Portfolio Choice and Benchmarking: The Case of the Unemployment Insurance Fund in Chile

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Abstract

A new Unemployment Insurance System based on individual accounts was launched in Chile on October 2002. One of the most interesting features of the system is given by the compensation scheme of the fund manager, which contains a performance-based incentive benchmarked to one of the default portfolios of the pension system (pension funds Type E, with a 100% investment in fixed-income securities).

This paper studies the portfolio choice problem of a fund manager which is subject to a similar performance-based compensation scheme. We model the portfolio choice problem of a risk averse portfolio manager that must finance an exogenous sequence of benefits, and whose terminal payoff depends upon the terminal value of the portfolio under management, relative to an exogenous benchmark portfolio. Our interest is on the consequences of the incentive scheme over the portfolio that is selected by the portfolio manager.

For the Black and Scholes [1973] economy we are able to determine the investment policy in closed form. We show that the riskiness of the portfolio depends on the composition of the benchmark, and that the fund manager is motivated to imitate the investment policy of the benchmark in some random scenarios.

JEL Classification: D81; G11; G18; and H55.

Keywords: Benchmark portfolio; Individual accounts; Portfolio choice; Unemployment Insurance.

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1 Introduction

A new unemployment insurance (UI) system based on individual accounts was launched in Chile on October 2002. This new system mandates all workers and employers, engaged in labor relationships started after October 2002, to save a fraction of the worker's salary in an individual account on a monthly basis.[1] The savings, plus the returns obtained, can be withdrawn from the individual account by the worker in case of unemployment, according to a pre-defined schedule of withdrawals and the balance of the individual account.[2]

The collection of contributions, investment management, and the payment of the corresponding benefits in this new system has been franchised to a single-purpose firm, known as the Unemployment Insurance Fund Manager (UIFM). The monthly revenue of this firm is based solely on a fee charged over the total funds under management, from the individual accounts of the workers that were employed during the previous month.

One of the main features of the new system is the presence of a compensation scheme that contains incentives based on the percentage fee that the UIFM can charge every month. The incentives are benchmarked against the pension funds that invest only in fixed-income securities.[3] In particular, the compensation scheme establishes an increase or decrease in the monthly base fee (0.05%) that can go up or down up to 10% when the UIFM beats or is beaten by the benchmark portfolio. The base of comparison corresponds to the annualized returns of the last 36 months.

The subject of this paper is to study the portfolio choice implications of the incentives embedded into this compensation scheme. To this end, we model the portfolio choice problem of a risk averse portfolio manager that has to pay random benefits during a given month. We compare the optimal portfolio choice of the manager with the one

[1] In the case of labor contracts of indefinite duration (i.e., those without a foreseeable horizon), the employee's contribution amounts to 0.6% of the salary, while the employer's contribution amounts to 1.6%. For the case of fixed duration or by task contracts, the employer' contribution amounts to 3%, while the employee contributes nothing. In both cases there is a maximum monthly salary of reference currently set at $3,000 (US) dollars.

[2] For the case of indefinite contracts there are additional contributions which are made to a so-called 'solidarity fund' by the employer (0.8%) and the Central Government (an amount which is set by law). The purpose of the solidarity fund is to finance a floor of withdrawals for workers that have been fired and have contributed at least 12 months.

[3] The Chilean Pension System based on individual accounts has adopted a life-cycle investment style based on funds targeted to different age groups. The system currently considers 5 different funds. The more conservative one (Type E) is only allowed to invest in fixed-income securities; see Ferreiro-Yazigi [9] for further references.

that results from a compensation scheme where these incentives are absent. We study the portfolio choice problem of the fund manager in a Black and Scholes [2] setting and provide the optimal portfolio policy in closed-form. For this particular environment, we are able to show that the optimal portfolio of the UIFM replicates the investment policy of the benchmark portfolio in some states of the world.

The literature on compensation schemes and incentives has been subject to important contributions during the recent years; see, e.g., Basak *et al.* [1], Carpenter [3], Goetzmann *et al.* [10], Hodder and Jackwerth [11], and Ross [18]. The paper which is closer in spirit to the present one is the one by Basak, Pavlova and Shapiro [1] that studies the problem of a fund manager facing a compensation scheme which is benchmarked against an exogenous portfolio, and that considers —as we do— the simplified context of the Black and Scholes setting. Although the basic underlying principles are the same, our paper considers a concrete compensation scheme that is worth studying. In particular, the compensation scheme under consideration explicitly considers bonuses and penalties starting from a base situation, and further, the economic environment faced by the fund manager is one in which the random payment of benefits is a relevant consideration.

We employ the martingale approach pioneered by Cox and Huang [4] and Karatzas *et al.* [13] to study the portfolio choice problem of the UIFM. This approach allows us to analytically characterize the investment policy that emerges in portfolio choice problems under very general economic environments.

The paper has the following structure. Section 2 presents the model, section 2.1 describes the economic environment, sections 2.2 and 2.3 analyze the portfolio choice problem with and without incentives in the compensation scheme, and section 3 provides some concluding remarks. All proofs are collected in the Appendix.

2 The model

This section develops a model that is aimed at capturing the relevant features contained in the compensation scheme faced by the fund manager in a given month. The main element of interest corresponds to the fact that fund manager's revenues are dictated by the annualized returns obtained by both the fund manager and the benchmark portfolio.

In order to establish a valid point of reference to which compare the effects of the incentives embedded in the compensation scheme, we first analyze the portfolio choice problem of the fund manager with a compensation scheme in which such incentives are absent.

2.1 Economic environment

(UNCERTAINTY) Time is continuous on a finite horizon $[0, T]$, $T > 0$. The uncertainty is generated by a Brownian Motion process, $W \in \mathbb{R}^d$, $d \in \mathbb{N}$, defined on a complete, filtered probability space, $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, where $\mathbb{P}$ denotes the probability measure defined over $(\Omega, \mathcal{F})$. Given that our interest is placed on the characterization of the situation of interest, in what follows we will point out only the most relevant technical conditions, and will assume that all processes to be introduced are well-defined. In particular, all processes to appear have a strong solution, and all expressions containing (in)equalities of random variables will be understood in the *almost sure* sense; that is, for any two given random variables $x(\omega)$, $y(\omega) : \Omega \mapsto \mathbb{R}$, the expression $x > y$ is to be understood in the sense that $\mathbb{P}(x > y) = 1$.

(CONSUMPTION SPACE) In the economy under analysis there is a single perishable consumption good that will be taken as the nummeraire. The consumption space is given by consumption bundles $c \equiv \{c_t \geq 0\}_{0 \leq t \leq T}$ that have the property of being integrable.[4]

(PREFERENCES) Fund manager's preferences are represented by a von Neumann Morgenstern index defined over the terminal value of the Unemployment Fund, $F_T \geq 0$, of the form: $V(F_T) \triangleq \mathbb{E}_0 [u(\phi(F_T))]$, where $\mathbb{E}_t [\cdot] \triangleq \mathbb{E} [\cdot | \mathcal{F}_t]$, $\forall t \in [0, T]$, corresponds to the mathematical expectation at time t, conditional on the information available at that time $(\mathcal{F}_t)$, $\phi(\cdot) : \mathbb{R}_+ \mapsto (0, 1)$ denotes the compensation scheme of the fund manager which is a function of F_T, and the mapping $u(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}$ is of class C^2, with $u'(\cdot) > 0$, $u''(\cdot) < 0$, which in addition satisfies Inada endpoint conditions.

(FINANCIAL MARKET) We consider a frictionless financial market where all sources of uncertainty (generated by W) can be hedged away; that is, we will consider a (dynamically) complete financial market with no transaction costs, and where the fund manager can rebalance its portfolio continuously; see Merton [14]. In particular, we will consider a financial market comprised of $d + 1$ assets, represented by d (locally) risky securities $(S \equiv \{S^i\}_{1 \leq i \leq d})$, and a (locally) riskless money-market account (B),[5] which dynamics is

[4] A process x is called p-integrable if $\mathbb{P}(\int_0^T x_t^p dt < \infty) = 1$. If $p = 1$ the process if called *integrable*.

[5] In an economic environment where the instantaneous interest rate is a function of the trajectories of W, the price of a long-term bond with maturity T is "locally" risky (i.e., between t and $t + dt$, the change in the price of the long-term bond depends on dW).

dictated by the following system:

$$\begin{aligned} dB_t &= r_t B_t dt, \quad B_0 = 1 \\ dS_t + \delta_t dt &= \mathbf{I}^S \left(\mu_t dt + \sigma_t dW_t\right), \quad S_0 \in \mathbb{R}^d_{++} \text{ given,} \end{aligned}$$

where $\mathbf{I}^S$ denotes a diagonal matrix of dimension d, that contains the price of each risky security on its diagonal, and the quantities

$$(r, \mu, \delta, \sigma) \equiv \{(r_t, \mu_t, \delta_t, \sigma_t) \in \mathbb{R}_+ \times \mathbb{R}^d_+ \times \mathbb{R}^d_+ \times \mathbb{R}^{d\times d}/\{0\}\}_{0\leq t\leq T}$$

are functions of the trajectories of W (or Itô processes) that satisfy the conditions that make B and S integrable processes. They represent, respectively, the instantaneous interest rate, the expected return of the risky securities, their corresponding dividend, and volatility matrix.

(WITHDRAWAL OF BENEFITS) The withdrawal of benefits throughout the investment horizon $[0, T]$ is represented by an $\mathbb{F}$-adapted process, $b \equiv \{b_t \in [0, \bar{b})\}_{0\leq t\leq T}$, $0 < \bar{b} < K < \infty$, whose dynamics is described by $db_t/b_t = \mu^b_t dt + \sigma^b_t dW_t$, $b_0 > 0$, where (μ^b, σ^b) are functions of the trajectories of W, that satisfy the necessary conditions for b to be an integrable process.

(UNEMPLOYMENT INSURANCE FUND) The dynamics of the Unemployment Insurance Fund, $F^\pi \equiv \{F^\pi_t \geq 0\}_{0\leq t\leq T}$, is generated by the investment plan chosen by the fund manager. For a given investment plan $\pi \equiv \{\pi_t \in \mathbb{R}^d\}_{0\leq t\leq T}$, where π denotes the dollar amount invested in each of the risky assets, the Unemployment Insurance Fund's dynamics is represented by the following system[6]

$$\text{(1)} \qquad \begin{cases} dF^\pi_t = [F^\pi_t - \pi'_t\mathbf{1}]r_t dt + \pi'_t[\left(\mathbf{I}^S\right)^{-1}(dS_t + \delta_t dt)] - b_t dt; \\ F_0 > 0; \;\; F^\pi_t \geq 0, \forall t \in [0, T]; \;\; F_T = F^\pi_T, \end{cases}$$

where the symbol "$'$" denotes transposition, $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^d$, $F^\pi_t - \pi'_t\mathbf{1}$ corresponds to the dollar amount invested in the money-market account (that provides a return of r_t), while the remainder, $\pi'_t\mathbf{1}$, corresponds to the dollar amount invested in the risky assets (that provides a return given by capital gains plus dividends, $(dS_{it} + \delta_{it}dt)/S_{it}$, for $i = 1, \ldots, d$).

[6]The dynamics in (1) implicitly assumes that the fund manager's revenue is received after T. To assume the opposite will amount to modify the expression in (2) so that the new terminal value of the UI fund will be given by $\hat{F}_T = (1 - \phi)F_T$. The analysis that follows remains unchanged by this re-normalization.

(INVESTMENT PLANS) In what follows, we will denote the set of investment plans that satisfy the system (1), along with the conditions $V(F_T) < \infty$ and F_t^{π} integrable, by $\mathcal{A}^{\pi}(F_0)$. An investment plan $\pi \in \mathcal{A}^{\pi}(F_0)$ will be called *admissible*. We will also denote the set of *optimal* investment plans by $\mathcal{A}^{*}(F_0)$, where $\pi^* \in \mathcal{A}^{*}(F_0)$, if and only if, $V(F_T^{\pi^*}) \geq V(F_T^{\pi})$, $\forall \pi \in \mathcal{A}^{\pi}(F_0)$.

2.2 The portfolio choice problem

We first analyze the portfolio choice problem of the fund manager by considering an incentive-free compensation scheme given by a constant function $\phi(F_T) = \phi F_T$. Under these circumstances, the fund manager's problem is given by the selection of an admissible investment plan $\pi \in \mathcal{A}^{\pi}(F_0)$ to maximize the utility index $V(F_T)$.

Making use of some well known results in the literature (Cox and Huang [4, 5], Karatzas *et. al* [13]), it is possible to show that an investment plan π is admissible, if and only if, the following "static" version of the budget constraint in (1) is satisfied by F_T,

$$\mathbb{E}_0\left[\int_0^T \xi_{0,t} b_t dt + \xi_{0,T} F_T\right] \leq F_0, \tag{2}$$

where

$$\xi_{s,t} \triangleq \exp\left(-\int_s^t \left(r_v + \frac{1}{2}\theta_v'\theta_v\right) dv - \int_s^t \theta_v' dW_v\right), \tag{3}$$

with $\theta \triangleq \sigma^{-1}(\mu - r\mathbf{1})$, stand for the stochastic discount factor (SDF), or Arrow-Debreu state price density of buying at time s a unit of the consumption good to be delivered at time $t \in [s, T]$, $0 \leq s \leq t$, which is, in addition, compatible with the absence of arbitrage opportunities. To simplify the notation we will make use of the following identity $\xi_{0,t} \equiv \xi_t$, $\forall t \in [0, T]$. Note that equation (2) is simply a condition between the initial value of the Unemployment Insurance Fund, F_0, and the present value of all disbursements.

In order to avoid trivial cases, we will assume that

$$\mathbb{E}_0\left[\int_0^T \xi_t \bar{b} dt\right] < F_0,$$

and will impose the following technical condition.

Assumption 1 (Novikov Condition) *The processes (μ, r, σ) are such that*

$$\mathbb{E}_0\left[\exp\left(\int_0^T \theta_t'\theta_t dt\right)\right] < \infty.$$

From the static representation of the budget constraint in (2), it is possible to re-state the fund manager's problem, this time, in terms of the terminal value of the Unemployment Insurance Fund, F_T, in the following manner:

$$\text{(P0)} \qquad \max_{F_T \geq 0} V(F_T), \quad \text{subject to: } \mathbb{E}_0 \left[\int_0^T \xi_t b_t dt + \xi_T F_T \right] \leq F_0.$$

Problem (P0) is a standard portfolio choice problem (Karatzas and Shreve [12], Ch. 3), from where we have that the optimal terminal value of the fund, F_T^*, can be obtained from the first-order conditions, $F_T^* \equiv \phi^{-1} J(\phi^{-1} y^* \xi_T)$, where $J(\cdot) \triangleq u'^{-1}(\cdot) : \mathbb{R}_{++} \mapsto \mathbb{R}_{++}$, and the unique real number $y^* > 0$ is the Lagrange multiplier for which the static budget constraint holds with equality.[7] The exact value of y^* is obtained as the unique solution $\mathcal{X}(y^*) = F_0$, where

$$\text{(4)} \qquad \mathcal{X}(y) \triangleq \mathbb{E}_0 \left[\int_0^T \xi_t b_t dt + \xi_T F_T^* \right].$$

Once F_T^* has been identified, it is possible to determine the investment plan that finances the optimal wealth process given by[8]

$$F_t^* = \mathbb{E}_t \left[\int_t^T \xi_{t,s} b_s ds + \xi_{t,T} F_T^* \right].$$

With regard to this, we have the following result.

Proposition 1 (Ocone and Karatzas [17]) *The optimal investment plan of the fund manager, π^*, is given by the expression*

$$\text{(5)} \qquad \begin{aligned} \pi_t^* = {} & \sigma_t^{-1} \theta_t \mathbb{E}_t \left[\xi_{t,T} \frac{F_T^*}{\gamma_T} \right] - \sigma_t^{-1} \mathbb{E}_t \left[\xi_{t,T} F_T^* \left(1 - \frac{1}{\gamma_T} \right) H_{t,T} \right] \\ & + \sigma_t^{-1} \sigma_t^b \mathbb{E}_t \left[\int_t^T \xi_{t,s} b_s ds \right] + \sigma_t^{-1} \mathbb{E}_t \left[\int_t^T \xi_{t,s} b_s (\Gamma_{t,s} - H_{t,s}) ds \right] \end{aligned}$$

where $\gamma_T \triangleq -F_T^* u''(F_T^*)/u'(F_T^*)$ denotes the relative risk aversion coefficient, and the quantities $H_{t,s}$ and $\Gamma_{t,s}$ are defined in the Appendix.

[7] This property follows from the strict monotonocity of $u(\cdot)$, while the existence and uniqueness of $y^* > 0$, follows from the limits $\lim_{y \uparrow \infty} \mathcal{X}(y) \triangleq \mathcal{X}(\infty) = \mathbb{E}_0[\int_0^T \xi_t b_t dt] < F_0$, $\lim_{y \downarrow 0} \mathcal{X}(y) \triangleq \mathcal{X}(0) = \infty > F_0$, and the continuity of $\mathcal{X}(\cdot)$. For more details, the reader is referred to the Chapter 3 in [12].

[8] The classical references on this topic are Karatzas and Shreve [12] and Ocone and Karatzas [17]. A more accesible presentation is provided by Detemple, García and Rindisbacher [7, 8].

The expression in (5) uncovers the dependence of the optimal investment plan with respect to two quantities: the present value (at time $t \in [0, T]$) of the desired terminal value of the fund,

$$\mathbb{E}_t \left[\xi_{t,T} F_T^* \right], \tag{6}$$

and the present value (at time $t \in [0, T]$) of the flow of benefits to be paid throughout the investment horizon,

$$\mathbb{E}_t \left[\int_t^T \xi_{t,s} b_s ds \right]. \tag{7}$$

Regarding the first two components in (5), the optimal investment plan shows that the investment in risky assets is adjusted by the relative risk aversion of the fund manager. This result is identical to the one obtained by Merton [14, 15] for the case of a stochastic investment opportunity set. In particular, the optimal investment plan contains a mean-variance component (first term in (5)), and a hedging term that accounts for the fluctuations in the price of providing the desired terminal values of the fund (second term in (5)).

The last two components in (5) show that the fund manager should replicate, to some extent, the movements in the exogenous benefit process (b). On the one hand, the third component in (5) corresponds to a position that replicates a claim that accounts for the present value of the benefit process, while on the other hand, the last term represents a hedging component that accounts for fluctuations in both the price and the amount of such a claim.

Remark 1 *In an economy characterized by a constant investment opportunity set, i.e., where (r, μ, σ, δ, μ^b, σ^b) are all constant, we have that $H_{t,s} = \Gamma_{t,s} = 0$. If in addition $u(\cdot) = \ln(\cdot)$, it follows that $\gamma_T = 1$, and therefore the optimal investment plan of the fund manager only considers the mean-variance efficient portfolio and the claim that replicates the payment of benefits (first and third terms in (5)).*

2.3 The incentives of the compensation scheme

Throughout this section we will introduce additional simplifications in order to analyze the investment plan of the fund manager in a tractable manner. In particular, we consider the incentives embedded in a compensation scheme by means of an observable benchmark portfolio, along with the following assumptions:

A1 the uncertainty is generated by a 1-dimensional Brownian motion process (i.e., $d = 1$);

A2 the parameters of the economy, $(r, \mu, \delta, \sigma, \mu^b, \sigma^b) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}/\{0\}$, are constant;

A3 the mapping $u(\cdot)$ is of the CRRA class, with $\gamma > 0$.

Corollary 1 *Under the assumptions (A1)-(A3), we have that the optimal* ***fraction*** *of wealth allocated in the risky asset, when ϕ is a constant, is given by*

$$\frac{\pi_t^*}{F_t^*} = \frac{(\mu - r)}{\gamma\sigma^2} f_t^* + \frac{\sigma^b}{\sigma}(1 - f_t^*),$$

where

$$f_t^* = \frac{1}{\mathbb{E}_t\left[\int_t^T \xi_{t,s} b_s ds\right] \Big/ \mathbb{E}_t\left[\xi_{t,T} F_T^*\right] + 1}$$

with

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^T \xi_{t,s} b_s ds\right] &= b_t \frac{\exp\left\{(\mu^b - r - \sigma^b\theta)(T-t)\right\} - 1}{\mu^b - r - \sigma^b\theta} \\
\mathbb{E}_t\left[\xi_{t,T} F_T^*\right] &= \phi^{1/\gamma - 1}(y^*)^{-1/\gamma} \xi_t^{-1/\gamma} e^{-(1-1/\gamma)(r+\theta^2/2\gamma(T-t)}.
\end{aligned}$$

Corollary 1 shows that under the simplifications contained in (A1)-(A3), the optimal investment plan, expressed as a fraction of wealth allocated in risky assets, is a weighted average of two quantities: the Merton's solution ($\pi_M \equiv (\mu - r)/\gamma\sigma^2$) and the ratio between the volatility of the benefit process and the risky asset (σ^b/σ). While the weighting function (f_t^*) is given by the fraction of optimal wealth,

$$F_t^* = \mathbb{E}_t\left[\int_t^T \xi_{t,s} b_s ds + \xi_{t,T} F_T^*\right],$$

that is accounted for the quantities in (6) and (7), respectively.

2.3.1 The benchmark portfolio

We introduce the benchmark portfolio, $Y \equiv \{Y_t \geq 0\}_{0 \leq t \leq T}$, against to which the performance of the Unemployment Insurance Fund will be measured. The dynamics of this benchmark is described by the system

$$\text{(8)} \qquad \begin{cases} dY_t/Y_t = (1 - \pi_Y)\, r dt + \pi_Y[(dS_t + \delta dt)/S_t] - b^Y dt; \\ Y_0 = F_0; \quad Y_t \geq 0, \forall t \in [0, T]; \end{cases}$$

where $(\pi_Y, b^Y) \in \mathbb{R} \times [0,1]$ denote the *fraction* (of total wealth) invested by the benchmark in the risky asset, and the fraction that is paid as benefits, respectively. This last quantity satisfies in addition the condition

$$\mathbb{E}_0 \left[\int_0^T \xi_t b^Y Y_t dt + \xi_T Y_T \right] = F_0. \tag{9}$$

2.3.2 The incentives and the compensation scheme

In order to capture the main features of the compensation scheme faced by the Unemployment Insurance Fund Manager, we introduce a stylized version of it that relates the fund manager's revenue with the difference in returns obtained by the Unemployment Insurance Fund, $R_T^F = \ln(F_T/F_0)$, and the benchmark portfolio, $R_T^Y = \ln(Y_T/Y_0)$, in the following manner:

$$\phi = \begin{cases} \phi_H & \text{if} \quad R_T^F \geq R_T^Y + \eta + \kappa \triangleq R^+ \\ \phi_M & \text{if} \quad R_T^F \in [R^-, R^+) \\ \phi_L & \text{if} \quad R_T^F < R_T^Y + \eta - \kappa \triangleq R^- \end{cases} \tag{10}$$

where $1 > \phi_H > \phi_M > \phi_L > 0$, $\eta \in \mathbb{R}$ denotes the predetermined component in the computation of the annualized returns, and $\kappa \in [0,1]$ stands for the excess in return that triggers the increase or decrease in the base fee, ϕ_M. In order to simplify the analysis, we have assumed that the difference in returns that triggers the bonus or penalty fee (ϕ_H and ϕ_L, respectively) can be represented as a symmetric band of size $\kappa > 0$, around the base return $R_T^Y + \eta$.

Given the compensation scheme presented above, the time T fund manager's revenue is given by:

$$\phi F_T = \phi_L F_T \mathbf{1}_{\{R_T^F < R^-\}} + \phi_M F_T \mathbf{1}_{\{R^- \leq R_T^F < R^+\}} + \phi_H F_T \mathbf{1}_{\{R_T^F \geq R^+\}}, \tag{11}$$

where $\mathbf{1}_E$ corresponds to the indicator function of the event $E \in \mathcal{F}_T$.

Remark 2 *Note that the dynamics of the benchmark portfolio can be replicated by a combination of B and S. That is, if we start from a neutral return base at the beginning of the month ($\eta = 0$), and consider the initial position of both portfolios ($F_0 = Y_0$), and the amount of the expected benefits, the fund manager has an admissible investment plan to track the benchmark portfolio as close as desired.*

The optimization problem of the fund manager, in the presence of the compensation scheme described in (10), can therefore be written as:

$$\text{(P1)} \qquad \max_{F_T \geq 0} \mathbb{E}_0 \left[u(\phi_L F_T \mathbf{1}_{\{R_T^F < R^-\}} + \phi_M F_T \mathbf{1}_{\{R^- \leq R_T^F < R^+\}} + \phi_H F_T \mathbf{1}_{\{R_T^F \geq R^+\}}) \right],$$
$$\text{subjet to: } \mathbb{E}_0 \left[\int_0^T \xi_t b_t dt + \xi_T F_T \right] \leq F_0.$$

A similar problem has been recently studied by Basak, Pavlova and Shapiro [1]. The main difference of problem (P0) with respect to previous studies is given by the presence of the withdrawal process, b, and the particular compensation scheme under analysis. Our results (Proposition 2 and Corollary 2), hence, follow closely the technique developed by these authors.

As is well known (Basak *et al.*), the presence of compensation schemes such as the one under analysis includes an additional difficulty with respect to standard portfolio choice problems, which is given by the fact that the terminal payoff,

$$\phi F_T = \phi_L F_T + (\phi_H - \phi_L) F_T \mathbf{1}_{\{R_T^F \geq R^+\}} + (\phi_M - \phi_L) F_T \mathbf{1}_{\{R^- \leq R_T^F < R^+\}},$$

creates a local non-concavity in the objective function. This non-concavity causes in the fund manager a "risk-loving" behavior in some subsets of the parameter space. This additional complexity can nevertheless be address by employing ξ_T as a state variable. By doing so we have the following result.

Proposition 2 *Consider assumptions (A1)-(A3) and assume further that: (1) the quantities (ϕ_L, ϕ_M, ϕ_H) are constant for any realization of ξ_T, and (2) that the parameter of the economy are such that $\theta/\pi_Y\sigma > \gamma > 1$. Then, the following expressions are local maximum of $V(F_T)$, for the intervals of ξ_T that are indicated,*

$$\begin{array}{lll} F_H(\xi_T) \triangleq \phi_H^{1/\gamma-1} (y^{**}\xi_T)^{-1/\gamma} & \text{for} & 0 \leq \xi_T < \xi_{H^+} \\ F_{k^+}(\xi_T) \triangleq k^+ (\xi_T)^{-\pi_Y\sigma/\theta} & \text{for} & \xi_H \leq \xi_T < \xi_a \\ F_M(\xi_T) \triangleq \phi_M^{1/\gamma-1} (y^{**}\xi_T)^{-1/\gamma} & \text{for} & \xi_a \leq \xi_T \leq \xi_{k^-} \\ F_{k^-}(\xi_T) \triangleq k^- (\xi_T)^{-\pi_Y\sigma/\theta} & \text{for} & (\xi_{k^-} \vee \xi_a) < \xi_T < \xi_b \\ F_L(\xi_T) \triangleq \phi_L^{1/\gamma-1} (y^{**}\xi_T)^{-1/\gamma} & \text{for} & \xi_b \leq \xi_T < \infty \end{array}$$

where $(a \vee b) = \max(a, b)$. The solution to the stochastic control problem (P1) is therefore given by

$$\text{(12)} \qquad \begin{aligned} F_T^{**} = {} & F_H \mathbf{1}_{\{0 \leq \xi_T < \xi_H\}} + F_{k^+} \mathbf{1}_{\{\xi_H \leq \xi_T < \xi_a\}} + F_M \mathbf{1}_{\{\xi_a \leq \xi_T \leq \xi_{k-}\}} \\ & + F_{k^-} \mathbf{1}_{\{(\xi_{k-} \vee \xi_a) < \xi_T < \xi_b\}} + F_L \mathbf{1}_{\{\xi_b \leq \xi_T < \infty\}}, \end{aligned}$$

with

$$\xi_{H^+} \triangleq (\phi_H^{1-1/\gamma}(y^{**})^{1/\gamma}k^+)^{1/(\pi_Y\sigma/\theta-1/\gamma)}, \quad \xi_{k^-} \triangleq (\phi_M^{1-1/\gamma}(y^{**})^{1/\gamma}k^-)^{1/(\pi_Y\sigma/\theta-1/\gamma)},$$
$$\xi_L \triangleq (\phi_L^{1-1/\gamma}(y^{**})^{1/\gamma}k^-)^{1/(\pi_Y\sigma/\theta-1/\gamma)},$$

where y^{**} is the value of the Lagrange multiplier for which the budget constraint holds with equality, and the quantities $(\xi_a, \xi_b, k^{\pm})$ are defined in the Appendix.

Proposition 2 illustrates the effects of the incentives embedded in the compensation scheme under analysis, for the case where $\pi_M = (\mu - r)/\gamma\sigma^2 > \pi_Y$, i.e., when the benchmark portfolio allocates a smaller fraction in risky assets that the Merton's solution; the solution of the opposite case is provided in the Appendix [see Proposition A.3].

In the former case (i.e., constant ϕ), the desired value of the terminal fund (F_T^*) was a strictly decreasing function of the state-price ξ_T. Now, the fund manager is willing to target a higher value than the one prescribed by the local conditions of optimality alone,

$$\text{(13)} \qquad \begin{cases} u'(\phi_H F_T^{**}) = \phi_H^{-1} y^{**}\xi_T & \text{for} \quad R_T^F \geq R^+ \\ u'(\phi_M F_T^{**}) = \phi_M^{-1} y^{**}\xi_T & \text{for} \quad R_T^F \in [R^-, R^+) \\ u'(\phi_L F_T^{**}) = \phi_L^{-1} y^{**}\xi_T & \text{for} \quad R_T^F < R^-. \end{cases}$$

This is due to the discontinuity in the compensation scheme faced by the fund manager. As an illustration, consider the first two terms in (12). In the event $\{\xi_T \leq \xi_H\}$, we have that the terminal value of the fund prescribed by the local conditions in (13) is such that $F_H(\xi_T) \geq F_{k^+}(\xi_T)$. In such a case, by setting $F_H(\xi_T)$ as the optimal choice the manager is not only being efficient in marginal terms ($F_H(\xi_T) \equiv u'^{-1}(\phi_H^{-1} y^{**}\xi_T)$), but it is also charging the high fee ϕ_H by doing so.[9] In contrast, for values of ξ_T in the range $\xi_H < \xi_T < \xi_a$, the local conditions of optimality suggests the manager to reduce the desired terminal level of the fund (i.e., $F_H(\xi_T) < F_{k^+}(\xi_T)$), but this would amount to give up the incentive to increase its revenues ($\phi_H > \phi_M$). This is why the manager targets a terminal fund (F_{k^+}) that allows him to retain the revenue bonus ($\phi_H > \phi_M$) in a broader range of events ($\{0 \leq \xi_T < \xi_a\}$) than those suggested by the local conditions in (13) alone (i.e., $\{0 \leq \xi_T < \xi_H\}$).

As in the previous case, (Corollary 1), the identification of the desired level of the terminal fund that the manager will pursue, F_T^{**}, is enough to determine the investment plan that finances this quantity. With regard to this, we have the following result.

[9] In the proof it is established the equivalency between the following events: $\{R_T^F \geq R^+\} \equiv \{F_T \geq F_{k^+}\}$.

Corollary 2 *Under the assumptions of Proposition 2, the optimal* ***fraction*** *invested in risky assets is given by*

$$\text{(14)} \qquad \frac{\pi_t^{**}}{F_t^{**}} = \frac{(\mu - r)}{\gamma\sigma^2} f_{1t}^{**} + (\pi_Y)\, f_{2t}^{**} + \frac{\sigma^b}{\sigma}(1 - f_{1t}^{**} - f_{2t}^{**}) - \frac{1}{\sigma\sqrt{T-t}} g_t^{**}$$

where

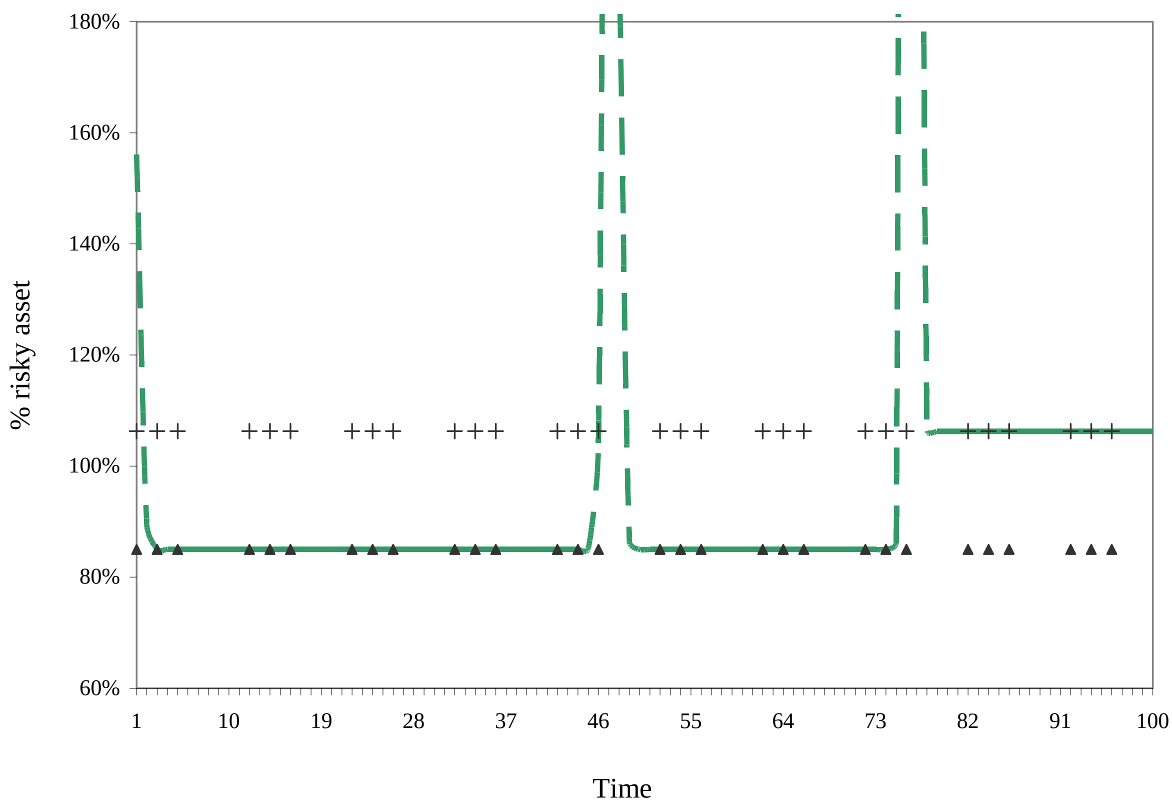
$$\begin{aligned}
f_{1t}^{**} &= \Big(A_1\xi_t^{-1/\gamma}\left[1-\Phi(e_1^-)\right] + A_3\xi_t^{-1/\gamma}\left[\max\{\Phi(e_3^+)-\Phi(e_3^-),0\}\right] \\
&\quad + A_5\xi_t^{-1/\gamma}\left[\Phi(e_5^+)\right]\Big)/F_t^{**}, \\
f_{2t}^{**} &= \Big(A_2\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_2^+)-\Phi\left(e_2^-\right)\right] + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_4^+)-\Phi(e_4^-)\}\right]\Big)/F_t^{**}, \\
F_t^{**} &= b_t\frac{\left[\exp\left\{(\mu^b - r - \sigma^b\theta)(T-t)\right\}-1\right]}{\mu^b - r - \sigma^b\theta} + A_1\xi_t^{-1/\gamma}\left[1-\Phi(e_1^-)\right] \\
&\quad + A_2\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_2^+)-\Phi\left(e_2^-\right)\right] + A_3\xi_t^{-1/\gamma}\left[\max\{\Phi(e_3^+)-\Phi(e_3^-),0\}\right] \\
&\quad + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_4^+)-\Phi(e_4^-)\}\right] + A_5\xi_t^{-1/\gamma}\left[\Phi(e_5^+)\right], \\
g_t^{**} &= (F_t^{**})^{-1}\Big\{A_1\xi_t^{-1/\gamma}\left[-\phi(e_1^-)\right] + A_2\xi_t^{-\pi_Y\sigma/\theta}[\phi(e_2^+)-\phi(e_2^-)] \\
&\quad + A_3\xi_t^{-1/\gamma}\left[\phi(e_3^+)-\phi(e_3^-)\right]\mathbf{1}_{\{\Phi(e_3^+)\geq\Phi(e_3^-)\}} + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\phi(e_4^+)-\phi(e_4^-)\right] \\
&\quad + A_5\xi_t^{-1/\gamma}\phi(e_5^+)\Big\},
\end{aligned}$$

$(\Phi(\cdot), \phi(\cdot))$ *correspond to the p.d.f. and c.d.f. of the standard normal distribution, respectively, and the quantities* $(A_i, e_i^{\pm})$, $i = 1, \ldots, 5$, *are defined in the Appendix.*

From the expression in (14) it follows that the optimal investment plan has four components: the Merton's solution (π_M), the investment policy of the benchmark portfolio (π_Y), the ratio of volatilities (σ^b/σ), and a function that depends upon the indicator functions associated to the expression for F_T^{**} (g_t^{**}). Depending on the configuration of parameters describing the economy, Basak *et al.* have shown that a fund manager subject to a similar compensation scheme has incentives to increase the relative volatility between π_t^{**}/F_t^{**} and π_Y, motivated by the possibility of increasing revenues ($\phi_H - \phi_M$), when the implicit option is "in-the-money" ($R_t^F - R_t^Y \geq \eta \pm \kappa$), or by the possibility of avoiding a penalty ($\phi_M - \phi_L$), when the implicit option is "out-the-money" ($R_t^F - R_t^F < \eta \pm \kappa$); for more details see Basak *et al*.

Since the realization of ξ_T is uncertain before time T, the effect of the incentives embedded in the compensation scheme will depend upon the trajectory followed by $\xi_{(\cdot)}(\omega)$: $[0, T] \mapsto [0, +\infty)$. Hence, as long as the trajectory followed by $\xi_{(\cdot)}$ assigns a relevant probability mass to the events $\{\xi_H \leq \xi_T \leq \xi_a\}$ and $\{(\xi_{k-} \vee \xi_a) < \xi_T < \xi_b\}$, the fund manager will be motivated to deviate its unconstrained investment plan towards the investment plan π_Y.

In what follows we present the results of a numerical exercise aimed at clarifying the most relevant aspects discussed until now. The figure below shows the optimal investment plan resulting from a trajectory of $\xi_{(\cdot)}$ that arrives monotonically at a value $\xi_T(\omega_0)$ which is higher than ξ_b, which in turn makes F_L the optimal choice [see Equation (12)]. The parameters have been chosen to magnify the effect that is being illustrated.[10] To simplify the analysis the figure considers a withdrawal/benefit process which is null at all times.



The levels marked with the symbols "+" and "▲" correspond to the Merton's solution (π_M) and the investment plan of the benchmark portfolio (π_Y), respectively, while the dashed line denotes the optimal choice of the fund manager characterized in Corollary 2.

[10]The parameters of the figure are as follows: $\pi_Y = 0.85$; $r = 0.25$; $\mu = 0.35$; $\delta = 0$; $\sigma = 0.028$; $\gamma = 1.2$; $F_0 = 5$; $\eta = 0$; $\kappa = 0.1$; $\phi_M = 0.05\%$; $\phi_H = 1.1\phi_M$; $\phi_L = 0.9\phi_M$. With these parameters one obtains $\xi_H = 0.954$; $\xi_a = 9.27$, $\xi_b = 37.7$. The trajectory corresponds to one where $\Delta W_{t_i} = 1.35$, with $i \in \{1, \ldots, 100\}$.

In the figure it is possible to see that the optimal investment plan has three peaks: one that is located at the very beginning of the investment horizon, and two that are located at the middle and near the end of the investment horizon. All of them can be explained by the position of $\xi_{(\cdot)}(\omega_0)$, and the quantities (ξ_H, ξ_a, ξ_b).[11] In the case of the first peak, the initial position of $\xi_0(= 1)$ is such that $\xi_H < \xi_0 < \xi_a$. In this scenario the fund manager faces a situation where the probabilities that the quantities F_H and F_{k^+} will become optimal at time T (conditional on $\mathcal{F}_0$) are 11% and 89%, respectively. Additionally, the fund manager must obtain a return of 10% ($\kappa = 0.1$) in excess of the benchmark portfolio in order to increase its revenue. The fund manager has therefore the incentive to increase the volatility of its relative position (i.e., compared with the one of the benchmark), which for the case where $\pi_M > \pi_Y$ can be attained by increasing the holding of the risky asset (Basak *et al.*). Given the trajectory under consideration, F_{k^+} ends up being optimal (almost surely) after the third period, and the fund manager optimally decides to imitate the investment plan of the benchmark portfolio (π_Y) in order to obtain the revenue bonus. This situation continues to be optimal until the trajectory of $\xi_{(\cdot)}$ gets closer to ξ_a, which gives rise to the second peak. At this point, the probabilities of F_{k^+} and F_{k^-} of becoming optimal at time T (conditional on $\mathcal{F}_t$) are 67% and 33%, respectively. In this scenario the fund manager has once again the incentive to increase the relative volatility of its position. This time though the bet does not pay off and the quantity F_{k^-}, associated with a normal level of revenue (ϕ_M), ends up being optimal (almost surely) after three periods. Under this new situation the fund manager optimally imitates the investment plan of the benchmark portfolio, in this occasion not to increase its revenues, but to avoid a punishment. As before, this situation continues to be optimal until the trajectory of $\xi_{(\cdot)}$ gets closer to ξ_b, which gives rise to the third peak. In this last case the fund manager battles to avoid a decrease in its revenue (ϕ_L). As in the previous cases, the fund manager decides to increase the relative volatility of its position (F_{k^-} and F_L have probabilities of becoming optimal at time T of 84% and 16%, respectively), but the bets does not pay off either (F_L becomes optimal {almost surely} after three periods), and a reduction in revenues materializes with probability one. Given this, the fund manager optimally decides to adopt its unconstrained optimal policy (Corollary 1), i.e., π_M.

One element that deserves to be emphasized is the "risk-loving" behavior of the fund manager. Even though Figure 1 was constructed with a value of γ relatively low ($\gamma =$

[11] Note that from the parameters we have that $\xi_a > \xi_{k^-}$, and thus F_M is never a part of the optimal solution.

1.2), it is possible to find high bets for small values of κ (closed to zero). This illustrates the precaution that must be exercised by policy makers in the design of these compensation schemes (for more examples in this line see Hodder and Jackwerth [11]).

3 Conclusions

In this paper we have analyzed the implications of a compensation scheme with benchmarking incentives. In line with the recent literature, we have shown the importance of these incentives. In particular, depending on the parameters of the economy, the fund manager will be motivated to imitate the portfolio policy of the benchmark portfolio to either obtain a gain, or avoid a loss depending on the specifics of the compensation scheme.

Going beyond the theoretical implications of the admittedly stylized model that we developed, there is a fact that should be of interest for policy makers regarding the newly launched system in Chile. In particular, the analysis suggests that the investment policy of the Unemployment Insurance Fund should to some extent be influenced by the nature of the flow of benefits that are paid over the month. This probably amounts to allow the investment in securities other than fixed-income alone. Given the high dependence of the Chilean economy from the copper price it is reasonable to allow the Unemployment Insurance Fund to hedge some of the business-cycle fluctuations through securities that are correlated with such an asset.

The present study was conducted for an existing compensation scheme with an exogenously provided benchmark. The study of how to select an appropriate benchmark is thus the next direction to pursue. Such an study will amount, in principle, to incorporate either the member's preferences, or the fiscal authority's interests.

Appendix

Proof of Proposition 1. From the dynamics in (1), and the definition of ξ in (3), an application of Itô lemma delivers

$$\mathbb{E}_t\left[\xi_T F_T + \int_0^T \xi_t b_t dt\right] - \mathbb{E}_0\left[\xi_T F_T + \int_0^T \xi_t b_t dt\right] = \int_0^t \xi_s\left[\sigma_s \pi_s - F_s \theta_s\right] dW_s.$$

Since the expression on the LHS is a martingale, there exists a process h_s such that

$$M_t \triangleq \int_0^t \xi_s\left[\sigma_s \pi_s - F_s \theta_s\right] dW_s = \int_0^t h_s dW_s.$$

Equating terms we obtain

$$\pi_t^* \equiv F_t^* \sigma_t^{-1} \theta_t + \xi_t^{-1} \sigma_t^{-1} h_t^*,$$

where we have made use of the fact that

$$F_t^* = \mathbb{E}_t \left[\int_t^T \xi_{t,s} b_s ds + \xi_{t,T} F_T^* \right]$$

corresponds to the optimal wealth process. That is, the process that finances the desired terminal value of the Unemployment Insurance Fund, taking into account the payment of future (uncertain) benefits.

Applying the Clark-Ocone (Ocone and Karatzas [17]) formula, it follows that

$$h_t^* = \mathbb{E}_t \left[\int_t^T \xi_s b_s \left\{ \sigma_t^b - \theta_t + \Gamma_{t,s} - H_{t,s} \right\} ds - \left[\xi_T F_T^* \left\{ \theta_t + H_{t,T} \right\} \left(1 - \frac{1}{\gamma_T} \right) \right] \right]$$

where

$$\begin{aligned} H_{t,s} &\triangleq \left[\int_t^s \mathcal{D}_t(r_v) dv + \int_t^s \mathcal{D}_t(\theta_v)' \theta_v dv + \int_t^s \mathcal{D}_t(\theta_v) dW_v \right] \\ \Gamma_{t,s} &\triangleq \left[\int_t^s \mathcal{D}_t(\mu_v^b) dv - \int_t^s \mathcal{D}_t(\sigma_v^b)' \sigma_v^b dv + \int_t^s \mathcal{D}_t(\sigma_v^b) dW_v \right] \end{aligned}$$

and $\mathcal{D}_t(\cdot)$ is the Malliavin derivative operator.[12] ■

Proof of Corollary 1. From the definition of f_t^* we have ($b_{t,s} = b_s / b_t$)

$$f_t^* = \frac{\phi^{1/\gamma - 1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\xi_{t,T}^{1-1/\gamma} \right]}{b_t \mathbb{E}_t \left[\int_t^T \xi_{t,s} b_{t,s} ds \right] + \phi^{1/\gamma - 1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\xi_{t,T}^{1-1/\gamma} \right]}$$

For the numerator we can write

$$\begin{aligned} & \phi^{1/\gamma - 1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\xi_{t,T}^{1-1/\gamma} \right] \\ = \; & \phi^{1/\gamma - 1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\exp \left\{ -(1 - 1/\gamma) \left([r + \theta^2/2](T-t) + \theta [W_T - W_t] \right) \right\} \right] \\ = \; & \phi^{1/\gamma - 1} y^{-1/\gamma} \xi_t^{-1/\gamma} \exp \left\{ \left(-(1 - 1/\gamma)(r + \theta^2 / 2\gamma) \right) (T-t) \right\} \end{aligned}$$

[12] The Malliavin derivative operatior is an extension of the classical notion, that extends the concept to functions of the trajectories of W. In the same way that the classical derivative measures the local change in the function, due to a local change in the underlying variable, the Malliavin derivative measures the change in the function (that depends on the trajectories of W) implied by a small change in the trajectory of W. The interested reader is referred to Detemple *et al.* [7] for a brief introduction to this operator, and to Nualart [16] for a comprehensive treatment.

while for the denominator we have

$$
\begin{aligned}
& b_t \mathbb{E}_t\left[\int_t^T \xi_{t,s} b_{t,s} ds\right] + \phi^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbf{E}_t\left[\xi_{t,T}^{1-1/\gamma}\right] \\
= \;& b_t \mathbb{E}_t\left[\int_t^T \left(\exp\left\{\mu^b - (r + (\theta^2 + \sigma_b^2)/2)(s-t) + (\sigma_b - \theta)(W_s - W_t)\right\}\right) ds\right] \\
& + \phi^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t\left[\exp\left((1-1/\gamma)\left(-(r+\theta^2/2\right)(T-t) - \theta[W_T - W_t]\right)\right] \\
= \;& b_t \frac{\exp\left\{(\mu^b - r - \sigma^b\theta)(T-t)\right\} - 1}{\mu^b - r - \sigma^b \theta} \\
& + \phi^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \exp\left\{-(1-1/\gamma)\left(r + \theta^2/2\gamma\right)(T-t)\right\}.
\end{aligned}
$$

■

Proof of Proposition 2. Consider the following set equalities:

$$
\{R_T^F \gtrless R^{\pm}\} \equiv \{F_T/F_0 \gtrless [Y_T/Y_0] \exp(\eta \pm \kappa)\} \equiv \{F_T \gtrless k^{\pm}(\xi_T)^{-\pi_Y \sigma/\theta} \triangleq F_{k^{\pm}}(\xi_T)\},
$$

where the last equality follows from $Y_T = A(\xi_T)^{-\pi_Y\sigma/\theta}$, with

$$
\begin{aligned}
k^{\pm} &= F_0 \exp\left(\left[r + \pi_Y\sigma(\theta/2 - r/\theta) - b^Y - \pi_Y^2\sigma^2/2\right] T + \eta \pm \kappa\right), \\
A &= F_0 \exp\left(\left[r + \pi_Y\sigma\theta - \pi_Y\sigma\left(r/\theta + \theta/2\right) - b^Y - \pi_Y^2\sigma^2/2\right] T\right).
\end{aligned}
$$

Then, we can re-write the expression in (11) as

$$
\phi F_T = \phi_L F_T 1_{\{F_T < F_{k^-}\}} + \phi_M F_T 1_{\{F_{k^-} \le F_T < F_{k^+}\}} + \phi_H F_T 1_{\{F_T \ge F_{k^+}\}}.
$$

Now, consider the following quantities:

$$
\begin{aligned}
\xi_{H^+} &\triangleq \left(\phi_H^{1-1/\gamma} y^{1/\gamma} k^+\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)}, \quad \xi_{H^-} \triangleq \left(\phi_H^{1-1/\gamma} y^{1/\gamma} k^-\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)}, \\
\xi_{k^+} &\triangleq \left(\phi_M^{1-1/\gamma} y^{1/\gamma} k^+\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)}, \quad \xi_{k^-} \triangleq \left(\phi_M^{1-1/\gamma} y^{1/\gamma} k^-\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)}, \\
\xi_L &\triangleq \left(\phi_L^{1-1/\gamma} y^{1/\gamma} k^-\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)},
\end{aligned}
$$

which correspond to the boundaries of the following intervals: $\{F_H \ge F_{k^+}\} \equiv \{\xi_T \le \xi_{H^+}\}$, $\{F_H \ge F_{k^-}\} \equiv \{\xi_T \le \xi_{H^-}\}$, $\{F_{k^-} \le F_M < F_{k^+}\} \equiv \{\xi_{k^+} < \xi_T \le \xi_{k^-}\}$, and $\{F_L < F_{k^-}\} \equiv \{\xi_T > \xi_L\}$, where $y > 0$ is a real number to be identified later on. For the set of parameters under consideration (i.e., $\pi_Y\sigma/\theta - 1/\gamma < 0$ and $1/\gamma - 1 < 0$) it follows that $\xi_{H^+} < \xi_{H^-} < \xi_{k^+} < \xi_{k^-} < \xi_L$, and $F_H(\xi_T) \triangleq \phi_H^{1/\gamma-1}(y\xi_T)^{-1/\gamma} < F_M(\xi_T) \triangleq \phi_M^{1/\gamma-1}(y\xi_T)^{-1/\gamma} < F_L(\xi_T) \triangleq \phi_L^{1/\gamma-1}(y\xi_T)^{-1/\gamma}$ for any ξ_T.

Let

$$\begin{aligned} F_T^{**} = F_H\mathbf{1}_{\{0\le\xi_T<\xi_{H^+}\}} + F_{k^+}\mathbf{1}_{\{\xi_{H^+}\le\xi_T<\xi_a\}} + F_M\mathbf{1}_{\{\xi_a\le\xi_T\le\xi_{k^-}\}} \quad & + F_{k^-}\mathbf{1}_{\{(\xi_{k^-}\vee\xi_a)<\xi_T<\xi_b\}} \\ & + F_L\mathbf{1}_{\{\xi_b\le\xi_T<\infty\}}, \end{aligned}$$

be the solution of problem (P1). Then, for any F_T that satisfies the static budget constraint (2), F_T^{**} must be such that the following condition holds (Basak *et al.* [1]),

$$\mathbb{E}_0\left[v(F_T^{**},\xi_T) - v(F_T,\xi_T)\right] \ge 0,$$

where

$$v(F_T,\xi_T) \triangleq u\left(\phi_L F_T 1_{\{F_T\le F_{k^-}\}} + \phi_M F_T 1_{\{F_{k^-}<F_T<F_{k^+}\}} + \phi_H F_T 1_{\{F_T\ge F_{k^+}\}}\right) - yF_T\xi_T.$$

In order to show the optimality of F_T^{**} we proceed to analyze the proposed solution for different intervals of ξ_T. We notice before that

In the interval $0 \le \xi_T \le \xi_{H^+}$ we have that $F_{k^-} < F_{k^+} \le F_H < F_M < F_L$, and therefore, the bonus fee is obtained by setting $F_T \in \{F_H,\ F_{k^+},\ F_M,\ F_L\}$, in which case $v(F_T,\xi_T) = u(\phi_H F_T) - yF_T\xi_T$. It thus follows that F_H is the global maximum in this case. In the interval $\xi_{H^+} < \xi_T \le \xi_{H^-}$ we have that $F_{k^-} \le F_H < F_{k^+} \le F_M < F_L$, and the bonus fee is obtained by setting $F_T \in \{F_{k^+},\ F_M,\ F_L\}$. It thus follows by the concavity of $v(\cdot,\xi_T)$ that $v(F_{k^+},\xi_T) > v(F_M,\xi_T)$ and so F_{k^+} is the global maximum in this case. The same logic applies for the interval $\xi_{H^-} < \xi_T \le \xi_{k^+}$ where $F_H < F_{k^-} < F_{k^+} \le F_M < F_L$, and so once again, F_{k^+} is the global maximum in this case. In the interval $\xi_{k^+} < \xi_T < \xi_{k^-}$ we have that $F_H < F_{k^-} < F_M < F_{k^+} < F_L$ so that the bonus fee is obtained by setting $F_T \in \{F_{k^+}, F_L\}$. In this case, however, the base fee (ϕ_M) is obtained by setting $F_T = F_M$. If F_{k^+} is to be the global maximum in this interval (since F_L is dominated, i.e., $v(F_{k^+},\xi_T) > v(F_L,\xi_T)$) it must be true that $v(F_{k^+},\xi_T) - v(F_M,\xi_T) \ge 0$. By inspection it follows that $v(F_{k^+},\xi_T) = v(F_M,\xi_T) + \ell_a(\xi_T)$, where

$$\text{(A.1)} \qquad \ell_a(\xi) \triangleq \frac{1}{1-\gamma}\left[\left(\frac{(\xi)^{\pi_Y\sigma/\theta}}{k^+\phi_H}\right)^{\gamma-1} - \gamma\left(\frac{y\xi}{\phi_M}\right)^{1-1/\gamma}\right] - yk^+(\xi)^{1-\pi_Y\sigma/\theta}.$$

For the set of parameters under consideration, we have that $\ell_a(\xi_{k^+}) > 0$, $\ell_a(\infty) = -\infty$, from where it follows that there is a value $\xi_T = \xi_a$, such that $\ell_a(\xi_a) = 0$. Therefore, in the interval $\xi_{H^+} < \xi_T < \xi_a$ F_{k^+} is the global maximum. If the set of parameters is such that $\xi_a < \xi_{k^-}$, it folows by the logic applied before in the case where F_H was the maximum that F_M will be the global maximum in the interval $\xi_a < \xi_T < \xi_{k^-}$. On the other hand, if $\xi_a > \xi_{k^-}$, F_M will not be part of the optimal solution. For the next interval, $(\xi_{k^-} \vee \xi_a) < \xi_T \leq \xi_L$, we have $F_M < F_{k^-} \leq F_L$. So F_M cannot be the global maximum either. In such interval F_{k^-} is the global maximum. Now, if $(\xi_a \vee \xi_{k^-}) < \xi_L$ in the interval $\xi_L < \xi_T < \infty$, we have that $F_{k^-} < F_L$. In order to propose F_{k^-} as the global maximum it must be true that $v(F_{k^-}, \xi_T) - v(F_L, \xi_T) \geq 0$. For this case we have $v(F_{k^-}, \xi_T) = v(F_L, \xi_T) + \ell_b(\xi_T)$, where

$$\text{(A.2)} \qquad \ell_b(\xi) \triangleq \frac{1}{1-\gamma}\left[\left(\frac{(\xi)^{\pi_Y\sigma/\theta}}{k^-\phi_M}\right)^{\gamma-1} - \gamma\left(\frac{y\xi}{\phi_L}\right)^{1-1/\gamma}\right] - yk^-\,(\xi)^{1-\pi_Y\sigma/\theta}.$$

From the set of parameters under configuration, we have that $\ell_b(\xi_L) > 0$. As in the case above, there is a value $\xi_T = \xi_b$ such that $\ell_b(\xi_b) = 0$. So that in the interval $\xi_L < \xi_T < \xi_b$ F_{k^-} is the global maximum. If $\xi_a > \xi_L$, we have that in the interval $\xi_a < \xi_T < \xi_b$ F_{k^-} is the global maximum. Therefore, in the interval $\xi_b \leq \xi_T < \infty$ F_L is the global maximum.

Finally, the real number $y = y^{**} > 0$ corresponds to the unique solution $\mathcal{X}(y^{**}) = F_0$, where we have replaced F_T^{**}, instead of F_T^* in (4). ∎

Proof of Corollary 2. Since the indicator functions in the expression for F_t^{**} do not belong to the domain of $\mathcal{D}_t(\cdot)$, we can alternatively determine π_t^{**} from the volatility component associated to

$$\begin{aligned}
F_t^{**} &= \mathbb{E}_t\left[\int_t^T \xi_{t,s} b_s ds + \xi_{t,T} F_T^{**}\right] \\
&= \frac{b_t}{\mu^b - r - \sigma^b\theta}\left[\exp\left\{(\mu^b - r - \sigma^b\theta)(T-t)\right\} - 1\right] \\
&\quad + A_1\xi_t^{-1/\gamma}\left[1 - \Phi(e_1^-)\right] + A_2\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_2^+) - \Phi\left(e_2^-\right)\right] \\
&\quad + A_3\xi_t^{-1/\gamma}\left[\max\{\Phi(e_3^+) - \Phi(e_3^-), 0\}\right] + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_4^+) - \Phi(e_4^-)\right] \\
&\quad + A_5\xi_t^{-1/\gamma}\left[\Phi(e_5^+)\right]
\end{aligned}$$

where the terms $(A_i, e_i^{\pm})$, $i = 1, \ldots, 5$, are define below.

From the expression in (1) we have that $dF_t^{**} = [F_t^{**} - \pi_t^{**}]rdt + \pi_t^{**}[\mu dt + \sigma dW_t] - b_t dt$, so that the expression next to the "dW" term, in the expression for dF^{**}, corresponds to the term $\pi_t^{**}\sigma$. An application of Itô's rule delivers the expression (with

$\partial^2 F_t^{**}/\partial\xi_t \partial b_t = 0$)

$$
\begin{aligned}
dF_t^{**} &= (\partial F_t^{**}/\partial t)dt + (\partial F_t^{**}/\partial b_t)db_t + (1/2)(\partial^2 F_t^{**}/(\partial b_t^2)dt + (\partial F_t^{**}/\partial \xi_t)d\xi_t \\
&\quad +(1/2)(\partial^2 F_t^{**}/(\partial \xi_t^2)dt,
\end{aligned}
$$

and hence our interest is placed on the second and fourth term in this expression (i.e., those containing de "dW" components). Such terms are given by the following expressions

$$
\begin{aligned}
\frac{\partial F_t^{**}}{\partial b_t} &= \frac{\exp\left\{(\mu^b - r - \sigma^b\theta)(T-t)\right\} - 1}{\mu^b - r - \sigma^b\theta}, \\
\frac{\partial F_t^{**}}{\partial \xi_t} &= -(1/\gamma)A_1\xi_t^{-1/\gamma-1}\left[1-\Phi(e_1^-)\right] + A_1\xi_t^{-1/\gamma}\left[-\phi(e_1^-)\right](\xi_t\theta\sqrt{T-t})^{-1} \\
&\quad - (\pi_Y\sigma/\theta)A_2\xi_t^{-\pi_Y\sigma/\theta-1}[\Phi(e_2^+) - \Phi\left(e_2^-\right)] + A_2\xi_t^{-\pi_Y\sigma/\theta}[\phi(e_2^+) - \phi(e_2^-)](\xi_t\theta\sqrt{T-t})^{-1} \\
&\quad - (1/\gamma)A_3\xi_t^{-1/\gamma-1}\left[\max\{\Phi(e_3^+) - \Phi(e_3^-), 0\}\right] \\
&\quad + A_3\xi_t^{-1/\gamma}\left[\phi(e_3^+) - \phi(e_3^-)\right]\mathbf{1}_{\{\Phi(e_3^+)\geq\Phi(e_3^-)\}}(\xi_t\theta\sqrt{T-t})^{-1} \\
&\quad - (\pi_Y\sigma/\theta)A_4\xi_t^{-\pi_Y\sigma/\theta-1}\left[\Phi(e_4^+) - \Phi(e_4^-)\right] + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\phi(e_4^+) - \phi(e_4^-)\right](\xi_t\theta\sqrt{T-t})^{-1} \\
&\quad - A_5(1/\gamma)\xi_t^{-1/\gamma-1}\left[\Phi(e_5^+)\right] + A_5\xi_t^{-1/\gamma}\phi(e_5^+)(\xi_t\theta\sqrt{T-t})^{-1},
\end{aligned}
$$

while $d\xi_t = -\xi_t(rdt + \theta dW_t)$. Comparing terms we obtain

$$
\frac{\pi_t^{**}}{F_t^{**}} = \frac{1}{\sigma F_t^{**}}\left(\sigma^b b_t \frac{\partial F_t^{**}}{\partial b_t} - \theta\xi_t\frac{\partial F_t^{**}}{\partial \xi_t}\right).
$$

Plugging in the expressions for $(\partial F_t^{**}/\partial\xi_t)$, $(\partial F_t^{**}/\partial b_t)$ and F_t^{**} we obtain equation (14):

$$
\begin{aligned}
\frac{\pi_t^{**}}{F_t^{**}} &= \frac{\sigma^b}{\sigma}\frac{\mathbb{E}_t\left[\int_t^T \xi_{t,s}b_s ds\right]}{F_t^{**}} \\
&\quad + \left(\frac{\mu - r}{\gamma\sigma^2}\right)\frac{\xi_t^{-1/\gamma}}{F_t^{**}}\left(A_1\left[1-\Phi(e_1^-)\right] + A_3\left[\max\{\Phi(e_3^+) - \Phi(e_3^-), 0\}\right] + A_5\left[\Phi(e_5^+)\right]\right) \\
&\quad + \pi_Y\frac{\xi_t^{-\pi_Y\sigma/\theta}}{F_t^{**}}\left(A_2[\Phi(e_2^+) - \Phi\left(e_2^-\right)] + A_4\left[\Phi(e_4^+) - \Phi(e_4^-)\right]\right) \\
&\quad - \frac{1}{\sigma\sqrt{T-t}F_t^{**}}\Big\{A_1\xi_t^{-1/\gamma}\left[-\phi(e_1^-)\right] + A_2\xi_t^{-\pi_Y\sigma/\theta}[\phi(e_2^+) - \phi(e_2^-)] \\
&\quad + A_3\xi_t^{-1/\gamma}\left[\phi(e_3^+) - \phi(e_3^-)\right]\mathbf{1}_{\{\Phi(e_3^+)\geq\Phi(e_3^-)\}} + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\phi(e_4^+) - \phi(e_4^-)\right] \\
&\quad + A_5\xi_t^{-1/\gamma}\phi(e_5^+)\Big\}
\end{aligned}
$$

In what follows we derive the second term in $\mathbb{E}_t\left[\xi_{t,T}F_T^{**}\right]$. The derivation of the remaining terms follows identical steps and is left as a proposed exercise for the interest reader.

$$\begin{aligned}
& \mathbb{E}_t\left[\xi_{t,T}F_{k^+}\mathbf{1}_{\{\xi_H\leq\xi_T<\xi_a\}}\right] \\
= \;& \mathbb{E}_t\left[\xi_{t,T}\left((\xi_T)^{-\pi_Y\sigma/\theta}\right)\mathbf{1}_{\{\xi_H\leq\xi_T<\xi_a\}}\right] \\
= \;& \xi_t^{-\pi_Y\sigma/\theta}k^+\mathbb{E}_t\left[\left((\xi_{t,T})^{1-\pi_Y\sigma/\theta}\right)\mathbf{1}_{\{\xi_H\leq\xi_T<\xi_a\}}\right] \\
= \;& \xi_t^{-\pi_Y\sigma/\theta}k^+\mathbb{E}_t\left[e^{-(1-\pi_Y\sigma/\theta)\left[\left(r+\theta^2/2\right)(T-t)+\theta(W_T-W_t)\right]}\mathbf{1}_{\{\xi_H\leq\xi_T<\xi_a\}}\right]
\end{aligned}$$

From the equivalency of the events $\{\xi_H\leq\xi_T<\xi_a\}$ and

$$\begin{aligned}
& \{\xi_H\leq\xi_t e^{-(r+\theta^2/2)(T-t)-\theta(W_T-W_t)}<\xi_a\} \\
& \equiv\{\ln(\xi_H/\xi_t)+(r+\theta^2/2)(T-t)\leq-\theta\left(W_T-W_t\right)<\ln(\xi_a/\xi_t)+(r+\theta^2/2)(T-t)\} \\
& \equiv\left\{d_2^+\triangleq\frac{\ln(\xi_t/\xi_H)-(r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}}\geq z>\frac{\ln(\xi_t/\xi_a)-(r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}}\triangleq d_2^-\right\}
\end{aligned}$$

where $z\sim\mathcal{N}(0,1)$, it follows that the second term can be written as

$$\begin{aligned}
& \xi_t^{-\pi_Y\sigma/\theta}k^+\int_{(d_2^-,d_2^+]}e^{(1-\pi_Y\sigma/\theta)\left[\left(-r-\theta^2/2\right)(T-t)-x\theta\sqrt{T-t}\right]}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right)dx \\
= \;& A_2\xi_t^{-\pi_Y\sigma/\theta}\int_{(d_2^-,d_2^+]}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(x+(1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t}\right)^2}{2}\right\}dx \\
= \;& A_2\xi_t^{-\pi_Y\sigma/\theta}\int_{(e_2^-,e_2^+]}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\}dy \\
= \;& A_2\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_2^+)-\Phi\left(e_2^-\right)\right]
\end{aligned}$$

where $A_2=k^+e^{-(r+\pi_Y\sigma/2\theta)(1-\pi_Y\sigma/\theta)(T-t)}$, $e_2^{\mp}\triangleq d_2^{\mp}-(1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t}$, and the function $\Phi(\cdot)$ corresponds to the c.d.f. of a standard normal variable.

Following analogous steps, it is possible to derive the remainder terms. After some algebra, one obtains the expression

$$\begin{aligned}
\mathbb{E}_t\left[\xi_{t,T}F_T^{**}\right] \;=\;& A_1\xi_t^{-1/\gamma}\left[1-\Phi(e_1^-)\right]+A_2\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_2^+)-\Phi\left(e_2^-\right)\right] \\
& +A_3\xi_t^{-1/\gamma}\left[\max\{\Phi(e_3^+)-\Phi(e_3^-),0\}\right]+A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_4^+)-\Phi(e_4^-)\right] \\
& +A_5\xi_t^{-1/\gamma}\left[\Phi(e_5^+)\right]
\end{aligned}$$

in the text, with

$$
\begin{aligned}
A_1(t) &= \phi_H^{1/\gamma-1}(y^{**})^{-1/\gamma}e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)+(1-1/\gamma)^2(T-t)\theta^2/2}, \\
A_2(t) &= k^+e^{-(r+\pi_Y\sigma/2\theta)(1-\pi_Y\sigma/\theta)(T-t)}, \\
A_3(t) &= \phi_M^{1/\gamma-1}(y^{**})^{-1/\gamma}e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)+(1-1/\gamma)^2(T-t)\theta^2/2}, \\
A_4(t) &= k^-e^{-(1-\pi_Y\sigma/\theta)\left[r+\theta^2/2\right](T-t)+(1-\pi_Y\sigma/\theta)^2(T-t)\theta^2/2}, \\
A_5(t) &= \phi_L^{1/\gamma-1}(y^{**})^{-1/\gamma}e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)+(1-1/\gamma)^2(T-t)\theta^2/2},
\end{aligned}
$$

and $(e_1^-, e_2^-, e_2^+, e_3^-, e_3^+, e_4^-, e_5^+)$ given by

$$
\begin{aligned}
e_1^- &= \frac{\ln(\xi_t/\xi_H) - (r+\theta^2/2)\,(T-t)}{\theta\sqrt{T-t}} - (1-1/\gamma)\,\theta\sqrt{T-t} \\
e_2^- &= \frac{\ln(\xi_t/\xi_a) - (r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}} - (1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t} \\
e_2^+ &= \frac{\ln(\xi_t/\xi_H) - (r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}} - (1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t} \\
e_3^- &= \frac{\ln(\xi_t/\xi_{k^-}) - (r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}} - (1-1/\gamma)\,\theta\sqrt{T-t} \\
e_3^+ &= \frac{\ln(\xi_t/\xi_a) - (r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}} - (1-1/\gamma)\,\theta\sqrt{T-t} \\
e_4^- &= \frac{\ln(\xi_t/\xi_b) - (r+\theta^2/2)\,(T-t)}{\theta\sqrt{T-t}} - (1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t} \\
e_4^+ &= \frac{\ln(\xi_t/\{\xi_a \vee \xi_{k^-}\}) - (r+\theta^2/2)\,(T-t)}{\theta\sqrt{T-t}} - (1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t} \\
e_5^+ &= \frac{\ln(\xi_t/\xi_b) - (r+\theta^2/2)\,(T-t)}{\theta\sqrt{T-t}} - (1-1/\gamma)\,\theta\sqrt{T-t}.
\end{aligned}
$$

■

Proposition A.3. *Consider assumptions (A1)-(A3) and assume further that: (1) the quantities (ϕ_L, ϕ_M, ϕ_H) are constant for any realization of ξ_T, and (2) the parameters of the economy are such that $1 < \gamma < \theta/\pi_Y\sigma$. Then, we have that the following expressions are local maxima of $V(F_T)$, for the indicated intervals*

$$
\begin{aligned}
&F_H(\xi_T) \triangleq \phi_H^{1/\gamma-1}\,(y^{**}\xi_T)^{-1/\gamma} && \textit{for} \quad \xi_H < \xi_T < \infty \\
&F_{k^+}(\xi_T) \triangleq k^+\,(\xi_T)^{-\pi_Y\sigma/\theta} && \textit{for} \quad \xi_c < \xi_T < \xi_{H^+} \\
&F_M(\xi_T) \triangleq \phi_M^{1/\gamma-1}\,(y^{**}\xi_T)^{-1/\gamma} && \textit{for} \quad \xi_{k^-} \le \xi_T \le \xi_c \\
&F_{k^-}(\xi_T) \triangleq k^-\,(\xi_T)^{-\pi_Y\sigma/\theta} && \textit{for} \quad \xi_d < \xi_T \le (\xi_{k^-} \vee \xi_c) \\
&F_L(\xi_T) \triangleq \phi_L^{1/\gamma-1}\,(y^{**}\xi_T)^{-1/\gamma} && \textit{for} \quad 0 \le \xi_T \le \xi_d
\end{aligned}
$$

The solution to problem (P1) is hence given by

$$F_T^{**} = F_H \mathbf{1}_{\{\xi_H < \xi_T < \infty\}} + F_{k^+} \mathbf{1}_{\{\xi_c < \xi_T < \xi_{H^+}\}} + F_M \mathbf{1}_{\{\xi_{k^-} \leq \xi_T \leq \xi_c\}} + F_{k^-} \mathbf{1}_{\{\xi_d < \xi_T \leq (\xi_{k^-} \vee \xi_c)\}} + F_L \mathbf{1}_{\{0 \leq \xi_T \leq \xi_d\}}, \quad (A.3)$$

where the constants (ξ_c, ξ_d) *are identified in the proof.*

Proof. The proof follows identical steps to those in the proof of Proposition 2. For the set of parameters under consideration, in this occasion we have $\{F_H \geq F_{k^+}\} \equiv \{\xi_T \geq \xi_H\}$, $\{F_{k^-} \leq F_M < F_{k^+}\} \equiv \{\xi_{k^-} \leq \xi_T < \xi_{k^+}\}$, and $\{F_L < F_{k^-}\} \equiv \{\xi_T < \xi_L\}$. Likewise, the following relations hold: $\xi_H > \xi_{k^+} > \xi_{k^-} > \xi_L$, and $F_H < F_M < F_L$.

Following the same logic, in the interval $\xi_T > \xi_H$ we have $v(F_T, \xi_T) = u(\phi_H F_T) - yF_T\xi_T$, from where it follows that F_H is the global maximum. In the interval $\xi_{k^+} \leq \xi_T < \xi_H$, it follows that $F_H < F_{k^+} \leq F_M$, so that F_H cannot be the global maximum. In this interval F_{k^+} is the global maximum. Next, in the interval $\xi_{k^-} \leq \xi_T < \xi_{k^+}$ we have that $F_{k^-} \leq F_M < F_{k^+}$. As in the previous proof, in order to postulate F_{k^+} as the global maximum it must be true that $v(F_{k^+}, \xi_T) - v(F_M, \xi_T) \geq 0$. From the expression $v(F_{k^+}, \xi_T) = v(F_M, \xi_T) + \ell_c(\xi_T)$, where $\ell_c(\cdot) \triangleq \ell_a(\cdot)$ behaves in the same manner ($\ell_c(\xi_{k^+}) > 0$), we have that there is a value $\xi_T = \xi_c$, such that $\ell_c(\xi_c) = 0$. Then, in the interval $\xi_c < \xi_T < \xi_H$ F_{k^+} is the global maximum. If we have $\xi_c > \xi_{k^-}$, then F_M is the global maximum in the interval $\xi_{k^-} \leq \xi_T < \xi_c$. Otherwise, F_M can never be the global maximum (i.e., $\mathbf{1}_{\{\xi_c \leq \xi_T \leq \xi_{k^-}\}} = 0$, $\forall \omega \in \Omega$). Next, in the interval $\xi_L \leq \xi_T \leq (\xi_c \vee \xi_{k^-})$ we have that $F_M < F_{k^-} \leq F_L$. So that F_{k^-} is the global maximum. Finally, in the interval $0 < \xi_T < \xi_L$ we have that in order to postulated F_{k^-} as the global maximum in this interval, it must be true that $v(F_{k^-}, \xi_T) - v(F_L, \xi_T) \geq 0$. For such a case we have $v(F_{k^-}, \xi_T) = v(F_L, \xi_T) + \ell_d(\xi_T)$, where $\ell_d(\cdot) \triangleq \ell_b(\cdot)$ behaves in the same manner ($\ell_d(\xi_L) > 0$), and as in the previous case, there is a value $\xi_T = \xi_d$, such that $\ell_d(\xi_d) = 0$. Hence, in the interval $0 < \xi_T \leq \xi_d$ F_L is the global maximum. ■

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Derivación de los ξF_i's. Para ξF_1 se tiene que

$$\begin{aligned} &\xi F_1 \\ =\ &\phi_H^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\left(\exp\left\{ -\left(1-1/\gamma\right) \left(\left[r+\theta^2/2 \right] (T-t) + \theta \left(W_T - W_t \right) \right) \right\} \right) 1_{\{\xi_T < \xi_H\}} \right] \end{aligned}$$

De la equivalencia

$$\begin{aligned} \{\xi_T &< \xi_H\} \\ &\equiv \left\{ e^{-(r+\theta^2/2)(T-t)-\theta(W_T-W_t)} < \xi_H/\xi_t \right\} \\ &\equiv \left\{ -(r+\theta^2/2)(T-t) - \theta(W_T - W_t) < \ln(\xi_H/\xi_t) \right\} \\ &\equiv \left\{ -z\theta\sqrt{T-t} < \ln(\xi_H/\xi_t) + \left[r+\theta^2/2 \right] (T-t) \right\} \\ &\equiv z > \frac{\ln(\xi_t/\xi_H) - \left[r+\theta^2/2 \right] (T-t)}{\theta\sqrt{T-t}} \triangleq d_1^- \end{aligned}$$

se tiene que ξF_1 puede expresarse como

$$\begin{aligned} &\phi_H^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_1^+)} \frac{1}{2\pi} e^{-(1-1/\gamma)\left(\left[r+\theta^2/2 \right](T-t) + x\theta\sqrt{T-t} \right) - x^2/2} dx \right] \\ =\ &A_1 \phi_H^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_1^+)} \frac{1}{2\pi} \exp\left\{ -(1-1/\gamma)^2 (T-t)\theta^2/2 - (1-1/\gamma)\, x\theta\sqrt{T-t} - x^2/2 \right\} \right. \\ =\ &A_1 \phi_H^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_1^+)} \frac{1}{2\pi} \exp\left\{ -\frac{\left(x + (1-1/\gamma)\,\theta\sqrt{T-t} \right)^2}{2} \right\} dx \right] \\ =\ &A_1 \phi_H^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{(-\infty,e_1^+)} \frac{1}{2\pi} \exp\left\{ -\frac{1}{2} y^2 \right\} dy \right] \\ =\ &A_1 \phi_H^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[1 - \Phi(e_1^+) + \phi(e_1^+) \right] \end{aligned}$$

en donde $A_1 = e^{-(1-1/\gamma)\left[r+\theta^2/2 \right](T-t) + (1-1/\gamma)^2 (T-t)\theta^2/2}$, $e_1^+ = d_1^+ - (1-1/\gamma)\,\theta\sqrt{T-t}$.

Para ξF_3 se tiene que

$$\begin{aligned} &\xi F_3 \\ =\ &\mathbb{E}_t \left[\xi_{t,T} F_M 1_{\{\xi_a \le \xi_T \le \xi_{k-}\}} \right] \\ =\ &\phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\xi_{t,T}^{1-1/\gamma} 1_{\{\xi_a \le \xi_T \le \xi_{k-}\}} \right] \\ =\ &\phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\exp\left\{ -\left(1-1/\gamma\right) \left(\left[r+\theta^2/2 \right] (T-t) + \theta \left(W_T - W_t \right) \right) \right\} 1_{\{\xi_a \le \xi_T \le \xi_{k-}\}} \right] \end{aligned}$$

De la equivalencia

$$
\begin{aligned}
\{\xi_a &\leq \xi_T \leq \xi_{k^-}\} \\
&\equiv \{(\xi_a/\xi_t) \leq e^{-(r+\theta^2/2)(T-t)-\theta(W_T-W_t)} \leq (\xi_{k^-}/\xi_t)\} \\
&\equiv \left\{ d_3^+ \triangleq \frac{\ln(\xi_t/\xi_a) - (r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}} \geq z \geq \frac{\ln(\xi_t/\xi_{k^-}) - (r+\theta^2/2)(T-t)}{\theta\sqrt{T-t}} \triangleq d_3^- \right\}
\end{aligned}
$$

se tiene que ξF_3 puede expresarse como

$$
\begin{aligned}
& \phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{[d_3^-, d_3^+]} e^{-(1-1/\gamma)\left(\left[r+\theta^2/2\right](T-t)+x\theta\sqrt{T-t}\right)} \left(\frac{1}{2\pi} e^{-x^2/2} \right) dx \right] \\
= \; & \phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{[d_3^-, d_3^+]} \frac{1}{2\pi} e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)-(1-1/\gamma)x\theta\sqrt{T-t}-x^2/2} dx \right] \\
= \; & A_3 \phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{[d_3^-, d_3^+]} \frac{1}{2\pi} e^{-(1-1/\gamma)^2(T-t)\theta^2/2-(1-1/\gamma)x\theta\sqrt{T-t}-x^2/2} dx \right] \\
= \; & A_3 \phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{[d_3^-, d_3^+]} \frac{1}{2\pi} \exp\left\{ -\frac{\left(x + (1-1/\gamma)\,\theta\sqrt{T-t}\right)^2}{2} \right\} dx \right] \\
= \; & A_3 \phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\int_{[e_3^-, e_3^+]} \frac{1}{2\pi} \exp\left\{ -\frac{y^2}{2} \right\} dy \right] \\
= \; & A_3 \phi_M^{1/\gamma-1} y^{-1/\gamma} \xi_t^{-1/\gamma} \left[\Phi(e_3^+) - \Phi(e_3^-) \right]
\end{aligned}
$$

en donde $A_3 = e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)+(1-1/\gamma)^2(T-t)\theta^2/2}$, $e_3^{\pm} = d_3^{\pm} - (1-1/\gamma)\,\theta\sqrt{T-t}$.

Para ξF_4 se tiene que

$$
\begin{aligned}
& \mathbb{E}_t \left[\xi_{t,T} F_{k^-} 1_{\{\xi_{k^-} < \xi_T < \xi_b\}} \right] \\
= \; & k^- \xi_t^{-\pi_Y\sigma/\theta} \mathbb{E}_t \left[\xi_{t,T}^{1-\pi_Y\sigma/\theta} 1_{\{\xi_{k^-} < \xi_T < \xi_b\}} \right] \\
= \; & k^- \xi_t^{-\pi_Y\sigma/\theta} \mathbb{E}_t \left[\xi_{t,T}^{1-\pi_Y\sigma/\theta} \exp\left\{ -(1-\pi_Y\sigma/\theta) \left(\left[r+\theta^2/2 \right] (T-t) + \theta\left(W_T - W_t\right) \right) \right\} 1_{\{\xi_{k^-} < \xi_T < \xi_b\}} \right]
\end{aligned}
$$

De la equivalencia

$$
\begin{aligned}
\{\xi_{k^-} &< \xi_T < \xi_b\} \\
&\equiv \left\{ d_4^+ \triangleq \frac{\ln(\xi_t/\xi_{k^-}) - \left[r+\theta^2/2\right](T-t)}{\theta\sqrt{T-t}} > z > \frac{\ln(\xi_t/\xi_b) - \left[r+\theta^2/2\right](T-t)}{\theta\sqrt{T-t}} \triangleq d_4^- \right\}
\end{aligned}
$$

se tiene que ξF_4 puede expresarse como

$$\begin{aligned}
& A_4 k^- \xi_t^{-\pi_Y\sigma/\theta} \left[\int_{(d_4^-,d_4^+)} \frac{1}{2\pi} e^{-(1-\pi_Y\sigma/\theta)^2(T-t)\theta^2/2-(1-\pi_Y\sigma/\theta)x\theta\sqrt{T-t}-x^2/2} dx \right] \\
= & A_4 k^- \xi_t^{-\pi_Y\sigma/\theta} \left[\int_{(d_4^-,d_4^+)} \frac{1}{2\pi} \exp\left\{ -\frac{\left(x+(1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t}\right)^2}{2} \right\} dx \right] \\
= & A_4 k^- \xi_t^{-\pi_Y\sigma/\theta} \left[\int_{(e_4^-,e_4^+)} \frac{1}{2\pi} \exp\left\{ -\frac{y^2}{2} \right\} dy \right] \\
= & A_4 k^- \xi_t^{-\pi_Y\sigma/\theta} \left[\Phi(e_4^+) - \phi(e_4^+) - \Phi(e_4^-) + \phi(e_4^-) \right]
\end{aligned}$$

en donde $A_4 = e^{-(1-\pi_Y\sigma/\theta)\left[r+\theta^2/2\right](T-t)+(1-\pi_Y\sigma/\theta)^2(T-t)\theta^2/2}$, $e_4^\pm = d_4^\pm - (1-\pi_Y\sigma/\theta)\,\theta\sqrt{T-t}$.

Para ξF_5 se tiene que

$$\begin{aligned}
& \mathbb{E}_t \left[\xi_{t,T} F_L 1_{\{\xi_T \geq \xi_b\}} \right] \\
= & \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\xi_{t,T}^{1-1/\gamma} 1_{\{\xi_T \geq \xi_b\}} \right] \\
= & \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \mathbb{E}_t \left[\exp\left\{ -(1-1/\gamma)\left(\left[r+\theta^2/2 \right](T-t) + \theta\left(W_T - W_t\right) \right) \right\} 1_{\{\xi_T \geq \xi_b\}} \right]
\end{aligned}$$

De la equivalencia

$$\begin{aligned}
\{\xi_T & \geq \xi_b\} \\
& \equiv \left\{ \exp\left\{ -\left(r+\theta^2/2\right)(T-t) - \theta\left(W_T - W_t\right) \right\} \geq \xi_b/\xi_t \right\} \\
& \equiv \left\{ z \leq \frac{\ln\left(\xi_t/\xi_b\right) - \left(r+\theta^2/2\right)(T-t)}{\theta\sqrt{T-t}} \triangleq d_5^+ \right\}
\end{aligned}$$

se tiene que ξF_5 puede expresarse como

$$\begin{aligned}
& \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_5^+]} \frac{1}{2\pi} e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)-(1-1/\gamma)x\theta\sqrt{T-t}-x^2/2} dx \right] \\
= & A_5 \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_5^+]} \frac{1}{2\pi} e^{-(1-1/\gamma)^2(T-t)\theta^2/2-(1-1/\gamma)x\theta\sqrt{T-t}-x^2/2} dx \right] \\
= & A_5 \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_5^+]} \frac{1}{2\pi} \exp\left\{ -(1-1/\gamma)^2(T-t)\,\theta^2/2 - (1-1/\gamma)\,x\theta\sqrt{T-t} - x^2/2 \right\} dx \right] \\
= & A_5 \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \left[\int_{(-\infty,d_5^+]} \frac{1}{2\pi} \exp\left\{ -\frac{\left(x+(1-1/\gamma)\,\theta\sqrt{T-t}\right)^2}{2} \right\} dx \right] \\
= & A_5 \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \left[\int_{(-\infty,e_5^+]} \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}y^2 \right\} dx \right] \\
= & A_5 \phi_L^{1/\gamma-1} \xi_t^{-1/\gamma} \left[\Phi(e_5^+) \right]
\end{aligned}$$

en donde $A_5 = e^{-(1-1/\gamma)\left[r+\theta^2/2\right](T-t)+(1-1/\gamma)^2(T-t)\theta^2/2}$, $e_5^+ = d_5^+ - (1-1/\gamma)\,\theta\sqrt{T-t}$. ■

Derivación de π^{**}. Una vez singularizado el proceso $\{F_t^{**};\ t \in [0,T]\}$, podemos identificar la volatilidad de dF_t^{**} mediante una aplicación del lema de Itô. El componente asociado a la volatilidad está dado por

$$\pi_t^{**}\sigma = (\partial F_t^{**}/\partial b_t)(b_t\sigma^b) + (\partial F_t^{**}/\partial \xi_t)(-\xi_t\theta).$$

Para el término $\partial F_t^{**}/\partial b_t$ tenemos

$$\frac{e^{(\mu^b-r-\sigma^b\theta)(T-t)}-1}{\mu^b-r-\sigma^b\theta}.$$

En tanto que para el término $\partial F_t^{**}/\partial \xi_t$ tenemos

$$\begin{aligned}
&A_1(-1/\gamma)\xi_t^{-1/\gamma}\left\{\xi_t^{-1}\left[1-\Phi(e_1^-)+\phi(e_1^-)\right]-(\partial\Phi(e_1^-)/\partial\xi_t)+(\partial\phi(e_1^-)/\partial\xi_t)\right\}\\
&+A_2\xi_t^{-\pi_\Upsilon\sigma/\theta}\left\{\xi_t^{-1}\left[\Phi(e_2^+)-\Phi\left(e_2^-\right)+\phi(e_2^-)\right]+\partial(\Phi(e_2^+)/\partial\xi_t)-(\partial\Phi\left(e_2^-\right)/\partial\xi_t)+(\partial\phi(e_2^-)/\partial\xi_t)\right\}\\
&+A_3\xi_t^{-1/\gamma}\left\{\xi_t^{-1}\left[\left(\Phi(e_3^+)-\Phi(e_3^-)\right)^+\right]+\left((\partial\Phi(e_3^+)/\partial\xi_t)-(\partial\Phi(e_3^-)/\partial\xi_t)\right)^+\right\}\\
&+A_4\xi_t^{-\pi_\Upsilon\sigma/\theta}\left\{\xi_t^{-1}\left[\Phi(e_4^+)-\phi(e_4^+)-\Phi(e_4^-)+\phi(e_4^-)\right]+(\partial\Phi(e_4^+)/\partial\xi_t)\right.\\
&\left.-(\partial\phi(e_4^+)/\partial\xi_t)-(\partial\Phi(e_4^-)/\partial\xi_t)+(\partial\phi(e_4^-)/\partial\xi_t)\right\}+A_5\xi_t^{-1/\gamma}\left\{\xi_t^{-1}\Phi(e_5^+)+(\partial\Phi(e_5^+)/\partial\xi_t)\right\}
\end{aligned}$$

en donde

$$\begin{aligned}
\Phi(y(x)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y(x)} e^{-s^2/2}ds\\
\frac{\partial\Phi(y(x))}{\partial x} &= \frac{1}{\sqrt{2\pi}}e^{-y(x)^2/2}\cdot(-y(x))\cdot y'(x)\\
\frac{\partial\phi(y(x))}{\partial x} &= \frac{1}{\sqrt{2\pi}}e^{-y(x)^2/2}\cdot(-y(x))\cdot y'(x)
\end{aligned}$$

■

Proposition 3 *n*

Corollary 3

$$\ell_a(\xi) \triangleq \frac{1}{\gamma-1}\left[\gamma\left(\frac{y\xi}{\phi_M}\right)^{1-1/\gamma} - \left(\frac{(\xi)^{\pi_Y\sigma/\theta}}{k^+\phi_H}\right)^{\gamma-1}\right] - yk^+ (\xi)^{1-\pi_Y\sigma/\theta}$$

$$\partial\ell_a/\partial\xi = \left(\frac{y}{\phi_M}\right)^{1-1/\gamma}\xi^{-1/\gamma} - \left(\frac{1}{k^+\phi_H}\right)^{\gamma-1}(\gamma-1)(\pi_Y\sigma/\theta)\,\xi^{(\gamma-1)(\pi_Y\sigma/\theta)-1}$$
$$-yk^+(1-(\pi_Y\sigma/\theta))\xi^{-\pi_Y\sigma/\theta}$$

$$\ell_b(\xi) \triangleq \frac{1}{\gamma-1}\left[\gamma\left(\frac{y\xi}{\phi_L}\right)^{1-1/\gamma} - \left(\frac{(\xi)^{\pi_Y\sigma/\theta}}{k^-\phi_M}\right)^{1-\gamma}\right] - yk^- (\xi)^{1-\pi_Y\sigma/\theta}.$$

Parámetros
$k^{\pm} \triangleq F_0 \exp\left(\left[r+\pi_Y\sigma(\theta/2-r/\theta) - b^Y - \pi_Y^2\sigma^2/2\right]T + \eta \pm \kappa\right)$

.1 Basak's derivation

Sea

$$W_t^* = [N(d(\gamma_M,\hat{\xi}))f_H^{(1/\gamma_M-1} + N(-d(\gamma_M,\xi_a)f_L^{(1/\gamma_;-1)}]Z(\gamma_M)(y\xi_t)^{-1/\gamma_M}$$
$$+[N(d(\kappa,\xi_a)) - N((d(\kappa,\hat{\xi}))]AZ(\kappa)\xi_t^{-1/\kappa}$$

$$\hat{\theta}_t W_t = W_t\theta^N + [N(d(\kappa,\xi_a)) - N(d(\kappa,\hat{\xi}))](\gamma_M/\kappa-1)A\theta^N Z(\kappa)\xi_t^{-1/\kappa}$$
$$+\{[\phi(d(\kappa,\xi_a)) - \phi(d(\kappa,\hat{\xi}))]AZ(\kappa)\xi_t^{-1/\kappa}$$
$$+[\phi(d(\gamma_M,\hat{\xi}))f_H^{(1/\gamma_M-1)} - \phi(d(\gamma_M,\xi_a))f_L^{(1/\gamma_M-1)}]Z(\gamma_M)(y\xi_t)^{-1/\gamma_M}\}$$
$$\times\frac{\gamma_M\theta^N}{\kappa\sqrt{T-t}}$$

La derivada de W_t^* está dada por

$$-\frac{\partial W_t^*}{\partial\xi_t}\frac{\xi_t\theta/\sigma}{W_t^*} = (\theta/\gamma\sigma)\left[N(d(\gamma_M,\hat{\xi}))f_H^{(1/\gamma_M-1} + N(-d(\gamma_M,\xi_a)f_L^{(1/\gamma_;-1)}\right]Z(\gamma_M)(y\xi_t)^{-1/\gamma_M}/W_t^*$$
$$+(\theta/\gamma\sigma)\left[(1+\gamma/\kappa-1)\left[N(d(\kappa,\xi_a)) - N((d(\kappa,\hat{\xi}))]\right]AZ(\kappa)\xi_t^{-1/\kappa}\right]/W_t^*$$
$$+(\theta/\sigma)\left[\phi(d(\gamma_M,\hat{\xi}))f_H^{(1/\gamma_M-1)} + \phi(-d(\gamma_M,\xi_a)f_L^{(1/\gamma_;-1)}\right]Z(\gamma_M)(y\xi_t)^{-1/\gamma_M}(W_t^*\kappa\sqrt{T$$
$$+(\theta/\sigma)[\phi(d(\kappa,\xi_a)) - \phi((d(\kappa,\hat{\xi}))]AZ(\kappa)\xi_t^{-1/\kappa}(W_t^*\kappa\sqrt{T-t})^{-1}$$
$$= (\theta/\gamma\sigma) + \left[(\theta/\gamma\sigma)(\gamma/\kappa-1)\left[N(d(\kappa,\xi_a)) - N((d(\kappa,\hat{\xi}))]\right]AZ(\kappa)\xi_t^{-1/\kappa}\right]/W_t^*$$
$$+(W_t^*\kappa\sqrt{T-t})^{-1}(\theta/\sigma)\{\left[\phi(d(\gamma_M,\hat{\xi}))f_H^{(1/\gamma_M-1)} + \phi(-d(\gamma_M,\xi_a)f_L^{(1/\gamma_;-1)}\right]Z(\gamma_M)(y\xi$$
$$+[\phi(d(\kappa,\xi_a)) - \phi((d(\kappa,\hat{\xi}))]AZ(\kappa)\xi_t^{-1/\kappa}\}$$

Supongamos que $b_0 = 0$. Entonces tenemos que

$$\begin{aligned} F_t^{**} &= A_1\xi_t^{-1/\gamma}\left[1-\Phi(e_1^-)\right] + A_2\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_2^+) - \Phi\left(e_2^-\right)\right] \\ &\quad + A_3\xi_t^{-1/\gamma}\left[\left(\Phi(e_3^+) - \Phi(e_3^-)\right)^+\right] + A_4\xi_t^{-\pi_Y\sigma/\theta}\left[\Phi(e_4^+) - \Phi(e_4^-)\right] \\ &\quad + A_5\xi_t^{-1/\gamma}\left[\Phi(e_5^+)\right] \end{aligned}$$

Definimos los procesos $R_t^F = \ln(F_t^{**}/F_0)$ y $R_t^Y = \ln(Y_t/Y_0)$. La pregunta relevante es si la cantidad $R_t^F - R_t^Y$ es monótona en ξ_t, de manera de poder graficar π_t^{**} como función de $R_t^F - R_t^Y$. Para esta expresión tenemos que $R_t^F - R_t^Y = \ln(F_t^{**}/Y_t)$. De la demostración de la Proposición 2 se tiene que $Y_t = A(\xi_t)^{-\pi_Y\sigma/\theta}$. Luego, podemos escribir

$$\begin{aligned} R_t^F - R_t^Y &= \ln\Big((A_1/A)\xi_t^{\pi_Y\sigma/\theta-1/\gamma}\left[1-\Phi(e_1^-)\right] + (A_2/A)\left[\Phi(e_2^+) - \Phi\left(e_2^-\right)\right] \\ &\quad + (A_3/A)\xi_t^{\pi_Y\sigma/\theta-1/\gamma}\left[\left(\Phi(e_3^+) - \Phi(e_3^-)\right)^+\right] + (A_4/A)\left[\Phi(e_4^+) - \Phi(e_4^-)\right] \\ &\quad + (A_5/A)\xi_t^{\pi_Y\sigma/\theta-1/\gamma}\left[\Phi(e_5^+)\right]\Big). \end{aligned}$$

La derivada de esta expresión, con respecto a ξ_t, está dada por

$$\begin{aligned} &\frac{Y_t}{F_t^{**}} \times \Big((A_1/A)(\pi_Y\sigma/\theta - 1/\gamma)\xi_t^{\pi_Y\sigma/\theta-1/\gamma-1}\left[1-\Phi(e_1^-)\right] \\ &+ (A_1/A)\xi_t^{\pi_Y\sigma/\theta-1/\gamma-1}[-\phi(e_1)](1/(\theta\sqrt{T-t})) \\ &+ 1_{\{\Phi(e_2^+)\geq\Phi(e_2^-)\}}(A_2/A)\left[\phi(e_2^+) - \phi\left(e_2^-\right)\right](1/(\theta\sqrt{T-t})) \\ &+ (A_3/A)(\pi_Y\sigma/\theta - 1/\gamma)\xi_t^{\pi_Y\sigma/\theta-1/\gamma-1}\left[\left(\Phi(e_3^+) - \Phi(e_3^-)\right)^+\right] \\ &+ 1_{\{\Phi(e_3^+)\geq\Phi(e_3^-)\}}(A_3/A)\xi_t^{\pi_Y\sigma/\theta-1/\gamma-1}\left[\phi(e_3^+) - \phi(e_3^-)\right](1/(\theta\sqrt{T-t})) \\ &+ 1_{\{\Phi(e_4^+)\geq\Phi(e_4^-)\}}(A_4/A)\left[\phi(e_4^+) - \phi(e_4^-)\right](1/(\theta\sqrt{T-t})) \\ &+ (A_5/A)(\pi_Y\sigma/\theta - 1/\gamma)\xi_t^{\pi_Y\sigma/\theta-1/\gamma-1}\left[\Phi(e_5^+)\right] \\ &+ (A_5/A)\xi_t^{\pi_Y\sigma/\theta-1/\gamma-1}\left[\phi(e_5^+)\right](1/(\theta\sqrt{T-t}))\Big) \end{aligned}$$

Para el caso de los términos que contienen la función $\Phi(\cdot)$ el signo de la expresión está dado por el término $\pi_Y\sigma/\theta - 1/\gamma$.

Notes on the proof of Propoposition 2.
In the interval $\xi_T \leq \xi_{H^+}$ we have $F_{k^-} < F_{k^+} \leq F_H < F_M < F_L$
In the interval $\xi_{H^+} < \xi_T \leq \xi_{H^-}$ we have $F_{k^-} \leq F_H < F_{k^+} < F_M < F_L$
In the interval $\xi_{H^-} < \xi_T \leq \xi_{k^+}$ we have $F_H < F_{k^-} < F_{k^+} \leq F_M < F_L$

In the interval $\xi_{k^+} < \xi_T \leq \xi_{k^-}$ we have that $F_H < F_{k^-} \leq F_M < F_{k^+} < F_L$
In the interval $(\xi_{k^-} \vee \xi_a) < \xi_T \leq \xi_L$ we have $F_H < F_M < F_{k^-} < F_{k^+} < F_L$
we also need to know the relationship between

$$\begin{aligned}
\xi_T &< \left(\phi_H^{1-1/\gamma} y^{1/\gamma} k^+\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)} = \xi_{H^+} \\
\xi_T &< \left(\phi_H^{1-1/\gamma} y^{1/\gamma} k^-\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)} = \xi_{H^-} \\
\xi_T &< \left(\phi_M^{1-1/\gamma} y^{1/\gamma} k^+\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)} = \xi_{k^+} \\
\xi_T &< \left(\phi_M^{1-1/\gamma} y^{1/\gamma} k^-\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)} = \xi_{k^-} \\
\xi_T &< \left(\phi_L^{1-1/\gamma} y^{1/\gamma} k^-\right)^{1/(\pi_Y\sigma/\theta - 1/\gamma)} = \xi_L
\end{aligned}$$