We recall that f *is concave in* $[0, +\infty[$ *if and only if* $f(k_1) \leq f(k_0) + f^{'}(k_0)(k_1 - k_0)$, *for all* $k_0, k_1 \geq 0$. In particulary, if $k_0 = k$ and $k_1 = 0$, we have $f(0) \leq f(k) + f^{'}(k)(0 - k)$, from which $0 \leq kf^{'}(k) \leq f(k) - f(0)$.

Alternately, if $f^{'}(0) < \infty$, by the inequality $w(0) \leq w(k)$ for all $k \geq 0$, we have $f(0) - 0 \cdot f^{'}(0) \leq f(k) - kf^{'}(k)$, from which $0 \leq kf^{'}(k) \leq f(k) - f(0)$.

Finally, consider the graph of a monotonically strictly increasing and concave function f with $f(0) > 0$. Geometrically we may intuit the inequality drawing in the plane $(k, f(k))$ the line which goes across the points $(0, f(0))$ and $(k, f(k))$ and the tangent line in the point $(k, f(k))$: the slope of the first line, $\frac{f(k)-f(0)}{k}$, will appear greater or equal to the slope $f^{'}(k)$ of the second line. By continuity of $f(k)$ on $k = 0$, we obtain the $\lim_{k \to 0} f(k) = f(0)$. Thus, from the previous inequality, $\lim_{k \to 0} kf^{'}(k) \leq \lim_{k \to 0}(f(k) - f(0)) = f(0) - f(0) = 0$.

Similarly to the Solow model we obtain that the time-one map of capital accumulation is

$k_{t+1} = G(k_t) = \frac{1}{1+n}((1 - \delta)k_t + s_w w(k_t) + s_r k_t f^{'}(k_t))$ (4.1).

Proposition 4.1.4 Given $n \geq 0$ and $0 \leq \delta \leq 1$, let $f(k)$ be a production function which satisfies the WIC. If the workers do not save less than shareholders (i.e. $s_w \geq s_r$) or $e_{f'}(k) \geq -1$ then G is monotonically increasing in k.

Proof We observe that $\frac{dG(k_t)}{dk_t} = \frac{1}{1+n}((1-\delta)-s_w kf''(k)+s_r(f'(k_t)+k_t f''(k_t)))$. Thus $\frac{dG(k_t)}{dk_t} \geq 0$ is equivalent to inequality $(s_w - s_r)kf''(k) \leq 1-\delta+s_r f'(k)$. From the assumptions $f'(k) > 0$, $1-\delta \geq 0$ and $s_r > 0$, we deduce that $(1-\delta+s_r f'(k) > 0)$. Being $f''(k) < 0$, if $s_w \geq s_r$, the left-hand side of inequality is negative and the inequality is satisfied trivially. Otherwise, rewriting the inequality in the following manner $s_w kf''(k) \leq (1-\delta)+s_r(kf^{''}(k)+f'(k))$, we notice that it is true if $(kf''(k)+f'(k) \geq 0)$, i.e. $e_{f'}(k) \geq -1$.

The following proposition investigates *the existence and the uniqueness of steady states.*

Proposition 4.1.5 Consider n and δ fixed and let $f(k)$ be a production function which satisfies the WIC. The following conditions hold:

- $k = 0$ if and only if $s_w = 0$ or $f(0) = 0$.
- There exists al least one positive steady state if ($s_r > 0$ and $\lim_{k\to 0} f'(k) = 0$) or if ($s_w > 0$ and $f'(0) < \infty$).
- There exists at most one positive steady state if ($s_r \geq s_w$).

Proof We observe that k is a steady state if and only if $k = G(k)$, that is

$$s_w w(k) + s_r kf'(k) = (n+\delta)k.$$

Thus $0 = G(0)$ if and only if $(s_w(f(0) - \lim_{k\to 0} kf'(k)) + s_r \lim_{k\to 0} kf'(k) = 0)$.

By a previous observation we have that $\lim_{k\to 0} kf'(k) = 0$, therefore $k = 0$ is a steady state if and only if $s_w f(0) = 0$.

Moreover the existence of a positive steady state k is equivalent to

$$s_w(\frac{f(k)}{k} - f'(k)) + s_r f'(k) = n + \delta.$$

We set $H(k) = s_w(\frac{f(k)}{k} - f'(k)) + s_r f'(k)$. By Bolzano's Theorem, being $H(k)$ continuous in interval $]0, +\infty[$, the range J of $H(k)$ is an interval. We notice that $J =]0, +\infty[$. As a matter of fact, if suppose that $\lim_{k\to\infty} f'(k) =$

$+\infty$, we may apply the Hôpital's Rule to the first of the conditions denoted above with (I), and we have $0 = \lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f'(k)$, from which $\lim_{k\to+\infty} H(k) = 0$. From the second relation of (I) and setting $f'(0) < +\infty$, we obtain that $\lim_{k\to 0} H(k) = +\infty$. Therefore, the equation $H(k) = n + \delta$ accepts at least one positive solution. Being $\frac{dH(k)}{dk} = s_w(\frac{kf'(k)-f(k)}{k^2} - f''(k)) + s_r f''(k)$ $= s_w(\frac{kf^{'}(k)-f(k)}{k^2}) + (s_r - s_w)f^{''}(k)$ and since $kf^{'}(k) - f(k) = -w(k) < 0$, if we suppose $s_r \geq s_w$, we deduce that $\frac{dH(k)}{dk} \leq 0$. Thus $H(k)$ is strictly monotonically decreasing and the equation $H(k) = n + \delta$ admits only one root.

Proposition 4.1.6 $k^\star$ is a steady state of Pasinetti-Kaldor iff, for given n and δ, the pairs (s_r, s_w) of savings rate describe the line $s_r + \frac{1-e_f(k^\star)}{e_f(k^\star)} s_w = 1$ in the (s_r, s_w)-diagram, where $e_f(k) = \frac{kf^{'}(k)}{f(k)}$.

Proof We observe that the total consumption per worker is $c(k) = f(k) - sw(k) - skf^{'}(k)$. If $k^\star$ is a steady state then $c(k^\star) = f(k^\star) - (n+\delta)k^\star$. We want the steady state $k^\star$, with different savings rate, which maximize $c(k^\star)$. Thus, setting $\frac{dc(k^\star)}{dk^\star} = 0$, we find $f^{'}(k^\star) = (n + \delta)$, that is $k^\star = f^{-1}((n + \delta))$. We call *Kaldor-Pasinetti equilibrium* the optimal steady state consumption (or the *golden rule for capital stock*). Replacing $(n + \delta)$ with $f^{'}(k^\star)$ in the right-hand side of the steady state condition $s_w w(k^\star) + s_r k^\star f^{'}(k^\star) = (n + \delta)k^\star$, we obtain $s_w w(k^\star) + s_r k^\star f^{'}(k^\star) = k^\star f^{'}(k^\star)$, that is $s_w(f(k^\star) - k^\star f^{'}(k^\star)) + s_r k^\star f^{'}(k^\star) = k^\star f^{'}(k^\star)$. Dividing both sides of the previous equation by $f(k^\star)$ and recalling the definition of $e_f(k)$, we have $s_r + \frac{1-e_f(k^\star)}{e_f(k^\star)} s_w = 1$. We notice that in the (s_r, s_w)-plane the last equation can be viewed as a line that

- has negative slope;
- goes across the point $(s_r, s_w) = (1, 0)$;
- is below or above the 45°-line $s_w = s_r$ depending on $e_f(k^\star)$ is less or greater than $\frac{1}{2}$.

The (s_r, s_w)-plane is coincident with the square $[0, 1]^2$.

4.2 The dynamics with fixed proportions

We consider the Leontief technology

$f_L(k) = min\{ak, b\} + c,\ a, b, c > 0.$

Let $k^\star = b/a$ be. We have

$$f_L(k) = \begin{cases} ak + c, & \text{if } k \leq k^\star, \\ b + c, & \text{if } k > k^\star; \end{cases} \text{ and } f_L^{'}(k) = \begin{cases} a, & \text{if } k \leq k^\star, \\ 0, & \text{if } k > k^\star. \end{cases}$$

The map G becomes

$$G_L(k) = \begin{cases} G_1(k) = \frac{1}{1+n}((1 - \delta + s_r a)k + s_w c), & \text{if } k \leq k^\star, \\ G_2(k) = \frac{1}{1+n}((1 - \delta)k + (b + c)s_w), & \text{if } k > k^\star. \end{cases}$$

We may say that:

- G_1 and G_2 are affine-linear maps strictly monotonically increasing;
- $G_1^{'} = \frac{1}{1+n}(1 - \delta + s_r a) > G_2^{'} = \frac{1}{1+n}(1 - \delta)$;
- $G_2^{'} < 1$: the map $G_2^{'}$ has always a fixed point k_2;
- G_1 has the fixed point k_1 if and only if $G_1^{'} < 1$, that is $n + \delta - s_r a > 0$;
- $G_1(0) = \frac{1}{1+n} s_w c < G_2(0) = \frac{1}{1+n}(b + c)s_w$.

Let k_1 be the fixed point for G_1. Then k_1 *is a fixed point also for* G *if and only if* $k_1 < k^\star$. Analogously, found the fixed point k_2 for G_2, we have that k_2 *is a fixed point also for* G *if and only if* $k^\star < k_2$ (See **Figure 3**).

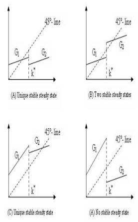

Figure 3: *The Maps G_1 and G_2*

Proposition 4.2.1 Let $G_1^{'} < 1$ be. We obtain that:

(i) the fixed point k_1 for G_1 is equal to $\frac{cs_w}{n+\delta-as_r}$;

(ii) k_1 is a fixed point also for G if and only if $bs_r + cs_w < (n+\delta)\frac{b}{a}$;

(iii) $G_1(k^\star) < k^\star$ if and only if $bs_r + cs_w < (n+\delta)\frac{b}{a}$.

Proof We solve the equation $G_1(k) = k$. We get

$\frac{1}{1+n}((1-\delta+s_r a)k + s_w c) = k$, from which

$(s_r a - n - \delta)k = -s_w c$. Thus $k_1 = \frac{cs_w}{n+\delta-as_r}$.

Moreover $k_1 < k^\star$ if and only if $\frac{cs_w}{n+\delta-as_r} < \frac{b}{a}$. From the assumption $G_1^{'} < 1$ we deduce $n + \delta - s_r a > 0$. Therefore $cs_r < -bs_w + (n+\delta)\frac{b}{a}$, from which $bs_r + cs_w < (n+\delta)\frac{b}{a}$.

The inequality $G_1(k^\star) < k^\star$ is equivalent to the following $\frac{1}{1+n}((1-\delta+s_r a)k^\star + s_w c) < k^\star$. We get before $(as_r - n - \delta)k^\star < -s_w c$, and after $s_r a k^\star - (n+\delta)k^\star < -s_w c$. We deduce the relation (iii). (i) and (ii) are equivalent.

Proposition 4.2.2 We get

(i) the fixed point of G_2 is $k_2 = \frac{(b+c)s_w}{n+\delta}$;

(ii) k_2 is the fixed point also for G if and only if $s_w > \frac{(n+\delta)b}{(b+c)a}$;

(iii) $G_2(k^\star) > k^\star$ if and only if $s_w > \frac{(n+\delta)b}{(b+c)a}$.

Proof Solving the equation $G_2(k) = k$, we obtain the following equivalent relations:

$\frac{1}{1+n}((1-\delta)k + (b+c)s_w) = k,$

$(1-\delta)k - (1+n)k = -(b+c)s_w,$

$-(n+\delta)k = -(b+c)s_w$, from which $k_2 = \frac{(b+c)s_w}{n+\delta}$.

Moreover $k_2 > k^\star$ if and only if $\frac{(b+c)s_w}{n+\delta} > \frac{b}{a}$, from which $s_w > \frac{(n+\delta)b}{(b+c)a}$. (iii) trivial. Obviously (ii) and (iii) are equivalent (See **Figure 4**).

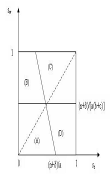

Figure 4: *Stability regions for the Leontief technology*

Remark 4.2.3 G_L has two fixed point if and only if $G_1(k^\star) < k^\star < G_2(k^\star)$, from which $G_1(k^\star) < G_2(k^\star)$. Then $\frac{1}{1+n}((1-\delta+s_r a)k^\star + s_w c) < \frac{1}{1+n}((1-\delta)k^\star + (b+c)s_w$. Thus $s_r < s_w$.

(A) G_L has only one fixed point: the fixed point of G_1, that is it holds the system

$$\begin{cases} bs_r + cs_w < (n+\delta)\frac{b}{a}, \\ s_w < \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

(B) G_L has two fixed points: the fixed point of G_1 and the fixed point of G_2, that is it holds the system

$$\begin{cases} bs_r + cs_w < (n+\delta)\frac{b}{a}, \\ s_w > \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

(C) G_L has only one fixed point: the fixed point of G_2, that is it holds the system

$$\begin{cases} bs_r + cs_w > (n+\delta)\frac{b}{a}, \\ s_w > \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

(D) G_L don't has fixed point, that is it holds the system

$$\begin{cases} bs_r + cs_w > (n+\delta)\frac{b}{a}, \\ s_w < \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

Remark 4.2.4 Now consider the case (B). Since $G_1(k^\star) < k^\star < G_2(k^\star)$, we get

$$G_1(k_1) < G_1(k^\star) < k^\star < G_2(k^\star) < G_2(k_2),$$

from which

$G_1(k_1) < G_2(k_2)$ for all pairs (k_1, k_2) such that $0 \leq k_1 \leq k^\star$ and $k_2 > k^\star$.

Thus G_L *is strictly monotonically increasing (and therefore injective) in the case (B).*

Remark 4.2.5 Look at case (D), that is $G_2(k^\star) < k^\star < G_1(k^\star)$. Then $G_L(G_2(k^\star)) = G_1(G_2(k^\star))$ and $G_L(G_1(k^\star)) = G_2(G_1(k^\star))$. Moreover, by relations

$$G_1(G_2(k^\star)) = \frac{(1-\delta+s_r a)(1-\delta)}{(1+n)^2} k^\star + \frac{(1-\delta+s_r a)(b+c)s_w}{(1+n)^2} + \frac{c s_w}{(1+n)},$$

$$G_2(G_1(k^\star)) = \frac{(1-\delta+s_r a)(1-\delta)}{(1+n)^2} k^\star + \frac{(1-\delta)c s_w}{(1+n)^2} + \frac{(b+c)s_w}{(1+n)},$$

we will show that $G_1(G_2(k^\star)) > G_2(G_1(k^\star))$, and thinking as before,

we may deduce that G_L *is injective on the interval* $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$.

As a matter of fact, we can write G_1 and G_2 such that:

$G_1(k^\star) = m_1 k^\star + n_1$ and $G_2(k^\star) = m_2 k^\star + n_2$, where $m_1 \geq 1 > m_2 > 0$ and $n_2 > n_1 > 0$.

We have

$$G_1(G_2(k^\star)) = m_1(m_2 k^\star + n_2) + n_1 = m_1 m_2 k^\star + m_1 n_2 + n_1,$$

$$G_2(G_1(k^\star)) = m_2(m_1 k^\star + n_1) + n_2 = m_1 m_2 k^\star + m_2 n_1 + n_2.$$

Let $n_2 = n_1 + \epsilon$ be, where $\epsilon > 0$. Then we may conclude observing that $m_1 n_2 + n_1 = m_1(n_1 + \epsilon) + n_1 = m_1 n_1 + m_1 \epsilon + n_1 > m_2 n_1 + n_2 = m_2 n_1 + n_1 + \epsilon$.

Proposition 4.2.6 We consider the case (D), i.e. $G_2(k^\star) < k^\star < G_1(k^\star)$. Let $K_\tau = (k_s)_{s=1,\ldots,\tau}$ be a cycle of order τ for G_L such that $k_s \neq k^\star$ for all $s = 1, \ldots, \tau$. Then K_τ is globally stable.

Proof By recurrence it proves that on the interval $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$

- each sth iterate G_L^s is injective;
- the τth iterate G_L^τ, presents a discontinuity either at $k^\star$ or at $G_L^{-s}(k^\star)$, $s = 1, \dots, \tau - 1$.

Thus G_L^τ shows at most τ discontinuities and we may find a partition $\{I_1, \dots, I_m\}$ of $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$ into m intervals I_s ($s = 1, \dots, m$ and $m \leq \tau + 1$) such that $G_L^\tau(k) = A_s + B_s k$, $s \in I_s$, where A_s and B_s are positive constants.

Let $(k_s)_{s=1,\dots,\tau}$ be a cycle of order τ. If we assume that $k_s \in I_s$ ($s = 1, \dots, \tau$), we obtain that $B_s < 1$. As a matter of fact, imposing $k_s = A_s + k_s B_s$, we have $(1 - B_s)k_s = A_s$. Being k_s and A_s positive, we deduce that $1 - B_s > 0$.Therefore we may say that *each trajectory starting in* $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$ *converges to* K_τ.

5 Complex Dynamics in a Pasinetti-Solow Model of Growth and Distribution: a Model of P.Commendatore

5.1 Introduction

Similarly to the paper of Böhm and Kaas (1999), the model of Commendatore (2005)

- is a two-class model, that is two distinct group of economic agents (workers and capitalists) exist, with constant propensities to save (Kaldor, 1956);
- labor and capital markets are perfectly competitive;
- the income sources of workers are wages and profits and the income of capitalists is only profits (Pasinetti, 1962);
- the time is discrete;
- there is a single good in the economy (one sector model).

Commendatore's model differs from the model of Böhm and Kaas in some assumptions:

- following Chiang (1973), workers not save in same proportions out of labor and income of capital;
- the production function is not with fixed proportions (Leontief technology) but it is a CES production function;
- likewise Samuelson-Modigliani (1966) that, following Pasinetti (1962), extend the Solow growth model (1956) to two-dimensions, the map that describes the accumulation of capital in discrete time is two-dimensional because it considers not only the different saving behaviour of two-classes but also their respective wealth (capital) accumulation.

5.2 The model: the economy, short-run equilibrium, steady growth equilibrium

Let $f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}}$ be the CES production function in intensive form, where k is the capital/labor ratio, $0 < \alpha < 1$ is the distribution coefficient, $-\infty < \rho < 1$ ($\rho \neq 0$), $\eta = \frac{1}{1-\rho}$ is the constant elasticity of substitution. We consider $f(k) > 0$. Therefore $f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}} = [\alpha k^{-\rho} + (1-\alpha)]^{\frac{1}{\rho}} k$. The terms k_w and k_c denote, respectively, workers' and capitalists' capital per worker, where $0 \leq k_w \leq k$, $0 \leq k_c \leq k$, $k = k_w + k_c$. The workers' saving out of wages are represented by $s_{ww}(f(k) - kf^{'}(k))$ and the workers' saving out of capital revenues consist in $s_{wP}f^{'}(k)k_w$, where $0 \leq s_{ww} \leq 1$, $0 \leq s_{wP} \leq 1$. Instead the capitalists' savings are $s_c f^{'}(k)k_c$, where $0 \leq s_c \leq 1$. We assume $s_c > max\{s_{ww}, s_{wP}\}$. Thus the aggregate savings correspond to

$$s(k_c, k_w) = s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w + s_c f^{'}(k_c).$$

Let n be the constant rate of growth of labor force, the following map

$$G(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k + i]$$

describes the rule of capital accumulation per worker, where i indicates gross investment per worker and $0 < \delta < 1$ is the constant rate of capital depreciation. In a short-run equilibrium G becomes

$$G(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k + s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w + s_c f^{'}(k_c)] \ (5.1),$$

from which we deduce the capitalist' process of capital accumulation

$G_w(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k_w + s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w]$

and the capitalist's rule of capital accumulation

$G_c(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k_c + s_c f^{'}(k)k_c]$.

In order to obtain the steady states of G_w and G_c, we imposing

$G_w(k_w, k_c) = k_w$ and $G_c(k_w, k_c) = k_c$.

We get

$(n+\delta)k_w = s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w$, $(\star)$

$(n+\delta)k_c = s_c f^{'}(k_c)$ $(\star\star)$

We find three types of equilibria: *Pasinetti equilibrium* (capitalists own positive share of capital), *dual equilibrium* (only workers own capital) and *trivial equilibrium* (the overall capital is zero).

5.2.1 Pasinetti equilibrium

Now we indicate a Pasinetti equilibrium with (k_w^P, k_c^P),

where, by definition, $k^P = k_w^P + k_c^P$. We prove the following

Proposition 5.2.1.1 For the Pasinetti Equilibrium the following conditions hold:

- $f^{'}(k^P) = \frac{n+\delta}{s_c}$,
- $k_w^P = \frac{s_{ww}}{s_c - s_{wP}} \frac{1-e_f(k^P)}{e_f(k^P)} k^P$,

- $k_c^P = (1 - \frac{s_{ww}}{s_c - s_{wP}} \frac{1-e_f(k^P)}{e_f(k^P)})k^P$.

Proof We start by the relation ($\star\star$). Since $k_c \neq 0$ then $(n+\delta) = s_c f^{'}(k)$, from which $f^{'}(k^P) = \frac{n+\delta}{s_c}$. In the left-hand side of ($\star$), we replace $(n+\delta)$ with $s_c f^{'}(k)$. We get

$$s_c f^{'}(k) k_w - s_{wp} f^{'}(k) k_w = s_{ww}(f(k) - f^{'}(k)k),$$

$$k_w f^{'}(k)(s_c - s_{wp}) = s_{ww}(f(k) - f^{'}(k)k),$$

$$k_w f^{'}(k)(s_c - s_{wp}) = s_{ww} f(k)[1 - \frac{f^{'}(k)k}{f(k)}],$$

$$k_w f^{'}(k)k(s_c - s_{wp}) = s_{ww} f(k)[1 - \frac{f^{'}(k)k}{f(k)}]k,$$

$$k_w \frac{f^{'}(k)k}{f(k)}(s_c - s_{wp}) = s_{ww}[1 - \frac{f^{'}(k)k}{f(k)}]k,$$

$$k_w e_f(k)(s_c - s_{wp}) = s_{ww}(1 - e_f(k))k,$$

$$k_w^P = \frac{s_{ww}}{s_c - s_{wp}} \frac{1-e_f(k)}{e_f(k)} k^P.$$

Since $k^P = k_w + k_c$, we have $k_c = k^P - k_w$, from which

$$k_c^P = k^P - \frac{s_{ww}}{s_c - s_{wp}} \frac{1-e_f(k)}{e_f(k)} k^P = [1 - \frac{s_{ww}}{s_c - s_{wp}} \frac{1-e_f(k)}{e_f(k)}]k^P.$$

5.2.2 Dual equilibrium

We indicate the dual equilibrium with (k_w^D, k_c^D), where $k^D = k_w^D + k_c^D$.

We prove the following

Proposition 5.2.2.1 The dual equilibria are given by the relations

$\frac{f(k^D)}{k^D} = \frac{n+\delta}{s_{ww}(1-e_f(k^D))+s_{wP}e_f(k^D)}$, $k_w^D = k^D$ and $k_c^D = 0$

Proof We rewrite the relation $(\star)$ replacing k_w^D with k^D and k with k^D.

We get

$$(n+\delta)k^D = s_{ww}(f(k^D) - f^{'}(k^D)k^D) + s_{wp}f^{'}(k^D)k^D,$$

from which

$$(n+\delta)k^D = s_{ww}f(k^D)(1 - \frac{f^{'}(k^D)k^D}{f(k^D)}) + s_{wp}\frac{f^{'}(k^D)k^D}{f(k^D)},$$

$$(n+\delta)\frac{k^D}{f(k^D)} = s_{ww}(1 - e_f(k^D)) + s_{wp}e_f(k^D),$$

$$\frac{f(k^D)}{k^D} = \frac{n+\delta}{s_{ww}(1-e_f(k^D)+s_{wp}e_f(k^D}.$$

5.2.3 Trivial equilibrium

(k_w^0, k_c^0) and $k^0 = k_w^0 + k_c^0$ where $k^0 = k_w^0 = k_c^0 = 0$.

Output elasticity

We see immediately that

$$e_f(k) = \frac{kf^{'}(k)}{f(k)} = (1-\alpha)(\alpha k^{-\rho} + 1 - \alpha)^{-1},$$

$$0 < e_f(k) \leq 1.$$

5.3 Meade's Relation For Pasinetti Equilibria

We introduce the *Meade's relation* for Pasinetti equilibria

$\frac{f(k)}{k} = \varphi(e_f(k)),$

where $\varphi(x) = (\frac{1-\alpha}{x})^{\frac{1}{\rho}}$.

We notice that for $\varphi(x)$ occurs:

- $\varphi^{'}(x) = \frac{(1-\alpha)}{\rho}(\frac{1-\alpha}{x})^{\frac{1}{\rho}-1}(-\frac{1}{x^2}) = -\frac{(1-\alpha)}{\rho}\frac{1}{x^2}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}$
- $\varphi^{''}(x) = -\frac{(1-\alpha)}{\rho}\{-2x^{-3}(\frac{1-\alpha}{\rho})^{\frac{1-\rho}{\rho}} + x^{-2}(\frac{1-\rho}{\rho})(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}-1}(1-\alpha)(-x^{-2})\}$

  $= \frac{(1-\alpha)}{\rho}x^{-3}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}(2 + \frac{1-\rho}{\rho})$

  $= (1+\rho)\frac{(1-\alpha)}{\rho^2}x^{-3}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}$

The former features of $\varphi(x)$ lead us to state that (See **Figure 5**)

Proposition 5.3.1 For the function $\varphi(x)$ is true that:

- it is strictly monotonic for all $\rho < 1$ and $\rho \neq 0$;
- it is strictly convex for all $0 < \rho < 1$ and strictly concave for all $\rho < -1$;
- it becomes the line $\varphi(x) = \frac{x}{1-\alpha}$ if $\rho = -1$.
- $\lim_{x\to 0}\varphi(x) = +\infty$ if $0 < \rho < 1$.

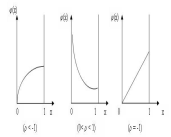

Figure 5: *The diagram of φ for different ρ.*

Proposition 5.3.2 Both workers and capitalists own a positive share of capital if and only if

$$0 < e_f^T < e_f(k^P) < 1,$$

where $e_f^T = \frac{s_{ww}}{s_c-(s_{wP}-s_{ww})}$.

Proof We observe that $k_w^P > 0$ is equivalent to say that ($e_f < 1$ and $s_c > s_{wp}$) or ($e_f > 1$ and $s_c < s_{wp}$).

We don't accept the second condition because the CES don't satisfies the inequality $e_f > 1$.

Moreover the inequality $k_c^P > 0$ holds iff $\frac{1-e_f}{e_f}\frac{s_{ww}}{s_c-s_{wP}} < 1$.

Thus is true that

$$\frac{1-e_f}{e_f} < \frac{s_c-s_{wP}}{s_{ww}},$$

from which

$$\frac{1}{e_f} < 1 + \frac{s_c-s_{wP}}{s_{ww}}, \ \frac{1}{e_f} < \frac{s_c-(s_{wP}-s_{ww})}{s_{ww}}. \text{ Q.E.D.}$$

Observed that

- Case (a): $s_{ww} = s_c$. Then $e_f^T = \frac{s_{ww}}{s_c}$;

- Case (b: $s_{ww} < s_c$. Then $s_c - (s_{wP} - s_{ww}) < s_c$;
- Case (c): $s_{ww} > s_c$. Then $s_c - (s_{wP} - s_{ww}) > s_c$;

we deduce that

$$e_f^T(Case(c)) < e_f^T(Case(a)) < e_f^T(Case(b)).$$

Proposition 5.3.3 We have $e_f(k^P) = (1-\alpha)^{\frac{1}{1-\rho}} \ (\frac{n+\delta}{s_c})^{\frac{\rho}{\rho-1}}$

Proof From definition of e_f we obtain that $\frac{f(k)}{k} = \frac{f^{'}(k)}{e_f(k)}$ and by Meade's relation $\frac{f(k)}{k} = \varphi(e_f(k))$ we get $\varphi(e_f(k^P)) = \frac{f^{'}(k^P)}{e_f(k^P)} = \frac{n+\delta}{s_c}\frac{1}{e_f(k^P)}$: *the intersection between the arc of hyperbola* $\Gamma : \frac{n+\delta}{s_c}\frac{1}{e_f(k^P)}$ *and the curve* $\varphi(e_f(k^P))$ *identifies the unique Pasinetti equilibrium.*

From $e_f(k^P) = \frac{f^{'}(k^P)}{\varphi(e_f(k^P))}$ and by definition of $\varphi(k)$ we have $(\frac{n+\delta}{s_c})\ (\frac{e_f(k^P)}{1-\alpha})^{\frac{1}{\rho}} = e_f(k^P)$. We obtain

$$(\frac{n+\delta}{s_c})^{\rho}(\frac{e_f(k^P)}{1-\alpha}) = (e_f(k^P))^{\rho},$$

$$(e_f(k^P))^{\rho-1} = \frac{1}{1-\alpha}(\frac{n+\delta}{s_c})^{\rho}.\ \text{Q.E.D.}$$

Commendatore (2005), generalizing a relation of Samuelson-Modigliani (1966) and Miyazaki (1991), shows that

Proposition 5.3.4 We assume that:

- $f^{'}(k)$ is monotonically increasing,
- $e_f(k) < 1$,
- $s_{ww} \leq s_{wP}$,
- $k^D > k^P$.

Then is true that

$$e_f^T > e_f(k^P),$$

where $e_f^T = \frac{s_{ww}}{s_c-(s_{wP}-s_{ww})}$ and $e_f(k) = \frac{kf^{'}(k)}{f(k)}$.

Proof We observe that *a CES production function satisfies the former two assumptions of proposition* first, then we prove that $\frac{f(k)}{k}$ *is monotonically decreasing if and only if* $f^{'}(k) < \frac{f(k)}{k}$. As a matter of fact, let $g(k) = \frac{f(k)}{k}$ be. We have that $g^{'}(k) = \frac{f^{'}(k)k-f(k)}{k^2} < 0$ if and only if $f'(k)k < f(k)$. Since $e_f(k) = \frac{f^{'}(k)k}{f(k)} < 1$ then the previous inequality is satisfied. Thus from the assumption $k^P < k^D$ we deduce $\frac{f(k^P)}{k^P} > \frac{f(k^D)}{k^D}$.

Moreover the *dual equilibrium* can be rewritten as follows

$$(n+\delta)k^D = s_{ww}(f(k^D) - f^{'}(k^D)k^D) + s_{wP}f^{'}(k^D)k^D,$$

$$(n+\delta)k^D = s_{ww}f(k^D) - s_{ww}f^{'}(k^D)k^D + s_{wP}f^{'}(k^D)k^D,$$

$$(n+\delta)k^D = s_{ww}f(k^D) + (s_{wP} - s_{ww})f^{'}(k^D)k^D,$$

$$(n+\delta) = s_{ww}\frac{f(k^D)}{k^D} + (s_{wP} - s_{ww})f^{'}(k^D),$$

$$\frac{(n+\delta)}{s_{ww}} = \frac{f(k^D)}{k^D} + \frac{s_{wP}-s_{ww}}{s_{ww}}f^{'}(k^D),$$

$$\frac{f(k^D)}{k^D} = \frac{(n+\delta)}{s_{ww}} - \frac{s_{wP}-s_{ww}}{s_{ww}}f^{'}(k^D).$$

Therefore $\frac{f(k^P)}{k^P} > \frac{(n+\delta)}{s_{ww}} - \frac{s_{wP}-s_{ww}}{s_{ww}}f^{'}(k^D)$.

Then, recalling that $s_{ww} \leq s_{wP}$ and $f^{'}(k^P) = \frac{n+\delta}{s_c}$, we have

$$s_{ww}\frac{f(k^P)}{k^P} > (n+\delta) - (s_{wP} - s_{ww})f^{'}(k^D) = s_c f'(k^P) - (s_{wP} - s_{ww})f^{'}(k^D),$$

and, observing that from the strict monotonicity of $f^{'}(k)$, the inequality $k^D > k^P$ implies $f^{'}(k^D) > f^{'}(k^D)$, we get

$s_{ww}\frac{f(k^P)}{k^P} > [s_c - (s_{wP} - s_{ww})]f^{'}(k^P)$. Q.E.D.

5.4 Meade's Relation For Dual Equilibria

In order to detect geometrically the dual equilibria we will use the following *Meade's relation* for dual equilibria

$\frac{f(k)}{k} = \theta(e_f(k))$,

where $\theta(x) = \frac{n+\delta}{s_{ww}(1-x)+s_{wP}x}$.

We observe that

- $\theta : [0,1] \to [0,1]$ and $\theta(x) > 0$ for all $x \in [0,1]$;
- $\theta(0) = \frac{n+\delta}{s_{ww}} > 0$ and $\theta(1) = \frac{n+\delta}{s_{wP}} > 0$;
- $\theta(x)$ is a continuous function in [0,1];
- $\theta^{'}(x) = (s_{ww} - s_{wP})\frac{\theta(x)^2}{n+\delta}$;
- $\theta^{''}(x) = \frac{2(s_{ww}-s_{wP})^2}{(n+\delta)^2}\theta(x)^3 \geq 0$;

Thus $\theta(x)$ is (See **Figure 6**)

- constant if $s_{ww} = s_{wP}$;
- strictly monotonically increasing if $s_{ww} > s_{wP}$;
- strictly monotonically decreasing if $s_{ww} < s_{wP}$;
- strictly convex if $s_{ww} \neq s_{wP}$.

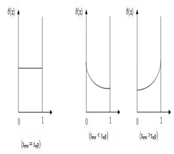

Figure 6: *The diagram of θ for different comparisons of s_{ww} with s_{wP}.*

Proposition 5.4.1 The dual equilibria are given by the set

$\{x \in [0,1] : \varphi(x) = \theta(x)\}$.

Proof We distinguish the following two cases:

- Case I: $\rho = -1$. Then $\varphi(x)$ becomes $(\frac{1-\alpha}{x})^{-1}$. Thus we must solve the equation (See **Figure 7**)

  $\frac{x}{1-\alpha} = \frac{n+\delta}{s_{ww}(1-x)+s_{wp}x}$.

  If $s_{ww} = s_{wP}$ then the equation $\varphi(x) = \theta(x)$ is equivalent to relation

  $\frac{x}{1-\alpha} = \frac{n+\delta}{s_{ww}}$,

  from which, trivially, it follows the solution $x = \frac{n+\delta}{s_{ww}}(1-\alpha)$. We notice that x is acceptable iff $x \in [0,1]$.

  If $s_{ww} \neq s_{wp}$, from the relation

  $x[s_{ww}(1-x) + s_{wP}x] = (n+\delta)(1-\alpha)$,

  we obtain that

$-s_{ww}x^2 + (s_{ww} + s_{wP})x = (n + \delta)(1 - \alpha).$

Thus

$s_{ww}x^2 - (s_{ww} + s_{wP})x + (n + \delta)(1 - \alpha) = 0.$

We set

$A = s_{ww},\ B = -(s_{ww} + s_{wP}),\ C = (n + \delta)(1 - \alpha),\ \Delta = B^2 - 4AC.$

We may conclude that *if* $\Delta \geq 0$ *then dual equilibria exist (two real repeated equilibria or two real distinct equilibria).*

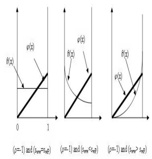

Figure 7: *The diagram of* φ *for* $\rho = -1$ *and the different diagrams of* θ*.*

- Case II: $(\rho < -1) \vee (0 < \rho < 1)$.

  We find the solutions of the equation (See **Figure 8** and **Figure 9**)

  $(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{n+\delta}{s_{ww}(1-x)+s_{ww}x}.$

  We may rewrite the previous equation such that (for details, see **Remark 5.4.2**)

  $\frac{1-\alpha}{(n+\delta)^{\rho}} = \frac{x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}}.$

Now we set $g(x) = \frac{x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}}$.

After some transformations (see **Remark 5.4.3**) we get

$g^{'}(x) = \frac{s_{ww}+(1-\rho)(s_{wP}-s_{ww})x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho+1}}$.

If $s_{wP} \geq s_{ww}$ then $g(x)$ *is strictly monotonically increasing in* $[0,1]$ and the range of $g(x)$ is

$$[0, \frac{1}{[s_{ww}+(s_{wP}-s_{ww})]^{\rho}}].$$

By *Bolzano's Theorem* and by the strictly monotonicity of $g(x)$ exists an unique solution of equation

$$g(x) = \frac{1-\alpha}{(n+\delta)^{\rho}}.$$

If $s_{ww} < s_{wp}$ then $g(x)$ can be monotonically decreasing and exists an unique dual equilibrium.

Notice that $g^{'}(x) = 0$ iff $s_{ww}+(1-\rho)(s_{wP}-s_{ww})x$, i.e., $x = -\frac{s_{ww}}{(1-\rho)(s_{wp}-s_{ww})}$.

Therefore the point $x^{\star} = \frac{s_{ww}}{(1-\rho)(s_{wP}-s_{ww})}$ may be the maximum or minimum for $g(x)$.

Observed that $g(x)$ is strictly concave (or strictly convex), also by Bolzano's Theorem, we obtain one or two dual equilibrium if and only if $\frac{1-\alpha}{(n+\delta)^{\rho}} \leq g(x^{\star})$.

We can say that an unique dual equilibrium exists if the line $y = \frac{1-\alpha}{(n+\delta)^{\rho}}$ intersects the graph of function $g(x)$ at $(x^{\star}, g(x^{\star}))$, being $g(x^{\star})$ the maximum of $g(x)$.

Instead, if $\frac{1-\alpha}{(n+\delta)^{\rho}} < g(x^{\star})$, then, by concavity of $g(x)$, the line $y = \frac{1-\alpha}{(n+\delta)^{\rho}}$ intersects the graph of $g(x)$ in two distinct points $(x^{'}, g(x^{'}))$ and $(x^{''}, g(x^{''}))$, i.e. there are two points $x^{'}$ and $x^{''}$ in $[0,1]$ such that $g(x^{'}) = g(x^{''}) = \frac{1-\alpha}{(n+\delta)^{\rho}}$.

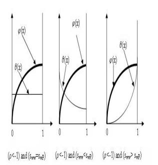

Figure 8: *The diagram of* φ *for* $\rho < -1$ *and the different diagrams of* θ.

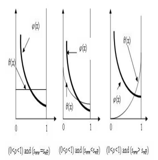

Figure 9: *The diagram of* φ *for* $0 < \rho < 1$ *and the different diagrams of* θ.

In the figures **10, 11, 12** we identify the steady-growth equilibria (*Pasinetti, Dual and Trivial*) for the cases (a)$s_{ww} = s_{wP}$, (b) $s_{ww} < s_{wP}$ and (c)$s_{ww} > s_{wP}$:

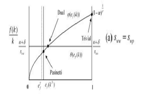

Figure 10: *Steady-growth equilibria identified for the case* $s_{ww} = s_{wP}$.

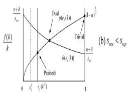

Figure 11: *Steady-growth equilibria identified for the case* $s_{ww} < s_{wP}$.

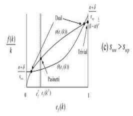

Figure 12: *Steady-growth equilibria identified for the case* $s_{ww} > s_{wP}$.

Remark 5.4.2

$$(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{n+\delta}{s_{ww}+(s_{wP}-s_{ww})x},$$

$$(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{n+\delta}{s_{ww}-s_{ww}x+s_{wP}x},$$

$$\frac{(1-\alpha)^{\frac{1}{\rho}}}{x^{\frac{1}{\rho}}} = \frac{n+\delta}{s_{ww}-s_{ww}x+s_{wP}x},$$

$$\frac{1-\alpha}{(n+\delta)^{\rho}} = \frac{x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}}$$

Remark 5.4.3

$$g^{'}(x) = \frac{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}-\rho x(s_{wP}-s_{ww})[s_{ww}+(s_{wP}-s_{ww})x]^{\rho-1}}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho}}$$

$$= \frac{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}\{1-\rho x(s_{wP}-s_{ww})[s_{ww}+(s_{wP}-s_{ww})x]^{-1}\}}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho}}$$

$$= \frac{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}\{1-\frac{\rho(s_{ww}-s_{wP})x}{s_{ww}+(s_{wP}-s_{ww})x}\}}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho}}$$

$$= \frac{s_{ww}+(s_{wP}-s_{ww})x-\rho x(s_{wP}-s_{ww})}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho-\rho+1}} = \frac{s_{ww}+(1-\rho)x(s_{wP}-s_{ww}))}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho+1}}.$$

We note that $e_f(k=0) = (1-\alpha)(1-\alpha)^{-1} = 1$, from which $\varphi(e_f(0)) = \varphi(1) = (1-\alpha)^{\frac{1}{\rho}}$. Thus *the intersection between the curve $\varphi(e_f(k))$ and the vertical line at 1 identifies the trivial equilibrium.*

5.5 Local stability analysis

5.5.1 The Jacobian evaluated at a Pasinetti equilibrium

In order to determine the local stability of the fixed points of our dynamical system we will linear approximate it with **the Hartman-Grobman Theorem**. We begin with the Jacobian matrix of the dynamical system evaluated at a Pasinetti-equilibrium:

$$J(k_w^P, k_c^P) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$J_{11} = \frac{1}{1+n}[1-\delta+(s_{wP}-s_{ww})f^{''}(k^P)k^P + s_{wP}(f^{'}(k^P) - f^{''}(k^P)k_c^P)],$$

$$J_{12} = \frac{1}{1+n}[(s_{wP}-s_{ww})f^{''}(k^P)k^P - s_{wP}f^{''}(k^P)k_c^P],$$

$$J_{21} = \frac{1}{1+n}[s_c f^{''}(k^P)k_c^P],$$

$$J_{22} = \frac{1}{1+n}[1-\delta+s_c(f^{'}(k^P) + f^{''}(k^P)k_c^P)].$$

After some transformations we obtain the *trace* of the Jacobian matrix at the Pasinetti-equilibrium

$$T(k_w^P, k_c^P) = \frac{n+\delta}{1+n}[\frac{2(1-\delta)}{n+\delta} + 1 + e_{f'}(k^P) + (\frac{s_{wP}e_f(k^P) - s_{ww}e_{f'}(k^P)}{s_c e_f(k^P)})],$$

and the *determinant* of the Jacobian matrix at the Pasinetti-equilibrium

$$D(k_w^P, k_c^P) = T(k_w^P, k_c^P)(\frac{1-\delta}{1+n}) - (\frac{1-\delta}{1+n})^2 + \frac{e_{f'}(k^P)(s_{wP} - s_{ww}) + s_{wP}}{s_c}(\frac{n+\delta}{1+n})^2.$$

For two-dimensional discrete time maps, to search the region of stability of Pasinetti-equilibrium and to study how here frontier is crossed, we will apply the following three conditions:

(1) $1 + T(k_w^P, k_c^P) + D(k_w^P, k_c^P) > 0$;

(2) $1 - T(k_w^P, k_c^P) + D(k_w^P, k_c^P) > 0$;

(3) $1 - D(k_w^P, k_c^P) > 0$.

The previous relations in the plane *trace-determinant* lead to *the triangle of stability* and they guarantee that the modulus of each eigenvalue of the Jacobian matrix, calculated at the Pasinetti-equilibrium, is less than one. From the characteristic equation we derive the eigenvalues of the Jacobian matrix evaluated at an equilibrium point. For the Pasinetti-equilibrium we have:

$\lambda_i^P = \frac{1}{2}(T(k_w^P, k_c^P) \pm \sqrt{(T(k_w^P, k_c^P))^2 - 4D(k_w^P, k_c^P)})$, where $i = 1, 2$.

Commendatore (2005), rewriting the stability conditions in terms of $e_f(k)$ and $e_{f'}(k)$, deduces very interesting relations.

Setting

$$e_{f'}^F = -2(\frac{1+n}{n+\delta})\frac{(n+2-\delta)s_c + (n+\delta)s_{wP}}{(n+2-\delta)(s_c - s_{ww}\frac{1}{e_f(k)}) + (n+\delta)(s_{wP} - s_{ww})},$$

and

$$\overline{e}_f = \frac{s_{ww}(n+2-\delta) - (s_{wp} - s_{ww})(n+\delta)}{s_c(n+2-\delta)},$$

from (1), after some transformations, we obtains the first relations:

- $e_{f'}(k) > e^F_{f'}$ if $e_f(k) > \overline{e}_f$;
- $e_{f'}(k) < e^F_{f'}$ if $e_f(k) < \overline{e}_f$.

In the $(e_f(k)), e_{f'}(k))$-plane the former inequality is satisfied by points which are above the diagram of $e^F_{f'}$ and at left of the right-line $e_f(k) = \overline{e}_f$. Analogously we will think for the last inequality. Moreover the condition (2) always holds if $e_f(k) < \overline{e}_f$ and it reduces to relation $e_f > e^T_f$.

We pose

$$e^N_{f'} = \frac{(s_c - s_{wP})(1+n)}{(s_{wP} - s_{ww})(n+\delta) + (1-\delta)(s_c - s_{ww}\frac{1}{e_f(k)})},$$

and

$$\overline{\overline{e}}_f = \frac{s_{ww}}{s_c + (s_{wP} - s_{ww})\frac{n+\delta}{1-\delta}}.$$

We have that the condition (3) is equivalent to the inequalities

- $e_{f'}(k) < e^N_{f'}$ for $e_f(k) > \overline{\overline{e}}_f$;
- $e_{f'}(k) > e^N_{f'}$ for $e_f(k) < \overline{\overline{e}}_f$.

We note that:

- $e^F_{f'}$ depends on $e_f \neq e_0$, where $e_0 = \frac{(n+2-\delta)s_{ww}}{(n+\delta)(s_{wP} - s_{ww}) + (n+2-\delta)s_c}$;
- $e^F_{f'}$ is continuous and monotonically strictly increasing in $X =]0, e_0[\cup]e_0, 1]$;
- $e^F_{f'}$ is never vanish in X;
- $\lim_{e_f \to e_0} e^F_{f'} = \infty$: in the $(e_f, e^F_{f'})$-plane the straight-line $e_f = e_0$ is an asymptote for $e^F_{f'}$;
- $\lim_{e_f \to 0} e^F_{f'} = 0$;

- $\lim_{e_f \to 1} e^F_{f'} = -2(\frac{1+n}{n+\delta})\frac{(n+2-\delta)s_c+(n+\delta)s_{wP}}{(n+2-\delta)(s_c-s_{ww})+(n+\delta)(s_{wP}-s_{ww})}$;
- $\lim_{e_f \to e^T_f} e^F_{f'} = -\frac{(n+2-\delta)s_c+(n+\delta)s_{wP}}{(n+\delta)(s_{wP}-s_{ww})} \begin{cases} < 0 & \text{if } s_{wP} > s_{ww}, \\ > 0 & \text{if } s_{wP} < s_{ww}; \end{cases}$
- by the **Theorem about Sign Permanence** the function $e^F_{f'}$ has constant sign on both convexes $]0, e_0[$ and $]e_0, 1]$, particularly $e^F_{f'}$ is positive on the left of e_0 and negative on the right of e_0. Moreover the *test-point* e^T_f lies on the left of e_0 if $s_{wP} < s_{ww}$ and on the right of e_0 if $s_{wP} > s_{ww}$.

Analogously for $e^N_{f'}$ we may say that:

- $e^N_{f'}$ depends on $e_f \neq e_{00}$, where $e_{00} = \frac{(1-\delta)s_{ww}}{(n+\delta)(s_{wP}-s_{ww})+(1-\delta)s_c}$;
- $e^N_{f'}$ is continuous and monotonically strictly decreasing in $X =]0, e_{00}[\cup]e_{00}, 1]$;
- $e^N_{f'}$ is never vanish in X;
- $\lim_{e_f \to e_0} e^N_{f'} = \infty$: in the $(e_f, e^N_{f'})$-plane the straight-line $e_f = e_{00}$ is an asymptote for $e^N_{f'}$;
- $\lim_{e_f \to 0} e^N_{f'} = 0$;
- $\lim_{e_f \to 1} e^N_{f'} = \frac{(s_c-s_{wP})(1+n)}{(s_{wP}-s_{ww})(n+\delta)+(1-\delta)(s_c-s_{ww})}$;
- $\lim_{e_f \to e^T_f} e^N_{f'} = \frac{s_c-s_{ww}}{s_{wP}-s_{ww}} \begin{cases} < 0 & \text{if } s_{wP} < s_{ww}, \\ > 0 & \text{if } s_{wP} > s_{ww}; \end{cases}$
- by the **Theorem about Sign Permanence** the function $e^N_{f'}$ has constant sign on both convexes $]0, e_{00}[$ and $]e_{00}, 1]$, particularly $e^N_{f'}$ is negative on the left of e_{00} and positive on the right of e_{00}. Moreover the *test-point* e^T_f lies on the left of e_{00} if $s_{wP} < s_{ww}$ and on the right of e_{00} if $s_{wP} > s_{ww}$.

5.5.2 The Jacobian matrix evaluated at a dual equilibrium

Setting $k^D_c = 0$ we calculate the Jacobian matrix at a dual equilibrium we obtain

$$J(k^D_w, k^D_c) = \begin{pmatrix} \frac{1}{1+n}[1-\delta+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D)] & \frac{1}{1+n}(s_{wP}-s_{ww})f^{''}(k^D)k^D \\ 0 & \frac{1}{1+n}(1-\delta+s_cf^{'}(k^D)) \end{pmatrix}.$$

Since the Jacobian matrix $J(k_w^D, k_c^D)$ is a diagonal matrix on $\Re$, then *the eigenvalues λ_1^D and λ_2^D are real and they correspond to diagonal elements of the matrix $J(k_w^D, k_c^D)$*. Therefore *the dual equilibrium can't lose stability through a Neimark-Saker bifurcation.* We recall that the dual equilibrium is stable if $-1 < \lambda_1^D < 1$ and $-1 < \lambda_2^D < 1$. The expression of eigenvalues depends on saving propensities s_{ww} and s_{wp} and that lead us to distinguish three cases:

- **Case I**: $s_{ww} = s_{wp}$. The eigenvalues become $\lambda_1^D = \frac{1}{1+n}[1-\delta+s_{wp}f^{'}(k^D)]$ and $\lambda_2^D = \frac{1}{1+n}[1-\delta+s_c f^{'}(k^D)]$. Since $f^{'}(k^D) > 0$ we deduce that *both eigenvalues are positive.* By the assumption $s_{wp} < s_c$ we obtain that $\lambda_1^D < \lambda_2^D$. Thus the stability conditions for dual equilibrium reduces to relation $\lambda_2^D < 1$, which holds for $k^D > k^P$. As a matter of fact, the inequality $\lambda_2^D < 1$ is equivalent to relation $\frac{1}{1+n}[1-\delta+s_{wp}f^{'}(k^D)] < 1$, from which we have firstly $f'(k^D) < \frac{n+\delta}{s_c}$ and secondly, by $f^{'}(k^P) = \frac{n+\delta}{s_c}$, $f^{'}(k^D) < f^{'}(k^P)$. Finally, by the property $f^{''}(k) < 0$ of CES production function, we deduce $k^D > k^P$. Commendatore (2005) explains the last inequality saying that *a stability loss involves a transcritical bifurcation which goes in the opposite direction to the one that concerns the Pasinetti equilibrium. Now, it is the dual equilibrium which loses stability and the Pasinetti equilibrium, already existing, that gains stability.*

- **Case II**: $s_{ww} < s_{wp}$. Since $f^{''}(k^D) < 0$ we notice that the term $(s_{wP} - s_{ww})f^{''}(k^D)k^D$ of eigenvalue λ_1^D is negative and λ_1^D could be itself negative. Everyone $\lambda_2^D > 0$ and $\lambda_2^D > max\{\lambda_1^D, 0\}$. Thinking as above, we deduce that $\lambda_2^D < 1$ for $k^P > k^D$. Moreover from inequality $\lambda_1^D > -1$ we obtain the following equivalent relations

  $$\frac{1}{1+n}[1-\delta+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D)] > -1,$$

  $$1-\delta+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D) > -1-n,$$

  $$(2+n-\delta)+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D) > 0,$$

  $$\frac{2+n-\delta}{f^{'}(k^D)} + (s_{wP}-s_{ww})\frac{f^{''}(k^D)k^D}{f^{'}(k^D)} + s_{wP} > 0,$$

  $$\frac{s_{wP}+\frac{2+n-\delta}{f^{'}(k^D)}}{s_{wP}-s_{ww}} + e_{f'}(k^D) > 0,$$

  $$e_{f'}(k^D) > \epsilon_F < -1,$$

  where $\epsilon_F = -\frac{s_{wP}+\frac{2+n-\delta}{f^{'}(k^D)}}{s_{wP}-s_{ww}}$.

We observe that *the stability of dual equilibrium may be lost through a transcritical bifurcation when* λ_2^D *crosses 1 or through a flip bifurcation when* $\lambda_1^D crosses -1$.

- **Case III**: $s_{ww} > s_{wp}$. We notice immediately that both eigenvalues are positive. As a matter of fact is sufficient to observe that the term $(s_{wP} - s_{ww})f^{''}(k^D)k^D$ of λ_1^D is positive. Moreover $\lambda_2^D < 1$ for $k^D > k^P$ and $\lambda_2^D < 1$ for $e_{f'}(k^D) > \epsilon^S < 0$, where

$$\epsilon^S = -\frac{\frac{n+\delta}{f'(k^D)} - s_{wP}}{s_{ww} - s_{wP}}.$$

We conclude that *the dual equilibrium may lose stability through a saddle-node (fold or tangent) bifurcation and two equilibria of dual type are created, one stable and the other unstable.*

5.5.3 The Jacobian matrix evaluated at a trivial equilibrium

We recall that if $f(k)$ is the CES production function then $f^{'}(0) = (1-\alpha)^{\frac{1}{\rho}}$, where $0 < \alpha < 1$ and $\rho < 1$ ($\rho \neq 0$), i.e. $0 < f^{'}(0) < \infty$. By definition of trivial equilibrium we have

$$J(k_w^0, k_c^0) = \begin{pmatrix} \frac{1}{1+n}(1 - \delta + s_{wP}f^{'}(0)) & 0 \\ 0 & \frac{1}{1+n}(1 - \delta + s_c f^{'}(0)) \end{pmatrix}.$$

Since the Jacobian matrix $J(k_w^0, k_c^0)$ is an upper triangular matrix on $\Re$, then the eigenvalues λ_1^0 and λ_2^0 are real and lie along the principal diagonal of the matrix $J(k_w^0, k_c^0)$. If we assume $s_{wp} < s_c$, we get $0 < \lambda_1^0 < \lambda_2^0$. Therefore the stability of trivial equilibrium depends on the inequality $\lambda_2^0 < 1$, i.e. $f^{'}(0) < \frac{n+\delta}{s_c}$. We recall that $f^{'}(k^P) = \frac{n+\delta}{s_c}$ and $f^{''}(k) < 0$. Then we derive the relation $k^P < 0 = k^0$, that can't occur. Thus *the trivial equilibrium is never stable.*

6 Conclusions

We conclude observing that Commendatore's model generalizes Böhm and Kaas (2000) model and Solow (1956) model. As a matter of fact

- setting $s_{ww} = s_{wP}$ and $k = k_w = k_c$ in (5.1)

  $G(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k + s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w + s_c f^{'}(k_c)],$

  we have the (4.1), i.e. from Commendatore's model we deduce Böhm and Kaas (2000) model;

- setting $s_w = s_r$ in (4.1)

  $k_{t+1} = G(k_t) = \frac{1}{1+n}((1-\delta)k_t + s_w w(k_t) + s_r k_t f^{'}(k_t)),$

  we obtain the (2.2), i.e. from Böhm and Kaas (2000) model we deduce the Solow (1956) model.

Appendix 1: Basic Concepts on the Family of Logistic Maps

The notion of logistic map plays a central role in many economic dynamic models with chaos, particularly in the Day's model (1982, 1983). We define the logistic map setting $f(x) = ax(1-x)$, where $a \geq 0$ and $x \in \Re$, and we find the fixed points of $f(x)$ solving the equation $ax(1-x) = x$. We obtain the product $x[(a-1) - ax] = 0$ that leads to solutions $x = 0$ and $x = (a-1)/a$ $(a \neq 1)$. We observe that $f'(x) = a - 2ax$ and if we evaluate $f'(x)$ at $x = 0$ and $x = (a-1)/a$ we have $f'(0) = a$ and $f'(a-1)/a) = 2 - a$. Thus we deduce that $x = 0$ is stable if $-1 < a < 1$ and $x = (a-1)/a$ is stable if $1 < a < 3$. If we see the logistic map as a dynamical system, i.e. $x_{t+1} = ax_t(1-x_t)$, where t is a discrete time $(t = 0, 1, \ldots)$, we can say that *if* $-1 < a < 1$ *the attractor* $x = 0$ *have as basin of attraction the set of point between* 0 *and* 1. Following Alligood et al. (1996), about the dynamic of growth of populations, the previous result means that *with small reproduction rates, small populations tend to die out.* Instead for $1 < a < 3$ the point $x = 0$ is unstable and $x = (a-1)/a$ is stable and we can say that *small populations grow to steady-state of* $x = (a-1)/a$ (See **Figure 13**).

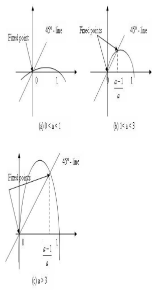

Figure 13: *Logistic Map*

We suppose that $x_t \in [0,1]$, $a \in [0,4]$ and we note that :

- $x_{t+1} = ax_t(1-x_t)$ is a concave quadratic function which maps $[0,1]$ onto itself for all $a \in [0,4]$;
- in the (x_t, x_{t+1})-plane $x_{t+1} = ax_t(1-x_t)$ represents an example of *unimodal map*, i.e. it has an unique point x^* which maximize $f(x_t, a)$, it is smooth and there are two points α and β such that $f(\alpha, a) = 0 = f(\beta, a)$, where $f(x_t, a) = ax_t(1-x_t)$;
- the one-dimensional map $f(x_t, \mu)$ is not invertible because, fixed x_{t+1}, exist two points x_t and $x_{t'}$ such that $x_{t+1} = f(x_t, a) = f(x_{t'}, a)$.

From the assumptions on a and x_t we deduce that

- $f^{'}(x_t, a) = a(1-x_t) - ax_t = 0$ if and only if $x^* = \frac{1}{2}$;
- $f(\frac{1}{2}, a) = \frac{a}{4} \leq (4)(\frac{1}{4}) \leq 1$.

The trajectories of dynamical system x_{t+1} depend on the value of a. As a matter of fact x_{t+1} presents (R.H. Day, 1982)

- monotonic contraction to 0 if $0 < a \leq 1$;
- monotonic growth converging to $x = \frac{a-1}{a}$ if $1 < a \leq 2$;

- oscillations converging to $x = \frac{a-1}{a}$ if $2 < a \leq 3$;
- continued oscillations if $3 < a \leq 4$.

Appendix 2: The Li-Yorke Theorem

In 1975 Li and Jorke published a work entitled "*Period three implies chaos* " which has collected favor among economists "because its simplicity as it requires only checking the existence of a period-3 orbit in order to deduce the existence of "chaos"" one-dimensional (Boldrin-Woodford (1990, 1992)). We simply stating the Li-Yorke theorem and refer to the original work for a demonstration (See **Figure 14**).

Theorem of Li-Yorke Let J be an interval in $\Re$ and let $f : J \to J$ be a continuous map. We consider the difference equation

$$x_{t+1} = f(x_t) \ (\star)$$

and we admit there exists a point $x \in J$ such that

$$f^3(x) \leq x < f(x) < f^2(x).$$

Then

- For every $k = 1, 2, 3, \ldots$, there exists a k-periodic solution such that $x_t \in J$ for all t.
- There is a countable set (containing no periodic points) $S \subset J$ for every $x_0 \in J$ the solution path of difference equation $(\star)$ remains in S and
  - for all $x, y \in S$, $x \neq y$,

    $\limsup_{t\to\infty} |f^t(x) - f^t(y)| > 0$, $\liminf_{t\to\infty} |f^t(x) - f^t(y)| = 0$;
  - for all periodic points x and all points $y \in S$,

    $\limsup_{t\to\infty} |f^t(x) - f^t(y)| > 0$.

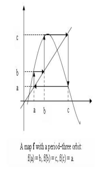

Figure 14: *A map with a period three orbit*

Appendix 3: A CES Production Function

We define *CES Production Function* , where the term *CES* stands for *Constant Elasticity of Substitution*, the following function

$$f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}},$$

being k the capital/labor ratio, $0 < \alpha < 1$ a constant, $-\infty < \rho < 1$ and $\rho \neq 0$ a parameter.

The main features of CES production function $f(k)$ are:

1. $f'(k) > 0$ for all $k \geq 0$ (*i.e. f(k) is increasing*);
2. $f''(k) < 0$ for all $k \geq 0$ (*i.e. f(k) is concave*);
3. $\lim_{\rho \to 0} f(k) = k^{1-\alpha}$ (*i.e. when* ρ *tends towards* 0 *the CES behaves as a Cobb-Douglas*);
4. $\lim_{\rho \to -\infty} f(k) = min\{1, k\} = \begin{cases} k, & \text{if } 0 < k < 1 \\ 1, & \text{if } k \geq 1 \end{cases}$;
5. $\lim_{\rho \to 1} f(k) = \alpha + (\alpha - 1)k$;

6. $0 < f'(0) < \infty$.

As a matter of fact:

- $f'(k) = \frac{1}{\rho}[\alpha + (1-\alpha)k^\rho]^{\frac{1}{\rho}-1}\rho(1-\alpha)k^{\rho-1}$

  $= (1-\alpha)k^{\rho-1}[\alpha + (1-\alpha)k^\rho]^{\frac{1}{\rho}-1}$

  $= (1-\alpha)k^{\rho-1}k^{1-\rho}[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}}$

  $= (1-\alpha)[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}} > 0$;

- $f''(k) = (1-\alpha)\frac{1-\rho}{\rho}[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}-1}(-\rho\alpha k^{-\rho-1})$

  $= \alpha(1-\alpha)(\rho-1)k^{-\rho-1}[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-2\rho}{\rho}} < 0$;

- $\lim_{\rho\to 0} f(k) = \lim_{\rho\to 0} e^{\frac{\ln[\alpha+(1-\alpha)k^\rho]}{\rho}} = \lim_{\rho\to 0} e^{\frac{(1-\alpha)k^\rho \ln k}{\alpha+(1-\alpha)k^\rho}}$

  $= \lim_{\rho\to 0} e^{\ln k^{1-\alpha}} = k^{1-\alpha}$;

- Because $\lim_{\rho\to-\infty} k^\rho$ is equal to 0 if $k > 1$ and it is equal to ∞ if $0 < k < 1$, then

  $\lim_{\rho\to-\infty} f(k) = \lim_{\rho\to-\infty} e^{\frac{\ln[\alpha+(1-\alpha)k^\rho]}{\rho}}$

  is equal to $e^0 = 1$ if $k > 1$ while it is equal to $e^{\ln k} = k$ if $0 < k < 1$.

Let $f(k)$ be a production function in intensive form. We set $e_f(k) = \frac{kf'(k)}{f(k)}$ and $e_{f'}(k) = \frac{kf''(k)}{f'(k)}$. If $f(k)$ is a CES production function we obtain that $e_f(k) = (1-\alpha)(\alpha k^{-\rho} + 1 - \alpha)^{-1}$ and $e_{f'}(k) = \alpha(\rho-1)[\alpha + (1-\alpha)k^\rho]^{-1}$. As a matter of fact

- $e_f(k) = \frac{f'(k)k}{f(k)} = \frac{(1-\alpha)[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1-\rho}{\rho}}k}{[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1}{\rho}}k} = (1-\alpha)[\alpha k^{-\rho} + (1-\alpha)]^{-1}$;

- $e_{f'}(k) = \frac{kf''(k)}{f'(k)} = \frac{\alpha(1-\alpha)(\rho-1)k^{-\rho-1}[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1-2\rho}{\rho}}k}{(1-\alpha)[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1-\rho}{\rho}}}$

  $= \alpha(\rho-1)k^{-\rho}[\alpha k^{-\rho} + (1-\alpha)]^{-1}$

  $= \alpha(\rho-1)k^{-\rho}k^{\rho}[\alpha + (1-\alpha)k^{\rho}]^{-1}$

  $= \alpha(\rho-1)[\alpha + (1-\alpha)k^{\rho}]^{-1}$.

Obviously, $e_{f'}(k) < 0$ for all $\rho < 1$ ($\rho \neq 0$) and for all $k \geq 0$.

Developing an observation of Commendatore (2005, p.16) we establish that (See **Figure 15** and **Figure 16**)

Proposition A3.1 If $f(k)$ is the CES production function then the inequality

$$e_{f'}(k) > -1$$

is true always for all $0 < \rho < 1$ and for all $k \geq 0$; while if $\rho < 0$ the inequality is verified only for those $k \in]0, k^{\star}[$, where $k^{\star} = (\frac{\alpha\rho}{\alpha-1})^{\frac{1}{\rho}}$ and $e_{f'}(k^{\star}) = -1$.

Proof Let $0 < \alpha < 1$ be. We observe that:

- $\frac{de_{f'}(k)}{dk} = \frac{\alpha\rho(\rho-1)(\alpha-1)k^{\rho-1}}{[\alpha+(1-\alpha)k^{\rho}]^2}$;
- $e_{f'}(k)$ is strictly increasing if $0 < \rho < 1$ and is strictly decreasing if $\rho < 0$;
- $\lim_{k\to 0} e_{f'}(k) = \begin{cases} (\rho-1) & \text{if } 0 < \rho < 1, \\ 0 & \text{if } \rho < 0; \end{cases}$
- $\lim_{k\to +\infty} e_{f'}(k) = \begin{cases} 0 & \text{if } 0 < \rho < 1, \\ (\rho-1) & \text{if } \rho < 0. \end{cases}$

Being $e_{f'}(k)$ continuous on the interval $]0, +\infty[$, by *Bolzano's Theorem* [1], the range J of $e_{f'}(k)$ is an interval, and, by *Theorem about limits of monotonically functions*[2], J is equal to $](\rho-1), 0[$ for all $\rho < 1$ ($\varrho \neq 0$).

[1] Let $g : X \subseteq \Re \to \Re$ be. If g is continuous on X and X is an interval, then $g(X)$ is an interval.(For a proof of the Bolzano's Theorem see Vincenzo Aversa (2006))

[2] Let $g : X \subseteq \Re \to \Re$ be. We suppose that $infX$ and $supX$ are points of accumulation for X. Then,

Now we consider $0 < \rho < 1$. Since $-1 < \rho - 1 = inf\{e_{f'}(k) : k \geq 0\} \leq e_{f'}(k)$, we obtain that $e_{f'}(k) > -1$.

After we fix $\rho < 0$ and we solve the equation $e_{f'}(k) = -1$. We have as an unique solution $k^\star = (\frac{\alpha\rho}{\alpha-1})^{\frac{1}{\rho}}$. Being $e_{f'}(k)$ strictly decreasing, for all $0 < k < k^\star$, $e_{f'}(k) > e_{f'}(k^\star) = -1$. Q.E.D.

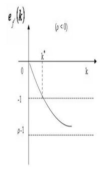

Figure 15: *The case $\rho < 0$*

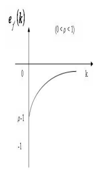

Figure 16: *The case $\rho < 1$*

- for $x \to infX$, $g(x) \to inf(g(X))$ if g is monotonically increasing, otherwise $g(x) \to sup(g(X))$ if g is monotonically decreasing;
- for $x \to supX$, $g(x) \to sup(g(X))$ if g is monotonically increasing, otherwise $g(x) \to inf(g(X))$ if g is monotonically decreasing.

(See Vincenzo Aversa (2006))

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