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# Monotone Comparative Statics for Games with Strategic Substitutes

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# Monotone Comparative Statics for Games With Strategic Substitutes

**Abstract**

This paper studies comparative statics of equilibria in models where the optimal responses under consideration are (weakly) decreasing in endogenous variables, and (weakly) increasing in exogenous parameters. Such models include parameterized games of strategic substitutes. The analysis provides a sufficient condition for existence of increasing equilibria at a higher parameter value. This condition is presented first for best-response functions; it can be translated easily to payoff functions with one-dimensional individual strategy spaces, and it has a natural analogue to best-response correspondences. The condition is tight in the sense that with a weakenened condition, the same result may not obtain. The results here apply to asymmetric equilibria, and are applied to two classes of examples – Cournot duopoly and tournaments. Moreover, sufficient conditions are presented to exhibit strong comparative statics of equilibria (that is, every equilibrium at a higher parameter value is greater than a given equilibrium at a lower parameter value), and to show existence of increasing equilibrium selections.

# 1  Introduction

As is well-known, games with strategic complements and strategic substitutes are found in many areas of economics.[2] Although comparative statics results for general games with strategic complements are well-developed,[3] results of similar generality are less commonly available for games with strategic substitutes, or in games in which best-response functions under consideration are weakly decreasing in endogenous variables.

For example, we are not aware of a general result for such games that can be applied to show increasing equilibria in a simple, parameterized, asymmetric, Cournot duopoly with linear demand, constant marginal cost, and the standard product order on strategy spaces, such as the following. Consider a linear inverse market demand curve given by $p = a - b(x_1 + x_2)$, where $x_1$ is the output of firm 1, and $x_2$ of firm 2. Suppose each firm has constant marginal cost $c$. Moreover, there is a subsidy of $t \leq c$ per unit, and this subsidy is split with

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[2]Such games are defined in Bulow, Geanakoplos, and Klemperer (1985), and as they show, models of strategic investment, entry deterrence, technological innovation, dumping in international trade, natural resource extraction, business portfolio selection, and others can be viewed in a more unifying framework according as the variables under consideration are strategic complements or strategic substitutes. Additional classes of examples are provided by Cournot oligopolies, bargaining games (Nash demand game), and as described in Dubey, Haimanko, and Zapechelnyuk (2006), include games of team projects with complementary or substitutable tasks, and tournaments.

[3]Some of this work can be seen in Topkis (1979), Lippman, Mamer, and McCardle (1987), Sobel (1988), Vives (1990), Milgrom and Roberts (1990), Zhou (1994), Milgrom and Shannon (1994), Milgrom and Roberts (1994), Shannon (1995), Villas-Boas (1997), Edlin and Shannon (1998), Echenique (2002), and Echenique and Sabarwal (2003), among others. Extensive bibliographies are available in Topkis (1998) and in Vives (1999).

an exogenously specified share $\frac{3}{5}$ for firm 1, and share $\frac{2}{5}$ for firm 2.[4] Thus, firm 1's marginal cost net of subsidy is $c - \frac{3}{5}t$, and that of firm 2 is $c - \frac{2}{5}t$. In this case, the unique equilibrium is given by $x^*(t) \equiv (x_1^*(t), x_2^*(t)) = (\frac{a-c+(\frac{9}{5}-1)t}{3b}, \frac{a-c+(2-\frac{9}{5})t}{3b})$.

With the standard product order on strategy spaces, this example does not fit the framework of Milgrom and Shannon (1994), because the profit functions are not quasi-supermodular.[5] Therefore, this game is not supermodular, and this example does not fit the framework of Topkis (1979), Sobel (1988), or Vives (1990). If the order on one of the strategy spaces is reversed, then it is known (see, for example, Milgrom and Shannon (1994), and a detailed application in Amir (1996)) that this example is a quasi-supermodular game. Of course, this does not imply that equilibria are increasing or weakly increasing in the original product order in $t$.[6] Moreover, asymmetric Cournot conditions rule out an application of Amir and Lambson (2000), and of the intersection point theorem of Tarski (1955).[7]

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[4]Alternatively, the parameter $t$ can be thought of as technological improvement, and $(\frac{3}{5}, \frac{2}{5})$ can be thought of as differential adaptation of technological improvement. A slightly more general example is presented later.

[5]Denote profit of firm 1 at $(x_1, x_2, t)$ by $f^1(x_1, x_2, t)$, and consider the values $a = 10, b = 1, c = 1, t = 0$, and consider $(x_1, x_2) = (3, 2)$, and $(x_1', x_2') = (4, 3)$. Then, $f^1(x_1', x_2, t) \geq f^1(x_1, x_2, t)$, but $f^1(x_1', x_2', t) < f^1(x_1, x_2', t)$.

[6]Indeed, as shown below, with slight variations in this game, it is easy to have the equilibrium strategy of either player increasing and that of the other player decreasing.

[7]Tarski's intersection point theorem applies to linearly ordered spaces. It is noteworthy that one trick that can work for the special duopoly case is to compose the reaction functions of the two firms. This yields an increasing function. In this case, an equilibrium can be shown to exist, and at least for one of the players, equilibrium can be shown to be increasing, but (in asymmetric Cournot) not necessarily for the other player. Indeed, as shown below, it is easy to formulate examples of simple Cournot duopolies where the equilibrium is increasing for one player, and decreasing for the other. The same point applies to techniques that apply when the best-response of one player depends only on the aggregate best-response of other players. Of

One result is available for games where best-responses of endogenous variables are weakly decreasing. As shown by Villas-Boas (1997), in such games, equilibria do not decrease when the exogenous parameter increases.[8]

This paper considers parameterized models in which best-responses under consideration are weakly decreasing in endogenous variables, and weakly increasing in parameters.[9] The analysis provides a sufficient condition for existence of increasing equilibria at a higher parameter value. This condition is presented first for best-response functions; it can be translated easily to payoff functions with one-dimensional individual strategy spaces, and it has a natural analogue to best-response correspondences. The condition applies to asymmetric equilibria.

Moreover, sufficient conditions are also presented to exhibit strong comparative statics of equilibria (that is, every equilibrium at a higher parameter value is greater than a given equilibrium at a lower parameter value), and to show existence of increasing equilibrium selections.

The condition on best-responses applies to games with strategic substitutes, finite number of players, finite-dimensional strategy spaces, continuous best-response functions, and

course, such techniques have been formulated primarily to prove existence theorems for Cournot oligopolies, and not necessarily to show increasing equilibria. See, for example, Selten (1970), Roberts and Sonnenschien (1976), Bamon and Fraysee (1985), Novshek (1985), Kukushkin (1994), and Amir (1996), and additional discussion in Vives (1999).

[8]Additionally, some aspects of non-monotone mappings that are increasing in some variables and decreasing in others are explored in Roy (2002). Also confer Roy and Sabarwal (2007/8) for another view of the result of Villas-Boas.

[9]This class of models includes those in which endogenous variables are strategic substitutes for each other.

partially ordered parameter spaces. The condition is tight in the sense that with a weakened condition, the same result may not obtain. The results apply to asymmetric equilibria, and are applied to two classes of examples – Cournot duopoly and tournaments, as described in Dubey, Haimanko, and Zapechelnyuk (2006).

Intuitively, in games of strategic substitutes, there are two opposing effects of an increase in the parameter value. The direct effect increases each player's best-response, but strategic substitutes imply that an increase in the best-response of other players has an additional indirect and opposite effect on each player's best-response. At a new parameter value, if this indirect effect does not dominate the direct effect, then a larger equilibrium exists.[10] The condition presented in this paper identifies a measure of this combined effect.

A limitation of the paper is that the results apply mainly to best-responses, and not to payoff functions. This work highlights that an appropriate measure of the combined direct and indirect effects on the best-responses plays a central role in identifying monotone comparative statics. One translation of this effect to payoff functions is pointed out, and a relationship of this effect to the standard concepts of quasi-supermodularity and single crossing property is explored. At present, however, a full identification of properties on payoff functions is unavailable.

Notice that as shown by Villas-Boas (1997), in the case of a Cournot oligopoly, when a new partial order can be chosen as well, then there exists a new partial order in which equilibria are increasing. The new partial order, however, might not necessarily be intuitive or relevant for natural parametric policy experiments. For example, for a Cournot oligopoly, the product order may be natural when considering the impact of taxes or subsidies on firm output, and

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[10]Notably, in games with strategic complements, both effects work in the same direction.

the existence of some other partial order under which equilibria are increasing might not be relevant. In games of strategic complements, the product order is used commonly for the same reason; that is, to investigate the impact of different parameters on each agent's choice. The results here apply to cases where a partial order is considered as fixed.

The paper proceeds as follows. Section 2 presents a sufficient condition on continuous best-response functions for the existence of increasing equilibria when these functions are weakly decreasing in endogenous variables, and weakly increasing in parameters, and applies this condition to two classes of examples. A translation to payoff functions with real-valued individual strategy spaces is presented. Section 3 presents three additional propositions based on the result in Section 2. The first proposition identifies a sufficient condition for strong comparative statics (that is, every equilibrium at a higher parameter value is greater than a given equilibrium at a lower parameter value). The second proposition identifies a condition for existence of increasing equilibrium selections. The third proposition extends the main theorem to best-response correspondences.

## 2    Existence of Increasing Equilibria

This section considers models in which the best-response functions under consideration are weakly decreasing in endogenous variables, and weakly increasing in parameters.

The model space for endogenous variables is assumed to be a partially ordered, non-empty, compact, convex subset of Euclidean space, denoted $(X, \preceq)$. The space for exogenous parameters is assumed to be a partially ordered set, denoted $T$.[11] An **admissible family**

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[11]When no confusion arises, the same symbol $\preceq$ denotes the partial order on $T$.

*of functions* is a function $g : X \times T \to X$ such that for every $t$, the function $g(\cdot, t)$ is weakly decreasing[12] and continous,[13] and for every $x$, the function $g(x, \cdot)$ is weakly increasing.[14] For each $t$, let $FP(t) = \{x \in X \mid x = g(x, t)\}$ be the fixed points of $g$ at $t$. Brouwer's theorem implies that for every $t$, $FP(t)$ is non-empty.

Notice that for a $N$-player game with strategic substitutes, the product of the best-response functions of the players satisfies the weakly decreasing property.

To develop a better understanding of the general result, it is helpful to view it explicitly first in the special case of a game with two agents, each with a decreasing best-response function, each with a one-dimensional strategy space, and with the partial order determined by the product order. This case is considered below, and for additional insight, in this case, a direct proof is provided as well.

Consider a game with two agents, indexed $i = 1, 2$. Agent $i$'s action space is a non-empty, compact, convex interval $X^i$ of the real numbers, and there is a partially ordered parameter space $T$. Agent $i$'s best-response function is $g^i : X^j \times T \to X^i$, with $i \neq j$. For each $i$ and $t$, suppose that $g^i(\cdot, t)$ is strictly decreasing, and for each $i$, and for each $x_j \in X^j$, suppose that $g^i(x_j, \cdot)$ is strictly increasing.[15] Let $X = X^1 \times X^2$, and with the product order (denoted $\leq$). Suppose $g(x_1, x_2, t) \equiv (g^1(x_2, t), g^2(x_1, t))$ is a continuous function in $(x_1, x_2)$, and let $FP(t)$ be the set of fixed points of $g$ at $t$.[16]

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[12] For every $x, y \in X$, $x \preceq y$, $\Rightarrow g(y, t) \preceq g(x, t)$.

[13] The assumption of continuity is made to guarantee existence of equilibrium via Brower's theorem.

[14] For every $t \preceq \hat{t}$, $\Rightarrow g(x, t) \preceq g(x, \hat{t})$.

[15] Note that for the general case considered in the paper, only weakly decreasing in endogenous variables and weakly increasing in parameters are required.

[16] For reference, notice that this class of games allows for multiple equilibria, and allows for agent conditions to be asymmetric.

**Proposition 1.** *Fix $t^* \in T$, let $x^* = (x_1^*, x_2^*) \in FP(t^*)$, and consider $\hat{t} \in T$ with $t^* \preceq \hat{t}$. Let*

$(\hat{y}_1, \hat{y}_2) = (g^1(x_2^*, \hat{t}), g^2(x_1^*, \hat{t}))$, *and let* $(\hat{x}_1, \hat{x}_2) = (g^1(\hat{y}_2, \hat{t}), g^2(\hat{y}_1, \hat{t}))$.

*If $x_1^* \leq \hat{x}_1$ and $x_2^* \leq \hat{x}_2$, then there is $\hat{x}^* = (\hat{x}_1^*, \hat{x}_2^*) \in FP(\hat{t})$ such that $x^* \leq \hat{x}^*$.*

**Proof.** [17] For notational convenience, in this proof only, for $i = 1, 2$, let $g_t^i(\cdot) \equiv g^i(\cdot, t)$, and let $g_t(\cdot) \equiv g(\cdot, t)$. Notice that $\hat{x}_1 \leq \hat{y}_1$, because $(g_{\hat{t}}^1)^{-1}$ is decreasing, and

$$(g_{\hat{t}}^1)^{-1}(\hat{y}_1) = x_2^* = g_{t^*}^2(x_1^*) \leq g_{\hat{t}}^2(x_1^*) = \hat{y}_2 = (g_{\hat{t}}^1)^{-1}(\hat{x}_1),$$

where the inequality follows from $g_t^2(x_1^*)$ is increasing in $t$. Moreover, notice that $(g_{\hat{t}}^1)^{-1}$ is defined on $[\hat{x}_1, \hat{y}_1]$.

The first condition in the hypothesis, along with $g_{\hat{t}}^2$ is decreasing implies that

$$g_{\hat{t}}^2(\hat{x}_1) \leq g_{\hat{t}}^2(x_1^*) = \hat{y}_2 = (g_{\hat{t}}^1)^{-1}(\hat{x}_1),$$

and the second condition in the hypothesis implies that

$$g_{\hat{t}}^2(\hat{y}_1) = \hat{x}_2 \geq x_2^* = (g_{\hat{t}}^1)^{-1}(\hat{y}_1).$$

Therefore, by the intermediate value theorem, there is $\hat{x}_1^* \in [\hat{x}_1, \hat{y}_1]$ such that $g_{\hat{t}}^2(\hat{x}_1^*) = (g_{\hat{t}}^1)^{-1}(\hat{x}_1^*)$. Let $\hat{x}_2^* = (g_{\hat{t}}^1)^{-1}(\hat{x}_1^*)$. Then $\hat{x}^* = (\hat{x}_1^*, \hat{x}_2^*) \in FP(\hat{t})$. Finally, notice that $x_1^* \leq \hat{x}_1 \leq \hat{x}_1^*$, and moreover, $\hat{x}_1^* \leq \hat{y}_1$ implies that $\hat{x}_2^* = (g_{\hat{t}}^1)^{-1}(\hat{x}_1^*) \geq (g_{\hat{t}}^1)^{-1}(\hat{y}_1) = x_2^*$, whence $x^* \leq \hat{x}^*$.

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The conditions in this proposition can be viewed explicitly, as follows. Starting from an existing equilibrium, $x^* = (x_1^*, x_2^*)$ at $t = t^*$, an increase in $t$ has two effects on $g^2(\cdot, \cdot)$. One effect is an increase in $g^2$, because best-response functions are increasing in $t$. (This is a

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[17]We are grateful to an anonymous referee for this version of the proof.

direct effect of an increase in $t$.) The other effect is a decrease in $g^2$, because an increase in $t$ increases $g^1(x_2^*, t)$, and $x_1$ and $x_2$ are strategic substitutes. (This is an indirect effect arising from the response of player 1 to an increase in $t$.) Similar statements are valid for player 1 as well. Taken together, the conditions in the theorem say that for each player, as long as the indirect strategic substitute effect does not dominate the direct parameter effect, there is a new equilibrium that is larger than $x^* = (x_1^*, x_2^*)$. A simple graphical illustration of these conditions is presented in figure 1.
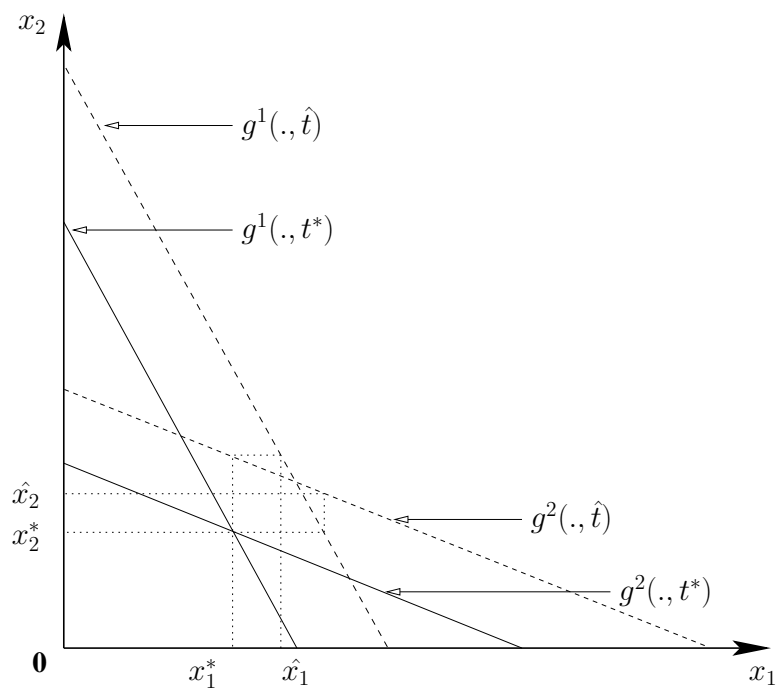


Figure 1: Existence of Increasing Equilibria

It is useful to note that if either condition is not satisfied, this result may not necessarily apply. This can be seen in the following simple example, and graphically in figure 2, where the first condition is violated but the second condition is satisfied. An alternative figure can be constructed similarly where the reverse is true.
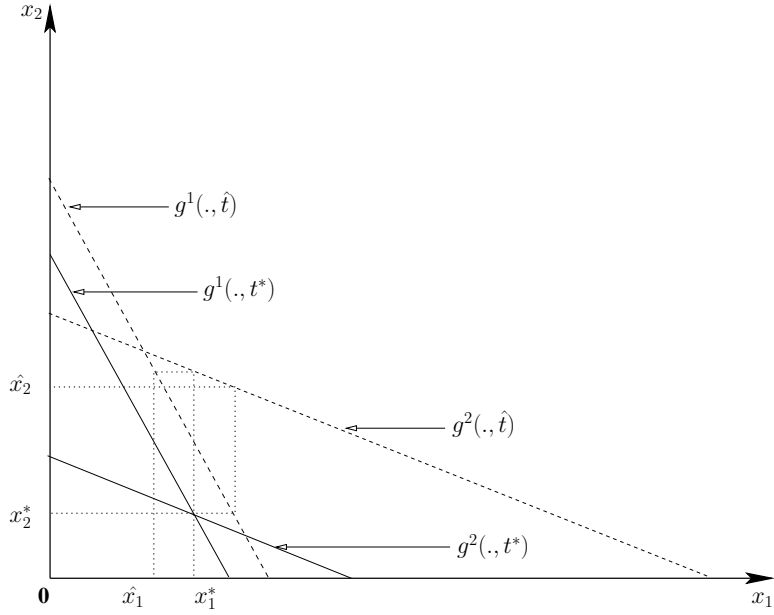
8

Figure 2: Violation of first condition

**Example 1.** Consider a standard Cournot duopoly with a linear inverse market demand curve given by $p = a - b(x_1 + x_2)$, where $x_1$ is output of firm 1, and $x_2$ of firm 2. Suppose each firm has constant marginal cost $c$. Moreover, there is a subsidy of $t \leq c$ per unit, and this subsidy is split with share $\xi \in [0,1]$ for firm 1, and share $1 - \xi$ for firm 2.[18] Thus, firm 1's marginal cost net of subsidy is $c - \xi t$, and that of firm 2 is $c - (1 - \xi)t$. In this case, the best-response function of firm 1 is $g^1(x_2, t) = \frac{a - c + \xi t - b x_2}{2b}$, and that of firm 2 is $g^2(x_1, t) = \frac{a - c + (1 - \xi)t - b x_1}{2b}$. It is easy to check that $g(x_1, x_2, t) \equiv (g^1(x_2, t), g^2(x_1, t))$ is a strictly decreasing function in $(x_1, x_2)$, it is strictly increasing in $t$, and the unique equilibrium

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[18]The example in the introduction is the case where $\xi = \frac{3}{5}$.

at $t$ is $x^*(t) \equiv (x_1^*(t), x_2^*(t)) = (\frac{a-c+(3\xi-1)t}{3b}, \frac{a-c+(2-3\xi)t}{3b})$. Consequently,

$$\xi < \tfrac{1}{3} \qquad \Leftrightarrow \quad x_1^*(t) \text{ is decreasing in } t, \text{ and } x_2^*(t) \text{ is increasing in } t,$$

$$\tfrac{1}{3} \leq \xi \leq \tfrac{2}{3} \quad \Leftrightarrow \quad x_1^*(t) \text{ is increasing in } t, \text{ and } x_2^*(t) \text{ is increasing in } t, \text{ and}$$

$$\tfrac{2}{3} < \xi \qquad \Leftrightarrow \quad x_1^*(t) \text{ is increasing in } t, \text{ and } x_2^*(t) \text{ is decreasing in } t.$$

Moreover, for $t^* \leq \hat{t}$, $\hat{x}_2 = g^2(g^1(x_2^*(t^*), \hat{t}), \hat{t}) = \frac{1}{12b}[4(a-c) + (6-9\xi)(\hat{t}-t^*) + (8-12\xi)t^*]$,

whence for $t^* \leq \hat{t}$, $x_2^*(t^*) \leq \hat{x}_2 \Leftrightarrow \xi \leq \tfrac{2}{3}$. Similarly, for $t^* \leq \hat{t}$, $\hat{x}_1 = g^1(g^2(x_1^*(t^*), \hat{t}), \hat{t}) =$

$\frac{1}{12b}[4(a-c) + (9\xi-3)(\hat{t}-t^*) + (12\xi-4)t^*]$, whence for $t^* \leq \hat{t}$, $x_1^*(t^*) \leq \hat{x}_1 \Leftrightarrow \xi \geq \tfrac{1}{3}$. Thus,

if $\xi < \tfrac{1}{3}$, then the first condition is violated, but the second is satisfied, and if $\xi > \tfrac{2}{3}$, then

the first condition is satisfied, but the second is violated.

A similar tradeoff between direct and indirect effects is useful in proving a more general

theorem.

**Theorem 1.** *Let $g : X \times T \to X$ be an admissible family of functions. Fix $t^* \in T$, and let*

*$x^* \in FP(t^*)$. Consider $\hat{t} \in T$ such that $t^* \preceq \hat{t}$, and let $\hat{y} = g(x^*, \hat{t})$, and $\hat{x} = g(\hat{y}, \hat{t})$.*

*If $x^* \preceq \hat{x}$, then there is $\hat{x}^* \in FP(\hat{t})$ such that $x^* \preceq \hat{x}^*$.*

**Proof.** Notice that $x^* \preceq \hat{y}$, because $g$ is weakly increasing in $t$. Moreover, for every $x$

in $[x^*, \hat{y}]$, $g(x, \hat{t}) \in [x^*, \hat{y}]$, and this can be seen as follows. Suppose $x^* \preceq x \preceq \hat{y}$. Then

$x \preceq \hat{y}$ implies that $g(x, \hat{t}) \succeq g(\hat{y}, \hat{t}) \succeq x^*$, where the first inequality follows from the fact

that $g(\cdot, \hat{t})$ is weakly decreasing, and the second follows from the condition in the theorem.

Moreover, $x^* \preceq x$ implies that $g(x, \hat{t}) \preceq g(x^*, \hat{t}) = \hat{y}$, where the inequality follows from weakly

decreasing $g(\cdot, \hat{t})$, and the equality follows from definition of $\hat{y}$. Therefore, the restriction of

$g(\cdot, \hat{t})$ to $[x^*, \hat{y}]$ is a map from $[x^*, \hat{y}]$ to $[x^*, \hat{y}]$. By Brouwer's theorem, there is $\hat{x}^* \in [x^*, \hat{y}]$

such that $g(\hat{x}^*, \hat{t}) = \hat{x}^*$, and consequently, there is $\hat{x}^* \in FP(\hat{t})$ such that $x^* \preceq \hat{x}^*$. ∎

Notice that for the special case considered in proposition 1, the conditions here specialize to those in proposition 1. The intuition for the general case is the same as for the special case.

The following example uses the condition in the previous theorem to exhibit increasing equilibria in cases where equilibria may be asymmetric, and might not necessarily be computable analytically.

**Example 2.** Consider games of tournaments.[19] Suppose a tournament has 3 players, where a parameterized reward $r(t)$ (with $0 \leq t \leq T$, and $r'(t) > 0$)[20] is shared by the players who succeed in the tournament. If one player succeeds, he gets $r(t)$ for sure, if two players succeed, each gets $r(t)$ with probability one-half, and if all players succeed, each gets $r(t)$ with probability one-third. Expected reward for player $i$ is

$$r(t)x_i(1 - x_j)(1 - x_k) + \frac{r(t)}{2}x_ix_j(1 - x_k) + \frac{r(t)}{2}x_ix_k(1 - x_j) + \frac{r(t)}{3}x_ix_jx_k.$$

The quadratic cost of effort $x_i$ is $\frac{c_i}{2}x_i^2$, and is allowed to be asymmetric across players. The payoff to player $i$ is expected reward minus cost of effort. It is easy to calculate that the best-response of player $i$ is

$$g^i(x_{-i}, t) = \frac{r(t)}{c_i}(1 - \frac{1}{2}(x_j + x_k) + \frac{1}{3}x_jx_k),$$

and this best-response is decreasing in other player actions, and increasing in $t$. Let the equilibrium at $t = t^*$ be given by $(x_1^*, x_2^*, x_3^*)$. Then it is easy to check that $g^i(x_{-i}^*, t) = \frac{r(t)}{r(t^*)}x_i^*$.

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[19]The version used here is the one presented in Dubey, Haimanko, and Zapechelnyuk (2006).

[20]As shown below, the best-response function depends on $\frac{r(t)}{c_i}$, where $c_i$ measures player $i$'s costs, and therefore, $r(t)$ can be viewed as a relative reward enhancement parameter, relative to a player's costs.

Consequently, for player $i$,

$$g^i(g^j(x^*_{-j}, t), g^k(x^*_{-k}, t), t) = \frac{r(t)}{c_i}(1 - \frac{1}{2}\frac{r(t)}{r(t^*)}(x^*_j + x^*_k) + \frac{1}{3}(\frac{r(t)}{r(t^*)})^2 x^*_j x^*_k).$$

When viewed as a function of $t$, the derivative of this function evaluated at $t^*$ is

$$\frac{r'(t^*)}{c_i}(1 - (x^*_j + x^*_k) + x^*_j x^*_k) = \frac{r'(t^*)}{c_i}(1 - x^*_j)(1 - x^*_k).$$

Given the domain restriction of strategies to the unit interval, this expression is positive exactly when both $x^*_j < 1$ and $x^*_k < 1$. Similar results hold for players $j$ and $k$. Consequently, if the equilibrium is not degenerate, (that is, no player wins the tournament for sure,) then the equilibrium increases with the parameter.

The idea of this example is that if an estimate of an equilibrium is available (perhaps because we observe a particular equilibrium under given economic conditions), then it can be concluded whether an increase in economic conditions will increase the equilibrium. Similar applications of the theorem can be made to other games when an estimate of an equilibrium is available, and best-response functions are computable. In particular, an application of this theorem does not require that best-response functions have analytically closed forms. Therefore, from a practical point of view, this theorem can have broader applications.

Notice that in the particular case of real-valued strategies, the condition in the theorem translates to a condition on payoff functions, as follows. For player $i = 1$, the condition in the theorem can be written as

$$\left. \frac{\partial}{\partial t} \left( g^1(g^2(x_{-2}, t), g^3(x_{-3}, t), \ldots, g^N(x_{-N}, t), t) \right) \right|_{(x^*, t^*)} \geq 0,$$

and for each player $i = 2, \ldots, N$, a similar condition with an appropriate change of index.[21]

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[21]Of course, to apply this version, we suppose that the derivative is well-defined; in particular, $(x^*, t^*)$ is in the interior.

Using the implicit function theorem, it is easy to calculate that

$$\left.\frac{\partial}{\partial t}\left(g^1(g^2(x_{-2},t),\ldots,g^N(x_{-N},t),t)\right)\right|_{(x^*,t^*)} \geq 0 \quad \Leftrightarrow \quad \left.f^1_{1,t}+\sum_{n=2}^{N}f^1_{1,n}\left(-\frac{f^n_{n,t}}{f^n_{n,n}}\right)\right|_{(x^*,t^*)} \geq 0.$$

Here, a superscript on a payoff function $f$ indexes a player, and subscripts denote partial derivatives. Thus, for example, $f^i_{i,n}=\frac{\partial^2 f^i}{\partial x_i \partial x_n}$, and $f^i_{i,t}=\frac{\partial^2 f^i}{\partial x_i \partial t}$.[22]

With an appropriate change of index, it follows that if for every $i=1,\ldots,N$,

$$\left.f^i_{i,t}+\sum_{n\neq i}f^i_{i,n}\left(-\frac{f^n_{n,t}}{f^n_{n,n}}\right)\right|_{(x^*,t^*)} \geq 0,$$

then the sufficient condition in the theorem is satisfied.

These conditions are a natural analogue to those in the proposition and theorem above. For a given player $i$, the term $f^i_{i,t}$ is positive, and captures the direct effect of an increase in $t$ on $i$'s marginal profits, $f^i_i$. The term $\sum_{n\neq i}f^i_{i,n}\left(-\frac{f^n_{n,t}}{f^n_{n,n}}\right)$ is negative, and captures the indirect effect of an increase in $t$. This indirect effect can be viewed as the sum of indirect effects, one for each competitor player $n\neq i$, where each competitor player's effect is given by $f^i_{i,n}\left(-\frac{f^n_{n,t}}{f^n_{n,n}}\right)$. In this last expression, $\left(-\frac{f^n_{n,t}}{f^n_{n,n}}\right)$ measures the (positive) change in player $n$'s best-response from an increase in $t$, and $f^i_{i,n}$ measures the (negative) change in $i$'s marginal profits from an increase in $n$'s best-response. As earlier, if the indirect effect does not dominate the direct effect, an increasing equilibrium is guaranteed.

This condition on payoff functions can be applied to the two examples considered earlier to yield the same results. Details are provided in the appendix.

The idea of competing direct and indirect effects helps relate the conditions here to those

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[22]Notice that in a parameterized game of strategic substitutes, for $n\neq i$, $f^i_{i,n}<0$ formalizes strategic substitutes, and for $n=i$, $f^i_{i,n}<0$ formalizes strict concavity in own argument. Moreover, $f^i_{i,t}>0$ formalizes increasing differences in $t$.

that arise in models with strategic complements. In those models, the direct and indirect effects work in the same direction, and therefore, once a parameter increases, both effects serve to move the new equilibrium set higher. Moreover, in those models, once increasing equilibria have been demonstrated, additionally higher parameter values serve to increase equilibria further, and do not reverse any increases. When direct and indirect effects work in opposite directions, increasing equilibria are no longer guaranteed. Moreover, even when the tradeoff between indirect and direct effects implies a larger equilibrium at a higher parameter value, that tradeoff might not necessarily hold at additionally higher parameter values, and therefore, a demonstration of a favorable tradeoff at a parameter value does not necessarily imply increasing equilibria at additionally higher parameter values.

# 3   Remarks and Extensions

This section presents three additional propositions based on ideas related to theorem 1. The first proposition identifies a sufficient condition for strong comparative statics (that is, every equilibrium at a higher parameter value is greater than a given equilibrium at a lower parameter value). The second proposition identifies a condition for existence of increasing equilibrium selections. The third proposition extends theorem 1 to best-response correspondences.

## 3.1   Strong Comparative Statics

This subsection presents a condition that strengthens the result in the previous section to show that all equilibria at $\hat{t}$ are greater than $x^*$.

**Proposition 2.** *Let $g$ be an admissible family of functions, fix $t^* \preceq \hat{t}$, and let $x^* \in FP(t^*)$.*

*Consider $h(x, \hat{t}) = g(g(x, \hat{t}), \hat{t})$, and let $\hat{x}_L = inf_X \left\{ x | h(x, \hat{t}) \preceq x \right\}$.*

*If $x^* \preceq \hat{x}_L$, then for every $\quad \hat{x}^* \in FP(\hat{t}), \quad x^* \preceq \hat{x}^*$, and*

*If $x^* \prec \hat{x}_L$, then for every $\quad \hat{x}^* \in FP(\hat{t}), \quad x^* \prec \hat{x}^*$.*

**Proof**. Notice that $h$ is weakly increasing in $x$, and therefore, by Tarski's theorem, $\hat{x}_L$ exists, and is the smallest fixed point of $h$ at $\hat{t}$. Moreover, the set of fixed points of $h$ at $\hat{t}$ is a complete lattice. The result now follows by noting that the set of fixed points of $g$ at $\hat{t}$ is a subset of the set of fixed points of $h$ at $\hat{t}$. ∎

Another condition that guarantees the conclusion of the second statement (strictly increasing equilibria) in this proposition is the following.

If $x^* \preceq \hat{x}_L$, and if $FP(\hat{t})$ is not a singleton, then for every $\hat{x}^* \in FP(\hat{t}), x^* \prec \hat{x}^*$.

To prove this statement, we use results in Dacic (1981), and Roy and Sabarwal (2007/8), which imply that in games with strategic substitutes, the set of fixed points is completely unordered; that is, no two elements are comparable. Therefore, if $x^* \preceq \hat{x}_L$, and if $FP(\hat{t})$ is not a singleton, then $x^* \notin FP(\hat{t})$, and the result follows.

## 3.2 Existence of Increasing Selections

This subsection presents a sufficient condition for existence of increasing equilibrium selections.

Suppose a game has $N \geq 2$ players, the payoff function of player $i$ is given by $f^i(x_i, x_{-i}, t)$, and her best-response function is given by $g^i(x_{-i}, t)$. Suppose that each $g^i$ is weakly decreasing and continuous in $x_{-i}$, and weakly increasing in $t$. (Thus, the product of best-responses

is an admissible family of functions.) Fix $\bar{x} \in X$ arbitrarily, and consider the function,

$\tilde{f}^i(x_i, t) = f^i(x_i, g_{-i}(\bar{x}, t), t)$, where $g_{-i}(\bar{x}, t)$ is the combined best-response of the non-$i$ players at $(\bar{x}, t)$.

**Proposition 3.** *Suppose for each $\bar{x} \in X$, and for each $i = 1, \ldots, N$, $\tilde{f}^i$ is quasi-supermodular in $x_i$ and satisfies the single crossing property in $(x_i, t)$. Then for every weakly increasing sequence $(t_n)$ in $T$, there is an increasing selection from $\{FP(t_n)\}$.*

**Proof**. If $\tilde{f}^i(x_i, t)$ is quasi-supermodular in $x_i$ and satisfies the single crossing property in $(x_i, t)$, then by Milgrom and Shannon (1994), $g^i(g_{-i}(\bar{x}, t), t) = \text{argmax}_{x_i} \tilde{f}^i(x_i, t)$ is increasing in $t$. In particular, when we fix $t^* \preceq \hat{t}$ arbitrarily, and consider a fixed point $x^*$ at $t^*$, then setting $\bar{x} = x^*$ shows that for $t^* \preceq \hat{t}$, $x_i^* = g^i(g_{-i}(x^*, t^*), t^*) \preceq g^i(g_{-i}(x^*, \hat{t}), \hat{t})$, thereby satisfying the condition in Theorem 1, and yielding a higher equilibrium at $\hat{t}$. Therefore, for every weakly increasing sequence $(t_n)$ in $T$, there is an increasing selection from $\{FP(t_n)\}$. $\blacksquare$

Notice that a limitation of this proposition is that its application requires knowledge of best-responses. On the other hand, an advantage is that knowledge of an equilibrium is not required.

## 3.3    An Extension to Correspondences

This subsection extends theorem 1 to the case of correspondences.[23]

Suppose the model space for endogenous variables is a partially ordered, non-empty, complete, sub-lattice of Euclidean space, denoted $(X, \preceq)$. Denote the induced set order

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[23]We are grateful to an anonymous referee for pointing out this extension. This subsection uses standard lattice terminology. See, for example, Topkis (1998).

on $P(X) \setminus \{\emptyset\}$ by $\sqsubseteq$.[24] The space for exogenous parameters is assumed to be a partially ordered set, denoted $T$.[25] An ***admissible family of correspondences*** is a correspondence $g : X \times T \twoheadrightarrow X$ such that for every $t$, $g(\cdot, t)$ is weakly decreasing,[26] upper hemi-continuous, non-empty valued, and convex-subcomplete-sublattice valued, and for every $x$, the function $g(x, \cdot)$ is weakly increasing.[27] For each $t$, let $FP(t) = \{x \in X \mid x \in g(x, t)\}$ be the fixed points of $g$ at $t$. Kakutani's theorem implies that for every $t$, $FP(t)$ is non-empty.

Notice that for a $N$-player game with strategic substitutes, the product of the best-response correspondences of the players satisfies the weakly decreasing property.

**Proposition 4.** *Let $g : X \times T \twoheadrightarrow X$ be an admissible family of correspondences. Fix $t^* \in T$, and let $x^* \in FP(t^*)$. Consider $\hat{t} \in T$ such that $t^* \preceq \hat{t}$, and let $\hat{y} = \sup_X g(x^*, \hat{t})$. If $x^* \preceq \inf_X g(\hat{y}, \hat{t})$, then there is $\hat{x}^* \in FP(\hat{t})$ such that $x^* \preceq \hat{x}^*$.*

**Proof**. Notice that $x^* \preceq \hat{y}$, because $g$ is weakly increasing in $t$, (hence $g(x^*, t^*) \sqsubseteq g(x^*, \hat{t})$,) $x^* \in g(x^*, t^*)$, and $\sup g(x^*, t^*) \preceq \hat{y}$. Moreover, for every $x$ in $[x^*, \hat{y}]$, $g(x, \hat{t}) \subset [x^*, \hat{y}]$, and this can be seen as follows. Suppose $x^* \preceq x \preceq \hat{y}$. Then $x \preceq \hat{y}$ implies that $\inf g(x, \hat{t}) \succeq \inf g(\hat{y}, \hat{t}) \succeq x^*$, where the first inequality follows from the fact that $g(\cdot, \hat{t})$ is weakly decreasing with respect to $\sqsubseteq$, and the second follows from the condition in the proposition. Moreover, $x^* \preceq x$ implies that $\sup g(x, \hat{t}) \preceq \sup g(x^*, \hat{t}) = \hat{y}$, where the inequality follows from weakly decreasing $g(\cdot, \hat{t})$, and the equality follows from definition of $\hat{y}$. Therefore, the restriction of $g(\cdot, \hat{t})$ to $[x^*, \hat{y}]$ is a map from $[x^*, \hat{y}]$ to $[x^*, \hat{y}]$. In $\Re^N$, a sublattice is complete if, and

---

[24]For non-empty subsets $A, B$ of $X$, $A \sqsubseteq B$ if for every $a \in A$, and for every $b \in B$, $a \wedge b \in A$, and $a \vee b \in B$, where the operations $\wedge, \vee$ are with respect to $\preceq$.

[25]When no confusion arises, the same symbol $\preceq$ denotes the partial order on $T$.

[26]For every $x, y \in X$, $x \preceq y$, $\Rightarrow g(y, t) \sqsubseteq g(x, t)$.

[27]For every $t \preceq \hat{t}$, $\Rightarrow g(x, t) \sqsubseteq g(x, \hat{t})$.

only if, it is compact, and therefore, by Kakutani's theorem, there is $\hat{x}^* \in [x^*, \hat{y}]$ such that

$\hat{x}^* \in g(\hat{x}^*, \hat{t})$. Consequently, there is $\hat{x}^* \in FP(\hat{t})$ such that $x^* \preceq \hat{x}^*$. ∎

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# Appendix

The condition on payoff functions presented in Section 2 above is applied to the two examples considered earlier.

**Example 1 (continued).** Consider the Cournot duopoly example. The profit of each firm is given by

$$f^1(x_1, x_2, t) = (a - b(x_1 + x_2))x_1 - (c - \xi t)x_1, \text{ and}$$
$$f^2(x_1, x_2, t) = (a - b(x_1 + x_2))x_2 - (c - (1 - \xi)t)x_2.$$

Therefore,

$$f^1_{1,2} = -b, \quad f^1_{1,t} = \xi, \quad f^2_{2,2} = -2b, \quad f^2_{2,t} = 1 - \xi,$$

and consequently,

$$f^1_{1,t} + f^1_{1,2}\left(-\frac{f^2_{2,t}}{f^2_{2,2}}\right)\bigg|_{(x^*,t^*)} > 0 \iff \xi > \tfrac{1}{3}, \text{ and}$$
$$f^2_{2,t} + f^2_{1,2}\left(-\frac{f^1_{1,t}}{f^1_{1,1}}\right)\bigg|_{(x^*,t^*)} > 0 \iff \xi < \tfrac{2}{3}.$$

It follows that equilibria increase in the range $\xi \in (\tfrac{1}{3}, \tfrac{2}{3})$, as earlier.

**Example 2 (continued).** Consider the tournaments example. The payoff to player $i$ is given by expected reward minus quadratic cost of effort $x_i$. That is,

$$f^i(x_i, x_j, x_k) = r(t)\pi^i(x_i, x_j, x_k) - \frac{c_i}{2}x_i^2,$$

where the expected reward per unit is given by

$$\pi^i(x_i, x_j, x_k) = x_i(1 - x_j)(1 - x_k) + \frac{1}{2}x_i x_j(1 - x_k) + \frac{1}{2}x_i x_k(1 - x_j) + \frac{1}{3}x_i x_j x_k.$$

Notice that

$$
\begin{aligned}
f^i_{i,j} &= r(t)\pi^i_{i,j}(x_k), & f^i_{i,k} &= r(t)\pi^i_{i,k}(x_j), & f^i_{i,t} &= r'(t)\pi^i_i(x_j, x_k), \\
-\frac{f^j_{j,t}}{f^j_{j,j}} &= \frac{r'(t)\pi^j_j(x_i, x_k)}{c_j}, & -\frac{f^k_{k,t}}{f^k_{k,k}} &= \frac{r'(t)\pi^k_k(x_i, x_j)}{c_k}, \\
x^*_i &= \frac{r(t^*)\pi^i_i(x^*_j, x^*_k)}{c_i}, & x^*_j &= \frac{r(t^*)\pi^j_j(x^*_i, x^*_k)}{c_j}, & x^*_k &= \frac{r(t^*)\pi^k_k(x^*_i, x^*_j)}{c_k}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& f^i_{i,t} + f^i_{i,j}(-\tfrac{f^j_{j,t}}{f^j_{j,j}}) + f^i_{i,k}(-\tfrac{f^k_{k,t}}{f^k_{k,k}})\bigg|_{(x^*,t^*)} > 0 \\
\iff & r'(t)\pi^i_i(x_j, x_k) + r'(t)\pi^i_{i,j}(x_k)x_j + r'(t)\pi^i_{i,k}(x_j)x_k\big|_{(x^*,t^*)} > 0 \\
\iff & \pi^i_i(x^*_j, x^*_k) + \pi^i_{i,j}(x^*_k)x^*_j + \pi^i_{i,k}(x^*_j)x^*_k > 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \pi^i_i(x^*_j, x^*_k) + \pi^i_{i,j}(x^*_k)x^*_j + \pi^i_{i,k}(x^*_j)x^*_k \\
= & (1 - x^*_j)(1 - x^*_k) + \tfrac{1}{2}\left(x^*_j(1 - x^*_k) + x^*_k(1 - x^*_j)\right) + \tfrac{1}{3}x^*_j x^*_k \\
& + \left[\tfrac{1}{3}x^*_k - \tfrac{1}{2}((1 - x^*_k) + x^*_k)\right]x^*_j + \left[\tfrac{1}{3}x^*_j - \tfrac{1}{2}((1 - x^*_j) + x^*_j)\right]x^*_k \\
= & (1 - x^*_j)(1 - x^*_k),
\end{aligned}
$$

implies that the condition on payoff functions is satisfied, if each of $x_i^*, x_j^*, x_k^*$ is less than 1. In other words, if the equilibrium is not degenerate, that is, no player wins the tournament for sure, then the equilibrium increases with the parameter, as earlier. The result extends to $N$ player tournaments. The notationally intensive details for that case are provided below. Additional examples of games where these results may apply can be found in Dubey, Haimanko, and Zapechelnyuk (2006), for example, games of team projects with substitutable tasks.

**Example 2 (continued).** Suppose a tournament has $N \geq 2$ players, where a parameterized reward $r(t)$ (with $0 \leq t \leq T$, and $r'(t) > 0$) is shared by the players who succeed in the tournament. If one player succeeds, she gets $r(t)$ for sure, if two players succeed, each gets $r(t)$ with probability one-half, and if all players succeed, each gets $r(t)$ with probability one-third. For player $i = 1$, the expected reward per unit is

$$
\begin{aligned}
\pi^1(x_1, \ldots, x_N) = \quad & x_1 \prod_{i_1 \in \{2, \ldots, N\}} (1 - x_{i_1}) \\
+ \quad & \tfrac{1}{2} \sum_{i_1=2}^{N} x_1 x_{i_1} \prod_{i_2 \in \{2, \ldots, N\} \setminus \{i_1\}} (1 - x_{i_2}) \\
+ \quad & \tfrac{1}{3} \sum_{i_1=2}^{N} \sum_{i_2=i_1+1}^{N} x_1 x_{i_1} x_{i_2} \prod_{i_3 \in \{2, \ldots, N\} \setminus \{i_1, i_2\}} (1 - x_{i_3}) \\
+ \quad & \tfrac{1}{4} \sum_{i_1=2}^{N} \sum_{i_2=i_1+1}^{N} \sum_{i_3=i_2+1}^{N} x_1 x_{i_1} x_{i_2} x_{i_3} \prod_{i_4 \in \{2, \ldots, N\} \setminus \{i_1, i_2, i_3\}} (1 - x_{i_4}) \\
+ \quad & \ldots \\
+ \quad & \tfrac{1}{N} x_1 x_2 \cdots x_N,
\end{aligned}
$$

and the expected reward is

$$ r(t)\pi^i(x_1, \ldots, x_N). $$

The quadratic cost of effort $x_1$ is $\frac{c_1}{2}x_1^2$, and it is allowed to be asymmetric across players. The payoff to player 1 is expected reward minus cost of effort. That is,

$$ f^1(x_1, \ldots, x_N) = r(t)\pi^1(x_1, \ldots, x_N) - \frac{c_1}{2}x_1^2. $$

Following the same argument as in the text, it follows that

$$
\left. f^1_{1,t} + \sum_{n=2}^{N} f^1_{1,n} \left( -\frac{f^n_{n,t}}{f^n_{n,n}} \right) \right|_{(x^*, t^*)} > 0
$$
$$
\Leftrightarrow \quad \pi^1_1(x^*_{-1}) + \sum_{n=2}^{N} x^*_n \pi^1_{1,n}(x^*_{-(1,n)}) > 0.
$$

Here, as usual, $x^*_{-(1,n)}$ is the vector $x^*$ without components 1 and $n$. The details below show that

$$
\pi^1_1(x^*_{-1}) + \sum_{n=2}^{N} x^*_n \pi^1_{1,n}(x^*_{-(1,n)}) = \prod_{n=2}^{N} (1 - x^*_n).
$$

A similar result holds for each player $i$, and therefore, it follows that if the equilibrium is not degenerate, then equilibrium increases with the parameter.

**Details.** Notice that

$$
\begin{aligned}
\pi_1^1(x_{-1}^*) = \quad & \prod_{i_1\in\{2,\ldots,N\}}(1-x_{i_1}^*) \\
+ \quad & \tfrac{1}{2}\sum_{i_1=2}^{N} x_{i_1}^* \prod_{i_2\in\{2,\ldots,N\}\setminus\{i_1\}}(1-x_{i_2}^*) \\
+ \quad & \tfrac{1}{3}\sum_{i_1=2}^{N}\sum_{i_2=i_1+1}^{N} x_{i_1}^* x_{i_2}^* \prod_{i_3\in\{2,\ldots,N\}\setminus\{i_1,i_2\}}(1-x_{i_3}^*) \\
+ \quad & \tfrac{1}{4}\sum_{i_1=2}^{N}\sum_{i_2=i_1+1}^{N}\sum_{i_3=i_2+1}^{N} x_{i_1}^* x_{i_2}^* x_{i_3}^* \prod_{i_4\in\{2,\ldots,N\}\setminus\{i_1,i_2,i_3\}}(1-x_{i_4}^*) \\
+ \quad & \ldots \\
+ \quad & \tfrac{1}{N}x_2^* \cdots x_N.
\end{aligned}
$$

Moreover, for each $n=2,\ldots,N$,

$$
\begin{aligned}
\pi_{1,n}^1(x_{-(1,n)}^*) = \quad & -\prod_{i_1\in\{2,\ldots,N\}\setminus\{n\}}(1-x_{i_1}^*) \\
+ \quad & \tfrac{1}{2}\prod_{i_2\in\{2,\ldots,N\}\setminus\{n\}}(1-x_{i_2}^*) \\
- \quad & \tfrac{1}{2}\sum_{i_1\in\{2,\ldots,N\}\setminus\{n\}} x_{i_1}^* \prod_{i_2\in\{2,\ldots,N\}\setminus\{i_1,n\}}(1-x_{i_2}^*) \\
+ \quad & \tfrac{1}{3}\sum_{i_2\in\{2,\ldots,N\}\setminus\{n\}} x_{i_2}^* \prod_{i_3\in\{2,\ldots,N\}\setminus\{i_2,n\}}(1-x_{i_3}^*) \\
- \quad & \tfrac{1}{3}\sum_{i_1\in\{2,\ldots,N\}\setminus\{n\}}\sum_{i_2\in\{i_1+1,\ldots,N\}\setminus\{n\}} x_{i_1}^* x_{i_2}^* \prod_{i_3\in\{2,\ldots,N\}\setminus\{i_1,i_2,n\}}(1-x_{i_3}^*) \\
+ \quad & \tfrac{1}{4}\sum_{i_2\in\{2,\ldots,N\}\setminus\{n\}}\sum_{i_3\in\{i_2+1,\ldots,N\}\setminus\{n\}} x_{i_2}^* x_{i_3}^* \prod_{i_4\in\{2,\ldots,N\}\setminus\{i_2,i_3,n\}}(1-x_{i_4}^*) \\
- \quad & \tfrac{1}{4}\sum_{i_1\in\{2,\ldots,N\}\setminus\{n\}}\sum_{i_2\in\{i_1+1,\ldots,N\}\setminus\{n\}}\sum_{i_3\in\{i_2+1,\ldots,N\}\setminus\{n\}} x_{i_1}^* x_{i_2}^* x_{i_3}^* \prod_{i_4\in\{2,\ldots,N\}\setminus\{i_1,i_2,i_3,n\}}(1-x_{i_4}^*) \\
+ \quad & \ldots \\
+ \quad & \tfrac{1}{N}\prod_{i_1\in\{2,\ldots,N\}\setminus\{n\}} x_{i_1}^*.
\end{aligned}
$$

Using the above expression for $\pi_{1,n}^1(x_{-(1,n)}^*)$, notice that the first term in $x_n^*\pi_{1,n}^1(x_{-(1,n)}^*)$, the one that has a coefficient of $-1$, is

$$
-x_n^* \prod_{i_1\in\{2,\ldots,N\}\setminus\{n\}}(1-x_{i_1}^*),
$$

and therefore, the sum of such terms, as $n$ varies over $2,\ldots,N$, can be written as

$$
-\sum_{i_1\in\{2,\ldots,N\}} x_{i_1}^* \prod_{i_2\in\{2,\ldots,N\}\setminus\{i_1\}}(1-x_{i_2}^*).
$$

Similarly, notice that the term in $x_n^*\pi_{1,n}^1(x_{-(1,n)}^*)$ that has a coefficient of $+\tfrac{1}{2}$ is

$$
\frac{1}{2}x_n^* \prod_{i_2\in\{2,\ldots,N\}\setminus\{n\}}(1-x_{i_2}^*),
$$

23

and therefore, the sum of such terms, as $n$ varies over $2, \ldots, N$, can be written as

$$\frac{1}{2} \sum_{i_1 \in \{2,\ldots,N\}} x_{i_1}^* \prod_{i_2 \in \{2,\ldots,N\} \setminus \{i_1\}} (1 - x_{i_2}^*).$$

Now notice that the sum of the previous two sums cancels the second term in the expression for $\pi_1^1(x_{-1}^*)$, the one with coefficient $\frac{1}{2}$.

Similarly, it can be calculated that the term in $x_n^* \pi_{1,n}^1(x_{-(1,n)}^*)$ that has a coefficient of $-\frac{1}{2}$, when summed as $n$ varies over $2, \ldots, N$, can be written as

$$-\frac{1}{2} \sum_{n=2}^{N} \sum_{i_1 \in \{2,\ldots,N\} \setminus \{n\}} x_n^* x_{i_1}^* \prod_{i_2 \in \{2,\ldots,N\} \setminus \{i_1,n\}} (1 - x_{i_2}^*)$$

$$= -\sum_{i_1=2}^{N} \sum_{i_2=i_1+1}^{N} x_{i_1}^* x_{i_2}^* \prod_{i_3 \in \{2,\ldots,N\} \setminus \{i_1,i_2\}} (1 - x_{i_3}^*),$$

where the equality follows from adding terms that appear exactly two times. In the same manner, the term in $x_n^* \pi_{1,n}^1(x_{-(1,n)}^*)$ that has a coefficient of $+\frac{1}{3}$, when summed as $n$ varies over $2, \ldots, N$, can be written, after adding terms that appear exactly two times, as

$$\frac{2}{3} \sum_{i_1=2}^{N} \sum_{i_2=i_1+1}^{N} x_{i_1}^* x_{i_2}^* \prod_{i_3 \in \{2,\ldots,N\} \setminus \{i_1,i_2\}} (1 - x_{i_3}^*).$$

Notice again that the sum of the last two double sums cancels the third term in the expression for $\pi_1^1(x_{-1}^*)$, the one with coefficient $\frac{1}{3}$. Similarly, it can be concluded that

$$\pi_1^1(x_{-1}^*) + \sum_{n=2}^{N} x_n^* \pi_{1,n}^1(x_{-(1,n)}^*) = \prod_{n=2}^{N} (1 - x_n^*),$$

as desired.