

# MPRA

Munich Personal RePEc Archive

## **Allais-anonymity as an alternative to the discounted-sum criterion in the calculus of optimal growth II: Pareto optimality and some economic interpretations**

Mabrouk, Mohamed  
ESSAI-Tunis

04. April 2006

Online at <http://mpra.ub.uni-muenchen.de/15475/>  
MPRA Paper No. 15475, posted 29. May 2009 / 17:28

Allais-anonymity as an alternative to the  
discounted-sum criterion in the calculus of optimal  
growth

II: Pareto optimality and some economic interpretations

**Mohamed Mabrouk**

*Ecole Supérieure de Statistique et d'Analyse de l'Information de Tunisie  
(Université 7 November à Carthage), 6 rue des métiers, Charguia II, Tunis,  
Tunisia*

*Faculté des Sciences Economiques et de Gestion (Université Tunis El Manar),  
Campus Universitaire, Tunis 1060, Tunisia*

*Personal address : 7 rue des Lys, El Menzah 5, Tunis 1004, Tunisia*

*m\_b\_r\_mabrouk@yahoo.fr*

*tel: 21621141575*

version April 4, 2006

A previous version of this paper circulated under the title "Intergenerational  
anonymity as an alternative to the discounted-sum criterion in the calculus of  
optimal growth: II Pareto-optimality and some economic interpretations"

### **Abstract**

This paper studies the Pareto-optimality of the consensual optimum established in "Allais-anonymity as an alternative to the discounted-sum criterion I: consensual optimality" ([Mabrouk 2006a]). For that, a Pareto-optimality criterion is set up by the application of the generalized Karush, Kuhn and Tucker theorem and thanks to the decomposition of  $l_\infty^*$ . That makes it possible to find sufficient conditions so that a bequest-rule path is Pareto-optimal. Through an example, it is then shown that the golden rule must be checked to achieve Allais-anonymous optimality.

The introduction of an additive altruism makes it possible to highlight the intergenerational-preference rate compatible with Allais-anonymous optimality. In this approach, it is not any more the optimality which depends on the intergenerational-preference rate, but the optimal intergenerational-preference rate which rises from Allais-anonymous optimality.

*JEL classification:* D90; C61; D71;D63; O41; O30.

*Keywords:* Intergenerational anonymity; Allais-anonymity; Intergenerational equity; Optimal growth; Technical change; Time-preference; Discounted-sum criterion; Consensual criterion; Pareto-optimality; OG economy.

# 1 Introduction

## 1.1 Motivation

This paper pursues two goals. First, we seek to set up a criterion of Pareto-optimality applicable to a situation with exogenous technical change, overlapping generations, bequests and infinite horizon (sections 2, 3 and 4), with an aim of judging efficiency of the consensual optimum (with a Allais-anonymous consensual criterion) partly characterized in the article "Allais-anonymity as an alternative to the discounted-sum criterion I: consensual optimality" [Mabrouk 2006a] and of which this article constitutes the prolongation.

The criterion of Pareto-optimality is obtained thanks to the direct application of the generalized theorem of Karush, Kuhn and Tucker and also thanks to the decomposition of  $l_\infty^*$  (see [Mabrouk 2006a]) which enables to calculate the adjoint variable of the program defining Pareto-optimality. Using a suitable adaptation of the variables, one realizes that this criterion is in fact nothing but a particular case of the Pareto-optimality criterion of [Cass 1972] or the Pareto-optimality criterion of [Balasko-Shell 1980], although the method implemented here differs by the fact that it has recourse to the tools of the theory of optimization. It appears indeed that the criterion used here is equivalent to these criteria in the case of regular bequests plans. On the other hand, it does not constitute a necessary condition of Pareto-optimality in the case of nonregular bequests plans, whereas it is the case for Cass and Balasko-Shell criteria. But the ambition here is not to establish a complete characterization of Pareto-optimality following the example of [Cass 1972] or [Balasko-Shell 1980]. However, although incomplete, the criterion suggested here does not require a condition of minimal curvature on the indifference curves and can be thus extended to the case of unbounded capital without involving differentiability problems for the sequence of utility functions (section 2).

Section 2 establishes the criterion of Pareto-optimality. Section 3 considers the case where the growth rate of the capital is not the maximum rate, case not taken into account by section 2, the optimum being then non-interior.

In absence of a general result on the Pareto-optimality of a consensual optimum, whereas there was such a result in the case without technical change [Mabrouk 2005], section 4 gives some sufficient conditions for a consensual optimum to be Pareto-optimal. That will make it possible in certain cases, as in the example of section 5, to partly characterize the optimal growth path which satisfies at the same time consensual optimality and Pareto-optimality.

The second goal is to highlight certain properties of the optimal growth path to draw some economic interpretations from them. Will be successively approached the comparison between golden-rule states (that are shown to coincide asymptotically with optima) with and without technical change in a discrete-time case (subsection 6.1), the analysis of the stability of the optimal path with introduction of an additive altruism (subsection 6.2) and finally, the comparison between Allais-anonymous and discounted-sum criteria (subsection 6.3).

All proofs are gathered in section 7 except those relating to the discrete-time

example (section 5) which can be found in appendix A.

The model used in this paper is exactly that of [Mabrouk 2006a]. We start by recalling its essential features, assumptions and the principal results. To have more details on the model, it is preferable to consult [Mabrouk 2006a].

## 1.2 Model, assumptions and results on consensual optimality

The economy is constituted by a succession of generations  $g_1, g_2, g_3, \dots$ , each generation being made up of only one individual who is at the same time consumer and producer. At the beginning of its active life, a given generation  $g_i$  inherits a quantity  $b_{i-1}$  of that good. Its only acts during its life are: to consume, produce, invest in order to increase its future consumption and, at the end of the active lifetime, to bequeath  $b_i$  to the descent. In doing so, generation  $g_i$  achieves a level of life-utility  $U_i(b_{i-1}, b_i)$ . Each utility function  $U_i$  is defined from  $D_i \subset \mathbb{R}_+^2$  to  $\mathbb{R}$ .  $D_i$  is strictly included in  $\mathbb{R}_+^2$ , closed and with a non-empty interior;  $U_i$  is concave and of class  $C_2$  on  $D_i$ ;  $U'_{ih} \succ 0$  ( $U'_{ih}$  and  $U'_{il}$  are respectively the derivatives of  $U_i$  with respect to its first and second variable).

For  $r \geq 0$ , denote  $l_\infty^r = \{B = (b_1, b_2, \dots) / b_i \in \mathbb{R} \text{ and } \sup_{i \geq 1} |b_i| e^{-ri} < +\infty\}$  the Banach space normed by  $\|B\|_r = \sup_{i \geq 1} |b_i| e^{-ri}$ .

The set  $D = \{K = (k_1, k_2, \dots) / \text{for all } i \geq 1 : (k_{i-1}, k_i) \in D_i\}$  is assumed to be strictly geometric of reason  $p \geq 0$  (see [Mabrouk 2006a] section 5 for the definition of strict geometricity) and  $G(D)$  strictly geometric of reason  $p_1 \geq 0$ ,  $G$  being the mapping that associates to  $K \in D$ ,  $G(K) = (U_i(k_{i-1}, k_i))_{i \geq 1}$ .

Denote  $\overset{\circ}{D}$  the interior of  $D$  in  $l_\infty^p$  with respect to the norm  $\|B\|_p = \sup_{i \geq 1} |b_i| e^{-pi}$ . It has been proved in [Mabrouk 2006a], section 6, that if  $G$  is linear at infinity at  $K \in \overset{\circ}{D}$  for the reasons  $(p, p_1)$  then  $G$  is Frechet-differentiable at  $K$ .

Consider a consensual criterion represented by a real valued, Frechet-differentiable functional  $\Psi$  on  $l_\infty^{p_1}$ . The consensual value of a state  $K \in D$  is given by  $\Psi(G(K))$ . Suppose also that  $\Psi$  is Allais-anonymous and sensitive to long run interest (definitions in [Mabrouk 2006a], section 7).

It has been proved in [Mabrouk 2006a], section 7, theorem 18, that if a steady state<sup>1</sup>  $K$  in  $\overset{\circ}{D}$  is a consensual optimum for the criterion  $\Psi$  then

$$u'_h e^{-p} + u'_l = 0 \tag{1}$$

where  $u'_h = \lim U'_{nh}(k_{n-1}, k_n) e^{-(p_1-p)n}$  and  $u'_l = \lim U'_{nl}(k_{n-1}, k_n) e^{-(p_1-p)n}$ .

Equation (1) is the bequest-rule and characterizes consensual optimality.

---

<sup>1</sup> definition 12 in [Mabrouk 2006a]

## 2 Pareto optimality

### 2.1 Introduction

The criterion  $\Psi$  not being strictly increasing, it is not sure that any solution of the first order condition (1) is Pareto optimal. That's why an efficiency criterion is needed. It is the objective of the present section.

Define  $D_l = \{B \in D / \text{for all } i \geq 1, U'_{il}(b_{i-1}, b_i) \leq 0\}$ . Observe that a bequests plan which is not in  $D_l$  is not of interest since it cannot be Pareto optimal. We will henceforth look for solutions in  $D_l$ .

Suppose <sup>2</sup>

$$\overset{\circ}{D} \cap D_l \neq \emptyset \quad (\text{A1})$$

Let

$$K \in \overset{\circ}{D} \cap D_l \quad (\text{A2})$$

Suppose

$$(U_i)_{i \geq 1} \text{ linear at infinity at } K \quad (\text{A3})$$

We will first consider the case where

$$U'_{nl}(k_{n-1}, k_n) < 0 \text{ for all } n \geq 1 \quad (\text{A4})$$

The latter assumption will be used to set regularity and then dropped.

Let  $B \in D$ . For  $i \geq 1$ , let  $T_i$  be the transformation which suppresses the  $i^{\text{th}}$  component of an element of  $l^\infty$ , replaces it by the next one and shifts all the following components backward. Let  $e_i$  be the sequence of  $l^\infty$  which components are all 0 except the  $i^{\text{th}}$  equal to 1.

Denote  $H_i(B) = T_i(G(B) - G(K))$ .

Under the above assumptions,  $G$  is Frechet-differentiable at  $K$ . This implies that  $H_i$  and  $U_i$  are also Frechet-differentiable at  $K$  and we have

$$\delta H_i(K) = T_i(\delta G(K))$$

and

$$\delta U_i(K) = e_i \mid \delta G(K)$$

where  $\delta$  preceding a transformation means its Frechet-differential.

The program  $P_i(K)$  which gives Pareto optimality, can be written

$$\begin{aligned} & \max_{B \in D} e_i \mid G(B) \\ & \text{subject to} \quad : \quad H_i(B) \geq 0 \end{aligned}$$

---

<sup>2</sup>  $\overset{\circ}{D} \cap D_l = \emptyset$  would mean that in the interior of  $D$ ,  $U'_{il} \geq 0$ . Thus, the optimum would not be interior to  $D$ . For example if  $U'_{il}$  is everywhere positive, there would not be a real conflict of interest between a generation and the following generations. The optimum would consist in always bequeathing the maximum.

## 2.2 Regularity of $K$ for the inequality $H_i(B) \geq 0$

To apply the Karush-Kuhn-Tucker theorem to  $P_i(K)$ , we have to make sure that  $K$  is a regular point of the inequality  $H_i(B) \geq 0$ . This means that  $H_i(K) \geq 0$  and that there is  $X \in l_\infty^p$  such that  $H_i(K) + \delta H_i(K) \cdot X \succ 0$  (which means that all components are strictly positive and that the sequence is strictly of reason  $p_1$ ).

Denote henceforth  $u'_{hn} = U'_{nh}(k_{n-1}, k_n)$  and  $u'_{ln} = U'_{nl}(k_{n-1}, k_n)$ .

Define  $R_\sharp$  and  $R_b$  as follows

$$R_\sharp = \limsup \frac{u'_{hn}}{-u'_{ln}}, \quad R_b = \liminf \frac{u'_{hn}}{-u'_{ln}}$$

According to proposition 10 of [Mabrouk 2006a], the sequences  $(u'_{hn})_{n \geq 1}$  and  $(u'_{ln})_{n \geq 1}$  are in  $l_\infty^{p_1-p}$ . We need to assume that either  $(u'_{hn})_{n \geq 1}$  or  $(u'_{ln})_{n \geq 1}$  are in  $s_\infty^{p_1-p}$  to set regularity.

**Proposition 1** *Under assumptions (A1, A2, A3 and A4),  $K$  is regular if either  $(u'_{hn})_{n \geq 1}$  or  $(u'_{ln})_{n \geq 1}$  is in  $s_\infty^{p_1-p}$  and if either  $R_\sharp \prec e^p$  or  $R_b \succ e^p$ .*

**Remark 2** *If  $p_1$  was not the strict reason of  $G(D)$ , there would be  $p'_1 \prec p_1$  such that  $(u'_{hn})_{n \geq 1}$  and  $(u'_{ln})_{n \geq 1}$  are in  $l_\infty^{p'_1-p}$  which is contrary to  $(u'_{hn})_{n \geq 1}$  or  $(u'_{ln})_{n \geq 1} \in s_\infty^{p_1-p}$ . Thus, we would not have regular points. Hence, the assumption of strict geometricity of  $G(D)$  is crucial for the necessity of the Pareto-optimality criterion given by proposition 3. The strict geometricity of  $D$ , as for it, is crucial for both necessity and sufficiency as far as we need definition sets with non empty interiors to use optimization theorems.*

Denote

$$L_+ = \left\{ \begin{array}{l} K \in D / \limsup \frac{u'_{hn}}{-u'_{ln}} \prec e^p \text{ and} \\ \text{either } (u'_{hn})_{n \geq 1} \text{ or } (u'_{ln})_{n \geq 1} \text{ is in } s_\infty^{p_1-p} \end{array} \right\}$$

and

$$L_- = \left\{ \begin{array}{l} K \in D / \liminf \frac{u'_{hn}}{-u'_{ln}} \succ e^p \text{ and} \\ \text{either } (u'_{hn})_{n \geq 1} \text{ or } (u'_{ln})_{n \geq 1} \text{ is in } s_\infty^{p_1-p} \end{array} \right\}$$

### 2.3 Necessity

Suppose

$$\overset{\circ}{D} \cap D_l \cap (L_- \cup L_+) \neq \emptyset \quad (A'1)$$

**Proposition 3** Under the assumptions (A'1),  $K \in \overset{\circ}{D} \cap (L_- \cup L_+)$  and (A3), if  $K$  is a Pareto-optimal bequests plan then, for all  $i \geq 1$ , we have:

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{li+j} e^{p_1}}{u'_{hi+j+1}} \right| < +\infty$$

### 2.4 Sufficiency

**Proposition 4** Under assumptions (A1), (A2) and (A3), let  $i$  such that if  $i > 1$  we have:  $\prod_{j=1}^{i-1} u'_{lj} \neq 0$ .

If

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{li+j} e^{p_1}}{u'_{hi+j+1}} \right| < +\infty$$

then  $K$  is solution of  $P_i(K)$ .

If  $K$  is such that for all  $i \geq 1$  we have  $D_2 U_i(k_{i-1}, k_i) < 0$ , then we have:

$$\left( \sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{p_1}}{u'_{h2+j}} \right| < +\infty \right) \implies \left( \sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{li+j} e^{p_1}}{u'_{hi+j+1}} \right| < +\infty \right) \text{ for all } i \geq 1$$

If  $K$  is a solution of  $P_i(K)$  for all  $i \geq 1$ , then  $K$  is a Pareto-optimal bequests plan. Thus, we can state:

**Proposition 5** Under the assumptions (A1), (A2), (A3) and (A4), if

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{p_1}}{u'_{h2+j}} \right| < +\infty$$

then  $K$  is a Pareto-optimal bequests plan.

**Remark 6** (a) The condition  $K \in \overset{\circ}{D}$  is needed only to ensure  $G$ 's differentiability at  $K$ . If, besides, we are sure that  $G$  is differentiable at  $K$ , we don't need anymore this interiority condition. (b) In propositions 4 and 5, one could omit the assumption of strict geometricity of  $G(D)$ .



From proposition 3 and proposition 5, we deduce the following theorem:

**Theorem 7** *Under the assumptions (A'1),  $K \in \overset{\circ}{D} \cap (L_- \cup L_+)$ , (A3), and (A4),  $K$  is a Pareto-optimal bequests plan if and only if*

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l_{1+j}} e^{p_1}}{u'_{h_{2+j}}} \right| \prec +\infty$$

This condition implies that, for "most" generations we have:

$$-U'_{nl}(k_{n-1}, k_n) e^{p_1} \prec U'_{n+1h}(k_n, k_{n+1})$$

which means that **if generation  $g_n$  decreases its bequest by one unit, it wins  $e^{p_1}$  times less than what is lost by generation  $g_{n+1}$** . That suggests that the agents can be all the more selfish as  $p_1$  is large, because the reduction of heritage by a generation, without damage for all the line, is all the more high as  $p_1$  is large. This idea will be specified in section 6.

### 3 If $K$ has not the maximum growth rate

#### 3.1 Introduction

We supposed above that  $K \in \overset{\circ}{D}$ . But since  $\overset{\circ}{D} \subset s_{\infty++}^p$ ,  $\liminf |k_n| e^{-pn} \succ 0$  and  $k_n$  grows at the maximum rate  $e^p$ . We then need another method to test Pareto optimality for a bequests plan  $K$  which doesn't grow at the maximum rate.

**Definition 8** *Let  $\pi \in [0, p]$ ,  $D_\pi = D \cap l_\infty^\pi$  and  $\pi_1 = \text{reason of } G(D_\pi)$ . For  $K \in \overset{\circ}{D}_\pi$ , define linearity at infinity in  $l_\infty^\pi$  exactly as in definition 8 of [Mabrouk 2006a] after having replaced  $p$  and  $p_1$  respectively by  $\pi$  and  $\pi_1$ .*

In this section, the condition of linearity at infinity refers to linearity at infinity in  $l_\infty^\pi$  for the reasons  $(\pi, \pi_1)$ .

### 3.2 Necessity:

**Proposition 9** *If a bequests plan is in  $D_\pi$  where  $\pi \in [0, p]$ , all bequests plans that Pareto-dominate it are also in  $l_\infty^\pi$ .*

Consequently, if a bequests plan  $K$  in  $D_\pi$  is Pareto-optimal in  $D_\pi$ , it is also Pareto-optimal in  $D$ .

Suppose  $\overset{\circ}{D}_\pi \neq \emptyset$  (using the appropriate norm  $\|\cdot\|_\pi$ ) and  $G(D_\pi)$  strictly geometric of reason  $\pi_1$ . Let  $K \in \overset{\circ}{D}_\pi$ . Since  $\overset{\circ}{D}_\pi \subset s_{\infty^{++}}^\pi$  this implies that  $K \in s_{\infty^{++}}^\pi$ . Observe that the growth rate of  $K$  is now  $e^\pi \prec e^p$ . Denote

$$L_{\pi+} = \left\{ \begin{array}{l} K \in D / \limsup \frac{u'_{hp}}{u'_{ln}} \prec e^\pi \text{ and} \\ \text{either } (u'_{hn})_{n \geq 1} \text{ or } (u'_{ln})_{n \geq 1} \text{ is in } s_{\infty^{\pi_1 - \pi}} \end{array} \right\}$$

and

$$L_{\pi-} = \left\{ \begin{array}{l} K \in D / \liminf \frac{u'_{hp}}{u'_{ln}} \succ e^\pi \text{ and} \\ \text{either } (u'_{hn})_{n \geq 1} \text{ or } (u'_{ln})_{n \geq 1} \text{ is in } s_{\infty^{\pi_1 - \pi}} \end{array} \right\}$$

Now change  $p$  by  $\pi$  in the proposition 3,

It then gives:

**Proposition 10** *Under the assumptions (A'1),  $K \in \overset{\circ}{D}_\pi \cap (L_{\pi-} \cup L_{\pi+})$  and (A3)<sup>3</sup>, if  $K$  is a Pareto-optimal bequests plan in  $D$  then for all  $i \geq 1$  we have:*

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{i+j} e^{\pi_1}}{u'_{h+i+j+1}} \right| \prec +\infty$$

### 3.3 Sufficiency

Let  $\pi \in [0, p]$  and  $K \in \overset{\circ}{D}_\pi$  such that  $G$  is linear at infinity at  $K$  in  $l_\infty^\pi$ . Denote  $\pi_1$  the reason of  $G(D_\pi)$ . As we have done above, change  $p$  by  $\pi$  and  $p_1$  by  $\pi_1$  in proposition 5, theorem 7 and in assumptions (A'1, A3, and A4).

It then gives:

**Proposition 11** *If*

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{\pi_1}}{u'_{h2+j}} \right| \prec +\infty$$

*then  $K$  is a Pareto-optimal bequests plan.*

---

<sup>3</sup>For (A'1) and (A3), use  $(\pi, \pi_1)$  instead of  $(p, p_1)$ .

**Remark 12** *As in proposition 5, one could omit the assumption of strict geometricity of  $G(D_\pi)$ .*

**Theorem 13** *Let  $K \in \overset{\circ}{D}_\pi \cap (L_{\pi-} \cup L_{\pi+})$ .  $K$  is a Pareto-optimal bequests plan if and only if*

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l_{1+j}} e^{\pi_1}}{u'_{h_{2+j}}} \right| \prec +\infty$$

**Remark:** *For a bequests plan  $K$  which has not a strict reason, we cannot apply the propositions and theorem of this section since there is not  $\pi$  such that  $K \in \overset{\circ}{D}_\pi$ .*

## 4 Is a consensus-optimal plan Pareto-optimal?

An optimal growth path has to be at the same time consensus-optimal and Pareto-optimal. We have then to select from the set of consensus-optima those which are Pareto-optimal.

There is not here a general result on the Pareto optimality of consensual optima or on the Pareto-optimality of bequest-rule plans as in the case without technical change [Mabrouk 2005].

Indeed, in the case of an Allais-anonymous consensual criterion (which is, I think, the more interesting case), it is not certain that there exists a consensual optimum that is Pareto-optimal or a bequest-rule plan that is Pareto-optimal.

Nevertheless, the following propositions should help answer the question of Pareto-optimality of a bequest-rule plan in some practical cases. They give sufficient conditions for Pareto-optimality.

**Proposition 14** *Let  $K$  be a bequest-rule plan. Take  $\Delta k_1 \succ 0$ . For  $n \geq 1$  define the sequence  $(\Delta k_n)$  as follows:*

$$U_{n+1}(k_n - \Delta k_n, k_{n+1} - \Delta k_{n+1}) = U_{n+1}(k_n, k_{n+1})$$

*If for all  $\Delta k_1 \succ 0$  the sequence  $(\frac{\Delta k_{n+1}}{\Delta k_n})$  is increasing, then  $K$  is Pareto-optimal.*

**Remark 15** *If we had only for every  $\Delta k_1 \succ 0$  an  $\varepsilon \succ 0$  such that  $\frac{\Delta k_{n+1}}{\Delta k_n} \succ e^{\varepsilon} + \varepsilon$ , this would be anyway a proof of the Pareto-optimality of  $K$ .*

**Proposition 16** Let  $\alpha$  be a real,  $m$  a positive integer and  $(r_n)$  a sequence of reals such that  $\alpha \succ 1$ ,  $\lim r_n = 0$  and  $\sum r_n \prec +\infty$ . If there is a real  $\pi$  in  $[0, p]$  and a steady state  $K$  in  $\overset{\circ}{D}_\pi$  such that  $G$  is linear at infinity at  $K$  in  $l_\infty^\pi$  and such that:

$$\frac{-U'_{nl}(K)}{U'_{nh}(K)} e^p = 1 - \frac{\alpha}{n+1} + r_{n+1} \text{ for all } n \geq m \quad (2)$$

then  $K$  is a Pareto-optimal bequest-rule plan.

## 5 A discrete-time example

### 5.1 Introduction

Consider the labor-saving technical change case with a unique period. The agent gets born in the beginning of the period, immediately inherits a capital  $h$  and begins to produce with this capital. At the end of the period, the agent consumes  $c$ , the capital is depreciated of  $ah$ , the agent bequeaths  $l$  and dies.

Consumption of generation  $g_i$  is  $c = F(L^{i-1}, h) - ah - (l - h)$  where  $F$  is the production function and  $L$  is the exogenous labor-saving technical change factor ( $L \succ 1$ ).

The satisfaction level achieved by the generation  $g_i$  is then

$$U_i(h, l) = u(c) = u(F(L^{i-1}, h) - ah - (l - h)) \quad (3)$$

with the constraint  $c \in [0, F(L^{i-1}, h)]$ .

Suppose that, on top of meeting standard assumptions of respectively production and utility functions,  $u$  and  $F$  are strictly concave, increasing, continuous and twice derivable on their definition domains. Then, we check easily that all needed assumptions on  $(U_i)_{i \geq 1}$  are fulfilled.

Suppose  $F$  homogenous ( $F(\lambda X, \lambda Y) = \lambda F(X, Y)$ ),  $F(1, 0) \geq 0$ ,  $u(0) \geq 0$  and  $\lim_{y \rightarrow 0} D_2 F(1, y) \succ a + L - 1$ .

There is not much lost of generality in the two latter assumptions since we don't change the problem when we add a constant to  $u$  and if  $\lim_{y \rightarrow 0} D_2 F(1, y) \leq a + L - 1$ , we will see further that the productivity of capital would be so low that it will not be interesting any more to accumulate.

Also without loss of generality, suppose, to simplify, that the start-up capital  $k_0$  is strictly positive.

Suppose that  $a \in ]0, 1[$ . It means that capital does depreciate, but it can never disappear completely from the only fact of its depreciation.

Lastly, suppose  $\lim_{y \rightarrow +\infty} D_2 F(1, y) \prec a$ . As we shall see, this guarantees the geometricity of bequests. It means that for the first generation, from a given level of accumulation, marginal productivity falls under the rate of capital depreciation. Consequently, at this level, it would not be worth accumulating any more.

Denote  $f(k) = F(1, k)$ . Then

$$U_i(h, l) = u \left( L^{i-1} f\left(\frac{h}{L^{i-1}}\right) - ah - (l - h) \right)$$

and the definition domain of  $U_i$  is

$$D_i = \left\{ (h, l) / L^{i-1} f\left(\frac{h}{L^{i-1}}\right) - ah - (l - h) \geq 0 \right\}$$

For subsections 5.3 and 5.4, we will adopt the following assumptions:

$$G(D) \text{ is strictly geometric of reason } p_1 \quad (4)$$

$$G \text{ is linear at infinity on } \overset{\circ}{D} \text{ for the reasons } (LogL, p_1) \quad (5)$$

where  $Log$  denotes the Napierian logarithm.

The proofs of this section are in appendix A.

## 5.2 Geometricity

**Proposition 17** *Under the assumptions of subsection 5.1 on  $u$  and  $F$ ,  $D$  is strictly geometric of reason  $L$ .*

**Proposition 18** *Under the assumptions of subsection 5.1 on  $u$  and  $F$ ,  $G(D) \subset l_{\infty}^{LogL}$ .*

The above proposition indicates that if  $G(D)$  admits a strict reason, it is lower than  $LogL$ . But it doesn't give the exact reason of  $G(D)$ . For this, it is necessary to specify  $u$ , what I do in subsection 5.5.

## 5.3 Consensual optimality

Denote

$$w^* = f'^{-1}(a + L - 1)$$

**Proposition 19** *Under assumptions of subsection 5.1, there is  $i \geq 1$  and*

$$(k_1^*, k_2^*, \dots, k_i^*)$$

*such that the plan  $K^* = (k_1^*, k_2^*, \dots, k_i^*, L^{i+1}w^*, L^{i+2}w^*, \dots)$  is interior to  $D$ .*

Let  $\chi \succ 0$  such that  $S(K^*, \chi) \subset D$ . Let  $K$  be a plan such that

$$K \in S(K^*, \chi) \quad (6)$$

and that

$$\lim \frac{k_n}{L^n} = w^* \quad (7)$$

then:

**Proposition 20** *For a consensual criterion meeting the assumptions of subsection 1.2 and under the assumptions of subsection 5.1 on  $u$  and  $F$  and assumptions (4), (5), (6) and (7),  $K$  is an interior bequest-rule plan.*

## 5.4 Pareto-optimality

Let  $\xi$  be a real and  $(r_n)$  a sequence of reals such that  $\xi \succ 1$ ,  $\lim r_n = 0$  and  $\sum r_n \prec +\infty$ . Denote

$$x_n = a + \frac{1}{\left(1 - \frac{\xi}{n+1} - r_{n+1}\right)} L - 1$$

We have  $\lim x_n = a + L - 1$ . Therefore, we can define a bequests plan  $K^{**}$  such that (6) is checked and that, from a given index, we have:  $\frac{k_{n-1}^{**}}{L^{n-1}} = f'^{-1}(x_n)$ .  $K^{**}$  is built of kind to meet the assumptions of propositions 20 and 16.

**Proposition 21** *Under the same assumptions that proposition 20,  $K^{**}$  is an interior Pareto-optimal bequest-rule path.*

## 5.5 Checking of assumptions (4) and (5) for two particular utility functions

Assumptions (4) and (5) doesn't necessarily hold for all functions  $u$  and  $f$ . For example, (4) doesn't hold for  $u(c) = \text{Log}c$ .<sup>4</sup>

I have studied the case of an hyperbolic function of utility :  $u(c) = \alpha c + 1 - \frac{1}{c+1}$  with  $\alpha \succ 0$  and the case  $u(c) = c^{1-\theta}$  with  $\theta$  in  $]0, 1[$ . In the first case, except for assumptions (4) and (5), it is easy to see that all the other desired

---

<sup>4</sup>It is probable that these questions are primarily of a mathematical nature. With better mathematics, it should be possible, I believe, to extend the results to these cases. For example, for the case  $u(c) = \text{Log}c$ , we could plunge  $G(D)$  in the space  $l_{\infty}^1(n) = \left\{ B = (b_1, b_2, \dots) / b_i \in \mathbb{R} \text{ and } \sup_{i \geq 1} \frac{|b_i|}{e^{p1^i}} \prec +\infty \right\}$  and use the norm  $\|B\| = \sup_{i \geq 1} \frac{|b_i|}{e^{p1^i}}$ . We have also to change the criterion  $\Psi$ .

assumptions hold. But in the case  $u(c) = c^{1-\theta}$ ,  $U_i$  is not differentiable at 0. However, one can check that for a bequests plan  $K$  which is "candidate" to be an optimal growth path, it is possible to find a strictly positive real  $\varepsilon$  such that

$$\inf_{\substack{i > 1 \\ B \in S(K, \varepsilon)}} c_i \succ 0$$

Thus, assumptions of subsection 1.2 are fulfilled on  $S(K, \varepsilon)$ , which is enough for the validity of the results.

In appendix A, I give four propositions that prove that conditions (4) and (5) are checked, respectively for the case  $u(c) = \alpha c + 1 - \frac{1}{c+1}$  (propositions 26 and 27) and  $u(c) = c^{1-\theta}$  (propositions 28 and 29).

## 5.6 The neutral case

Consider now the same discrete-time example, but with neutral technical change (see [Solow 1956]), and with a Cobb-Douglas production function<sup>5</sup>. The utility of generation  $g_i$  is in this case :  $U_i(h, l) = u(N^{i-1}fh^\eta - ah - (l - h))$ , where  $\eta$  is the share of the income of the capital in the total production and  $N$  is the exogenous neutral technical change factor ( $N \succ 1$ ).

Although this case is not presented in detail here (because of the length of the calculus), it is interesting to quote the following results:

- As in the LS case, asymptotically, the bequest-rule path doesn't depend on the utility function, as long as the needed assumptions hold.
- Capital and production grow at the rate  $N^{\frac{1}{1-\eta}}$ , faster than technical change, whereas growth rate is the same for capital, production and technical change in the LS case.

# 6 Some implications of the bequest rule

## 6.1 Golden rule

The following analysis is based on the example of section 5.

The equation (7) indicates that the marginal productivity of capital  $mpc = D_2F(L^n, k_n) = L^n f'(\frac{k_n}{L^n})$  tends to  $a + L - 1$ . The equation  $mpc^* = a + L - 1$  replaces the golden rule  $mpc_0 = a$  characterizing the optimal path without technical change ([Mabrouk 2005]). This means that the optimal level of capital with technical change is always lower than the one without technical change,

---

<sup>5</sup>Contrary to the LS case, marginal productivity has to tend to 0 for infinite capital, which is the case for the Cobb-Douglas function. Otherwise, economy would grow faster than geometrically.

more precisely, the level of capital which would be optimal if technical progress had suddenly stopped.

Why isn't it optimal to reach the level that guarantees  $mpc_0 = a$ , despite that, at the level  $mpc^* = a + L - 1$ , an increase of capital by one unit implies a net increase of production?

The answer is that it would be too expensive for a father to bequeath a capital meeting the golden rule of his son. Indeed, with technical change, the golden-rule level of capital "flees". The source of non-optimality in trying to catch up with it, is that the effort made by a generation to enhance the satisfaction of its heir, hurts its own satisfaction more than what this generation gains from a similar effort by its predecessor.

In the neutral case, like in the LS case, the optimal marginal productivity of capital ( $mpc^*$ ) is equal to  $a + N - 1$ , where  $N$  denotes the rate of neutral technical change.

In addition, asymptotically, the optimal growth path doesn't depend on the choice of utility function, as long as the needed assumptions hold.

## 6.2 Welfare analysis

I try here to assess to what extent a behavior led by personal incentives is coherent with the optimal growth path. The observation that bequest constitutes ultimately the current shape of transmission of capital from generation to generation, led me to give to bequests a crucial role in the model. Although other bequest-motives exist and other kinds of altruism can be used (see [Saez-Marti-Weibull 2005, Lakshmi 2002, Barro 1974]), the proposed welfare analysis is limited to the introduction of a father-son altruism, additively separable from selfish utility and supposed to justify bequests.

Let's call "spontaneous equilibrium" the state of the economy achieved when agents behave according to their personal incentives. I do not use the name "competitive equilibrium" because it refers to the general equilibrium approach and to the assumptions which are attached to it, in particular the assumption of competition and price-taker behavior. The latter assumption is not adapted to a model which disregards the space dimension of the economy and where each agent is alone and cannot thus consider that the prices are imposed to him.

The question then amounts to find out what must be the intensity of the father-son altruism so that the spontaneous equilibrium it achieves coincides with the optimal growth path.

As in [Mabrouk 2005], personal incentives are modeled by a utility function of the form:

$$V_n(h, l) = U_n(h, l) + A_n(l)$$

where  $U_n$  is the selfish component of generation  $g_n$ 's utility and  $A_n$  is the altruistic component.  $A_n$  represents the feelings of generation  $g_n$  for its heir and is supposed to be increasing with respect to  $l$ ,  $C_1$  and strictly concave. Consequently, the altruism expressed by  $A_n$  is limited to the agent's family. That's



why it was called "familial altruism" in [Mabrouk 2005]. Hence, despite the presence of this familial altruism component,  $V_n$  can still be considered as an expression of personal incentives.

Let  $K$  be an interior steady-state optimal growth path and suppose that

$$U'_{nl}(k_{n-1}, l_{\max n}(k_{n-1})) + A'_n(l_{\max n}(k_{n-1})) \leq 0 \quad (8)$$

and that

$$U'_{nl}(k_{n-1}, l_{\min n}(k_{n-1})) + A'_n(l_{\min n}(k_{n-1})) \geq 0 \quad (9)$$

Assumptions (8) and (9) make sure that each of  $U_n$  and  $A_n$  is operative when  $g_n$  is to behave according to its personal incentives.

**Proposition 22** *Under assumptions (8), (9) and assumptions of subsection 1.2, if the interior steady-state optimal growth path  $K$  coincide with a spontaneous equilibrium then:*

$$u'_h = a' e^p \quad (10)$$

where  $e^p$  is the maximum growth rate of capital,  $u'_h = \lim u'_{hn} e^{-(p_1-p)n}$ ,  $u'_l = \lim u'_{ln} e^{-(p_1-p)n}$ ,  $a' = \lim A'_n e^{-(p_1-p)n}$  and  $e^{p_1}$  the maximum growth rate of utility.

The left hand-side of equation (10) is the increase of selfish utility resulting from an increase of heritage by one unit. The right hand-side is  $e^p$  times the increase of altruistic utility resulting from an increase of bequest by one unit.

To interpret this, observe that the ratio

$$s = \frac{u'_h}{a'} \quad (11)$$

measures selfishness with respect to the heir. Indeed, if one was to choose between an increase of his own heritage by one unit and the increase of his heir's heritage by one unit, one should assess the selfishness ratio  $s$ . If  $s \geq 1$ , the increase of own heritage is better and conversely if  $s \leq 1$ .

Thus, we can write (10) in another way:

$$s = e^p \quad (12)$$

what tells that one can be all the more selfish as technical progress, or more precisely, maximum capital growth rate, is significant. Much more, it is imperative to be more selfish if technical progress increases.

If  $s > e^p$ , agents being too selfish, we can expect accumulation to be insufficient.

If  $s < e^p$ , agents are not selfish enough but it is not certain whether the economy would go over-accumulated, which means that marginal productivity is lower than the capital depreciation ratio  $a$ . As said above (subsection 6.1)

the source of non-optimality could be that agents damage themselves in trying to catch up with the level of capital meeting the golden rule of next generation. The consequence is that everybody is worse off!

To some extent, relation (12) indicates that technical progress compensates for selfishness.

### 6.3 Comparison between discounted-sum and Allais-anonymous criteria

In this subsection, we place ourselves under the assumptions of subsection 1.2 and the assumption (A3).

Another way to present the relation between selfishness and technical change is to define the altruistic component of utility as next generation utility discounted by an own-generation-preference rate  $\rho$  ([Groth 2003, Heidjra-van der Poeg 2002, Barro 1974]):

$$A_n = \frac{1}{\rho} V_{n+1} = \frac{1}{\rho} (U_{n+1} + A_{n+1}) \quad (13)$$

Define as myopic spontaneous equilibrium the state of the economy achieved when each generation  $g_n$  maximizes  $U_n + \frac{1}{\rho} (U_{n+1} + A_{n+1})$  considering that  $(k_{n+1}, k_{n+2}, \dots)$  don't depend on its control variable:  $k_n$ , and define as rational-expectations spontaneous equilibrium the state of the economy achieved when each generation takes fully into account the changes of behavior of all posterior generations, induced by the variation of its own control variable.

**Proposition 23** *Let  $K^m$  be an interior steady-state myopic spontaneous equilibrium. Then, on the path  $K^m$ :*

$$u'_{hn+1} + \rho u'_{ln} = 0 \quad (14)$$

*Moreover, the asymptotic selfishness ratio and the asymptotic marginal rate of substitution between heritage and bequest are both equal to:*

$$\frac{\rho}{e^{p_1}} e^p$$

The interpretation of equation (12) is then that the "optimal own-generation-preference rate" is  $e^{p_1}$ , the maximum growth rate of utility.

Consider now the discounted-sum criterion:

$$\Phi(G(K)) = \sum_1^{+\infty} \frac{U_n(k_{n-1}, k_n)}{\rho^n}$$

So that the definition domain of  $\Phi$  contains  $l_\infty^{p_1}$ , it is necessary to have  $\rho > e^{p_1}$ .

**Proposition 24** *Let  $K^d$  be an interior steady-state consensual optimum for the criterion  $\Phi$ . Then, the path  $K^d$  checks equation (14) and the asymptotic marginal rate of substitution between heritage and bequest is the same than that of  $K^m$ .*

**Proposition 25** *Let  $K^r$  be a rational-expectations spontaneous equilibrium with  $\rho$  as own-generation-preference rate. Then equation (14) is checked on the path  $K^r$ .*

Proposition 25 shows that, using the same own-generation-preference rate  $\rho$ , equation (14) characterizes also rational-expectations equilibria.

Consequently, since equation (14) characterizes at the same time rational-expectations equilibria, discounted-sum optima and myopic equilibria, the three concepts are equivalent (It is not worth being rational!). Thus, since  $e^{p_1}$  is the optimal own-generation-preference rate, the condition  $\rho \succ e^{p_1}$  shows that **the optimality defined with a discounted-sum criterion will never coincide with that defined with an Allais-anonymous criterion.**

Economic intuition suggests that the discounted-sum criterion constitutes a short-run criterion compared to the Allais-anonymous criterion, supporting the close generations compared to those remote. I have not a general formal proof for that but, nevertheless, it is easy to check in the discrete-time example of section 5. Indeed, the discounted-sum optimum  $K^d$  checks

$$\frac{k_n^d}{L^n} \longrightarrow w$$

where  $w$  is such that  $f'(w) = a + \frac{\rho}{e^{p_1}}e^p - 1$ . But  $\frac{\rho}{e^{p_1}}e^p \succ e^p$  implies that  $w \prec w^*$  ( $w^*$  is defined in subsection 5.3). As a result, for  $n$  large enough,  $U_n(k_{n-1}^d, k_n^d) \prec U_n(k_{n-1}^{**}, k_n^{**})$ , where  $K^{**}$  is the bequest-rule path defined in subsection 5.4. Moreover, the form of  $\Phi$  implies, in an obvious way, that  $K^d$  is Pareto-optimal. We can thus deduce that close generations are necessarily better off in the discounted-sum optimum  $K^d$  while distant generations are better off in the Allais-anonymous optimum  $K^{**}$ .

## 7 Proofs

### 7.1 Proofs for section 2

**Proof of proposition 1:** Observe that  $H_i(K) = 0$ . Therefore, the question amounts to find  $X \in l_\infty^p$  such that  $\delta H_i(K) \cdot X \succ 0$ .

According to lemma 22 [Mabrouk 2006a],  $s_{\infty++}^{p_1} = l_{\infty+}^{p_1}$ . As a result,  $\delta H_i(K) \cdot X \succ 0 \Leftrightarrow \delta H_i(K) \cdot X \in s_{\infty++}^{p_1}$ . If we take, for convenience,  $x_0 = 0$ , we can write for  $n \in [1, i-1]$ :

$$(\delta H_i(K) \cdot X)_n = u'_{hn}x_{n-1} + u'_{ln}x_n$$

and for  $n \geq i$ :

$$(\delta H_i(K) \cdot X)_n = u'_{hn+1}x_n + u'_{ln+1}x_{n+1}$$

also, we have :  $(u'_{hn})_{n \geq 1} \in s_\infty^{p_1-p} \Leftrightarrow \liminf |u'_{hn}| e^{-(p_1-p)n} \succ 0$ , similarly for  $(u'_{ln})_{n \geq 1}$ .

**If**  $R_\# \prec e^p$ :

There is  $\varepsilon \succ 0$  such that

$$\frac{e^p}{R_\# + \varepsilon} - 1 \succ 0$$

and  $N$  integer such that  $n \geq N \Rightarrow$

$$\frac{u'_{hn}}{-u'_{ln}} \prec R_\# + \varepsilon$$

This gives

$$\frac{e^p}{\frac{u'_{hp}}{-u'_{lp}}} - 1 \succ \frac{e^p}{R_\# + \varepsilon} - 1 \succ 0$$

Take  $x_1 \prec 0$  and  $x_i \prec 0$ . For  $n \in [2, i-1] \cup [i+1, N]$  choose  $x_n \prec 0$  such that

$$x_n \prec \frac{u'_{hn}}{-u'_{ln}} x_{n-1}$$

Thus,  $u'_{hn}x_{n-1} + u'_{ln}x_n \succ 0$  for  $n \in [1, i-1] \cup [i+1, N]$  (with  $x_0 = 0$ ).

Take  $x_n = x_N e^{(n-N)p}$  for  $n \geq N$ . Thus,  $(\delta H_i(K) \cdot X)_{n-1} =$

$$\begin{aligned} u'_{hn}x_{n-1} + u'_{ln}x_n &= u'_{hn} \left( x_{n-1} + \frac{u'_{ln}}{u'_{hn}} x_n \right) \\ &= -x_N u'_{hn} e^{(n-1)p} \left( \frac{-u'_{ln}}{u'_{hn}} e^p - 1 \right) \succ 0 \end{aligned}$$

and, if we suppose that  $(u'_{hn})_{n \geq 1} \in s_\infty^{p_1-p}$

$$\begin{aligned} &\liminf [u'_{hn}x_{n-1} + u'_{ln}x_n] e^{-(n-1)p_1} \\ &\geq -x_N e^{p_1-p} \liminf u'_{hn} e^{n(p-p_1)} \left( \frac{e^p}{R_\# + \varepsilon} - 1 \right) \succ 0 \end{aligned}$$

We have then found  $X$  in  $l_\infty^p$  such that  $\delta H_i(K) \cdot X \succ 0$ . If we suppose, instead of  $(u'_{hn})_{n \geq 1} \in s_\infty^{p_1-p}$ , that  $(u'_{ln})_{n \geq 1} \in s_\infty^{p_1-p}$ , we have the same result.

**If**  $R_b \succ e^p$

There is  $\varepsilon \succ 0$  such that

$$1 - \frac{e^p}{R_b - \varepsilon} \succ 0$$

and  $N$  integer such that  $n \geq N \implies$

$$\frac{u'_{hn}}{-u'_{ln}} \succ R_b - \varepsilon$$

This gives

$$1 - \frac{e^p}{\frac{u'_{hn}}{-u'_{ln}}} \succ 1 - \frac{e^p}{R_b - \varepsilon} \succ 0$$

Take  $x_1 \prec 0$  and for  $n \in [2, i-1]$  choose  $x_n$  such that

$$x_n \prec \frac{u'_{hn}}{-u'_{ln}} x_{n-1}$$

Take  $x_i \succ 0$ . For  $n \in [i+1, N]$  choose  $x_n \succ 0$  such that

$$x_n \prec \frac{u'_{hn}}{-u'_{ln}} x_{n-1}$$

Thus,  $u'_{hn}x_{n-1} + u'_{ln}x_n \succ 0$  for  $n \in [1, i-1] \cup [i+1, N]$  (with  $x_0 = 0$ ).  
Take  $x_n = x_N e^{(n-N)p}$  for  $n \geq N$ . Thus,  $(\delta H_i(K) \cdot X)_{n-1} =$

$$\begin{aligned} u'_{hn}x_{n-1} + u'_{ln}x_n &= u'_{hn} \left( x_{n-1} + \frac{u'_{ln}}{u'_{hn}} x_n \right) \\ &= x_N u'_{hn} e^{(n-1)p} \left( 1 - \frac{-u'_{ln}}{u'_{hn}} e^p \right) \succ 0 \end{aligned}$$

and, if we suppose that  $(u'_{hn})_{n \geq 1} \in s_\infty^{p_1-p}$

$$\begin{aligned} &\liminf [u'_{hn}x_{n-1} + u'_{ln}x_n] e^{-(n-1)p_1} \\ &\geq x_N e^{p_1-p} \liminf u'_{hn} e^{n(p-p_1)} \left( 1 - \frac{e^r}{R_b - \varepsilon} \right) \succ 0 \end{aligned}$$

We have  $\delta H_i(K) \cdot X \succ 0$ . Similarly if  $(u'_{ln})_{n \geq 1} \in s_\infty^{p_1-p}$  ■

**Proof of proposition 3:** Suppose that (A'1) holds. Let  $K \in \overset{\circ}{D} \cap D_l \cap (L_- \cup L_+)$  such that (A3) and (A4) hold. According to the Karush-Kuhn-Tucker theorem, if  $K$  is a solution of  $P_i(K)$  then there is  $\lambda^* \in l_\infty^{p_1}$  such that, for all  $\Delta K$  in  $l_\infty^p$

$$\delta U_i(K) \cdot \Delta K + \langle \lambda^* | \delta H_i(K) \cdot \Delta K \rangle = 0 \quad (15)$$

with  $\lambda^* \geq 0$ .

Now apply lemma 2 of [Mabrouk 2006a]:

$$\lambda^* = \lambda + \beta$$

where  $\lambda = (\lambda_n)_{n \geq 1} \in l_1^{p_1}$  and  $\beta$  is such that its restriction to  $c_{p_1}$  is proportional to  $\delta_\infty^{p_1}$ .

Let  $\Delta K$  be in  $c_0^p$ . We see that  $\delta U_i(K) \cdot \Delta K$  is in  $c_0^{p_1}$ , and since the sequences  $(u'_{hn})_{n \geq 1}$  and  $(u'_{ln})_{n \geq 1}$  are in  $l_\infty^{p_1-p}$  (see proposition 10 of [Mabrouk 2006a]),

$$\begin{aligned} & \lim_{n \rightarrow +\infty} |u'_{hn} \Delta k_{n-1} + u'_{ln} \Delta k_n| e^{-np_1} \leq \lim_{n \rightarrow +\infty} |u'_{hn} \Delta k_{n-1}| e^{-np_1} + \\ & |u'_{ln} \Delta k_n| e^{-np_1} \\ = & \lim_{n \rightarrow +\infty} |u'_{hn}| e^{-n(p_1-p)} |\Delta k_{n-1}| e^{-np} + \left| u'_{ln} e^{-n(p_1-p)} \right| e^{-n(p_1-p)} |\Delta k_n| e^{-np} \\ \leq & \lim_{n \rightarrow +\infty} \left[ \|U'_h\|_{p_1-p} \Delta k_{n-1} e^{-np} + \|U'_l\|_{p_1-p} \Delta k_n e^{-np} \right] = 0 \end{aligned}$$

thus,  $\delta H_i(K) \cdot \Delta K$  is also in  $c_0^{p_1}$ . It implies that  $\langle \beta \mid \delta U_i(K) \cdot \Delta K \rangle = 0$  and  $\langle \beta \mid \delta H_i(K) \cdot \Delta K \rangle = 0$ .

Replace  $\lambda^*$  by  $\lambda + \beta$  in (15):

$$\delta U_i(K) \cdot \Delta K + \langle \lambda \mid \delta H_i(K) \cdot \Delta K \rangle = 0$$

for all  $\Delta K$  in  $c_0^p$ . By development and identification, we obtain:

$$\begin{aligned} \lambda_{i-n} &= (-1)^n \frac{u'_{hi} \dots u'_{hi-n+1}}{u'_{li-1} \dots u'_{li-n}} \text{ for } n \in [1, i-1] \\ \lambda_{i+n} &= (-1)^{n+1} \frac{u'_{li} \dots u'_{li+n}}{u'_{hi+1} \dots u'_{hi+n+1}} \text{ for } n \in [0, +\infty[ \end{aligned} \quad (16)$$

Since  $\lambda$  is in  $l_1^{p_1}$  we have  $\sum_{n=1}^{+\infty} |\lambda_n| e^{p_1 n} < +\infty$ . Replace  $(\lambda_j)$  by its value in (16):

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{li+j} e^{p_1}}{u'_{hi+j+1}} \right| < +\infty \quad (17)$$

If  $K$  is Pareto-optimal, the inequality (17) holds for all  $i \geq 1$ .

Now drop the assumption (A4). We show the same way than in [Mabrouk 2005] that (17) is still a necessary condition for Pareto-optimality:

Let  $J = \{j / U'_{nl}(k_{j-1}, k_j) = 0\}$ . If  $J$  is up-bounded, let  $q = \max J$ .

If  $K$  is Pareto-optimal, the bequests plan extracted from  $K$  and beginning at the generation  $g_{q+1} : (k_{q+1}, k_{q+2}, \dots)$  is also necessarily Pareto-optimal when we take  $(k_0, k_1, \dots, k_q)$  as fixed parameters. If  $K$  is in  $\overset{\circ}{D} \cap (L_- \cup L_+)$ , the extracted plan is also in  $\overset{\circ}{D}_{k_q} \cap (L_- \cup L_+)$  and  $(k_q, k_{q+1}) \in \overset{\circ}{D}_{q+1}$ <sup>6</sup>. Since  $\prod_{j=q+1}^{+\infty} U'_{nl}(k_{j-1}, k_j) \neq 0$ , it verifies necessarily the condition (17), but with  $g_{q+1}$  as first generation. This gives, for all  $i \geq 1$ :

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{lq+i+j} e^{p_1}}{u'_{hq+i+1+j}} \right| < +\infty$$

---

<sup>6</sup>  $D_{k_q}$  is the set  $\{B = (b_{q+1}, b_{q+2}, \dots) \in l_{\infty+}^p / \forall i \geq 1 : (b_{q+i-1}, b_{q+i}) \in D_{q+i}\}$

Multiply the above inequality with

$$\prod_{j=0}^{q-1} \left| \frac{u'_{i+j} e^{p_1}}{u'_{hi+1+j}} \right|$$

we find again the inequality (17).

If  $J$  is not up-bounded, there is episodically a  $q$  such that  $U'_{ni}(k_{q-1}, k_q) = 0$ . Consequently, in the sum  $\sum_{p=0}^{+\infty} \prod_{n=0}^p \left| \frac{u'_{i+n} e^{p_1}}{u'_{hi+1+n}} \right|$  there is a finite number of nonzero terms. Thus, for all  $i \geq 1$  the sum converges ■

**Proof of proposition 4:** The first step is to show the stationarity of the Lagrangian. For  $K \in \overset{\circ}{D}$  and  $i \geq 1$  define the Lagrangian  $L_i$  from  $D \times l_1^{p_1}$  to  $\mathbb{R}$  as follows  $L_i(B, \mu) = U_i(B) + \langle \mu | H_i(B) \rangle$ <sup>7</sup>, and suppose, if  $i \succ 1$ , that  $\prod_{j=1}^{i-1} u'_{ij} \neq 0$ . The system (16) defines a sequence  $\lambda$ .

If

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{i+j} e^{p_1}}{u'_{hi+j+1}} \right| < +\infty$$

then we can see that  $\lambda \in l_1^{p_1}$ . Like  $U_i$  and  $H_i$ ,  $L_i$  is differentiable with respect to  $B$  at  $B = K$ . Its differential, computed at  $\mu = \lambda$  and  $B = K$  is  $\delta L_i(K, \lambda) = \delta U_i(K) + \langle \lambda | \delta H_i(K) \rangle$ . Since  $\lambda \in l_1^{p_1}$ , for  $\Delta K \in l_\infty^p$  we have

$$\begin{aligned} \delta L_i(K, \lambda) \cdot \Delta K &= \delta U_i(K) | \Delta K + \lambda | \langle \delta H_i(K) \cdot \Delta K \rangle \\ &= \delta U_i(K) | \Delta K + \lim_{n \rightarrow +\infty} (\lambda_1, \lambda_2, \dots, \lambda_n) | \langle \delta H_i(K) \cdot \Delta K \rangle \end{aligned}$$

Replace  $(\lambda_j)$  by their values in (16), we obtain

$$\delta U_i(K) | \Delta K + (\lambda_1, \lambda_2, \dots, \lambda_n) | \langle \delta H_i(K) \cdot \Delta K \rangle = \lambda_n u'_{in+1} \Delta k_{n+1}$$

which tends to 0 when  $n$  tends to infinity.

Thus  $\delta L_i(K, \lambda) = 0$ . In other words, the Lagrangian  $L_i$  is stationary at  $(K, \lambda)$ .

The second step is to deduce optimality from stationarity.

Let  $i$  be such that if  $i \succ 1$  we have:  $\prod_{j=1}^{i-1} u'_{ij} \neq 0$ . From the system (16), we deduce that  $\lambda_j \geq 0$  for all  $j \geq 1$ . Thus,  $L_i$  is concave with respect to  $B$ . As a result, for all  $B \in D$  and  $\alpha \in ]0, 1[$  we have:

$$L_i((1 - \alpha)K + \alpha B, \lambda) \geq (1 - \alpha)L_i(K, \lambda) + \alpha L_i(B, \lambda)$$

then

$$\frac{L_i(K + \alpha(B - K), \lambda) - L_i(B, \lambda)}{\alpha} \geq L_i(B, \lambda) - L_i(K, \lambda)$$

<sup>7</sup>This is possible since  $l_1^{p_1} \subset l_\infty^{p_1^*}$  and  $H_i(B) \in l_\infty^{p_1}$ .

Since  $L_i$  is Frechet-differentiable at  $K$ , when  $\alpha$  tends to 0 we obtain:

$$\delta L_i(K, \lambda) \mid (B - K) \geq L_i(B, \lambda) - L_i(K, \lambda) \quad (18)$$

We know that if

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{li+j} e^{p_1}}{u'_{hi+j+1}} \right| < +\infty$$

we have  $\delta L_i(K, \lambda) = 0$ .

We then deduce from (18) that for all  $B \in D$ :

$$L_i(B, \lambda) - L_i(K, \lambda) \leq 0 \quad (19)$$

We now show that  $K$  solves  $P_i(K)$ . Suppose there is  $B$  in  $D$  such that  $U_i(B) \succ U_i(K)$  and  $H_i(B) \geq 0$ . We have  $H_i(K) = 0$  so  $H_i(B) \geq H_i(K)$ . Since  $\lambda \geq 0$  we have  $\lambda \mid H_i(B) \geq \lambda \mid H_i(K)$ . Finally  $U_i(B) + \lambda \mid H_i(B) \succ U_i(K) + \lambda \mid H_i(K)$ . This contradicts (19)■

## 7.2 Proofs for section 3

**Proof of proposition 9:** If we show that if a plan  $B$  Pareto-dominates a plan  $K$ , then  $b_n \leq k_n$  for all  $n \geq 1$ , this will prove the proposition. Suppose there is  $n \geq 1$  such that  $b_{n+1} \succ k_{n+1}$ . According to the assumptions on  $U$ , it is easy to show that we then have  $b_n \succ k_n$ . Hence, by induction,  $b_1 \succ k_1$  and the first generation  $g_1$  is better off in  $K$ , which contradicts that  $B$  Pareto-dominates  $K$ ■

## 7.3 Proofs for section 4

**Proof of proposition 14:** Observe that, because of the concavity of  $U_n$ , we have

$$\frac{\Delta k_{n+1}}{\Delta k_n} \succ \frac{D_1 U_{n+1}(K)}{D_2 U_{n+1}(K)} = e^p$$

$\Delta k_n$  is the decrease of bequest generation  $g_n$  would have to carry out to maintain its utility level if  $g_1$  lowers its bequest by a quantity  $\Delta k_1$ . Suppose that the sequence  $(\Delta k_n)$  is defined for all  $n \geq 1$ .  $(\frac{\Delta k_{n+1}}{\Delta k_n})$  would then converge to  $l \succ e^p$ , which means that  $(\Delta k_n)$  goes out of  $l^\infty$ . This contradicts  $D \subset l^\infty$ .

Thus, for all decrease of first generation's bequest  $\Delta k_1$ , there is a generation  $g_n$  which cannot maintain its utility level. We can make the same reasoning for the decrease of any generation's bequest. Thus  $K$  is Pareto-optimal.■

**Proof of proposition 16:** We see clearly that  $K$  is a bequest-rule plan. Equation (2) shows that for  $n$  large enough we have  $D_2 U_n(k_{n-1}, k_n) < 0$ . When  $K$  is Pareto-optimal starting from a given index, it is also Pareto-optimal starting from  $n = 1$  (see [Mabrouk 2005]). For this reason, we will suppose without



loss of generality that the inequality  $D_2U_n(k_{n-1}, k_n) < 0$  holds starting from  $n = 1$ . Let  $\pi_1$  be the reason of  $G(D_\pi)$ . According to proposition 11 ( section 3), if

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{\pi_1}}{u'_{h2+j}} \right| < +\infty$$

then  $K$  is Pareto-optimal.

Suppose first that  $p = p_1$ . For  $n \geq 1$ , we have:

$$\prod_{j=0}^n \left| \frac{u'_{l1+j} e^{p_1}}{u'_{h2+j}} \right| = \frac{-u'_{l1} e^p}{u'_{h2+n}} \prod_{j=2}^{n+1} \frac{-u'_{lj} e^p}{u'_{hj}}$$

For  $n \geq 1$  denote

$$\Pi_n = \prod_{j=2}^{n+1} \frac{-u'_{lj} e^p}{u'_{hj}}$$

and take  $\Pi_0 = 1$ . Observe that the equation (2) implies that the series  $\sum_{n=0}^{+\infty} \Pi_n$  fulfills Rabee-Duhamel criterion since

$$\frac{\Pi_n}{\Pi_{n-1}} = \frac{-u'_{ln+1} e^p}{u'_{hn+1}}$$

Thus, it converges.

But

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{p_1}}{u'_{h2+j}} \right| = \sum_{n=0}^{+\infty} \frac{-u'_{l1} e^p}{u'_{h2+n}} \Pi_n$$

Since  $K$  is steady state, we can write  $u'_h = \lim u'_{hn} e^{-(p_1-p)n} = \lim u'_{hn}$ . As a result, since  $\sum_{n=0}^{+\infty} \Pi_n$  converges, the series

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{p_1}}{u'_{h2+j}} \right|$$

also converges<sup>8</sup>. Since  $\pi_1 \leq p_1$  the series

$$\sum_{n=0}^{+\infty} \prod_{j=0}^n \left| \frac{u'_{l1+j} e^{\pi_1}}{u'_{h2+j}} \right|$$

converges and  $K$  is Pareto-optimal.

If  $p \neq p_1$ , multiply  $U_n$  by  $e^{(p-p_1)n}$ . Again  $D$  and  $G(D)$  have the same reason. Apply then the above proof to see that  $K$  is still Pareto-optimal. ■

---

<sup>8</sup>It is enough to consider the remainder of Cauchy to prove it.

## 7.4 Proofs for section 6

**Proof of proposition 22:** Generation  $g_n$  solves

$$\max_l V_n(k_{n-1}, l)$$

which is characterized by the equation

$$u'_{\text{in}} + a'_n = 0$$

Since the spontaneous equilibrium and the interior steady-state optimal growth path coincide,  $a' = \lim a'_n e^{-(p_1-p)n}$  exists and is equal to  $-u'_l$ . It remains only to use the bequest rule (1) to see that  $u'_h = a' e^p$  ■

**Proof of proposition 23 ;** Generation  $g_n$  maximizes

$$U_n(k_{n-1}^m, k_n^m) + \frac{1}{\rho} [U_n(k_n^m, k_{n+1}^m) + A_{n+1}(k_{n+1}^m, k_{n+2}^m, \dots)]$$

with respect to  $k_n^m$ , what gives equation (14). For a steady state,  $\lim \frac{u'_{hn}}{u'_{hn+1}} = e^{p-p_1}$ . Then, since  $a'_{\text{in}} = \frac{1}{\rho} u'_{hn+1}$ , the selfishness ratio is  $s_n = \frac{u'_{hn}}{a'_{\text{in}}} = \rho \frac{u'_{hn}}{u'_{hn+1}}$  (equation (11)) and  $s = \lim s_n = \rho e^{p-p_1}$ . The marginal rate of substitution between heritage and bequest is  $r_n = \frac{u'_{hn}}{-u'_{\text{in}}} = \frac{u'_{hn}}{u'_{hn+1}} \frac{u'_{hn+1}}{-u'_{\text{in}}}$ . Thanks to equation (14), this gives  $r_n = \frac{u'_{hn}}{u'_{hn+1}} \rho \rightarrow \rho e^{p-p_1}$  ■

**Proof of proposition 24 ;** Interior consensual optima for the criterion  $\Phi$  check the relation:

$$\delta\Phi(G(K)) = 0 \tag{20}$$

With the help of proposition 14 of [Mabrouk 2006a] and noting that  $\frac{\partial\Phi}{\partial p_1 \infty} = 0$ , (20) gives equation (14). The proof for the asymptotic marginal rate of substitution between heritage and bequest is similar to that of the myopic case ■

**Proof of proposition 25:** Equation (13) implies  $A_{n+1} = \rho A_n - U_{n+1}$ . By induction

$$\frac{A_{n+1}}{\rho^{n+1}} = \frac{A_1}{\rho} - \sum_{j=2}^{n+1} \frac{U_j}{\rho^j}$$

For each value of  $A_1$  there is a sequence  $(A_n)_{n \geq 1}$  that fulfills equation (13). We have to choose the value of  $A_1$  that fits best our economic question. That is why it is necessary to choose  $A_1$  such that  $\frac{A_1}{\rho} - \sum_{j=2}^{+\infty} \frac{U_j}{\rho^j} = 0$ . Otherwise, the sequence  $(A_n)_{n \geq 1}$  would grow at the rate  $\rho$  which is larger than  $e^{p_1}$ , the maximum growth rate of  $(U_n)_{n \geq 1}$ . This would imply that the selfish part of the

utility  $\frac{U_n}{V_n}$  tends to zero, which contradicts the idea of own-generation-preference.

We then have  $A_1 = \sum_{j=2}^{+\infty} \frac{U_j}{\rho^{j-1}}$  and, by induction

$$V_n = U_n + A_n = \sum_{j=0}^{+\infty} \frac{U_{n+j}}{\rho^j}$$

Denote  $\Phi_n(G(K)) = \sum_{j=0}^{+\infty} \frac{U_{n+j}(K)}{\rho^j}$ . Generation  $g_n$  will choose the capital to

bequeath to  $g_{n+1}$  anticipating the behavior of  $g_{n+1}$ ,  $g_{n+2}$ ... Denote  $K^{n+1}(h) = (k_{n+1}^n(h), k_{n+2}^n(h), k_{n+3}^n(h) \dots)$  the awaited response of the posterior generations, to a bequest  $k_n = h$  by  $g_n$ . Denote  $(h, k_{n+1}^n(h), k_{n+2}^n(h), k_{n+3}^n(h) \dots)$  by  $(h, K^{n+1}(h))$

and  $(U_n(k_{n-1}, h), U_{n+1}(h, k_{n+1}), U_{n+2}(k_{n+1}, k_{n+2}) \dots)$   
by  $G_n(k_{n-1}, h, K^{n+1}(h))$ .

$g_n$  solves

$$\max_h \Phi_n(G_n(k_{n-1}, h, K^{n+1}(h)))$$

The first order condition gives (rigorously, we should prove that  $k_{n+j}^n(h)$ , the expectation functions consistent with rational-expectations equilibria, are derivable):

$$u'_{\ln} + \frac{1}{\rho} u'_{h_{n+1}} + \frac{k_{n+1}^n}{\rho} \left( u'_{\ln+1} + \frac{1}{\rho} u'_{h_{n+2}} \right) + \frac{k_{n+2}^n}{\rho^2} \left( u'_{\ln+2} + \frac{1}{\rho} u'_{h_{n+3}} \right) + \dots = 0 \quad (21)$$

Inheriting  $h$ ,

$g_{n+1}$  bequeaths  $h_1$  on the basis of its anticipations  $k_{n+2}^{n+1}(h_1), k_{n+3}^{n+1}(h_1) \dots$

Rational expectations imply  $k_{n+1}^n(h) = h_1$ . We can then write  $k_{n+2}^n(h) = k_{n+2}^{n+1}(k_{n+1}^n(h))$ ,  $k_{n+3}^n(h) = k_{n+3}^{n+1}(k_{n+1}^n(h)) \dots$  Thus,  $k_{n+2}'^n = k_{n+2}^{n+1} k_{n+1}'^n$ ,  $k_{n+3}'^n = k_{n+3}^{n+1} k_{n+1}'^n \dots$  and (21) gives

$$u'_{\ln} + \frac{1}{\rho} u'_{h_{n+1}} + \frac{k_{n+1}^n}{\rho} \left[ \frac{k_{n+2}^{n+1}}{\rho} \left[ \left( u'_{\ln+1} + \frac{1}{\rho} u'_{h_{n+2}} \right) + \left( u'_{\ln+2} + \frac{1}{\rho} u'_{h_{n+3}} \right) + \frac{k_{n+3}^{n+1}}{\rho} \left( u'_{\ln+3} + \frac{1}{\rho} u'_{h_{n+4}} \right) \dots \right] \right] = 0 \quad (22)$$

But, since  $g_{n+1}$  solves

$$\max_{h_1} \Phi_{n+1}(G_{n+1}(k_n, h_1, K^{n+2}(h_1)))$$

we can replace  $n$  by  $n+1$  in (21):

$$\left( u'_{\ln+1} + \frac{1}{\rho} u'_{h_{n+2}} \right) + \frac{k_{n+2}^{n+1}}{\rho} \left[ \left( u'_{\ln+2} + \frac{1}{\rho} u'_{h_{n+3}} \right) + \frac{k_{n+3}^{n+1}}{\rho} \left( u'_{\ln+3} + \frac{1}{\rho} u'_{h_{n+4}} \right) \dots \right] = 0 \quad (23)$$

(22) and (23) imply

$$u'_{\ln} + \frac{1}{\rho} u'_{h_{n+1}} = 0$$

■

## A Proofs for the discrete-time example

**Proof of proposition 17:**

$$\begin{aligned} U_i(h, l) &= u \left( L^{i-1} f \left( \frac{h}{L^{i-1}} \right) - ah - (l - h) \right) \\ &= u \left[ L^{i-1} \left( f \left( \frac{h}{L^{i-1}} \right) - a \frac{h}{L^{i-1}} - \left( \frac{l}{L^{i-1}} - \frac{h}{L^{i-1}} \right) \right) \right] \end{aligned}$$

We see that

$$U_i(h, l) = (UL^{i-1})_1 \left( \frac{h}{L^{i-1}}, \frac{l}{L^{i-1}} \right)$$

where  $(UL^{i-1})_1$  is the function obtained by replacing  $u$  by  $uL^{i-1}$  in the equation (3) defining  $U_1$ .

This implies that

$$(h, l) \in D_i \iff \left( \frac{h}{L^{i-1}}, \frac{l}{L^{i-1}} \right) \in D_1 \quad (24)$$

therefore

$$D_i = L^{i-1}(D_1)$$

where  $L^{i-1}(\cdot)$  denotes the homothety with center  $O$  and factor  $L^{i-1}$ .

Denote  $\partial D_i$  the upper frontier of  $D_i$ .  $\partial D_i = \{(h, l)/l = l_{\max i}(h)\}$  where  $l_{\max i}(h) = \sup \{l/(h, l) \in D_i\}$ .

The concavity of  $f$  implies that  $D_i$  is convex and  $\partial D_i$  concave. Since  $\partial D_i = L^{i-1}(\partial D_1)$ ,  $\partial D_i$  and  $\partial D_1$  have the same asymptotic directions. This asymptotic direction is  $\lim_{h \rightarrow +\infty} \frac{f(h) + (1-a)h}{h} = 1 + \lim_{y \rightarrow +\infty} D_2 F(1, y) - a < 1$ .

Denote  $\Delta_L$  the straight line  $\{l = Lh\}$ .

$L > 1$ ,  $l_{\max i}(0) > 0$ ,  $\partial D_i$  concave and its asymptotic direction is strictly smaller than 1 implies  $\Delta_L$  cuts  $\partial D_i$  one time for each  $i \geq 1$ . Let  $M_i$  be the intersection point between  $\Delta_L$  and  $\partial D_i$  and  $(w_{i-1}, x_{i-1})$  the coordinates of  $M_i$ .

We have

$$L(\Delta_L) = \Delta_L \text{ and } L(\partial D_i) = \partial D_{i+1}$$

then

$$L(M_i) = L(\Delta_L \cap \partial D_i) = L(\Delta_L) \cap L(\partial D_i) = \Delta_L \cap \partial D_{i+1} = M_{i+1}$$

which implies

$$w_i = Lw_{i-1} \text{ and } x_i = Lx_{i-1}$$

But  $M_i \in \Delta_L \implies x_{i-1} = Lw_{i-1}$  hence  $x_{i-1} = w_i$  and the coordinates of  $M_i$  are  $(w_{i-1}, w_i) = L^{i-1}w_0.(1, L)$ .

Now let  $K \in D$ . Suppose there is  $i \geq 1$  such that  $k_{i-1} \leq w_{i-1}$ .

Since  $M_i \in \partial D_i$ , with an heritage  $w_{i-1}$ , generation  $g_i$  could not bequeath more than  $w_i$ . It follows that with an heritage  $k_{i-1}$  smaller than  $w_{i-1}$ , generation  $g_i$  could not bequeath more than  $w_i$ . So,  $k_i \leq w_i$ .

Consequently, for all  $j \geq i$  we have  $k_j \leq w_j = L^j w_0$ . Hence  $\sup k_j e^{-j \text{Log} L} < +\infty$  and  $K \in l_\infty^{\text{Log} L}$ .

Suppose now that for all  $i \geq 1$  we have  $k_{i-1} \succ w_{i-1}$ . Since the line  $\Delta_L$  comes out of  $D_i$  at the point  $M_i = (w_{i-1}, w_i)$ , the point  $(k_{i-1}, Lk_{i-1})$ , which belongs to  $\Delta_L$ , is out of  $D_i$ . Thus  $g_i$  with an heritage  $k_{i-1}$  cannot bequeath as much as  $Lk_{i-1}$ . Then,  $k_i \prec Lk_{i-1}$ . This implies  $k_i \prec L^i k_0$  for all  $i \geq 1$  and we have also  $K \in l_\infty^{\text{Log} L}$ .

This proves that  $D \subset l_\infty^{\text{Log} L}$ .

We show now that the interior of  $D$  in  $l_\infty^{\text{Log} L}$  is not empty, which also implies that  $\text{Log} L = \inf \{\alpha / D \subset l_\infty^\alpha\}$ . Take  $\xi \succ 0$  and  $k_1$  such that

$$\{k_0\} \times [k_1 - \xi, k_1 + \xi] \subset D_1 \text{ and } [k_1 - \xi, k_1 + \xi] \times [Lk_1 - L\xi, Lk_1 + L\xi] \subset D_2$$

Denote  $K = (k_1, Lk_1, L^2k_1, \dots)$ .

According to (24),

for all  $i \geq 0$ ,  $L^{i-1} [k_1 - \xi, k_1 + \xi] \subset D_i$  and

$\times_{i=1}^{+\infty} L^{i-1} [k_1 - \xi, k_1 + \xi] = S(K, \xi)$ . This shows that  $S(K, \xi) \subset D$ . Thus,

$K$  is interior to  $D$  and  $\overset{\circ}{D}$  is not empty ■

**Proof of proposition 18:** As seen in the proof of proposition 17,  $U_i(h, l) = (UL^{i-1})_1(\frac{h}{L^{i-1}}, \frac{l}{L^{i-1}})$ .

Moreover,  $u$  concave  $\implies u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y)$  for all  $x, y$  in  $[0, +\infty[$  and  $\lambda$  in  $[0, 1]$ . Take  $y = 0$  and  $\lambda = \frac{1}{L}$ , then  $u(\frac{x}{L}) \geq \frac{1}{L}u(x)$ . Denote  $c = \frac{x}{L}$ . Thus,  $L u(c) \geq u(Lc)$  for all  $c$  in  $[0, +\infty[$ .

For  $i \geq 2$ , this implies

$$\begin{aligned} U_i(h, l) &= (UL^{i-1})_1(\frac{h}{L^{i-1}}, \frac{l}{L^{i-1}}) \leq L(UL^{i-2})_1(\frac{h}{L^{i-1}}, \frac{l}{L^{i-1}}) \\ &= LU_{i-2}(\frac{h}{L}, \frac{l}{L}) \end{aligned}$$

We then easily prove that  $G(D) \subset l_\infty^{\text{Log} L}$  ■

**Proof of proposition 19:** Observe that  $f$  strictly concave,  $f(0) \geq 0$ ,  $\lim_{y \rightarrow +\infty} f'(y) \prec a$  and  $f'(w^*) = a + L - 1$  imply that  $f(w^*) \succ (a + L - 1)w^*$ . Moreover if we denote  $x$  the solution of  $f(x) = (a + L - 1)x$ , we have  $w^* \prec x$ .

$f(w^*) \succ (a + L - 1)w^*$  implies that there is  $\xi \succ 0$  such that for all  $(y, z)$  in  $[w^* - \xi, w^* + \xi]^2$ ,  $f(x) - ax - (Ly - x) \succ 0$ . Hence, if  $(k_{j-1}, k_j)$  is such that  $(\frac{k_{j-1}}{L^{j-1}}, \frac{k_j}{L^j}) \in [w^* - \xi, w^* + \xi]^2$ ,  $(k_{j-1}, k_j) \in D_j$ . Thus

$$[L^{j-1}(w^* - \xi), L^{j-1}(w^* + \xi)] \times [L^j(w^* - \xi), L^j(w^* + \xi)] \subset D_j \quad (25)$$

Denote  $\overline{K}$  the maximum bequests plan (obtained with zero consumption). We easily prove that  $\frac{k_j}{L^j} \rightarrow x$ . Thus, the inequality  $w^* \prec x$  shows that asymptotically, the plan  $(L^j w^*)$  is below  $\overline{K}$ . Starting from  $k_0$ , one can then reach the plan  $(L^j w^*)$  by bequeathing a little less than the maximum bequest to be interior to the definition domain. Suppose that generation  $g_{i+1}$  is the one that reaches the plan  $(L^j w^*)$  and that the path is  $(k_1^*, k_2^*, \dots, k_i^*)$ . Then, with (25), we deduce that the plan  $(k_1^*, k_2^*, \dots, k_i^*, L^{i+1} w^*, L^{i+2} w^*, \dots)$  is interior to  $D$  ■

**Proof of proposition 20:** According to theorem 18 of [Mabrouk 2006a], a steady state  $K$  in  $\overset{\circ}{D}$  is a consensual optimum if and only if

$$\frac{u'_h}{L} + u'_l = 0$$

Thus, if an interior steady state  $K$  checks

$$\lim \frac{u'_{hn}}{-u'_{ln}} = L$$

it is a consensual optimum.

But

$$\frac{u'_{hn}}{-u'_{ln}} = f'\left(\frac{k_{n-1}}{L^{n-1}}\right) - a + 1$$

and (7)  $\implies \lim f'\left(\frac{k_{n-1}}{L^{n-1}}\right) = a + L - 1$ .

Then

$$\frac{u'_{hn}}{-u'_{ln}} \longrightarrow L$$

Consequently, the interiority of  $K$  being warranted by (6),  $K$  is an interior steady state consensual optimum. ■

**Proof of proposition 21:** Since  $\lim f'^{-1}(x_n) = w^*$ , (7) is also checked and  $K^{**}$  is a consensual optimum.

We have

$$\begin{aligned} \frac{-u'_{ln} L}{u'_{hn}} &= \frac{L}{f'\left(\frac{k_{n-1}^{**}}{L^{n-1}}\right) - a + 1} \\ &= 1 - \frac{\xi}{n+1} + r_{n+1} \end{aligned}$$

We can then apply proposition 16 and conclude that  $K^{**}$  is also Pareto-optimal ■

The following propositions set conditions (4) and (5) respectively for the case  $u(c) = \alpha c + 1 - \frac{1}{c+1}$  (propositions 26 and 27) and  $u(c) = c^{1-\theta}$  (propositions 28 and 29).

**Proposition 26** *In the case  $u(c) = \alpha c + 1 - \frac{1}{c+1}$ , assumption (4) is checked with  $p_1 = \text{Log}L$ .*

Proof : We have

$$U_i(b_{i-1}, b_i) = u \left[ L^{i-1} \left( f\left(\frac{b_{i-1}}{L^{i-1}}\right) - a \frac{b_{i-1}}{L^{i-1}} - \left( \frac{b_i}{L^{i-1}} - \frac{b_{i-1}}{L^{i-1}} \right) \right) \right]$$

Let  $c_i = L^{i-1} \left( f\left(\frac{b_{i-1}}{L^{i-1}}\right) - a \frac{b_{i-1}}{L^{i-1}} - \left( \frac{b_i}{L^{i-1}} - \frac{b_{i-1}}{L^{i-1}} \right) \right)$ . Since  $\left| \frac{b_{i-1}}{L^{i-1}} \right| \leq \|B\|$  and  $\left| \frac{b_i}{L^{i-1}} \right| \leq L\|B\|$ ,  $\frac{c_i}{L^i}$  is bounded. We see easily that  $U_i(b_{i-1}, b_i) = u(c_i)$  is also in  $l_\infty^{\text{Log}L}$ . Then  $G(D) \subset l_\infty^{\text{Log}L}$ . Let  $(kL^n)$  be a sequence in  $D$ . Let  $c = \frac{1}{L} [f(k) - ak - (Lk - k)]$ . Since  $\overset{\circ}{D} \neq \emptyset$ , we can choose  $k$  such that  $[f(k) - ak - (Lk - k)] \succ 0$ . Then

$$\begin{aligned} c_i &= [L^{i-1} (f(k) - ak - (Lk - k))] \\ &= L^i c \\ \frac{u(c_i)}{L^i} &= \alpha c + \frac{1}{L^i} - \frac{1}{L^i (cL^i + 1)} \\ &\longrightarrow \alpha c \text{ when } i \longrightarrow +\infty \end{aligned}$$

and the sequence  $G((kL^n)) = (u(c_i))$  is convergent and strictly of reason  $\text{Log}L$  ■

**Proposition 27** *In the case  $u(c) = \alpha c + 1 - \frac{1}{c+1}$ , assumption (5) holds.*

Proof: Let's start with two preliminary remarks. First, since  $f'$  is positive and decreasing,  $f'(x) \longrightarrow f'_\infty \geq 0$  when  $x \longrightarrow +\infty$ . According to Taylor's formula, for  $x, y \geq h_0$ , there is  $z$  in  $[x, y]$  such that

$$f''(z) = \frac{f'(x) - f'(y)}{x - y}$$

Make  $x \longrightarrow +\infty$ . It comes that

$$\lim_{z \rightarrow +\infty} f''(z) = 0$$

Secondly, since maximum consumption is the production  $L^{i-1} f\left(\frac{h}{L^{i-1}}\right)$ , minimum bequest is  $l = (1 - a)h$ . Hence, with a start-up capital  $k_0 \succ 0$ ,  $b_n$  is strictly positive for all  $B \in D$  and  $n \geq 1$ .

In addition,  $u'(c) = \alpha + \frac{1}{(1+c)^2}$ ,  $u''(c) = \frac{-2}{(1+c)^3}$  and:

$$\begin{aligned} U''_{ih^2}(b_{i-1}, b_i) &= u''(c_i) \left[ f' \left( \frac{h}{L^{i-1}} \right) - a + 1 \right]^2 + u'(c_i) \frac{f'' \left( \frac{h}{L^{i-1}} \right)}{L^{i-1}} \quad (26) \\ U''_{ihl}(b_{i-1}, b_i) &= -u''(c_i) \left[ f' \left( \frac{h}{L^{i-1}} \right) - a + 1 \right] \\ U''_{il^2}(b_{i-1}, b_i) &= u''(c_i) \end{aligned}$$

$K^{**} \in \overset{\circ}{D} \implies$  there is  $\beta' \succ 0$  such that  $S(K^{**}, \beta') \subset \overset{\circ}{D}$ . Since  $\overset{\circ}{D} \subset s_{\infty++}^p$  ([Mabrouk 2006a], proposition 5),  $K^{**} \in \overset{\circ}{D} \implies \liminf \frac{k_n^{**}}{L^n} \succ 0$ . Then  $\inf \frac{k_n^{**}}{L^n} \succ 0$ . Take  $\beta'' = \frac{1}{2} \inf \frac{k_n^{**}}{L^n}$ . For all  $B$  in  $S(K^{**}, \beta'')$ ,  $\inf \frac{b_n}{L^n} \geq \frac{\beta''}{2}$ . Let  $\beta = \inf(\beta', \beta'')$ . Then for all  $B$  in  $S(K^{**}, \beta)$ ,  $B \in \overset{\circ}{D}$  and  $\inf \frac{b_n}{L^n} \geq \frac{\beta}{2} \succ 0$ . We then see that  $|f'(\frac{b_n}{L^n})|$  and  $|f''(\frac{b_n}{L^n})|$  are bounded when  $B \in S(K^{**}, \beta)$ . Let  $M'$  be an upper bound.

Because of the concavity of  $f$ ,  $f'(y) \succ a + L - 1$  implies that  $f(w^*) - (a + L - 1)w^* \succ 0$ . Since  $f$  is continuous, there is  $\gamma \succ 0$  such that

$$\inf_{x \in [w^* - \gamma, w^* + \gamma]} \{f(x) - (a + L - 1)x\} \succ 3L\gamma$$

and since  $\lim \frac{k_n^{**}}{L^n} = w^*$  there is  $N$  such that

$$n \geq N \implies \left[ \frac{k_n^{**}}{L^n} - \frac{\gamma}{2}, \frac{k_n^{**}}{L^n} + \frac{\gamma}{2} \right] \subset [w^* - \gamma, w^* + \gamma]$$

For  $B \in S(K^{**}, \frac{\gamma}{2})$  and  $n \geq N + 1$ , we then have

$$\left( \frac{b_{n-1}}{L^{n-1}}, \frac{b_n}{L^n} \right) \in [w^* - \gamma, w^* + \gamma]^2$$

thus

$$\frac{1}{L} \left[ f\left(\frac{b_{n-1}}{L^{n-1}}\right) - (a + L - 1) \frac{b_{n-1}}{L^{n-1}} \right] \succ 3\gamma \quad (27)$$

and

$$-\frac{b_n}{L^n} + \frac{b_{n-1}}{L^{n-1}} \geq -2\gamma \quad (28)$$

(27) + (28)  $\implies$

$$\frac{1}{L} \left[ f\left(\frac{b_{n-1}}{L^{n-1}}\right) - (a - 1) \frac{b_{n-1}}{L^{n-1}} - \frac{b_n}{L^n} \right] \succ \gamma$$

The left hand-side is  $\frac{c_n}{L^n}$ . We then have, for all  $B \in S(K^{**}, \frac{\gamma}{2})$  and  $n \geq N + 1$ :

$$\frac{c_n}{L^n} \succ \gamma \succ 0$$

Let  $\varepsilon = \inf(\beta, \frac{\gamma}{2})$ . With the help of equations (26), we obtain for all  $B \in S(K^{**}, \varepsilon)$  and  $n \geq N + 1$ :

$$\begin{aligned} |U''_{nh^2}(b_{n-1}, b_n)| &\leq \frac{2(M' + a + 1)^2}{(1 + \gamma L^n)^3} + \frac{(1 + \alpha)M'}{L^{n-1}}, \\ |U''_{nhl}(b_{n-1}, b_n)| &\leq \frac{2(M' + a + 1)}{(1 + \gamma L^n)^3} \text{ and} \\ |U''_{nl^2}(b_{n-1}, b_n)| &\leq \frac{2}{(1 + \gamma L^n)^3} \end{aligned}$$



These inequalities show that we can find 3 reals  $M_1$ ,  $M_2$  and  $M_3$  such that

$$\begin{aligned} L^{n-2} |U''_{nh^2}(b_{n-1}, b_n)| &\leq M_1, \\ 2L^{n-1} |U''_{nhl}(b_{n-1}, b_n)| &\leq M_2 \text{ and} \\ L^n |U''_{nl^2}(b_{n-1}, b_n)| &\leq M_3 \end{aligned}$$

Thus

$$\begin{aligned} &\sup_{n \geq N+1} \left[ \frac{L^{2(n-1)} |U''_{nh^2}(b_{n-1}, b_n)| + 2L^{2n-1} |U''_{nlh}(b_{n-1}, b_n)| +}{L^{2n} |U''_{nl^2}(b_{n-1}, b_n)|} \right] L^{-n} \\ &\leq M_1 + M_2 + M_3 \end{aligned}$$

For  $n$  in  $[1, N]$ , we can use the inequalities :  $|u''(c)| \leq 2$  and  $|u'(c)| \leq 1 + \alpha$  for all  $c \geq 0$ . In this way, we find easily a real  $M''$  such that, for  $n$  in  $[1, N]$ :

$$\left[ \frac{L^{2(n-1)} |U''_{nh^2}(b_{n-1}, b_n)| + 2L^{2n-1} |U''_{nlh}(b_{n-1}, b_n)| +}{L^{2n} |U''_{nl^2}(b_{n-1}, b_n)|} \right] L^{-n} \leq M''$$

For  $B \in S(K^{**}, \varepsilon)$  and  $X \in l_{\infty}^{LogL}$  such that  $\|X\| \leq 1$ , we can write  $\|\Theta(B, X)\|_{LogL} =$

$$\begin{aligned} &\sup_{n \geq 1} \left[ \frac{x_{n-1}^2 U''_{nh^2}(b_{n-1}, b_n) + 2x_{n-1}x_n U''_{nlh}(b_{n-1}, b_n) +}{x_n^2 U''_{nl^2}(b_{n-1}, b_n)} \right] L^{-n} \\ &\leq \sup_{n \geq 1} \left[ \frac{L^{2(n-1)} |U''_{nh^2}(b_{n-1}, b_n)| + 2L^{2n-1} |U''_{nlh}(b_{n-1}, b_n)| +}{L^{2n} |U''_{nl^2}(b_{n-1}, b_n)|} \right] L^{-n} \\ &\leq \sup(M'', M_1 + M_2 + M_3) = M \end{aligned}$$

Thus, linearity at infinity holds ■

**Proposition 28** *In the case  $u(c) = c^{1-\theta}$ , assumption (4) is checked with  $p_1 = LogL^{(1-\theta)}$ .*

Proof: We have seen above that the sequence  $(c_n)_{n \geq 1}$  is in  $l_{\infty}^{LogL}$ . Thus,  $u((c_n)_{n \geq 1})$  is in  $l_{\infty}^{LogL^{(1-\theta)}}$  and  $G(D) \subset l_{\infty}^{LogL^{(1-\theta)}}$ . When  $(c_n)_{n \geq 1}$  is strictly of reason  $LogL$ ,  $u((c_n)_{n \geq 1})$  is strictly of reason  $LogL^{(1-\theta)}$ . Hence, (4) holds with  $p_1 = LogL^{(1-\theta)}$  ■

**Proposition 29** *In the case  $u(c) = c^{1-\theta}$ , assumption (5) holds.*

Proof:  $u'(c) = (1-\theta)c^{-\theta}$ ,  $u''(c) = \frac{-\theta(1-\theta)}{c^{1+\theta}}$ . As in the case  $u(c) = \alpha c + 1 - \frac{1}{c+1}$ , we build a sphere  $S(K^{**}, \varepsilon)$  such that there is  $\beta \succ 0$ ,  $\gamma \succ 0$  and  $N$  integer such that:

$$\varepsilon \succ 0, S(K^{**}, \varepsilon) \subset \overset{\circ}{D}$$

and

$$\inf_{B \in S(K^{**}, \varepsilon), n \geq 1} \frac{b_n}{L^n} \succ \frac{\beta}{2} \succ 0 \quad (29)$$

and such that for all  $B \in S(K^{**}, \varepsilon)$  and  $n \geq N + 1$ :

$$\frac{c_n}{L^n} \succ \gamma \succ 0 \quad (30)$$

Moreover,  $S(K^{**}, \varepsilon) \subset \overset{\circ}{D} \implies$  for all  $B \in S(K^{**}, \varepsilon)$  and  $n \geq 1$  we have  $c_n \succ 0$ . But

$$c_n = \left[ f\left(\frac{b_{n-1}}{L^{n-1}}\right) - (a-1)\frac{b_{n-1}}{L^{n-1}} - L\frac{b_n}{L^n} \right] L^{n-1}$$

Thus

$$m_n = \min_{(x,y) \in \left[ \frac{k_n^{**}}{L^{n-1}} - \varepsilon, \frac{k_n^{**}}{L^{n-1}} + \varepsilon \right] \times \left[ \frac{k_n^{**}}{L^n} - \varepsilon, \frac{k_n^{**}}{L^n} + \varepsilon \right]} [f(x) - (a-1)x - Ly] L^{n-1} \succ 0$$

Denote

$$m = \min_{1 \leq n \leq N} m_n$$

$m$  is strictly positive and, for all  $B \in S(K^{**}, \varepsilon)$  and  $1 \leq n \leq N$  we have

$$c_n \geq m \quad (31)$$

Thanks to inequalities (29), (31) and to equations (26), we see that we can find  $M''$  such that, for all  $B \in S(K^{**}, \varepsilon)$  and  $X \in l_\infty^{LogL}$  such that  $\|X\| \leq 1$ :

$$\sup_{1 \leq n \leq N} \left[ \left| \frac{x_{n-1}^2 U''_{nh^2}(b_{n-1}, b_n) + 2x_{n-1}x_n U''_{nhl}(b_{n-1}, b_n) + x_n^2 U''_{nl^2}(b_{n-1}, b_n)}{L^{-n(1-\theta)}} \right| \right] \leq M''$$

Thanks to (29) and (30), we have also, for all  $B \in S(K^{**}, \varepsilon)$  and  $n \geq N + 1$ :

$$|U''_{nh^2}(b_{n-1}, b_n)| \leq \frac{\theta(1-\theta)}{\gamma^\theta L^{n(1+\theta)}} (M' + a + 1)^2 + \frac{(1-\theta)M'}{\gamma^\theta L^{n\theta} L^{n-1}},$$

$$|U''_{nhl}(b_{n-1}, b_n)| \leq \frac{\theta(1-\theta)}{\gamma^\theta L^{n(1+\theta)}} (M' + a + 1) \text{ and}$$

$$|U''_{nl^2}(b_{n-1}, b_n)| \leq \frac{\theta(1-\theta)}{\gamma^\theta L^{n(1+\theta)}}$$

We can write this in another way:

$$\begin{aligned} |U''_{nh^2}(b_{n-1}, b_n)| &\leq \frac{M_1}{L^{(1+\theta)n}}, \\ 2|U''_{nhl}(b_{n-1}, b_n)| &\leq \frac{M_2}{L^{(1+\theta)n}} \text{ and} \\ |U''_{nl^2}(b_{n-1}, b_n)| &\leq \frac{M_3}{L^{(1+\theta)n}} \end{aligned}$$

Thus, for all  $B \in S(K^{**}, \varepsilon)$  and  $X \in l_\infty^{LogL}$  such that  $\|X\| \leq 1$ :

$$\begin{aligned} & \sup_{n \geq N+1} \left[ \begin{array}{c} x_{n-1}^2 U''_{nh^2}(b_{n-1}, b_n) + 2x_{n-1}x_n U''_{nlh}(b_{n-1}, b_n) + \\ x_n^2 U''_{nl^2}(b_{n-1}, b_n) \end{array} \left| L^{-n(1-\theta)} \right. \right] \\ & \leq M_1 + M_2 + M_3 \end{aligned}$$

As a result, for all  $B \in S(K^{**}, \varepsilon)$  and  $X \in l_\infty^{LogL}$  such that  $\|X\| \leq 1$ ,  $\|\Theta(B, X)\|_{LogL} =$

$$\begin{aligned} & \sup_{n \geq 1} \left[ \begin{array}{c} x_{n-1}^2 U''_{nh^2}(b_{n-1}, b_n) + 2x_{n-1}x_n U''_{nlh}(b_{n-1}, b_n) + \\ x_n^2 U''_{nl^2}(b_{n-1}, b_n) \end{array} \left| L^{-n(1-\theta)} \right. \right] \\ & \leq \sup(M'', M_1 + M_2 + M_3) = M \end{aligned}$$

and linearity at infinity holds ■

## References

- [Balasko-Shell 1980] Y. Balasko, K. Shell, The OLG model, I: the case of pure exchange without money, *Journal of Economic Theory* 23(1980)281-306.
- [Barro 1974] R. Barro, Are government bonds net wealth?, *Journal of Political Economy* 82(1974) 1096-1117.
- [Cass 1972] D. Cass, On capital overaccumulation in the aggregative, neoclassical model of economic growth: a complete characterization, *Journal of Economic Theory* 4(1972)200-223.
- [Groth 2003] C. Groth, *Course of advanced macroeconomics 2003*, Institute of Economics, University of Copenhagen, [www.econ.ku.dk](http://www.econ.ku.dk).
- [Heidjra-van der Poeg 2002] B. Heijdra, R. van der Poeg, *Foundations of modern macroeconomics*, Oxford, 2002.
- [Lakshmi 2002] K.R. Lakshmi, Two-sided altruism, Lindhal equilibrium and Pareto optimality in OLG models, Working paper, University of Hawaii-Manoa, 2002.
- [Le Van-Saglam 2004] C. Le Van, H.C. Saglam, Optimal growth models and the Lagrange multiplier, *Journal of Mathematical Economics* 40(2004)393-410.
- [Luenberger 1997] D. Luenberger, *Optimization by vector spaces*, Wiley-Interscience, New Ed edition 1997.
- [Mabrouk 2005] M. Mabrouk, Optimal growth path in an OLG economy without time-preference assumptions, *Annales Maghrébines de l'Ingénieur ENIT-Tunis* (in press), available at <http://econwpa.wustl.edu/eps/ge/papers/0510/0510006.pdf>
- [Mabrouk 2006a] M. Mabrouk, Allais-anonymity as an alternative to the discounted-sum criterion

in the calculus of optimal growth I: consensual optimality.

- [Molina-Abraldes-Pintos-Clapes 2003] A. Molina-Abraldes, J. Pintos-Clapès, A complete characterization of Pareto optimality for general OLG economies, *Journal of Economic Theory* 113(2003)235-252.
- [Saez-Marti-Weibull 2005] M. Saez-Marti, J. W. Weibull, Discounting and altruism to future decision makers, *Journal of Economic Theory* (in press).
- [Solow 1956] R. Solow, A contribution to the theory of optimal growth, *Quarterly Journal of Economics* 70(1956) 65-94.