A Simple Hypothesis Test for Heteroscedasticity

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Abstract: *The scope of this paper is the presentation of a simple hypothesis test that enables to discern heteroscedastic data from homoscedastic i.i.d. gaussian white noise. The main feature will be a test statistic that's easy applicable and serves well in committing such a test. The power of the statistic will be underlined by examples where it is applied to stock market data and time series from deterministic diffusion a chaotic time series process. It will turn out that in those cases the statistic rejects with a high degree of confidence the random walk hypothesis and is therefore highly reliable. Furthermore it will be discussed, that the test in most cases also may serve as a test for independence and heteroscedasticity in general. This will be exemplified by independent and equally distributed random numbers.*

1. Introduction

The history of this paper is basically, that it was originally part of [1] and was used to show that DAX and Model time series obey similar degrees of heteroscedasticity. However for the sake of economy it was left out of [1] since hetero-scedasticity could also be easily demonstrated graphically. Nevertheless the used test procedure still has got a charm with respect to its simplicity and should therefore be presented in the following. It also contributes another measure in addition to the approaches shown in [2] and can be used complementary.

2. Heteroscedasticity

Heteroscedasticity is a common feature observed in certain time series e.g. financial time series like interest rates or stock returns. It happens to occur when a lot of large changes follow abruptly a series of moderate changes.

Definition 4.2.1(Heteroscedasticity)
Define the m-sample variance estimator at sample point k of a sample of N realizations of a variable $x_1, x_2 \ldots .., x_N$ as:

$$\hat{\sigma}_{m,k}^2 = \frac{1}{m-1}\sum_{i=k}^{m+k}\left(x_i - \hat{\mu}_{m,k}\right)^2$$

$$\text{where } \hat{\mu}_{m,k} = \frac{1}{m}\sum_{i=k}^{m+k} x_i$$

with $1<k<N$ and $n+k<N \ \forall \ k$

is defined as the m-sample mean estimator at sample point k. And define the sample variance estimator by:

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \hat{\mu}_N\right)^2$$

$$\text{where } \hat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$$

is defined as the sample mean estimator. In this context heteroscedasticity means that for N samples there exists significant many values k so that the m-sample estimators of non overlapping sample buckets differ significantly from the sample variance.

To have a clear discern between homoscedastcity and heteroscedasticity of course one needs a hypothesis test. By generating a test statistic, it can be defined what it means that there exist significantly many sample estimators that are distinct from the sample variance.

3. The Test Statistic

The hypothesis of heteroscedasticity will be tested against the Null Hypothesis of a homoscedastic Gaussian random process. To test for heteroscedasticity in time bucket τ the following test statistic is suggested:

$$\mathrm{T}(m,\alpha) = \frac{m\sum_{k=1}^{\mathrm{int}(N/m)} I_{k,m}(\alpha)}{N}$$

Where I_k is an indicator function for the non overlapping buckets k = 1,2,3,…N/m indicating:

$$
I_{k,m}(\alpha)\begin{cases} =1 \quad if \quad \dfrac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} < \dfrac{1}{m-1}\mathrm{X}_{\alpha/2}(m) \\ or \quad if \quad \dfrac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} > \dfrac{1}{m-1}\mathrm{X}_{1-\alpha/2}(m) \\ =0 \quad if \quad \dfrac{1}{m-1}\mathrm{X}_{\alpha/2}(m) \\ < \dfrac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} < \dfrac{1}{m-1}\mathrm{X}_{1-\alpha/2}(m) \end{cases}
$$

Where $X_\alpha(m)$ is the α-Quantile of the χ^2 distribution with m degrees of freedom.

The test is motivated by the fact that one can show under the assumption of a homoscedastic random process that:

$$
\frac{\hat{\sigma}^2_{k,m}}{\hat{\sigma}^2_N} \cong \frac{1}{1-m}\chi^2(m)
$$

where ≅ means is distributed as. I.e. under the assumption of a homoscedastcity gaussian random process the ratio of the sample variance and a m-sample variance should behave like a χ^2 distributed random variable with m degrees of freedom multiplied by 1/(m-1).

Thus the Indicator function always indicates on the test level α if the m-sample variance differs significantly from its expected value at either the upside or the downside that should be measured best by the sample variance. Finally we conclude that:

$$
\mathrm{T} \cong \frac{1}{n}B(n,p) : n = \mathrm{int}(N/m),\ p = \alpha
$$

I.e. T follows a binominal distribution multiplied by 1/n with parameters n=int(N/m), which is the number of time buckets yielded by the choice of time bucket length m, and α is the chosen test level. To commit a statistical test regarding T, one has to check whether the observed quantity of as significant indicated variance changes exceeds a certain 1-β-quantile of the binominal distribution multiplied by 1/N or not:

$$
T > \frac{1}{n}\mathrm{B}_{\beta-1}\left(\mathrm{int}(N/m),\alpha\right) \textit{ reject null}
$$

$$
T < \frac{1}{n}\mathrm{B}_{\beta-1}\left(\mathrm{int}(N/m),\alpha\right) \textit{ not reject null}
$$

The value β will then be the level of confidence of the test.

4. Examples

Table 1 and 2 show the results of the test statistic T for the DAX and Model time series of [1]. Test parameters chosen where α=β= 0.01. The null hypothesis of a homoscedastic i.i.d Gaussian random process is rejected for every time bucket significantly at a confidence level of one percent.

TimeBuckets	Lower-T	Upper-T	Sum-T	P-Value	Reject
50	10,00%	75,00%	85,00%	2,50%	TRUE
60	15,15%	75,76%	90,91%	3,03%	TRUE
70	14,29%	82,14%	96,43%	3,57%	TRUE
80	12,00%	72,00%	84,00%	4,00%	TRUE
90	18,18%	72,73%	90,91%	4,55%	TRUE
100	15,00%	80,00%	95,00%	5,00%	TRUE
110	16,67%	77,78%	94,44%	5,56%	TRUE
120	25,00%	75,00%	100,00%	6,25%	TRUE
130	20,00%	73,33%	93,33%	6,67%	TRUE
140	21,43%	78,57%	100,00%	7,14%	TRUE
150	23,08%	69,23%	92,31%	7,69%	TRUE
160	16,67%	66,67%	83,33%	8,33%	TRUE
170	18,18%	63,64%	81,82%	9,09%	TRUE
180	27,27%	63,64%	90,91%	9,09%	TRUE
190	20,00%	70,00%	90,00%	10,00%	TRUE
200	20,00%	70,00%	90,00%	10,00%	TRUE
220	22,22%	66,67%	88,89%	11,11%	TRUE
250	25,00%	62,50%	87,50%	12,50%	TRUE

Table 1

TimeBuckets	Lower-T	Upper-T	Sum-T	P-Value	Reject
50	17,31%	48,08%	65,38%	1,92%	TRUE
60	16,28%	44,19%	60,47%	2,33%	TRUE
70	18,92%	45,95%	64,86%	2,70%	TRUE
80	18,75%	50,00%	68,75%	3,13%	TRUE
90	21,43%	46,43%	67,86%	3,57%	TRUE
100	23,08%	50,00%	73,08%	3,85%	TRUE
110	17,39%	47,83%	65,22%	4,35%	TRUE
120	23,81%	47,62%	71,43%	4,76%	TRUE
130	25,00%	55,00%	80,00%	5,00%	TRUE
140	22,22%	50,00%	72,22%	5,56%	TRUE
150	23,53%	47,06%	70,59%	5,88%	TRUE
160	25,00%	50,00%	75,00%	6,25%	TRUE
170	26,67%	53,33%	80,00%	6,67%	TRUE
180	28,57%	50,00%	78,57%	7,14%	TRUE
190	15,38%	38,46%	53,85%	7,69%	TRUE
200	30,77%	46,15%	76,92%	7,69%	TRUE
220	27,27%	45,45%	72,73%	9,09%	TRUE
250	30,00%	50,00%	80,00%	10,00%	TRUE

Table 2

Table 1, 2 Results for model time series of the test statistic T for various time buckets by observing N=2000 sample date points for the model time series and DAX time series respectively. Upper-T, Lower-T percentage of samples for which the m-sample bucket variance differs significantly from its expected value indicated by I at either the upside or the downside respectively. Sum-T total percentage of m-sample bucket variance being significantly different from its expected value. P-Value is the 1-β Quantil of the Binomial Distribution B(n,p) Multiplied by 1/N with β = 0,01

From the proof of general limit theorem in [3], one could conjecture, that the intermediate distribution of squares of i.i.d. random variables is a χ^2 distributed random variable until also these sums of i.i.d. random variables converge to a normal distribution.

Therefore the test should also serve as a test for independence and a homoscedastcity in general. Finally table 3 shows the test results independent and equally on the interval [0,1] distributed random numbers. Please note that the test statistic only rejects the homoscedastcity and independence hypothesis for the smallest time bucket.

TimeBuckets	UpperP	LowerP	LowerP+UpperP	P-Value	Reject
20	0,00%	4,00%	4,00%	1,00%	TRUE
30	0,00%	0,00%	0,00%	1,52%	FALSE
40	0,00%	0,00%	0,00%	2,00%	FALSE
50	0,00%	0,00%	0,00%	2,50%	FALSE
60	0,00%	0,00%	0,00%	3,03%	FALSE
70	0,00%	0,00%	0,00%	3,57%	FALSE
80	0,00%	0,00%	0,00%	4,00%	FALSE
90	0,00%	0,00%	0,00%	4,55%	FALSE
100	0,00%	0,00%	0,00%	5,00%	FALSE
110	0,00%	0,00%	0,00%	5,56%	FALSE
120	0,00%	0,00%	0,00%	6,25%	FALSE
130	0,00%	0,00%	0,00%	6,67%	FALSE
140	0,00%	0,00%	0,00%	7,14%	FALSE
150	0,00%	0,00%	0,00%	7,69%	FALSE
160	0,00%	0,00%	0,00%	8,33%	FALSE
170	0,00%	0,00%	0,00%	9,09%	FALSE
180	0,00%	0,00%	0,00%	9,09%	FALSE
190	0,00%	0,00%	0,00%	10,00%	FALSE
200	0,00%	0,00%	0,00%	10,00%	FALSE
210	0,00%	0,00%	0,00%	11,11%	FALSE
220	0,00%	0,00%	0,00%	11,11%	FALSE
230	0,00%	0,00%	0,00%	12,50%	FALSE
240	0,00%	0,00%	0,00%	12,50%	FALSE
250	0,00%	0,00%	0,00%	12,50%	FALSE

Table 1, 2 Results for model time series of the test statistic T for various time buckets by observing N=2000 sample date points of an i.i.d random variable equally distributed on the interval [0,1].

5. Summary and Conclusions

The presented method is straight forward and shows, when applied, significant results indicating its power. It can be used in addition to the methods of [2] but should also work stand alone. Furthermore the test seems to have power in general to verify homoscedastcity and independence at the same time.

6 References

[1] Guido Venier "A New Model for Stock Price Movements" JAES "Journal of Applied Economic Sciences" http://www.jaes.uv.ro. Volume III Issue2(4) Fall2008

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[3] Heinz Bauer: Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie. 4. Auflage. DeGruyter, Berlin 1991