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NASH EQUILIBRIA APPLIED TO SPOT-FINANCIAL EQUILIBRIA IN GENERAL EQUILIBRIUM MARKET MODELS

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Abstract. We consider a two period pure exchange economy with a finite number of states of nature at the second date. The economy consists of a real asset structure and a finite set of durable goods in the initial period that depreciate; we suppose that there is only one single good available in each state of nature at the second date. In this paper, we demonstrate that the spot-financial equilibrium can be obtained as a Nash equilibrium of a market game in which the strategies of the players consist in suggesting prices and quantities for both goods and assets.

Key words: Incomplete markets, market games, Nash equilibrium, strategic outcome functions

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1 Introduction.

The first studies of games in the economics literature were the papers by Cournot (1838), Bertrand (1883) and Edgeworth (1925) but these were seen as special models that did little to change the way economists thought about most problems. The idea of a general theory of games was introduced by von Neumann and Morgenstern in their famous 1944 book "Theory of Games and Economic Behaviour", which proposed that most economic questions should be analized as games. Nash (1950) established what came to be known as the noncooperative concept of *Nash equilibrium* of a game in normal form. This is a natural generalization of the equilibria studied in specific models by Cournot and Bertrand where the strategies of the players are simply their choices of quantities and prices respectively, and it is the starting point for most economic analyses.

Since Nash contribution to the existence of equilibrium points in noncooperative games there has been a growing literature on the strategic approaches to economic equilibrium. In order to prove existence of Walras equilibrium, Arrow and Debreu (1954) and Debreu (1962) extended Nash's model of a game in normal form and of a Nash equilibrium to generalized games: by adding a fictitious price player, who controls the price vector and whose payoff function is the value of excess demand, they introduce "feasibility" to a game in normal form; thus, they considered an abstract economy (which is a pseudo-game) and a social equilibrium. Walras equilibria were obtained then as Nash equilibria of a pseudo-game that included the market participant. However, there is no indication as to how prices are formed.

The formation of prices plays a central role in any discussion of the market process: which of the given economic agents sets the price vector and in what way? This question gives rise to a series of papers on strategic market games where, in addition to the consumer behavior, a price mechanism is formulated. Several types of price-forming mechanisms have been described in which prices depend on the actions of the traders, avoiding the classical assumption that individuals must regard prices as fixed.

The majority of these works proposed strategic market game models of exchange economies in the spirit of Cournot and Bertrand to study the relationship between the Nash equilibria and the competitive equilibria. On one hand, and following the Cournot tradition, Shapley (1976), Shapley and Shubik (1977), Dubey and Shapley (1994) and Dubey and Geanakoplos (2003) proposed games where money is introduced as the stipulated medium of exchange, either as one of the intrinsically valuable commodities or as "inside" fiat money. These games adhere completely to the Nash format, i.e., no price player is involved, and it is shown that their Nash equilibria converge to Walras equilibria.

On the other hand, the analysis of Bertrand was extended by Schmeidler (1980) and Dubey (1982) who established the coincidence of Nash equilibria and Walras equilibria. In particular, Schmeidler (1980) provided a strategic market game model where the exchange mechanism is characterized by a strategic outcome function that maps agents' selections of strategies to allocations. For it, it was proposed a game in strategic form in which the choice of a price vector is a part of the strategy choice for each player and proved that the Nash equilibrium of the game is precisely the competitive equilibrium of the Arrow-Debreu pure exchange economy.

The purpose of this paper is to extend this Schmeidler's result to a two period financial economy with an incomplete market structure and in particular, to analyze the strategic approach to spot-financial equilibrium.

The classical Arrow-Debreu model was extended to take account of uncertainty as Debreu (1959) proposed in Chapter 7 of his "Theory of Value". In Debreu's analysis, the concept of uncertainty is integrated into a general equilibrium context by introducing a finite set of states of the world and viewing commodities as differentiated by state, that is, commodities can be differentiated not only by their physical properties and location in space and time but also by their location in "state". In the Arrow-Debreu model of general equilibrium it is assumed that markets are *complete*: there is a market and an associated price for each good in each state of nature. All commodities are traded simultaneously, no matter when they are consumed or under what state of the world. All consumers face only a budget constraint, which is defined across the states, and one price system.

However, as point out by Magill and Shafer (1991) and Magill and Quinzii (1996), the Arrow-Debreu model introduces an idealized market structure that under time and uncertainty can lead to unfeasible redistributions of resources because of the behavioral imperfections of agents. Hence, a more general market model must be considered with real and financial markets in which the structure of markets is incomplete. In these models with incomplete markets the allocation of resources is modelled by a market structure consisting of a system of spot markets for real goods coupled with a system of financial markets where assets are traded.

The characteristic feature of a model with incomplete markets is that consumers face a multiplicity of budget sets and price systems at different times and under different states of nature. Consumers must hold assets to transfer wealth among budget constraints: individuals can buy or sell assets and, after the state of nature is revealed, they trade in the spot market with income derived from the sale of their initial endowments plus the deliveries of the revenues they receive as a result of their portfolio holdings. Therefore, in the context of this finance economy, there are markets for income in each state and agents can transfer income as they want across the states. Whenever the number of assets is less than the number of states, agents will have only a limited ability to redistribute their income across the states. In this case, the financial markets are said to be *incomplete*.

Incomplete markets have been extensively researched as regards existence and properties of equilibria (see Hart (1975), Geanakoplos y Polemarchakis (1985), Geanakoplos (1990) and Geanakoplos, Magill, Quinzii y Dreeze (1990)). It has been shown, for both the pure exchange and the production case, that competitive allocations are not necessary Pareto optimal.

In this paper we consider a two period pure exchange economy with a finite number of states of nature at the second date. The economy consists of a real asset structure and a finite set of durable goods in the initial period that depreciate. We suppose that there is only one single good available in each state of nature at the second date. Thus, the assets yield payoffs denominated in a single *numéraire* commodity and this precisely makes irrelevant to consider in our economy nominal or real assets. In any case, individuals can buy or sell short any amount of the *numéraire* assets, in some limited collection. The outstanding hypothesis is that it is assumed bounded short sales. On other side, agents are assumed to dispose non-negative amounts of the single good in any state at date one. In this model the financial market structure is incomplete and, under standard assumptions on initial endowments and preferences, there always exists spot-financial equilibrium.

This paper concentrates on highlight that, in this context, the spot-financial equilibrium can be obtained as a Nash equilibrium of a game in which the strategies of the players consist in suggesting prices and quantities for goods and assets. Our approach follows the analysis of Schmeilder (1980). In this case, the main difficulty to overcome is to guarantee that date one commodity-bundles are non-negative. Thus, we construct a game in normal form where the strategy profile of each player consists of suggesting prices and amounts for both commodities and assets to be traded. In this game, in addition to the strategy outcome function for the commodities in the first period, it is defined another for the assets and from both, it is deduced the second date commodity outcome function. The strategy outcome functions are defined in such a way that, in equilibrium, they map agents strategies to non-negative consumptions at the second date. Then, we demonstrate that Nash equilibria of the associated game coincide with the spot-financial equilibria for the finance economy.

The paper is organized along with two other sections. In section 2 we describe a two period economy with incomplete market structure and in section 3 we define the associated game and our main result is stated and proven.

2 The model: a two period finance economy.

We consider a two period exchange economy E under uncertainty with spot and financial markets. The economy consists of a finite number of agents, a finite number of goods and a finite number of real assets.

To capture both time and uncertainty we consider a model with two time periods, $t = 0, 1$ and a set $S = \{1, \ldots, S\}$ of possible states of nature at date $t = 1$. For convenience, date $t = 0$ is often included as state $s = 0$, so that in total there are $S + 1$ states. Let there be only one good in each state $s \in S$ at date $t = 1$.

Let $L = \{1, \ldots, L\}$ be the set of goods at date $t = 0$. These L goods are supposed to be possible durable, i.e., they are not entire consumed at the initial date and can be used in $t = 1$. At the second period, durable goods may have depreciated. Their depreciation is denominated in terms of the single date 1 good in each state. Precisely, as there is a unique good at $t = 1$, we consider it as the *numeraire* good in each state of nature.

Let $\Upsilon_s : \mathbb{R}_+^L \to \mathbb{R}$ be the depreciation structure depending on the state $s \in S$ that occurs in the second period. The depreciation Υ_s is an increasing function in the commodity bundle which implies that, if it is differentiable, its partial derivatives are non-negative. The interpretation is clear: from time to time goods change in their value and thus monotonicity implies that for each good this change is directly proportional to its amount. Indeed, Υ_s is a concave function; this means that variations in the depreciation are in inverse proportion to the amounts. Latter on, when we state the hypothesis on the model, we will single out assumptions on the depreciation structure.

The commodity space is $\mathbb{R}^n$ with $n = L + S$. The consumption set of agent i is taken to be $X_i = \mathbb{R}_+^n$. It is convenient to write the consumption set as

$X_i = \mathbb{R}_+^L \times \mathbb{R}_+^S$. A consumption bundle is a vector $x = (x_0, x_1, \ldots, x_s) \in \mathbb{R}_+^L \times \mathbb{R}_+^S$ where $x_0 = (x_{0\,1}, \ldots, x_{0\,L})$ denotes date 0 consumption of the L goods and x_s represents the consumption of the *numeraire* good in each state $s \in \{1, \ldots, S\}$.

Let $J = \{1, \ldots, J\}$ be the set of real assets. In this model there are $J < S$ real assets that can be freely traded at date $t = 0$. A real asset j is a contract that can be purchased for the price q_j at $t = 0$ and promises to deliver at date $t = 1$ an amount A_s^j of units of the good in each state s, for $s \in S$. As there is only one good consumption in each state, the assets are always *numeraire* assets. A real asset is thus characterized by its return vector $A^j = (A_1^j, \ldots, A_S^j) \in \mathbb{R}_+^S$. The exogenous date 1 returns of the J real assets are summarized by the $S \times J$ matrix

$$A = \begin{pmatrix} A_1^1 & \ldots & A_1^J \\ \vdots & \ddots & \vdots \\ A_S^1 & \ldots & A_S^J \end{pmatrix}$$

Then, the matrix A represents the real asset structure of the economy. We suppose that the rank of A is maximum. Let $\langle A \rangle$ denote *the subspace of transfers* generated by A, that is, the subspace of $\mathbb{R}^S$ spanned by the J columns of A. As $\langle A \rangle = \mathbb{R}^J \neq \mathbb{R}^S$, then the asset structure is *incomplete*.

The set of agents is $I = \{1, \ldots, I\}$. Each agent $i \in I$ has an initial endowment of goods in each state, $\omega^i = (\omega_0^i, \omega_1^i, \ldots, \omega_S^i) \in \mathbb{R}_+^L \times \mathbb{R}_+^S$, where $\omega_0^i = (\omega_{0\,1}^i, \ldots, \omega_{0\,L}^i)$ and $\omega_{0\,l}^i$ denotes her initial endowment at date 0 of good l, and an initial endowment of real assets $\delta^i = (\delta_1^i, \ldots, \delta_J^i) \in \mathbb{R}_+^J$, with $\delta_j^i \geq 0$. The preference relation of agent i is represented by a utility function $u^i : \mathbb{R}_+^L \times \mathbb{R}_+^S \to \mathbb{R}$.

Therefore, the exchange economy with the real asset structure A is defined by

$$E = \left(\mathbb{R}_+^L \times \mathbb{R}_+^S, (u^i, \omega^i, \delta^i, A), i = 1, \ldots I\right).$$

The economy is denoted by $E(u, \omega, \delta, A)$.

In this economy, there exists a sequential market structure for the trade of goods and real assets. On one hand, transactions in real assets occur before the state of nature is known: each agent i, for a price vector $q = (q_1, \ldots, q_J)$, trades the J real assets. Let $z^i = (z^i_1, \ldots, z^i_J) \in \mathbb{R}^J$ denote the i^{th} agent portfolio. Each coordinate z^i_j represents the number of units of each of the J real asset purchased (if $z^i_j > 0$) or sold (if $z^i_j < 0$) by agent i. A portfolio plan $z = (z^1, \ldots, z^I)$ is feasible if $\sum_{i=1}^{I} z^i_j = \sum_{i=1}^{I} \delta^i_j$, for all $j \in J$.

On the other hand, agent i chooses a consumption bundle $x^i = (x^i_0, \ldots, x^i_S)$ and makes a choice of consumption today $x^i_0 = (x^i_{0\,1}, \ldots, x^i_{0\,L}) \in \mathbb{R}^l_+$ versus a consumption in the future $(x^i_1, \ldots, x^i_S) \in \mathbb{R}^S_+$, as well as a choice over the relative amounts of consumption in the different states at date 1. A consumption plan $x = (x^1 \ldots, x^I)$ is feasible if:

$$\sum_{i=1}^{I} x^i_0 \leq \sum_{i=1}^{I} \omega^i_0, \text{ and}$$

$$\sum_{i=1}^{I} x^i_s \leq \sum_{i=1}^{I} \omega^i_s + \sum_{i=1}^{I} \sum_{j=1}^{J} A^j_s \cdot z^i_j + \sum_{i=1}^{I} \Upsilon_s(x^i_0), \text{ for all } s \in S.$$

Hence, there is a market structure modelled by a collection of spot markets, for the trading of real goods, together with a system of financial markets for trading real assets. The financial markets allow each agent to redistribute income across the states; only by holding assets an agent can transfer wealth between budget constraints. The spot markets are supposed for each of the L goods at date 0 and in each state s at date 1. The main feature of a spot market is that its payment is made at date 1 in state s (if $s \geq 1$). Thus, under a system of spot markets, agents face a multiplicity of $S+1$ budget constraints, at different times and under different states of nature.

Let $p = (p_0, p_1, \ldots, p_S) \in \mathbb{R}^{L+S}_{++}$ denote the vector of spot prices, where $p_0 = (p_{0\,1}, \ldots, p_{0\,L})$ and $p_{0\,l}$ represents the price payable for a unit of good l at date 0. Note that at $t = 1$ we assume a single good, so, without loss of generality, we can put $p_s = 1$ for all $s = 1, \ldots, S$. Hence, from now on $p = p_0$.

A price system in this economy is therefore a collection (p, q) formed by a spot price vector p and a price vector $q = (q_1, \ldots, q_J)$ for real assets traded at $t = 0$. Let $\triangle^{L+J-1} = \left\{ (p,q) \in \mathbb{R}^L_{++} \times \mathbb{R}^J_{++} \mid \sum_{l=1}^{L} p_l + \sum_{j=1}^{J} q_j = 1 \right\}$. For a price system (p, q), the budget set of agent i is given by

$$
\begin{aligned}
B^i(p,q) \quad = \quad & \{(x^i, z^i) \in (\mathbb{R}^L_+ \times \mathbb{R}^S_+) \times \mathbb{R}^J \mid p \cdot x^i_0 + q \cdot z^i \leq p \cdot \ \omega^i_0 + q \cdot \delta^i \\
& \text{and } x^i_s \leq \omega^i_s + \sum_{j=1}^{J} A^j_s \cdot z^i_j + \Upsilon_s(x^i_0), \text{ for all } s \in S\}.
\end{aligned}
$$

Definition 2.1 *A **spot-financial equilibrium** for the finance economy E is a pair of actions and prices $((x, z), (p, q))$, where (x, z) is a consumption-portfolio bundle and (p, q) is a price system, such that*

(i) $(x^i, z^i) \in arg\ max\{u^i(x_i) \mid (x^i, z^i) \in B^i(p, q)\}, \ i = 1, \ldots, I,$

(ii) $\sum_{i=1}^{I} x^i_0 = \sum_{i=1}^{I} \omega^i_0,$

(iii) $\sum_{i=1}^{I} z^i_j = \sum_{i=1}^{I} \delta^i_j,$ *for all* $j \in J,$

(iv) $\sum_{i=1}^{I} x^i_s = \sum_{i=1}^{I} \omega^i_s + \sum_{i=1}^{I} \sum_{j=1}^{J} A^j_s \cdot z^i_j + \sum_{i=1}^{I} \Upsilon_s(x^i_0),$ *for all* $s \in S,$

On the whole, the existence of equilibrium in a finance economy as the one here described is not guarantee. The assumptions we state below on short sales, initial endowments, preferences and depreciation structure are enough to ensure the existence of spot-financial equilibrium (see for example Magill and Quinzii (1996)).In point of fact, in the standard incomplete markets framework-see Radner (1972) for instance- a bound on short sales eliminates any discontinuity and guarantees the existence of equilibrium. It is really true that all these hypothesis are not necessary to guarantee the equilibrium existences but we will use them to prove our main result.

$(A.1)$ (*Bounded short sales*). There exists a positive real number M such that $z_j^i > -M$ for all $i \in I$ and for all $j \in J$.

$(A.2)$ (*Strict positivity of endowments*). $\omega^i \gg 0$ and $\delta^i \gg 0$ for all $i \in I$.

$(A.3)$ (*Monotonicity*). For each $i \in I$ the utility function u^i is monotone.

$(A.4)$ (*Convexity*). For each $i \in I$ the utility function u^i is strictly quasi-concave.

$(A.5)$ (*Inada*). Each $i \in I$ prefers an interior commodity bundle to any consumption in the frontier of $\mathbb{R}^n_+$.

$(A.6)$ (*Differentiability*). For each $i \in I$ the utility function u^i is differentiable.

$(A.7)$ (*Depreciation structure*). For each state $s \in S$, Υ_s is differentiable and concave.

Observe that, under bounded short sales (assumption $(A.1)$), the budget set $B^i(p,q)$ is compact for all $(p,q) \in \triangle^{L+J-1}$. Then, there exists a well defined set of actions bundle that maximizes u^i over $B^i(p,q)$.

The concavity assumption of the depreciation structure guarantees that the budget set $B^i(p,q)$ is convex for any price system (p,q). In fact, $B^i(p,q)$ define a system of equations, one linear constraint and S constraints such that $x_s^i \leq \omega_s^i + \sum_{j=1}^{J} A_s^j \cdot z_j^i + \Upsilon_s(x_0^i)$. Thus, if (x,z) y (x',z') are in $B^i(p,q)$ and $\alpha \in (0,1)$ then $\alpha \cdot (x,z) + (1-\alpha)\cdot(x',z')$ is also in $B^i(p,q)$ because

$$\begin{aligned}\alpha \cdot x_s + (1-\alpha)\cdot x'_s \quad \leq \quad & \omega_s + \sum_{j=1}^{J} A_s^j \cdot (\alpha \cdot z_j^i + (1-\alpha)\cdot z_j^{'\,i}) + \\ & \alpha \cdot \Upsilon_s(x_0^i) + (1-\alpha)\cdot \Upsilon_s(x_0^{'\,i}).\end{aligned}$$

By the concavity of the depreciation structure, $\Upsilon_s(\alpha \cdot x_0^i + (1-\alpha)\cdot x_0^{'\,i}) \geq \Upsilon_s(x_0^i) + (1-\alpha)\cdot \Upsilon_s(x_0^{'\,i})$. Hence,

$$\begin{aligned}\alpha \cdot x_s + (1-\alpha)\cdot x'_s \quad \leq \quad & \omega_s + \sum_{j=1}^{J} A_s^j \cdot (\alpha \cdot z_j^i + (1-\alpha)\cdot z_j^{'\,i}) + \\ & \Upsilon_s(\alpha \cdot x_0^i + (1-\alpha)\cdot x_0^{'\,i}).\end{aligned}$$

The convexity of $B^i(p,q)$ together with the strict quasi-concavity of the utility function guarantees that the commodity bundle which maximizes the utility subject to the constraints embodied in the budget set exists (the budget set, by (C.1) is compact) and is unique for any price system (p,q). Let $d_c^i(p,q)$ denote i^{th} agent demand for each price pair (p,q). On the other hand, once the commodity demand is fixed, there exists a unique vector $d_a^i(p,q)$ that represents the real asset demand of agent i and for which the S inequations are saturated $(d_c^i(p,q))_s \leq \omega_s^i + \sum_{j=1}^{J} A_s^j \cdot z_j^i + \Upsilon_s((d_c^i(p,q))_0)$ are saturated, i.e, are satisfied with equal. We know that this inequation system has solution and, given the unique commodity demand, the rank of the S equation system $\sum_{j=1}^{J} A_s^j = (d_c^i(p,q))_s - \omega_s^i - \Upsilon_s((d_c^i(p,q))_0)$ is $J < S$. Thus $S-J$ equations are a linear combination of the other J and, therefore, this becomes a Cramer equation system with a unique solution.

3 Spot-financial equilibrium and Nash.

Let $E = \left(\mathbb{R}_+^L \times \mathbb{R}_+^S, (u^i, \omega^i, \delta^i, A), i = 1, \ldots I\right)$ be a finance economy. The purpose of this section is to relate the spot-financial market equilibria of this economy to the Nash equilibria of a game in strategic form.

Given the exchange economy above $E(u,\omega,\delta,A)$, we construct a game in normal form. Let $\Gamma = \{\Theta^i, \pi^i\}_{i=1,\ldots,n}$ be a $n-$person game where Θ^i is the strategy set and π^i the payoff function of a player i.

The set of strategies for a player i is defined by

$$\begin{aligned}\Theta^i \quad = \quad & \{(x^i, z^i, p^i, q^i) \in \mathbb{R}_+^L \times \mathbb{R}^J \times \triangle^{L+J-1} \mid p^i \cdot x_0^i + q^i \cdot z^i = p^i \cdot \omega_0^i + q^i \cdot \delta^i \\ & \text{and } \omega_s^i + \sum_{j=1}^{J} A_s^j \cdot z_j^i + \Upsilon_s(x_0^i) \geq 0, \text{ for all } s \in S\}.\end{aligned}$$

Note that there is a single good at $t=1$, so a player i will choose goods and spot prices at date 0. Hence, from now on in the game, $(x^i, z^i, p^i, q^i) = (x_0^i, z^i, p_0^i, q^i)$.

Therefore, a player i's strategy θ^i is defined by a consumption bundle x^i at $t = 0$, a portfolio z^i, a spot price vector p^i and a real asset price vector q^i, that is, $\theta^i = (x^i, z^i, p^i, q^i) \in \mathbb{R}^L_+ \times \mathbb{R}^J \times \triangle^{L+J-1}$, such that x^i and z^i belong to the budget set of agent i under the price system (p^i, q^i). Let $\Theta = \prod_{i=1}^{n} \Theta^i$be the set of possible strategy profiles.

Given a strategy profile $\theta = (\theta^1, \ldots, \theta^n) \in \Theta$, where $\theta^i = (x^i, z^i, p^i, q^i)$, let $A^i(\theta)$ denote the set of agents who propose the same prices as player i in the strategy profile θ. That is,

$$A^i(\theta) = \{k \in \{1, \ldots, I\} \mid p^k = p^i \text{ and } q^k = q^i\}$$

Agents who propose different prices, either for goods and/or assets do not trade at all. Therefore, trade of goods and assets only takes place among members of $A^i(\theta)$.

Let $f_0^i : \Theta \to \mathbb{R}^L_+$ be the commodity bundle defined by

$$f_0^i(\theta) = x^i - \frac{\sum\limits_{k \in A_i(\theta)} \left(x^k - \omega_0^k\right)}{Card(A^i(\theta))}$$

where $Card(A_i(\theta))$ denotes the cardinality of the set $A_i(\theta)$, let $f_a^i : \Theta \to \mathbb{R}^J$ be the real asset bundle with components

$$f_{a\ j}^i(\theta) = z_j^i - \frac{\sum\limits_{k \in A_i(\theta)} \left(z_j^k - \delta_j^k\right)}{Card(A^i(\theta))}$$

for $j \in J$, and let $f_1^i(\theta) = \omega_s^i + \sum\limits_{j=1}^{J} A_s^j \cdot f_{a\ j}^i(\theta) + \Upsilon_s(f_0^i(\theta))$ for all $s \in S$.

Now we define the function f^i as

$$f^i(\theta) = \left(f_0^i(\theta)\ ,\ f_1^i(\theta)\right) \text{ if } f_0^i(\theta) \geq 0 \text{ and } f_1^i(\theta) \geq 0$$

and otherwise

$$f^i(\theta) = \left(\omega_0^i - \epsilon \cdot \overline{1}_L \ , \ \left(\omega_s^i + \sum_{j=1}^{J} A_s^j \cdot \delta_j^i + \Upsilon_s(\omega_0^i)\right)_{s \in S} - \epsilon \cdot \overline{1}_S\right)$$

where $\overline{1}_L = (\overbrace{1, \ldots, 1}^{L})$, $\overline{1}_S = (\overbrace{1, \ldots, 1}^{S})$ and $\epsilon > 0$ is such that

$$\left(\omega_0^i - \epsilon \cdot \overline{1}_L \ , \ \left(\omega_s^i + \sum_{j=1}^{J} A_s^j \cdot \delta_j^i + \Upsilon_s(\omega_0^i)\right)_{s \in S} - \epsilon \cdot \overline{1}_S\right)$$

is non-negative.

The function $f^i : \Theta \to \mathbb{R}_+^L \times \mathbb{R}_+^S$ denotes the outcomes of player i at dates 0 and 1 under each strategy profile $\theta \in \Theta$. Then, the i^{th} agent payoff function $\pi_i : \Theta \to \mathbb{R}$ is defined by

$$\pi^i(\theta) = u^i\left(f^i(\theta)\right).$$

For a profile θ, let θ^{-i} denote a strategy selection for all players but i. So we write $\theta = (\theta^{-i}, \theta^i)$. A strategy profile $\theta^* = (\theta^{-i\,*}, \theta^{i\,*})$ is a **Nash equilibrium** if for all players i,

$$u^i\left(f^i(\theta^{*\,-i}, \theta^{*\,i})\right) \geq u^i\left(f^i(\theta^{*\,-i}, \theta^i)\right), \ \text{ for all } \theta^i \in \Theta^i.$$

Theorem 3.1 *Let E be an exchange economy with the real asset structure A satisfying assumptions $(A.1) - (A.7)$, with $N \geq 3$. Let Γ be the game in normal form associated with E. Then*

I. If θ^* is a Nash equilibrium of the game Γ, then $p^{*\,i} = p^*$ and $q^{*\,i} = q^*$ for all $i \in N$ and $(f_0^i(\theta^*), f_a^i(\theta^*), p^*, q^*)$ is a spot-financial equilibrium of the economy E.

II. If $(x^{i\,*}, z^{i\,*}, p^*, q^*)$ is a spot-financial equilibrium of E then the strategy profile $\theta^{i\,*} = (x^{*\,i}, z^{*\,i}, p^{*\,i}, q^{*\,i})$ for each $i = 1, \ldots, I$ is a Nash equilibrium of Γ.

Demonstration.

I. Let θ^* be a Nash equilibrium of Γ. We will see that $(f_0^i(\theta^*), f_a^i(\theta^*), p^*, q^*)$ is an equilibrium of E.

First, we will see that if $\theta^* = (\theta^{1\,*}, \ldots, \theta^{n\,*})$, with $\theta^{*\,i} = \left(x^{*\,i}, z^{*\,i}, p^{*\,i}, q^{*\,i}\right)$, is a Nash equilibrium of Γ, then $f_0^i(\theta^*) \geq 0$ and $f_1^i(\theta^*) \geq 0$ for all $i = 1, \ldots, I$. For it suppose that there exists an agent i such that $f_0^i(\theta^*) < 0$ or $f_1^i(\theta^*) < 0$. Then, $f^i(\theta^*) = \left(\omega_0^i - \epsilon \cdot \bar{1}_L\ ,\ \left(\omega_s^i + \sum_{j=1}^{J} A_s^j \cdot \delta_j^i + \Upsilon_s(\omega_0^i)\right)_{s\in S} - \epsilon \cdot \bar{1}_S\right)$. Consider that player i chooses $\tilde{\theta}^i = (\omega_0^i, \delta^i, p, q)$ with $p \neq p^{*\,k}$ or $q \neq q^{*\,k}$ for any $k \neq i$. Note that in this case $A^i(\theta^*) = \{i\}$, so $f_0^i(\theta^{*\,-i}, \tilde{\theta}^i) = \omega_0^i$ and $f_a^i(\theta^{*\,-i}, \tilde{\theta}^i) = \delta^i$. Then the outcome is

$$f^i(\theta^{*\,-i}, \tilde{\theta}^i) = \left(\omega_0^i,\ \left(\omega_s^i + \sum_{j=1}^{J} A_s^j \cdot \delta_j^i + \Upsilon_s(\omega_0^i)\right)_{s\in S}\right)$$

By the monotonicity of u^i, it follows that $u^i\left(f^i(\theta^{*\,-i}, \tilde{\theta}^i)\right) > u^i\left(f^i(\theta^*)\right)$, that is, $\pi^i\left(\theta^{*\,-i}, \tilde{\theta}^i\right) > \pi^i(\theta^*)$, so θ^* would not be a Nash equilibrium of Γ. Hence, if θ^* is a Nash equilibrium of Γ, then $f_0^i(\theta^*) \geq 0$ and $f_1^i(\theta^*) \geq 0$, which implies that $f^i(\theta^*) = \left(f_0^i(\theta^*), f_1^i(\theta^*)\right)$ for all $i \in I$. Therefore, under the Nash equilibrium strategy profile, θ^*, the commodity and real asset bundle that player i obtains is $(f_0^i(\theta^*), f_a^i(\theta^*))$.

Next, we will prove that for any different agents, $i, k \in I$, the payoff that agent i gets with $f^i(\theta^*)$ is greater or equal than her indirect utility at prices $(p^{*\,k}, q^{*\,k})$ proposed by agent k. We will distinguish two cases: (i) if $k \in A^i(\theta^*)$ and (ii) $k \notin A^i(\theta^*)$.

If (i) occurs, then $(p^{*\,i}, q^{*\,i}) = (p^{*\,k}, q^{*\,k})$. In this case, $Card(A^i(\theta^*)) - 1 \neq 0$ because $A^i(\theta^*) \geq 2$. Player i could receive her demands of goods and real assets at prices $(p^{*\,k}, q^{*\,k})$ by choosing her strategy $\theta^i = \left(x^i, z^i, p^{*\,k}, q^{*\,k}\right)$ where

$$x^i = \frac{1}{Card(A^i(\theta^*)) - 1}\left[\eta \cdot Card(A^i(\theta^*)) + \sum_{r\in A^i(\theta^*)\setminus\{i\}} x^{*\,r} - \sum_{r\in A^i(\theta^*)} \omega_0^r\right]$$

and

$$z^i = \frac{1}{Card(A^i(\theta^*)) - 1}\left[\mu \cdot Card(A^i(\theta^*)) + \sum_{r\in A^i(\theta^*)\setminus\{i\}} z^{*\,r} - \sum_{r\in A^i(\theta^*)} \delta^r\right]$$

It is easy to see that $\theta^i \in \Theta^i$. Note that

$$\begin{aligned}(f_0^i(\theta^{*\,-i},\theta^i), f_a^i(\theta^{*\,-i},\theta^i)) &= (d_c^i(p^{*\,i},q^{*\,i}), d_a^i(p^{*\,i},q^{*\,i})) = \\ &= (d_c^i(p^{*\,k},q^{*\,k}), d_a^i(p^{*\,k},q^{*\,k})).\end{aligned}$$

Since θ^* is a Nash equilibrium then

$$u^i(f^i(\theta^{*\,-i},\theta^{*\,i})) \geq u^i(f^i(\theta^{*\,-i},\theta^i)) = v^i(p^{*\,k},q^{*\,k})$$

where $v^i(p^{*\,k},q^{*\,k})$ is the indirect utility function of agent i.

Now we will prove that for all $r \in A^i(\theta^*)$, $(f_0^r(\theta^*), f_a^r(\theta^*)) \in B^r(p^{*\,k},q^{*\,k})$. Suppose that there exists an agent $r \in A^i(\theta^*)$ such that,

$$p^{*\,k}\cdot f_0^r(\theta^*) + q^{*\,k}\cdot f_a^r(\theta^*) > p^{*\,k}\cdot \omega_0^r + q^{*\,k}\cdot \delta^r$$

As $\sum_{r\in A^i(\theta^*)} f_0^r(\theta^*) = \sum_{r\in A^i(\theta^*)} \omega_0^r$ and $\sum_{r\in A^i(\theta^*)} f_a^r(\theta^*) = \sum_{r\in A^i(\theta^*)} \delta^r$, then it has to exist another agent $r' \in A^i(\theta^*)$ such that

$$p^{*\,k}\cdot f_0^{r'}(\theta^*) + q^{*\,k}\cdot f_a^{r'}(\theta^*) < p^{*\,k}\cdot \omega_0^{r'} + q^{*\,k}\cdot \delta^{r'}.$$

and $f^{r'}(\theta^*) = \left(f_0^{r'}(\theta^*), f_1^{r'}(\theta^*)\right)$. By monotonicity of u^i (assumption $(A.3)$), $u^{r'}(f^{r'}(\theta^*)) < v^{r'}(p^*,q^*)$, which is a contradiction. Then, $(f_0^r(\theta^*), f_a^r(\theta^*)) \in B^r(p^{*\,k},q^{*\,k})$, for all $r \in A^i(\theta^*)$. As $u^r(f^r(\theta^*)) \geq v^r(p^{*\,k},q^{*\,k})$ for all demands $(d_c^r(p^{*\,k},q^{*\,k}), d_a^r(p^{*\,k},q^{*\,k}))$, then $(f_0^r(\theta^*), f_a^r(\theta^*)) = (d_c^r(p^{*\,k},q^{*\,k}), d_a^r(p^{*\,k},q^{*\,k}))$ for all $r \in A^i(\theta^*)$. Hence, we conclude that

$$(f_0^i(\theta^*), f_a^i(\theta^*)) = (d_c^i(p^{*\,k},q^{*\,k}), d_a^i(p^{*\,k},q^{*\,k})).$$

If (ii) occurs, then $(p^{*\,i},q^{*\,i}) \neq (p^{*\,k},q^{*\,k})$. In this case, player i could still receive her demands of goods and real assets at prices $(p^{*\,k},q^{*\,k})$ by choosing her strategy $\theta^i = \left(x^i, z^i, p^{*\,k}, q^{*\,k}\right)$ where

$$x^i = \frac{1}{Card(A^k(\theta^*))} \cdot \left[\eta \cdot (Card(A^k(\theta^*)) + 1) + \sum_{r \in A^k(\theta^*)} x^{*\,r} - \sum_{r \in A^k(\theta^*) \bigcup \{i\}} \omega_0^r\right]$$

and

$$z^i = \frac{1}{Card(A^k(\theta^*))} \cdot \left[\mu \cdot (Card(A^k(\theta^*)) + 1) + \sum_{r \in A^k(\theta^*)} z^{*\,r} - \sum_{r \in A^k(\theta^*) \bigcup \{i\}} \delta^r\right]$$

It is easy to see that $\theta^i \in \Theta^i$. Note that

$$(f_0^i(\theta^{*\,-i}, \theta^i), f_a^i(\theta^{*\,-i}, \theta^i)) = (d_c^i(p^{*\,k}, q^{*\,k}), d_a^i(p^{*\,k}, q^{*\,k})).$$

Since θ^* is a Nash equilibrium, it follows that

$$u^i(f^i(\theta^{*\,-i}, \theta^{*\,i})) \geq u^i(f^i(\theta^{*\,-i}, \theta^i)) = v^i(p^{*\,k}, q^{*\,k}).$$

So $u^i(f^i(\theta^{*\,-i}, \theta^{*\,i})) \geq v^i(p^{*\,k}, q^{*\,k})$.

Now we will see that if θ^* is a Nash equilibrium, then all players propose the same prices:

If there exists an agent i such that $Card(A^i(\theta^*)) \geq 2$, then $A^k(\theta^*) = \{1, \ldots, I\}$ for every k.

For it, and by contradiction, suppose that there exists an agent k such that $k \notin A^i(\theta^*)$, so $(p^{*\,k}, q^{*\,k}) \neq (p^{*\,i}, q^{*\,i})$. Since $\sum_{r \in A^i(\theta^*)} f_0^r(\theta^*) = \sum_{r \in A^i(\theta^*)} \omega_0^r$ and $\sum_{r \in A^i(\theta^*)} f_a^r(\theta^*) = \sum_{r \in A^i(\theta^*)} \delta^r$, there are two cases:

(a) There exists $r \in A^i(\theta^*)$ such that $p^{*\,k} \cdot f_0^r(\theta^*) + q^{*\,k} \cdot f_a^r(\theta^*) < p^{*\,k} \cdot \omega_0^r + q^{*\,k} \cdot \delta^r$.

(b) For any $r \in A^i(\theta^*)$, $p^{*\,k} \cdot f_0^r(\theta^*) + q^{*\,k} \cdot f_a^r(\theta^*) = p^{*\,k} \cdot \omega_0^r + q^{*\,k} \cdot \delta^r$.

If (a) is the case, there exists (y_c, y_a) such that

$$p^{*\,k} \cdot (f_0^r(\theta^*) + y_c) + q^{*\,k} \cdot (f_a^r(\theta^*) + y_a) = p^{*\,k} \cdot \omega_0^r + q^{*\,k} \cdot \delta^r \ .$$

Since $(p^*, q^*) \in \triangle^{L+J-1}$ and by the monotonicity of u^i, then $v^r(p^{*\,k}, q^{*\,k}) > u^r(f^r(\theta^*))$ a contradiction with $u^r(f^r(\theta^*)) \geq v^r(p^{*\,k}, q^{*\,k})$.

Suppose that (b) occurs, then $p^{*\,k} \cdot f_0^r(\theta^*) + q^{*\,k} \cdot f_a^r(\theta^*) = p^{*\,k} \cdot \omega_0^r + q^{*\,k} \cdot \delta^r$, for any $r \in A^i(\theta^*)$. We will prove that $(p^{*\,i}, q^{*\,i}) = (p^{*\,k}, q^{*\,k})$. Since $r \in A^i(\theta^*)$, it follows (by (i)) that $(f_o^r(\theta^*), f_a^r(\theta^*)) = (d_c^r(p^{*\,k}, q^{*\,k}), d_a^r(p^{*\,k}, q^{*\,k}))$. Given that $Card(A^i(\theta^*)) \geq 2$, $(f_o^r(\theta^*), f_a^r(\theta^*)) = (d_c^r(p^{*\,i}, q^{*\,i}), d_a^r(p^{*\,i}, q^{*\,i}))$ for all $r \in A^i(\theta^*)$. Then

$$\left(d_c^r(p^{*\,k}, q^{*\,k}), d_a^r(p^{*\,k}, q^{*\,k})\right) = \left(d_c^r(p^{*\,i}, q^{*\,i}), d_a^r(p^{*\,i}, q^{*\,i})\right).$$

Let L^r be the agent r's Lagrangean function defined by

$$\begin{aligned} L^r(x^r, z^r, \lambda) &= u^r(x^r) + \lambda_0^i \left(p^i \cdot \omega_0^r + q^i \cdot \delta^r - p^i \cdot x_0^r - q^i \cdot z^r\right) + \\ &+ \sum_{s=1}^{S} \lambda_s^i \left(\omega_s^r + \sum_{j=1}^{J} A_s^j \cdot z_j^r + \Upsilon_s(x_0^r) - x_s^r\right) \end{aligned}$$

where we set the superindex i to refer the vector of Lagrange multipliers associated with the maximization problem at prices (p^i, q^i). We will write $\lambda^k = (\lambda_0^k, \lambda_1^k, \ldots, \lambda_S^k)$ the vector of Lagrange multipliers associated with the maximization problem at prices (p^k, q^k). Then, from the first order conditions for the problem at prices (p^i, q^i), we get that

(1) $\dfrac{\partial u^r}{\partial x_0^r} = \lambda_0^i \cdot p^i - \displaystyle\sum_{s=1}^{S} \lambda_s^i \cdot \dfrac{\partial \Upsilon_s}{\partial x_0^r}$

(2) $\dfrac{\partial u^r}{\partial x_s^r} = \lambda_s^i$, s=1,...,S

(3) $-\lambda_0^i \cdot q_j^i + \displaystyle\sum_{s=1}^{S} \lambda_s^i A_s^j = 0$, for each $j \in J$.

In the same way, for her problem at prices of goods p^k and prices of real assets q^k, we obtain that

(1) $\dfrac{\partial u^r}{\partial x_0^r} = \lambda_0^k \cdot p^k - \displaystyle\sum_{s=1}^{S} \lambda_s^k \cdot \dfrac{\partial \Upsilon_s}{\partial x_0^r}$

(2) $\dfrac{\partial u^r}{\partial x^r_s} = \lambda^k_s,$ s=1,...,S

(3) $-\lambda^k_0 \cdot q^k_j + \sum_{s=1}^{S} \lambda^k_s A^j_s = 0,$ for each $j \in J$.

Since the solution in both problems (the demand) is the same, we get from (2) that $\lambda^i_s = \lambda^k_s$. Then, by (1) we obtain that $p^{*\,i} = \gamma \cdot p^{*\,k}$, with $\gamma = \dfrac{\lambda^k_0}{\lambda^i_0}$, and by (3) that $q^{i\,*}_j = \gamma \cdot q^{*\,k}_j$. Since $(p^{i\,*}, q^{i\,*})$ and $(p^{*\,k}, q^{*\,k})$ are in the simplex $\triangle^{L+J-1}$, it follows that $(p^{*\,k}, q^{*\,k}) = (p^{*\,i}, q^{*\,i})$, a contradiction with $(p^{*\,k}, q^{*\,k}) \neq (p^{*\,i}, q^{*\,i})$ supposed above.

Hence, it only remains to demonstrate that there exists an agent i for which $Card(A^i(\theta^*)) \geq 2$ to conclude that in Nash equilibrium all players propose the same prices. For it, and by contradiction, suppose that $A^i(\theta^*) = \{i\}$ for every i. In this case, $(f^i_0(\theta^*), f^i_a(\theta^*)) = (\omega^i_0, \delta^i)$ for every i. Consider any agent i. There exists a (p, q) in $\triangle^{L+J-1}$ such that $d^i_0(p, q) = \omega^i_0$ and $d^i_a(p, q) = \delta^i$. Since $I \geq 3$, $Card\{(p^{*\,k}, q^{*\,k}) \mid k \neq i\} \geq 2$ and there exists $k \neq i$ such that $(p^{*\,k}, q^{*\,k}) \neq (p^{*\,i}, q^{*\,i})$. Then by definition of (p, q), $v^i(p^{*\,k}, q^{*\,k}) > v^i(p, q)$. By (ii), $u^i(f^i(\theta^*)) \geq v^i(p^{*\,k}, q^{*\,k})$, so $u^i(f^i(\theta^*)) > v^i(p, q)$ a contradiction. Then $Card(A^i(\theta^*)) \geq 2$.

II. Let (x^*, z^*, p^*, q^*) be an equilibrium of E.

Define $\theta^{i\,*} = (x^{*\,i}, z^{*\,i}, p^*, q^*)$ for every i. Then $(f^i_0(\theta^*), f^i_a(\theta^*)) = (x^{*\,i}, z^{*\,i})$. Also $(x^{*\,i}, z^{*\,i}) \in \left(d^i_c(p^*, q^*), d^i_a(p^*, q^*)\right)$.

Let $\theta^i = (x^{*\,i}, z^{*\,i}, p, q)$ with $(p, q) \neq (p^*, q^*)$. In this case the bundle is $(f^i_0(\theta^{*\,-i}, \theta^i), f^i_a(\theta^{*\,-i}, \theta^i)) = (\omega^i_0, \delta^i)$, so $\pi^i(\theta^{*\,-i}, \theta^i) = u^i(f^i(\theta^{*\,-i}, \theta^i)) \leq \pi^i(\theta^*) = u^i(f^i(\theta^*)) = v^i(p^*, q^*)$. On other hand, let $\theta^i = (x^i, z^i, p^*, q^*)$. Then $\pi^i(\theta^{*\,-i}, \theta^i) = u^i(f^i(\theta^{*\,-i}, \theta^i)) \leq \pi^i(\theta^*) = u^i(f^i(\theta^*)) = v^i(p^*, q^*)$.

Therefore, given the strategy profile θ^*, no agent i can get greater payoffs by choosing a strategy different from $\theta^{*\,i}$, while the other players choose $\theta^{*\,-i}$. Hence, θ^* is a Nash equilibrium of Γ.

Q.E.D.

To sum up, in this paper we extend the analysis of Schmeidler (1980) to a two period finance economy with $S \geq 1$ states of nature at the second date characterized by durable goods that depreciate and an incomplete *numeraire* asset structure. For it we have constructed a game in normal form associated to the economy where each player strategy profile is given by prices and quantities of goods and assets and the strategy outcome functions defined map strategies to portfolio and non-negative consumption bundles in both periods. We demonstrate that Nash equilibria of the associated game coincide with spot-finacial equilibria of the underlying economy.

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