Characterization of the Generalized Top-Choice Assumption (Smith) set

Andrikopoulos, Athanasios
University of Ioannina

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CHARACTERIZATION OF THE GENERALIZED TOP-CHOICE ASSUMPTION (SMITH) SET

ATHANASIOS ANDRIKOPOULOS

Abstract. In this paper, I give a characterization of the Generalized Top-Choice Assumption set of a binary relation in terms of choice from minimal negative consistent superrelations. This result provides a characterization of Schwart's set in tournaments.

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Key words: Negative Consistency, Generalized Top-Choice Assumption (Smith) set, Generalized Optimal-Choice Axiom (Schwartz) set.

1. Introduction

A fundamental problem of rational choice theory is to determine whether a choice is optimal relative to some preference relation or not. So a rational agent who knows his preference relation, chooses the maximal elements according to this relation, in every feasible set presented for choice. In other words, the optimal choice set of a choice process consists of the maximal alternatives in the feasible set according to the viewpoint of a binary relation. However, when will the set of optimal choices be non-empty? If the feasible set is finite and the binary relation is acyclic, then the set of optimal choices is always non-empty; When the optimal choice set is empty[1], the crucial question which has been arisen, is what to count as a choice. That is, what sets of alternatives may be considered as reasonable solutions? To answer this question, several methods (solution theories) for constructing non-empty choice sets have been proposed. Such a solution is the *Generalized Top-Choice Assumption set* ($\mathcal{GETCHA}$ *set*), introduced by Schwartz in [4], which is a generalization of the optimal choice set. Smith in [5] introduces a generalization of Condorcet Criterion that is satisfied when pairwise election are based on simple majority choices. He uses the notion of *dominant set*, that is, any candidate in this set is collectively preferred to any candidate not into this set. But Smith does not discuss the idea of a smallest dominant set. Fishburn in [3] narrows Smith's generalization of the Condorcet Criterion to the smallest dominant set and calls it *Smith's Condorcet Principle*. Schwartz in [4] discusses the Smith's Condorcet Principle as a possible standard for optimal collective choice and he call it $\mathcal{GETCHA}$.

[1]This problem is common in the analysis of pairwise majority voting, in the choice of a winning sport team, in the aggregation of multiple choice criteria, in committee selection, in the choice under uncertainty, etc.

The $\mathcal{GETCHA}$ set (*Smith set*)[2] is the choice set from a given set specified by the $\mathcal{GETCHA}$ condition. To address the absence of maximal elements, Schwartz [4] gives another general solution for constructing non-empty choice sets which is called *Generalized Optimal-Choice Axiom* ($\mathcal{GOCHA}$). The choice set from a given set specified by the $\mathcal{GOCHA}$ condition is the union of minimal sets which each one of them has the following property: no alternative outside this set is preferable to an alternative inside it.

In this paper, I show that an alternative belongs to the $\mathcal{GETCHA}$ set of a binary relation if and only if it is maximal for a minimal negative consistent superrelation. This result provides a characterization of the $\mathcal{GOCHA}$ set (*Schwartz set*)[3] in tournaments.

2. Notation and definitions

Let X be a non-empty universal set of alternatives, and let $R \subseteq X \times X$ be a binary binary relation on X. We will say that R is a subrelation of R', and R' a superrelation of R, denoted $R \subseteq R'$, when for all $x, y \in X$, xRy implies $xR'y$. The *complement* of R is denoted by R^c, that is for all $x, y \in X, R^c = \{(x, y)|(x, y) \notin R\}$. We sometimes abbreviate $(x, y) \in R$ as xRy. For any $x \in X$, $Rx = \{y \in X|yRx\}$ and $xR = \{y \in X|xRy\}$ denote respectively the *upper contour set* and *lower contour set* of R at x. The *asymmetric part* $P(R)$ of R is given by: $(x, y) \in P(R)$ if and only if $(x, y) \in R$ and $(y, x) \notin R$. $\mathcal{M}(R)$ denote the elements of X that are R-*maximal* in X, i.e., $\mathcal{M}(R) = \{x \in X|$ for all $y \in X, yRx$ implies $xRy\}$. We say that R is *transitive* if for all $x, y, z \in X$, $(x, z) \in R$ and $(z, y) \in R$ implies that $(x, y) \in R$. The *transitive closure* of R is denoted by $\overline{R}$, that is for all $x, y \in X, (x, y) \in \overline{R}$ if there exist $k \in \mathbb{N}$ and $x_0, ..., x_K \in X$ such that $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_{_K} = y$. A subset $Y \subseteq X$ is an R-cycle if, for all $x, y \in Y$, we have $(x, y) \in \overline{R}$ and $(y, x) \in \overline{R}$. We say that R is *acyclic* if there does not exist an R-cycle. Suzumura [6] provides the following definition, which generalize the notions of transitivity and acyclicity: The binary relation R is *consistent*, if for all $x, y \in X$, for all $k \in \mathbb{N}$, and for all $x_0, x_1, ..., x_K \in X$, if $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_K = y$, then $(y, x) \notin P(R)$. If in the definition of consistent binary relation above we replace R with R^c, we get the notion of a *negative consistent* binary relation. As binary relations are subsets of $X \times X$, they are naturally partially ordered by set-inclusion. A *chain*, denoted $\mathcal{C}$, is a class of relations such that $B, B' \in \mathcal{C}$ implies $B \subseteq B'$ or $B' \subseteq B$. A class $\mathcal{B}$ of relations is *closed downward* if, for all chains $\mathcal{C}$ in $\mathcal{B}$, $\bigcap\{B|B \in \mathcal{C}\} \in \mathcal{B}$.

Let Ω be a family of non-empty subsets of X that represents the different feasible sets presented for choice. A choice function is a mapping that assigns to each choice situation a subset of it:

[2]The Smith set also appears in the literature as *weak top cycle*.

[3]The Smith set is also sometimes confused with the Schwartz set because in tournaments (asymmetric and complete binary relations) both sets coincide.

$C : \Omega \to X$ such that for all $A \in \Omega$, $C(A) \subseteq A$.

The traditional choice-theoretic approach takes behavior as rational if it can be explained as the outcome of maximizing a binary relation R. In this direction, best choices can be expressed as the maximization of the individuals's preferences over a set of alternatives. That is, for every $A \in \Omega$, $C(A) = \mathcal{M}(R/A)$ ($\mathcal{M}(R/A)$ denote the elements of X that are R-maximal in A). An $A \in \Omega$ is *R-undominated* iff for no $x \in A$ there is a $y \in X \setminus A$ such that yRx. An R-undominated set is *minimal* if none of its proper subsets has this property. The set A is *R-dominant* if and only if xRy for each $x \in A$ and each $y \in X \setminus A$. An R-dominant set is *minimal* if none of its proper subsets is an R-dominant subset of X. To deal with the case where the set of maximal choices $C(A)$ is empty, Schwartz [4, Definition in page 141] has been proposed the following general solutions:
Generalized Top Optimal-Choice Axiom ($\mathcal{GETCHA}$): For each $A \in \Omega$, $C(A)$ is equivalent to the minimum R-dominated subsets of A.
Generalized Optimal-Choice Axiom ($\mathcal{GOCHA}$): For each $A \in \Omega$, $C(A)$ is equivalent to the union of minimum R-undominated subsets of A. The *$\mathcal{GETCHA}(R)$ set* (resp. *$\mathcal{GOCHA}(R)$ set*) is the choice set from a given set specified by the $\mathcal{GETCHA}$ (resp. $\mathcal{GOCHA}$) condition according to R.

3. Main Result

The main result in this paper establishes a binary characterization of the choices generated by negative consistent superrelations. Let $\mathcal{R}_N$ denote the negative consistent superrelations of R, i.e., $\mathcal{R}_N = \{R \subseteq R_N \mid R_N \text{ is negative consistent}\}$, and let R_{N*} denote the elements of $\mathcal{R}_N$ that are minimal with respect to set inclusion.
Zorn's Lemma: If every chain of a partially ordered set has a lower bound, then E has a minimal element.

In order to prove the main result of this paper, we need the following proposition which is a simplification of the dual version of the definition of Duggan [1, Definition 4] for consistent binary relations.

Proposition 1. A binary relation R is negative consistent if and only if $P(R) \subseteq P((\overline{R^c})^c)$.

Proof. Suppose that R fulfills the definition of negative transitivity and for $x, y \in X$, $(x, y) \in P(R)$. By way of contradiction, we assume that $(x, y) \notin P((\overline{R^c})^c)$. We have two cases: either $(x, y) \notin (\overline{R^c})^c$, or $(x, y) \in (\overline{R^c})^c$ and $(y, x) \in (\overline{R^c})^c$. In the first case we have $(x, y) \in \overline{R^c}$, that is, there exists a natural number n and alternatives $x_1, x_2, ..., x_n \in X$ such that

$$x = x_1 R^c x_2 ... x_{n-1} R^c x_n = y.$$

Thus, negative consistency yields $(y, x) \notin P(R^c)$ which contradicts the hypothesis that $(x, y) \in P(R)$. For the second case, where $(x, y) \notin \overline{R^c}$ and $(y, x) \notin \overline{R^c}$, it follows that $(x, y) \in I(R)$ which leads to a contradiction too.

To see the converse, first suppose $P(R) \subseteq P((\overline{R^c})^c)$ and take n and $x_1, x_2, ..., x_n \in X$ such that

$$x = x_1 R^c x_2 ... x_{n-1} R^c x_n = y.$$

Thus, $(x, y) \in \overline{R^c}$, implying $(x, y) \notin (\overline{R^c})^c$, and by supposition $(x, y) \notin P(R)$. Therefore, $(y, x) \notin P(R^c)$, as required. □

The following proposition is the dual of the Proposition 5 in [1].

Proposition 2. The class of all negative consistent binary relations is closed downward.

Proof. Let $\mathcal{B}$ be the class of all negative consistent binary relations. To prove that $\mathcal{B}$ is closed downward, take a chain $\mathcal{C}$ in $\mathcal{B}$, let $C = \bigcap_{B_i \in \mathcal{C}} B_i$, and take $(x, y) \in P(C)$. We prove that $(x, y) \in P((\overline{C^c})^c)$. We proceed by the way of contradiction, suppose that $(x, y) \notin P((\overline{C^c})^c)$, then there are two cases to consider: (i) $(x, y) \notin (\overline{C^c})^c$; (ii) $(x, y) \in (\overline{C^c})^c$ and $(y, x) \in (\overline{C^c})^c$. In the first case, if $(x, y) \notin (\overline{C^c})^c$, then, $(x, y) \in \overline{C^c}$. Thus, there exist $x_0, x_1, ..., x_K \in X$ such that

$$x = x_0,\ (x_{k-1}, x_k) \in C^c \text{ for all } k \in \{0, ..., K\} \text{ and } x_K = y.$$

But then, for each $k \in \{1, ..., K\}$, there is a $B_k \in \mathcal{C}$ such that $(x_{k-1}, x_k) \in B_k^c$. Since $\mathcal{C}$ is a chain, $\widetilde{\mathcal{B}} = \{B_k | k = 1, 2, ..., K\}$ contains a relation, B_λ, minimum with respect to set-inclusion. Hence,

$$x = x_0 B_\lambda^c x_1 ... x_{K-1} B_\lambda^c x_K = y.$$

On the other hand, since $(x, y) \in P(C)$, there is $B_\mu \in \mathcal{C}$ such that $(x, y) \in P(B_\nu)$ for each $B_\nu \subseteq B_\mu$. We have the following two subcases to consider: (i_a) $B_\lambda \subseteq B_\mu$; ($\mathrm{i}_\beta$) $B_\mu \subseteq B_\lambda$. For ($\mathrm{i}_a$), we have

$$x = x_0 B_\lambda^c x_1 ... x_{K-1} B_\lambda^c x_K = y \text{ and } (x, y) \in P(B_\lambda).$$

Since B_λ is negative consistent, it must be that $(y, x) \notin P(B_\lambda^c)$. Hence, $(y, x) \in B_\lambda$ or $(x, y) \in I(B_\lambda)$, contradicting $(x, y) \in P(B_\lambda)$. Now consider the subcase (i_β). Since $B_\mu \subseteq B_\lambda$, we have

$$x = x_0 B_\mu^c x_1 ... x_{K-1} B_\mu^c x_K = y \text{ and } (x, y) \in P(B_\mu).$$

This is a contradiction as well.

We come now to the second case, that of $(x, y) \in (\overline{C^c})^c$ and $(y, x) \in (\overline{C^c})^c$. In this case, we have $(x, y) \in I(C)$ which contradicts that $(x, y) \in P(B_\mu)$. □

The next two propositions are used in the proof of Theorem 5 below. The proof of the Proposition 3 uses the technique of Lemma 1 in [2]

Proposition 3. Let R be a binary relation on X. For each $x \in X$, there exists a negative consistent superrelation $R_{C(x)} \supseteq R$ such that $\overline{R^c_{C(x)}} x = \overline{R^c} x \setminus \{x\}$.

Proof. Let us define $Y = \overline{R^c}x \cup \{x\}$. Denote by $\mathcal{R}$ be the set of negative consistent superrelations $R_N \subseteq X \times X$ of R which satisfies the following property (c):

(c) For each $z, y \in X$, if $(z, y) \notin R_N$, then $x = y$ or $(y, x) \in \overline{R^c_N}$.

Since $X \times X$ lies in $\mathcal{R}$, this set is non-empty. Let $\mathcal{C}$ be a chain in $\mathcal{R}$, and let $\mathcal{D} = \bigcap \mathcal{C}$. Since the class of negative consistent relations is closed downward (Proposition 2), $\mathcal{D}$ is negative consistent. Moreover, $\mathcal{D}$ satisfies the condition (c). To see that, take any $s, t \in X$ such that $(t, s) \notin \mathcal{D}$, so there exists $R_N \in \mathcal{C}$ such that $(t, s) \notin R_N$. Hence, $s = x$ or $(s, x) \in \overline{R^c_N} \subseteq \overline{\mathcal{D}^c}$. Therefore, by Zorn's lemma, $\mathcal{R}$ has an element, say $R_{C(x)}$, that is minimal with respect to set inclusion. To prove that $\overline{R^c_{C(x)}}x = Y \setminus \{x\}$, it suffices to show that $\Lambda = (Y \setminus \{x\}) \setminus \overline{R^c_{C(x)}}x = \emptyset$. Now suppose to the contrary that there exists a point $y \in \Lambda$. Then, there exists a natural number n and alternatives $y_1, y_2, ..., y_n \in X$ such that

$$y = y_1 R^c y_2 ... y_{n-1} R^c y_n = x \quad \text{and} \quad (y, x) \notin \overline{R^c_{C(x)}}$$

Since $x \neq y$, we may assume that the elements $y_1, y_2 ..., y_n$ are distinct. Now define

$$Q_N = R_{C(x)} \setminus \{(y_1, y_2), ...(y_{n-1}, y_n)\}.$$

Then, we have $R \subseteq Q_N \subset R_{C(x)}$. The first inclusion is easy: For each $k \in \{1, ..., n-1\}$, $(y_k, y_{k+1}) \notin R$. For the second inclusion, it suffices to show that there is at most one $k \in \{1, ..., n-1\}$ such that $(y_k, y_{k+1}) \in R_{C(x)}$. Indeed, if for each $k \in \{1, ..., n-1\}$ we let $(y_k, y_{k+1}) \notin R_{C(x)}$, we obtain $(y, x) \in \overline{R^c_{C(x)}}$, a contradiction. Furthermore, Q_N satisfies the condition (c). Indeed, assume that $s, t \in X$ are such that $(t, s) \notin Q_N$. There are two cases to consider: (i) $(t, s) \notin R_{C(x)}$; (ii) $(t, s) = (y_k, y_{k+1})$ for some $i \in \{1, ..., n-1\}$. In the first case, by construction we have $x = s$ or $(s, x) \in \overline{R^c_{C(x)}} \subset \overline{Q^c_N}$. In the second case, there must exist $k \in \{1, ..., n-1\}$ such that $s = y_{k+1}$. If $k = n-1$, then $s = y_n = x$. Otherwise, $s = y_{k^*+1}$ for some $k^* \in \{1, ..., n-2\}$. Since $(y_{k^*}, y_{k^*+1}) \in Q^c_N$,..., $(y_{n-1}, y_n) = (y_{n-1}, x) \in Q^c_N$, we conclude that $(s, x) \in \overline{Q^c_N}$. Therefore, by minimality of $R_{C(x)}$, it is clear that Q_N is not negative consistent. Thus, there exists a natural number m and alternatives $z_0, z_1, ..., z_m \in X$ such that

$$\mu = z_0 Q^c_N z_1 ... z_{m-1} Q^c_N z_m = \nu \quad \text{and} \quad (\nu, \mu) \in P(Q^c_N).$$

Since $R_{C(x)}$ is negative consistent and $Q^c_N = R^c_{C(x)} \cup \{(y_1, y_2), ...(y_{n-1}, y_n)\}$, there must exist $\kappa = 1, ..., n-1$ and $\lambda = 0, 1, ..., m-1$ such that $(y_\kappa, y_{\kappa+1}) = (z_\lambda, z_{\lambda+1})$. Consider the smallest κ for which there exist such $\mu, \nu, m, z_0, ..., z_m$, and λ. We show that there is no $j \in \{1, ..., n-1\}$ such that $(z_{\mathrm{mod}[\lambda(m+1)+m+\lambda, m+1]}, z_\lambda) = (y_j, y_{j+1})$. We proceed by the way of contradiction. Suppose that $y_{j+1} = z_\lambda R^c z_{\lambda+1} = y_{k+1}$. Since the elements $y_1, ..., y_n$ are distinct, it follows that $\kappa \neq j$ and so $\kappa < j$. But then, from $y_\kappa = z_\lambda = y_{j+1}$ we conclude that $\kappa = j+1$ which is impossible. Thus, from $(z_{\mathrm{mod}[\lambda(m+1)+m+\lambda,m+1]}, z_\lambda) \in Q^c_N$ we deduce that $(z_{\mathrm{mod}[\lambda(m+1)+m+\lambda,m+1]}, z_\lambda) \in R^c_{C(x)}$. Since $R_{C(x)} \in \mathcal{R}$ and $(z_{\mathrm{mod}[\lambda(m+1)+m+\lambda,m+1]}, z_\lambda) \notin R_{C(x)}$, we have $x = z_\lambda$ or $(z_\lambda, x) \in \overline{R^c_{C(x)}}$. Using $z_\lambda = y_\kappa \neq x$, we exclude the first case. Hence,

$$y = y_1 R^c y_2 ... R^c y_k \overline{R^c_{C(x)}} x.$$

Now define

$$\Gamma_N = R_{C(x)} \setminus \{(y_1, y_2), ...(y_{\kappa-1}, y_\kappa)\}.$$

As in the proof of Q_N, we conclude that $R \subseteq \Gamma_N \subset R_{C(x)}$. Furthermore, for all $s, t \in X$, if $(t,s) \notin \Gamma_N$, then similarly to the case of the relation Q_N we can prove that $x = s$ or $(s,x) \in \overline{\Gamma^c_N}$. Thus, Γ_N satisfies the condition (c). Finally, because of choice of κ we conclude that Γ_N is negative consistent. Hence, $\Gamma_N \in \mathcal{R}$, contradicting the minimality of $R_{C(x)}$. This contradiction establish that $\Lambda = \emptyset$ and completes the proof. □

Proposition 4. Let X be a nonempty set of alternatives and let R be a binary relation over X. Then, the $\mathcal{GETCHA}(R)$ set is equivalent to $\mathcal{M}(\overline{([P(R)]^c)^{-1}})$.

Proof. Let $x \in \mathcal{GETCHA}(R)$. We have two cases to consider: (i) For each $y \in Y$ there holds $(x,y) \in R$; (ii) There exists $y_0 \in Y$ such that $(x, y_0) \notin R$. In the first case, we have $(y,x) \notin P(R)$ which implies that $(x,y) \in ([P(R)]^c)^{-1} \subseteq \overline{([P(R)]^c)^{-1}}$. Hence, $x \in \mathcal{M}(\overline{([P(R)]^c)^{-1}})$. In the second case, since $(x, y_0) \notin R$, it follows that $y_0 \in \mathcal{GETCHA}$, for otherwise $(x, y_0) \in R$ which is impossible. Let $A_x = \{t \in \mathcal{GETCHA}(R) | (x,t) \in \overline{R^c}\}$. We have that $A_x \neq \emptyset$, because otherwise, for each $t \in \mathcal{GETCHA}(R)$, $(x,t) \notin \overline{R^c}$. But then, $(x,t) \in R$, which implies that $\{x\} \subset \mathcal{GETCHA}$ is an R-dominant subset of X, a contradiction because of the minimal character of $\mathcal{GETCHA}(R)$. Let $G = \mathcal{GETCHA}(R) \setminus A_x$. We prove that $G = \emptyset$. We proceed by the way of contradiction. Suppose that $G \neq \emptyset$. Then, for each $t \in A_x$ and each $s \in G$ we have $(t,s) \in R$ for suppose otherwise, $(t,s) \in R^c$ which implies that $(x,s) \in \overline{R^c}$ contradicting $s \in G$. Therefore, $A_x \subset \mathcal{GETCHA}(R)$ is an R-dominant subset of X, again a contradiction. Hence, $A_x = \mathcal{GETCHA}(R)$. Since, $y_0 \in \mathcal{GETCHA}(R)$ we conclude that $(x, y_0) \in \overline{R^c}$. Similarly, we can prove that $(y_0, x) \in \overline{R^c}$. Hence, since $R^c \subseteq [P(R)]^c$ we conclude that x and y_0 belong to a $([P(R)]^c)^{-1}$-cycle. On the other hand, for each $y \in Y \setminus \mathcal{GETCHA}(R)$, as in the case (i), we deduce that $(x,y) \in \overline{([P(R)]^c)^{-1}}$. Hence in any case we have $(y,x) \notin P((\overline{([P(R)]^c)^{-1}})$ which implies that $x \in \mathcal{M}(\overline{([P(R)]^c)^{-1}})$.

To prove the converse, take any $x \in \mathcal{M}(\overline{([P(R)]^c)^{-1}})$. We show that $x \in \mathcal{GETCHA}(R)$. We will consider two cases:

Case 1: For each $y \in X$ there holds $(y,x) \notin \overline{([P(R)]^c)^{-1}}$. In this case we have $(x,y) \in P(R) \subseteq R$. Hence, x is an R-dominant element of X which implies that $\mathcal{GETCHA}(R) = \{x\}$.

Case 2. There exists $y \in X$ such that $(x,y) \in \overline{([P(R)]^c)^{-1}}$ and $(y,x) \in \overline{([P(R)]^c)^{-1}}$. In this case, x belongs to a $[P(R)]^c$-cycle. Let $\mathcal{C}(x)$ be a $[P(R)]^c$-cycle containing x that is maximal in the sense that it is not a proper subset of any other $[P(R)]^c$-cycle. We prove that $\mathcal{C}(x) = \mathcal{GETCHA}$. Suppose on the contrary, that $(t,z) \notin R$ for some $t \in \mathcal{C}(x)$ and $z \in X \setminus \mathcal{C}(x)$; to deduce a contradiction. It follows that $(t,z) \in [P(R)]^c$ which implies that $(x,z) \in \overline{[P(R)]^c}$. Hence, $(z,x) \in \overline{([P(R)]^c)^{-1}}$. Since $(z,x) \notin P((\overline{([P(R)]^c)^{-1}})$ we conclude that $(x,z) \in \overline{([P(R)]^c)^{-1}}$. Hence, $\mathcal{C}(x) \cup \{z\}$ is a $[P(R)]^c$-cycle, a contradiction. □

The next result shows the connection between the $\mathcal{GETCHA}(R)$ set and the choice sets generated from negative consistent superrelations.

Theorem 5. Let X be a nonempty set of alternatives and let R be a binary relation over X. Then, the $\mathcal{GETCHA}(R)$ set is equivalent to the union of maximal elements of all minimal negative consistent superrelations of R.

Proof. Let $R_{N^*} \in \mathcal{R}_{N^*}$ be minimal, take any $x \in \mathcal{M}(R_{N^*})$. We prove that $x \in \mathcal{GETCHA}(R)$. Suppose to the contrary that $x \notin \mathcal{GETCHA}(R)$, then by Proposition 4 there exists $y \in X$ such that $(y,x) \in P(\overline{([P(R)]^c)^{-1}})$. It follows that $(x,y) \notin \overline{([P(R)]^c)^{-1}}$ which implies that $(y,x) \in P(R)$. Hence, $(y,x) \in R \subseteq R_{N^*}$. Therefore by $(y,x) \notin P(R_{N^*})$ we conclude that $(x,y) \in R_{N^*}$. Let us define $R_{N^{**}} = R_{N^*} \setminus (x,y)$. Since $(x,y) \notin R$, we conclude that $R \subseteq R_{N^{**}} \subset R_{N^*}$ and $R_{N^{**}}$ is non negative consistent (the assumption that $R_{N^{**}}$ is negative consistent contradicts to the fact that R_{N^*} is minimal with respect to set-inclusion). Hence, there exist $s,t \in X$, $\lambda \in \mathbb{N}$, and $z_0, z_1, ..., z_\Lambda \in X$ such that $s = z_0$, $(z_{\lambda-1}, z_\lambda) \in R^c_{N^{**}}$ for all $\lambda \in \{1,...,\Lambda\}$, $z_\Lambda = t$ and $(t,s) \in P(R^c_{N^{**}}) \subseteq R^c_{N^{**}}$. Since R_{N^*} is negative consistent and $R^c_{N^{**}} = R^c_{N^*} \cup \{(x,y)\}$, there must exists $\lambda_0 \in \{1,...,\Lambda\}$ such that $(z_{\lambda_0-1}, z_{\lambda_0}) = (x,y)$ and for all $\lambda \in \{1,...,\Lambda\}$ with $\lambda \neq \lambda_0$, $(z_{\lambda-1}, z_\lambda) \in R_{N^{**}}$ if and only if $(z_{\lambda-1}, z_\lambda) \in R_{N^*}$. It then follows that $(z_{\lambda_0}, z_{\lambda_0-1}) \in \overline{R^c_{N^*}}$. Therefore, $(y,x) \in \overline{R^c_{N^*}} \subset \overline{R^c} \subset \overline{[P(R)]^c}$. But then, $(x,y) \in \overline{([P(R)]^c)^{-1}}$ contradicting $(y,x) \in P(\overline{([P(R)]^c)^{-1}})$. This contradiction confirms the claim.

To prove the converse, take any $x \in \mathcal{GETCHA}(R)$. We show that there exists $R_{N^*} \in \mathcal{R}_N$ such that $x \in \mathcal{M}(R_{N^*})$. Let $R_{C(x)}$ be as in Proposition 3. First, observe that x is $R_{C(x)}$-maximal in X. Indeed, suppose to the contrary that there exists $y \in X$ such that $(y,x) \in P(R_{C(x)}) \subseteq P((\overline{R^c_{C(x)}})^c)$. Since $(x,y) \notin R_{C(x)} \supseteq R$, it follows that $y \in \mathcal{GETCHA}(R)$, for otherwise $(x,y) \subseteq R \subseteq R_{C(x)}$ which is impossible. From $x \in \mathcal{GETCHA}(R)$ by using the proof of Proposition 4 we conclude that $(y,x) \in \overline{R^c}$. Therefore, $y \in \overline{R^c}x \setminus \{x\} = \overline{R^c_{C(x)}}x$, contradicting $(y,x) \in P((\overline{R^c_{C(x)}})^c) \subseteq (\overline{R^c_{C(x)}})^c$. Hence, x is $R_{C(x)}$-maximal in X. If $R_{C(x)}$ is minimal with respect to set-inclusion in X, then the proof is over. Otherwise, there exists at least one negative consistent superrelation Q such that $R \subseteq Q \subset R_{C(x)}$. Let $\mathcal{Q}$ be the set of negative consistent superrelations Q satisfying the latter condition. Let $\mathcal{C}$ be a chain in $\mathcal{Q}$, and let $\mathcal{D} = \bigcap \mathcal{C}$. Evidently, $R \subseteq$

$\mathcal{D} \subset R_{C(x)}$. Since the class of negative consistent relations is closed downward (proposition 2), $\mathcal{D}$ is negative consistent. Therefore, by Zorn's lemma, $\mathcal{Q}$ has an element, say $\widetilde{Q}$, that is minimal with respect to set inclusion. We prove that x is $\widetilde{Q}$-maximal. We proceed by way of contradiction. Let $y \in X$ such that $(y,x) \in P(\widetilde{Q})$. Since $x \in \mathcal{GETCHA}(R)$ and $(x,y) \notin R$, as above, we conclude that $y \in \overline{R^c}x \setminus \{x\} = \overline{R^c_{C(x)}}x \subset \overline{\widetilde{Q}^c}x$, contradicting $(y,x) \in P(\widetilde{Q}) \subseteq P((\overline{\widetilde{Q}^c})^c) \subseteq (\overline{\widetilde{Q}^c})^c$. The proof is over. □

References

[1] Duggan, J. (1999). A general extension theorem for binary relations, *Journal of Economic Theory*, **86**, 1-16.

[2] Duggan, J. (2007). A systematic approach to the construction of non-empty Choice sets, *Social Choice and Welfare*, **28**, 491-506.

[3] Fishburn, P., (1977). Condorcet Social Choice Functions, *SIAM Journal on Applied Mathematics*, Vol. **33**, No. 3, pp. 469-489.

[4] Schwartz, T., The logic of Collective Choice, New York: Columbia University Press.

[5] Smith, J. (1973). Aggregation of Preferences with Variable Electorate, *Econometrica*, Vol. **41**, No. 6, pp. 1027-1041.

[6] Suzumura, K. (1976). Remarks on the theory of collective. *Economica*, **43**, 381-390.