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Abstract: In a smooth pure exchange economy with fixed total resources we investigate whether the smooth selection property holds when endowments are redistributed across consumers through a continuous (non local) redistribution policy. We show that if the policy is regular then there exists a unique continuous path of equilibrium prices which support it. If singular economies are involved in the redistribution, then an analogous result can be obtained if the singular policy is the projection of a path transversal to the set of critical equilibria.

Keywords: Equilibrium manifold, regular equilibria, catastrophes, structural stability, smooth selection of prices

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1 Introduction

In a smooth pure exchange economy with fixed total resources, suppose that endowments are redistributed across agents according to a redistribution policy. Define such a policy by a continuous map $\gamma : [0,1] \to \Omega(r)$, $\gamma(0) = \omega_0$, $\gamma(1) = \omega_1$, where $\Omega(r)$ denotes the set of endowments with fixed total resources $r \in \mathbb{R}^l$ and x (y) denotes the initial (final) allocation, respectively. If we write, using standard vector notation to denote the aggregate excess demand function, the equilibrium condition as $z(p(t), \gamma(t)) = 0$, $t \in [0,1]$, very natural questions are whether $p(t)$ is locally unique and it is changing continuously while the parameter ω is varying. As highlighted by [6, p. 315], an affirmative answer to these two questions is crucial for comparative statics analysis and planning theory (the advantage of using a continuous policy would be invalidated by discontinuities of prices). In particular, the use of the second welfare theorem becomes problematic with multiplicity and instability of prices (see [5] and references therein).

In order to provide an answer to these questions, the traditional approach uses the implicit function theorem (IFT). Since IFT holds locally, this approach cannot deal with arbitrary variations of regular endowments. Moreover, IFT depends on a regularity condition, the non-singularity of the Jacobian matrix: it cannot be applied if the endowments changed are singular economies. Summarizing: Multiplicity of prices and catastrophes may originate indeterminacy and ambiguity. In Kehoe's words [6, p. 333]: "it does not seem that any sort of comparative statics methodology is applicable in such circumstances".

A geometric understanding of the relationship between endowments and price variations can be provided by the equilibrium manifold approach [2]. If we denote by S the set of normalized prices, the *equilibrium manifold* $E(r) \subset S \times \Omega(r)$ is the set of prices and endowments such that aggregate excess demand is zero. Let us denote by $\pi : E(r) \to \Omega(r)$ the *natural projection*, i.e. the restriction to the equilibrium manifold $E(r)$ of the projection $(p, \omega) \mapsto \omega$ from $S \times \Omega(r)$ into $\Omega(r)$ (see [2] and Section 3). The structure of the equilibrium manifold and the properness property of the natural projection (see Section 3) entail the existence of *smooth selections* of equilibrium prices: i.e., an infinitesimal change of regular endowments implies an infinitesimal change of the corresponding equilibrium price vectors. It is worth noting that this smooth selection property (SSP) is not a theory of equilibrium selection (for the existence of a continuous random selection see [9, p. 348] and [10]). As Balasko points out [2, p. 94], this structural stability property "should not be confused with stability in the sense of tatonnement".

When endowments are redistributed according to $\gamma(t)$, SSP holds when $\gamma(t)$ is a *local* and *regular* policy. By local we mean that $\gamma(t)$ belongs to a sufficiently small neighborhood U_x of the initial economy $x = \gamma(0)$ and by regular we mean that $\gamma(t)$ does not cross *singular economies* (see [2] or Section 3). In fact if a continuous local policy $\gamma(t)$ should cross the set of singular economies, it could give rise to *catastrophes* (see [1, 4]), thus determining discontinuities of prices.

In this paper we are interested in whether non local or regular redistributions of endowments across consumers can be supported by continuous price changes. This question is deeply related (see Remark 4.2) to the issue, raised and tackled by Balasko, of finding the largest domain of definition of the smooth selection mappings (see [2, p. 94 and 190]). In the case of two goods and an arbitrary number of consumers or in the case of two consumers and an arbitrary number of goods, he shows that the largest definition sets for the equilibrium price selections coincide with the connected component of the set of regular equilibria. When the number of consumers and goods are arbitrary, Balasko shows [2, Theorem 7.3.10] that the selection is defined if one takes as domain of definition the connected component of an open and dense subset of the space of economies $\Omega(r)$.

The main difference between our work and Balasko's issue is that we are not looking for the largest domain of definition but we are concerned with a path representing a redistribution of endowments. In this case one can extend the domain of definition of the selection, restricted to this path, to the connected component of the set of regular equilibria (see Theorem 4.1 when the redistribution only involves regular endowments).

This result can be proven by using the *arc lifting property* (ALP), a property known in the mathematics of covering spaces (see Proposition 2.1). Let us denote with $\tilde{\gamma}(t) = (p(t), \gamma(t))$ a continuous path on the equilibrium manifold. ALP says, roughly speaking, that if we start from an initial equilibrium (p, ω') and we reallocate endowments to ω'' through a redistribution policy $\gamma(t)$, where $\gamma(0) = \omega'$ and $\gamma(1) = \omega''$, then there exists a (unique) equilibrium path $\tilde{\gamma}(t) \in E(r)$ such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$ (where the endpoints of $\gamma(t)$ belong to the same connected component of $\Omega(r)$). Therefore ALP extends to the connected component what only holds locally by SSP, overcoming the local condition imposed by IFT and providing theoretical foundations to the assumption that when regular endowments are changed smoothly, the corresponding equilibrium price vectors will also change smoothly.

When the policy is not regular, i.e., when the redistribution concerns singular economies, the existence of a lift on $E(r)$ of a continuous policy becomes an issue. By using the geometric construction by [7, 8], in Theorem 4.3 we show under what conditions it is still possible to get a continuous path of equilibrium price vectors $p(t)$ while we change endowments according to a policy which encounters catastrophes. The intuition is that, in order to obtain the desired result, one needs to construct a path transversal to the set of critical equilibria and to project it onto the space of endowments. In [7] the authors define the length between two regular equilibria as the number of intersection points of the path connecting them with the set of critical equilibria and show the existence of a minimal path according to this definition of distance. In [8] it is shown that there exists a Riemannian metric on the equilibrium manifold $E(r)$ such that a minimal geodesic connecting two (sufficiently close) regular equilibria intersects the set of critical equilibria in a finite number of points. This metric can be regarded as an algorithm to calculate the optimal path $\tilde{\gamma} : [0,1] \to E(r)$ connecting two regular equilibria. Under this perspective, Theorems 4.1 and 4.3 can provide foundations to apply the tools developed by [7, 8] to construct an optimal policy.

The structure of this paper is the following. Sections 2 and 3 recall some mathematical results and the economic model. Section 4 is devoted to the proof of our main results.

2 Mathematical preliminaries

We start this section by introducing some mathematical results on covering spaces and lifting properties. Unless otherwise specified all the topological spaces involved are subsets of Euclidean spaces and all maps between them are assumed to be smooth. We refer the reader to [11] for the standard material on coverings. Let $\tilde{X}$ and X be two (not necessarily connected) topological spaces. A map $p : \tilde{X} \to X$ is called a *covering map* if it satisfies the following conditions:

(a) p is surjective;

(b) each $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p diffeomorphically onto U.

The neighbourhood U is said to be *well-covered* for p and the cardinality of the fiber of x, i.e. the set $p^{-1}(x)$, is called the *number of sheets* of the covering. Observe that this number could be infinite as for the map $p : \mathbb{R} \to S^1$ from the real numbers to the unit circle $S^1 \subset \mathbb{R}^2$ defined by $p(t) = (\cos t, \sin t)$.

Property (b) immediatly implies that p is a local diffeomorhism and hence an open map. Therefore a injective covering map is a diffeomorphism. We point out that there exist surjective local diffeomorphisms which are not a covering map. For example if one restricted the map of the previous example to the interval $(0, 4\pi)$, one would obtain a surjective local diffeomorphism but the point $(1, 0)$ does not admit a well-covered neighbourhood.

Let $p : \tilde{X} \to X$ be any (smooth) map and let Y be a connected topological space. A *lift* of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$. In the following propositions we state some standard properties of covering spaces which play a crucial role in the present paper. We recall that an arc on X is a map $\alpha : I \to X$, where $I = [0,1]$. The points $\alpha(0)$ and $\alpha(1)$ are called the starting and final points of α.

Proposition 2.1 (ALP: arc lifting property) *Given a covering space $p : \tilde{X} \to X$, let $\alpha : I \to X$ be an arc with starting point x_0 and let $\tilde{x}_0$ be any point in the fiber of x_0. There exists a unique lift $\tilde{\alpha} : I \to \tilde{X}$ of α with starting point $\tilde{x}_0$.*

In the previous proposition the existence of a lift relies on properties of a covering maps. Once a lift is given its uniqueness depends only on the fact that p is a local diffeomorphism. We state this observation in the following proposition.

Proposition 2.2 *Let $p : \tilde{X} \to X$ be a local diffeomorphism, Y a connected topological space, $y_0 \in Y$, $\tilde{x}_0 \in \tilde{X}$ and $f : Y \to X$ a map such that $f(y_0) = x_0 = p(\tilde{x}_0)$. A lift $\tilde{f} : Y \to \tilde{X}$ of f such that $\tilde{f}(y_0) = \tilde{x}_0$ is unique.*

A natural question is to understand when a surjective local diffeomorphism is a covering map. One can prove that a local diffeomorphism $p : \tilde{X} \to X$ such that every arc on X can be lifted is in fact a covering map. Since this property is generally impossible to handle, one can impose some sufficient topological conditions in order to have a covering map as in the following proposition.

Proposition 2.3 *Let $p : \tilde{X} \to X$ be a proper surjective local diffeomorphism. Then p is a finite covering map.*

Recall that a continuous map $p : \tilde{X} \to X$ is proper if the preimage of any compact set of X is a compact set of $\tilde{X}$.

3 The model

We refer the reader to [2] for the economic set-up briefly described in this section. We consider a pure exchange economy with fixed total resources. Let m and l be, respectively, the (finite) number of agents and commodities. Let $S = \{p \in \mathbb{R}^l \mid p_i \geq 0, i = 1, 2, \ldots, l-1, p_l = 1\}$ be the set of prices normalized by the numeraire convention. Let $r \in \mathbb{R}^l$ be the vector of fixed total resources and denote by $\Omega(r)$ the set of initial endowments with fixed total resources, i.e., $\Omega(r) = \{\omega \in \mathbb{R}^{lm} \mid \sum_{i=1}^m \omega_i = r\}$. Under the standard smooth assumptions on preferences (see [2, Ch. 2]), the problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e. consumer i's demand. Define the equilibrium manifold, denoted by $E(r)$, the set of pairs of prices and endowments such that aggregate net demand is zero, i.e.,

$$E(r) = \{(p, \omega) \in S \times \Omega(r) \mid \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\}.$$

The set $E(r)$ is globally diffeomorphic to $\mathbb{R}^{l(m-1)}$ (see [2, Ch. 5]). Let $\pi : E(r) \to \Omega(r)$ be the *natural projection*, i.e. the restriction to $E(r)$ of the projection $S \times \Omega(r) \to \Omega(r)$, such that $(p, \omega) \mapsto \omega$. The map π is smooth, proper and surjective.

One can define the set of *critical equilibria*, denoted by $E_c(r)$, as the pairs $(p, \omega) \in E(r)$ such that the derivative of π is not onto [1]. The set $E_c(r)$ is a closed subset of measure zero of the equilibrium manifold $E(r)$ (see [3]). The set of *singular economies*, denoted by Σ, is the image via π of the set $E_c(r)$. The set Σ is a closed (by properness of π) and a measure zero set in $\Omega(r)$ (by Sard's theorem). Let us define the regular economies $R = \Omega(r) \setminus \Sigma$ as the regular values of the map π.
We state as a theorem the following important result due to Balasko.

Theorem 3.1 (Balasko [2]) *The map $\pi_{|_{\pi^{-1}(R)}} : \pi^{-1}(R) \to R$ is a finite covering.*

Proof: By the inverse function theorem $\pi_{|_{\pi^{-1}(R)}}$ is a local diffeomorphism. Since $\pi_{|_{\pi^{-1}(S)}} : \pi^{-1}(S) \to S$ is still proper and surjective for any subset $S \subset \Omega(r)$, the result follows by Proposition 2.3. □

The economic meaning of this property, known as smooth selection property (SSP) [2, p. 94], is that in a neighborhood of a regular economy, smooth changes of the parameter ω imply smooth changes of the corresponding equilibrium price vectors, namely there exists a supporting equilibrium price vector sufficiently close to the initial one. As highlighted in the Introduction, SSP is not a theory of equilibrium selection: i.e., if endowments are locally redistributed smoothly across consumers then the selected equilibrium price vector will change smoothly.

4 Main results

In the sequel we denote a redistribution policy of endowments as a continuous map $\gamma : [0, 1] \to \Omega(r)$, where $\omega_0 = \gamma(0)$ and $\omega_1 = \gamma(1)$. Since we are interested in whether arbitrary redistributions of endowments across consumers can be supported by continuous price changes, we define a redistribution policy $\gamma(t)$:

- *regular* if $\gamma(t) \subset R$;
- *singular* if $\gamma(t) \in \Sigma$ for some $t \in [0, 1]$.

A first very natural question is to investigate whether SSP holds when the policy is regular. The following theorem provides an affirmative answer.

Theorem 4.1 *Let $\gamma : I \to R$ be a regular policy connecting $\omega_0 = \gamma(0)$ and $\omega_1 = \gamma(1)$ and let p_0 be the supporting equilibrium price vector associated with ω_0. Then there exists a unique lift $\tilde{\gamma} : I \to \pi^{-1}(R)$ of γ.*

Proof: It follows by Proposition 2.1 and Theorem 3.1. □

Remark 4.2 If there are two commodities or two consumers the theorem is trivial since Balasko has shown [2, p. 191] that the largest domain of definition of selections of price equilibria coincide with the connected component of the set of regular equilibria. In the general case, it is shown [2, Theorem 7.3.10] that one can take as domain of definition the connected component of an open and dense subset of the space of economies. It is in this general case where Theorem 4.1 applies, by providing a further insight of the structural stability of the equilibrium manifold. In fact it shows that the domain of definition, if restricted to a policy, can be extended to the connected component of the set of regular equilibria.

When the policy is singular the main issue becomes the existence of a lift. Therefore we tackle the problem using a different strategy. We construct a minimal path on $E(r)$ which connects two regular equilibria (p, ω) and (p', ω') and which intersects $E_c(r)$ on a finite number of points. The existence of such a path has been shown by [7], where the (minimal) path represents a solution to the problem of minimizing the number of intersection points with the set of critical equilibria, and by [8], where the path is a minimal geodesic according to a Riemannian metric constructed on $E(r)$. The policy is found by projecting this path onto the space of endowments $\Omega(r)$. Theorem 4.3 shows under what conditions this policy admits a (unique) lift. In the theorem we consider minimal paths as in [7], but the construction holds for any path transversal to $E_c(r)$ and so even for geodesics as in [8].

Theorem 4.3 *Let $\tilde{\gamma} : I \to E(r)$ be a minimal arc connecting two reqular equilibria x and y, where $x, y \in \pi^{-1}(R)$. Then $\tilde{\gamma}$ is uniquely determined by its projection $\gamma = \pi(\tilde{\gamma})$ and by a finite number of its points.*

Proof: Let $C = E_c(r) \cap \tilde{\gamma}(I)$. Since $\tilde{\gamma}$ is a minimal path either $C = \emptyset$ or C is a finite number of points. If $C = \emptyset$ then the conclusion follows by Proposition 2.2 applied to the local diffeomorphism $\pi : E \setminus E_c(r) \to \pi(E \setminus E_c(r))$. If C is nonempty set $C = \{c_1, c_2, \ldots, c_k\}$. Then there exist $0 < t_1 \leq \ldots \leq t_k < 1$ such that $c_i = \tilde{\gamma}(t_i)$, $i = 1, \ldots, k$. Choose $\xi_i = \tilde{\gamma}(s_i)$, with $i = 1, \ldots, k-1$, with $t_j < s_j < t_{j+1}$, $j = 1, \ldots, k-1$ such that $\tilde{\gamma}(s_j) \in E \setminus E_c(r)$. Consider the following subarcs of $\tilde{\gamma}$: $\tilde{\gamma}_x^{c_1}, \tilde{\gamma}_{c_1}^{\xi_1}, \tilde{\gamma}_{\xi_1}^{c_2}, \ldots, \tilde{\gamma}_{\xi_{k-1}}^{c_k}, \tilde{\gamma}_{c_k}^{y}$ connecting x with c_1, c_1 with ξ_1,..., ξ_{k-1} with c_k, and c_k with y. By applying again Proposition 2.2 to the local diffeomorphism $\pi : E \setminus E_c(r) \to \pi(E \setminus E_c(r))$ it follows that $\tilde{\gamma}_x^{c_1} \setminus \{c_1\}, \tilde{\gamma}_{c_1}^{\xi_1} \setminus \{c_1\}, \tilde{\gamma}_{\xi_1}^{c_2} \setminus$

$\{c_2\}, \ldots, \tilde{\gamma}^{c_k}_{\xi_{k-1}} \setminus \{c_k\}, \tilde{\gamma}^{y}_{c_k} \setminus \{c_k\}$ are the unique lifts of $\pi(\tilde{\gamma}^{c_1}_{x}) \setminus \{\pi(c_1)\}, \pi(\tilde{\gamma}^{\xi_1}_{c_1}) \setminus \{\pi(c_1)\}, \pi(\tilde{\gamma}^{c_2}_{\xi_1}) \setminus \{\pi(c_2)\}, \ldots, \pi(\tilde{\gamma}^{c_k}_{\pi(\xi_{k-1})}) \setminus \{\pi(c_k)\}, \pi(\tilde{\gamma}^{\pi(y)}_{\pi(c_k)}) \setminus \{\pi(c_k)\}$ passing through the points $\{\xi_1, \ldots, \xi_{k-1}\}$, respectively. Then, by a continuity argument, $\tilde{\gamma}$ is the unique lift of $\gamma = \pi(\tilde{\gamma})$ passing through the finite set of points $C \cup \{\xi_1, \ldots, \xi_{k-1}\}$.
□

Some features that characterize our geometric construction deserve a few comments. First, Theorem 4.1 should not be regarded as a particular case of Theorem 4.3, when $C = \emptyset$. In Theorem 4.1 the redistribution policy is regular, i.e. $\gamma(t) \subset R$, while in Theorem 4.3 it is not. Moreover it can be singular even at points which are the projections of regular equilibria belonging to $\tilde{\gamma}(t)$. Second, under additional assumptions on the equilibrium price dynamics , this construction can potentially be extended to construct an algorithm to find an optimal redistribution policy. Roughly speaking, the idea is the following. Given an initial and final economy ω and ω', the social planner wants to redistribute endowments across consumers to move the economy toward the target ω'. He/she constructs a Riemannian metric which embodies the objectives of the redistribution policy and he (she) then finds a geodesic $\tilde{\gamma}(t)$ joining $x = (p, \omega)$ and $y = (p', \omega')$. The redistribution policy is then $\pi(\tilde{\gamma}(t)) = \gamma(t)$, which represents the optimal choice among the infinite policies joining ω and ω' Observe that this policy, which is optimal from the perspective of the equilibrium manifold, can appear quite counterintuitive in the space of the endowments (for example, its self-intersections can be omeomorphic to intervals). Finally, when dealing with singularities it is necessary to fix as many supporting price vectors as many connected components crossed by the path on $E(r)$. This is due to the potential discontinuity which may arise when $\gamma(t)$ crosses singular economies. A minimal path allows to minimize the number of supporting price vectors to be fixed.

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