DOMINANT STRATEGY MECHANISMS IN NONATOMIC TRANSFERABLE UTILITY ECONOMIES: CHARACTERIZATION AND EXISTENCE

by

Louis Makowski (UC, Davis)

and

Joseph M. Ostroy (UCLA)

UCLA Dept. of Economics
Working Paper #421
October 1986

ABSTRACT

This paper extends the characterization of efficient dominant strategy mechanisms with quasi-linear preferences to models with a continuum of agents. Our results represent the nonatomic analogs of Groves schemes, which characterize dominant strategy mechanisms in finite agent models. The key concept in our analysis is that of an infinitesimal agent's marginal product. Efficient dominant strategy mechanisms satisfy the rule that an agent's payoff must equal his/her marginal product plus a lump sum. Unlike the finite agent case where lump sums typically cannot be chosen to achieve budget-balance, in the nonatomic model budget-balance can be attained. We also show that if an "individual rationality" restriction is imposed, lump sums must be zero, and we relate this to the possibility of designing mechanisms for public as compared to private goods.

DOMINANT STRATEGY MECHANISMS IN NONATOMIC TRANSFERABLE UTILITY ECONOMIES: CHARACTERIZATION AND EXISTENCE*

by

Louis Makowski and Joseph M. Ostroy (October 1986)

1. Introduction

This paper extends the characterization of dominant strategy mechanisms with quasi-linear preferences to models with a continuum of agents. As in the finite agents case (Vickrey [1961], Clarke [1971], Groves and Loeb [1975], Green and Laffont [1977], Walker [1978], Holmstrom [1979]), we show that with suitable qualifications to meet exigencies of nonatomic models, dominant strategy mechanisms are equivalent to Groves' demand-revealing mechanisms.

Hammond [1979] has characterized dominant strategy mechanisms with a continuum of agents without the quasi-linearity restriction, but the similarities between his work and ours may not be transparent. This is because he focused primarily on the dominant strategy problem as it relates to the pricing of allocations in continuum models, whereas we have followed a path originating in the above-cited literature on demand-revealing mechanisms for finite agent models where commodity pricing is not stressed. In a companion paper, we shall be more explicit about the connections with Hammond's results as well as related work by Schmeidler and Vind [1972], Varian [1976], Kleinberg [1980], Champsaur-Laroque [1981], McLennan [1981] and Mas-Colell [1983].

---

*Research supported by a grant from the National Science Foundation.

addition to the total gains from trade attributable to that agent's characteristics, plus a lump sum independent of that agent's characteristics. This equivalence will also be the basis for our continuum extension of finite demand-revealing mechanisms. Indeed, our characterization result is stated in MP terms; it says

> if and only if a mechanism rewards each agent with his/her marginal product, plus perhaps a lump sum, will it exhibit the dominant strategy property.

Thus, within the class of environments having quasi-linear preferences, the MP reward principle is of considerable generality for inducing revelation. It applies to finite agent or nonatomic models with private and/or public goods.

But what is the MP of a single agent in a nonatomic model? A too facile interpretation of the influence of sets of measure zero might lead one to conclude that in a nonatomic economy a single agent's MP is zero. Such reasoning ignores the possibility of doing "calculus over individuals", and it is such a calculus that we find essential to the characterization of nonatomic dominant strategy mechanisms. A nonatomic agent's marginal product is a derivative. It measures the limiting rate of change of a non-infinitesimal group's *per capita* contribution as the membership of the group shrinks to that of a single agent. In addition to its role in dominant strategy mechanisms, the concept of an infinitesimal agent's marginal product and a calculus over individuals is central to the work of Aumann and Shapley [1974] on the value for nonatomic games. It also underlies our work on perfect competition (Artzner and Ostroy [1983] and Ostroy [1984]).

From the characterization result we see that the existence of dominant strategy mechanisms is equivalent to the existence of budget-balancing mechanisms that reward all individuals with their marginal products.

environments in which the gains-from-trade function is linearly homogeneous. Because private goods [public goods] models do [do not] lead to linearly homogeneous gains-from-trade functions, we shall, with the aid of the individual rationality hypothesis, be able to draw a distinction between the incentive compatibility properties of public and private goods. (See Roberts [1976] for a similar conclusion, also based on individual rationality.)

We call attention to two technical features of this paper, one relating to the description of an agent's type and the other to the number of types. In some settings, e.g., the costless public good model, it is enough to describe an individual's type by his/her utility function on the fixed domain of possible public projects; this model abstracts away individuals' possibly different input requirements to produce the collective output. In other settings, e.g., private goods, when endowments or production possibilities may also be misrepresented, an individual's type requires the specification of not only his individual's tastes, but also his domain of feasible net trades. Because our goal is to provide an analysis including both of these settings, we shall describe an agent's type by a utility function on a possibly varying (effective) domain. Tastes and feasible net trades will be subsumed in the specification of "tastes". (In the language of the paper, we shall distinguish between (general) dominant strategy mechanisms and (less general) demand-revealing mechanisms. In the latter, only an individual's tastes, not his domain of feasible net trades, are subject to misrepresentation. Our desire to treat general dominant strategy mechanisms will also give rise to an assumption in the sequel -- that the set of types be *feasibly connected* -- that does not usually appear in the demand-revealing literature.)

classical adding-up question in the MP theory of distribution.

2. The Model

An individual's characteristics are given by a function $v: R^{\ell} \to R \cup \{-\infty\}$. The function v specifies both the tastes of the individual (his/her utility indicator) as well as the set of (net) trades feasible for that individual. These feasible trades are defined by the *effective domain* of v, the set $Y_v = \{y \in R^{\ell}: v(y) > -\infty\}$, which we assume to be closed in R^{ℓ}. Recalling the first technical feature mentioned in the Introduction, notice that this formulation is general enough to allow individuals to misrepresent both their tastes and their endowments or production possibilities (if any).

Let V be the set of possible individual characteristics. We shall not provide a self-contained set of hypothesis on V, e.g., we shall omit conditions guaranteeing that certain maxima be achieved below. However, it is useful to assume explicitly that V is a class of C^1 functions; i.e., for each $v \in V$ there is an open set containing its effective domain and a C^1 function $\hat{v}$ on this set such that $\hat{v} = v$ on the latter's effective domain.

An *economy* will be described by a (positive) measure μ on V. We shall confine our attention to those μ with finite support, i.e., to finite type economies. Such a μ can be written as

$$\mu = \sum_{v \in V\#} \alpha_v \delta_v,$$

where δ_v is the measure with unit mass at v and zero elsewhere, V# is a finite subset of V, and α_v is interpreted as the Lebesgue measure of the agents of type v. The set of all such measures with finite support is denoted M#[V].

$$\int m(\mu,v)d\mu(v) = 0.$$

However, unless otherwise stated, we shall make no aggregate feasibility restrictions on the money allocation.

It will be convenient to denote the product of all economies and all types by

$$\Omega = M\#[V] \times V.$$

Define a *mechanism* as a mapping $f\colon \Omega \to R^{\ell} \times R$, where $f(\mu,w) = (y(\mu,w),m(\mu,w))$ and $\forall\mu\ \forall v \in \text{supp}\ \mu\ \ (y(\mu,v))$ is aggregately feasible. Notice $f(\mu,\cdot)$ is defined for all $w \in V$, not just those $v \in \text{supp}\ \mu$. The significance of this relates to the second technical feature mentioned in the Introduction: insuring the asymptotic meaningfulness of our continuum model. This will be amplified on in Section 3A below.

A mechanism $f\colon \Omega \to R^{\ell} \times R$ exhibits the *dominant strategy* (DS) property if

(DS): $\forall\ \mu\ \forall\ v \in \text{supp}\ \mu\ \forall\ w,\ \ u(f(\mu,v),v) \geq u(f(\mu,w),v).$

Recalling the definition of u, this says that the utility an individual receives from the allocation mechanism by reporting his/her characteristics truthfully, $v(y(\mu,v)) + m(\mu,v)$, is at least as large as the utility an agent of that type could obtain by reporting other characteristics, $v(y(\mu,w)) + m(\mu,w)$. Notice if f satisfies DS then v has no incentive to misrepresent herself as a w, even for $w \notin \text{supp}\ \mu$.

REMARK 1: If v and $w \in \text{supp}\ \mu$ and allocations are net trades, DS implies the no-envy or fair property of net trades. If $\text{supp}\ \mu$ were always equal to V, then the fair property of net trades (see, e.g., Schmeidler and Vind [1972]) for all μ is clearly equivalent to the DS property.

There is also a prior question. Let

$$\Delta = \{(\mu,v): \mu \in M\#[V] \ \text{ and } \ v \in \text{supp } \mu\} \subset \Omega.$$

Why not simply ignore those $(\mu,w) \notin \Delta$ since in μ the announcement w can be true at best for a set of measure zero?

Confining permissible announcements of an individual's type to those already in the support of the distribution would place the continuum model at a decided disadvantage compared to the finite agent model where an individual can announce any type. Since in this important respect the finite agent model would be more demanding, the construction of limiting arguments linking the finite and continuum models would be hindered or precluded. We want to study the properties of models with a large number of agents in which a change in any one individual's characteristics has a very small impact on the distribution -- in the limit, none -- while also retaining the flexibility of finite agent models which allows an individual to announce any type as his/her own. To combine these features of the finite and the infinite, we have assumed $f(\mu,\cdot)$ is defined even for $w \notin \text{supp } \mu$. But for such w, $f(\mu,w)$ cannot be defined arbitrarily and still preserve the model's asymptotic meaningfulness. Thus we shall restrict our attention to mechanisms f that are "continuous" in the sense specified below.

The (norm) distance between two distributions $\mu = \Sigma_{v\in \text{supp } \mu}\, \alpha_v\, \delta_v$ and $\mu' = \Sigma_{v\in \text{supp } \mu'}\, \alpha'_v\, \delta_v$ is,

$$\begin{aligned} ||\mu'-\mu|| = {} & \sum_v \{|\alpha'_v-\alpha_v|: v \in \text{supp } \mu' \cap \text{supp } \mu\} \\ & + \sum_v \{\alpha'_v: v \in \text{supp } \mu' \setminus \text{supp } \mu\} \\ & + \sum_v \{\alpha_v: v \in \text{supp } \mu \setminus \text{supp } \mu'\}. \end{aligned}$$

exchange economies admitting efficiency prices. Let

$$(y(p,\alpha,v),m(p,\alpha,v)) \in \arg\max \{v(y) + m: py + m = \alpha\}.$$

It is well-known that $y(p,\cdot,v)$ does not vary with α, so define $e(p,v) = y(p,\cdot,v)$.

The economy μ is said to admit efficiency prices if there is a $p(\mu)$ such that for the y-optimal allocation, $y(\mu,v) = e(p(\mu),v)$. Suppose $p: M\#[V] \rightarrow 2^{R^{\ell}}$ is a *single-valued* efficiency price mapping whose range is contained in a compact set in R^{ℓ}_{++}. These hypotheses may be shown to imply that p is μ-uniformly continuous. A restriction on the curvature of preferences such as V is C^2 and $\nabla^2 v$ is negative definite (therefore strictly concave) and bounded away from zero could be used to show that $y(\mu,v) = e(p(\mu),v)$ is uniformly continuous on Δ.

Restricting ourselves to continuous mechanisms guarantees that the DS property for the continuum model can be given an asymptotic interpretation. In particular, suppose that instead of a continuum, agents are of size $t > 0$. The DS property would then require that

$$u(f(\mu,v),v) \geq u(f(\mu + t(\delta_w - \delta_v),w),v),$$

where $\mu + t(\delta_w - \delta_v)$ is the effect on the distribution of agents characteristics after an agent of type v and size t anounces type w. The continuity of u with respect to $f(\mu,v)$ and the continuity of $f(\mu,v)$ with respect to μ implies that if the DS property holds in the continuum then for any $\epsilon > 0$ there is a t_ϵ such that if $t < t_\epsilon$ then

$$u(f(\mu,v),v) + \epsilon \geq u(f(\mu + t(\delta_w - \delta_v),w),v).$$

That the DS property holds exactly in the limit implies that it holds approximately for large finite agent models, where the degree of adding (or removing) an infinitesimal individual of type w. To this end, let

$$y'(\mu,v;d) = \lim_{t\to 0_+} \frac{y(\mu + td;v) - y(\mu,v)}{t}.$$

About models with a continuum of agents, it is frequently said that the influence of one agent on another is nil. While there are models in which $y'(\mu,v;d) = 0$ (e.g., see the papers by Tideman-Tullock, Green and Laffont, Rob and Mitsui mentioned in the Introduction, which all use the costless collective good model), these are exceptional. More typically, "nil" should be understood to mean that the impact of w's behavior on v is infinitesimal relative to v's scale.

Now, assuming the existence of the directional derivatives $y'(\mu,v;d)$ as well as the (already assumed) differentiability of v, we can compute $MP(\mu,w)$ using the chain rule:

(1) for $w \notin \text{supp}\ \mu$:

$$\begin{aligned} MP(\mu,w) &= g'(\mu;\delta_w) \\ &= \lim_{t\to 0_+} \left[\int \frac{v(y(\mu+t\delta_w,v)-v(y(\mu,v))d\mu(v)}{t} + \frac{tw(y(\mu+t\delta_w,w))}{t}\right] \\ &= \int \nabla v(y(\mu,v))\cdot y'(\mu,v;\delta_w)d\mu(v) + w(y(\mu,v)) \\ &= \xi_+(\mu,w) + w(y(\mu,w)) \end{aligned}$$

(2) for $w \in \text{supp}\ \mu$:

$$\begin{aligned} MP(\mu,w) &= -g'(\mu;-\delta_w) \\ &= -\left[\lim_{t\to 0_+} \int \frac{v(y(\mu-t\delta_w,v))-v(y(\mu,v))d\mu(v)}{t} - \frac{tw(y(\mu-t\delta_w))}{t}\right] \end{aligned}$$

(R.3) $y'(\mu,v;d+\tilde{d}) = y'(\mu,v;d) + y'(\mu,v;\tilde{d})$, $\forall(\mu,v) \in \Delta\ \forall d,\tilde{d} \in D(\mu)$,

(R.4) $y'(\cdot,\cdot;d)$ is continuous on Δ, $\forall d = \delta_w$, $w \in V$.

(R.5) $\forall w,v$, $\forall$ sequences $\{\mu_n\} \subset M\#[V]$ such that $\mu_n \to \mu$, if $y(\mu_n,w) \notin Y_v$, $\forall n > N$, then $y(\mu,w) \notin Y_v$.

Letting $d = \delta_w$ and $\tilde{d} = -d$, and observing that $y'(\mu,v);0) = 0$, conditions (R.2) and (R.3) imply that two-sided directional derivatives exist for $y(\mu,v)$ in every direction in supp μ. Notice (R.4) is a simple rather than uniform continuity condition; its meaning is that for any $(\mu,v) \in \Delta$ and any sequence $\{(\mu_n,v)\} \subset \Delta$ such that $\mu_n \to \mu$, $y'(\mu_n,v,d) \to y'(\mu,v,d)$.

Condition (R.5) says that all y-allocations to agents that the mechanism calls for and that are not in v's effective domain, are away from the boundary of v's effective domain. This requirement rules out the possibility of $v(\cdot)$ being discontinuous (relative to the mechanism), for allocations approaching the boundary of v's effective domain. It is a strong assumption. But it is only needed to prove our results for general dominant strategy mechanisms; it is not required to prove any of our results for (less general) demand-revealing mechanisms -- where $Y_v = Y_w$ for all v and w.

REMARK 3: As in Remark 2, we sketch some sufficient conditions for a regular demand-revealing mechanism in the case of an exchange economy (so only tastes, not endowments, are subject to misrepresentation). Let $p(\mu)$ be the efficiency price vector for μ and $e(p(\mu),v) = y(\mu,v)$. The influence of w on v can now be traced through w's influence on prices. Denote by $p'(\mu;d)$, $d = \delta_w$, the directional derivative of the efficiency immediately implies that $\xi_+(\mu,w) = \xi_-(\mu,w)$ when $w \in \text{supp } \mu$.

Choose any $(\mu,w) \in \Omega$ and any sequence $\{(\mu_n,w)\} \subset \Omega$ such that $\mu_n \to \mu$. From the above argument we have

$$MP(\mu_n,w) = \xi(\mu_n,w) + w(y(\mu_n,w)).$$

By (R.1), $y(\mu_n,w) \to y(\mu,w)$; and since w is continuous on its closed effective domain, $w(y(\mu_n,w)) \to w(y(\mu,w))$. (Notice that for any $(\mu,w) \in \Omega$, even if $w \notin \text{supp } \mu$, $w(y(\mu,w)) > -\infty$. (Proof: Since Δ is dense in Ω there exists a sequence $\mu_n \to \mu$ with $w \in \text{supp } \mu_n$. By assumption, $y(\mu_n,w)$ is feasible for w. Hence $\lim y(\mu_n,w) = y(\mu,w)$ is feasible for w since w's effective domain is closed. N.B.: By contrast $v(y(\mu,w))$ may equal $-\infty$; this possibility will be discussed and dealt with in Section 3C. below.)

Because $\mu_n,\mu \in M\#[V]$ and $\mu_n \to \mu$, it follows that $\mu_n(\{\text{supp } \mu_n \cap \text{supp } \mu\}) \to \mu$. By hypothesis v is C^1 and by (R.3) y' is continuous on Δ. Therefore

$$\begin{aligned}\lim \xi(\mu_n,w) &= \lim \int \nabla v(y(\mu_n,v))y'(\mu_n,v;\delta_w)d\mu_n \\ &= \lim \int_{\text{supp}\mu_n \cap \text{supp}\mu} \nabla v(y(\mu_n))y'(\mu_n,v;\delta_w)d\mu_n \\ &= \int \nabla v(y(\mu,v))y'(\mu,v;\delta_w)d\mu \\ &= \xi(\mu,w).\end{aligned}$$

Since $MP(\mu,w) = \xi(\mu,w) + w(y(\mu,w))$, we have $MP(\mu_n,w) \to MP(\mu,w)$.||

3C. The MP of an Individual Who Misrepresents His Type

We shall show that any mechanism f is $DSPO_Y$ if and only if it always rewards all types with their MP's, plus perhaps a lump sum. Since any type $v \in \text{supp } \mu$ may claim he is really some other type $w \in V$, as a and $MP(\mu,w) = MP(\mu,w;w)$ is simply that in the latter the agent who truthfully announces w receives $w(y(\mu,w))$ while in the former the agent of type v who announces w receives $v(y(\mu,w))$. These conclusions are formalized in

LEMMA 2: *Let* f *be a regular,* PO_Y *mechanism. Then,* $\forall(\mu,v) \in \Delta$ $\forall w$,

$$MP(\mu,w;v) = MP(\mu,w) - w(y(\mu,w)) + v(y(\mu,w)) = \xi(\mu,w) + v(y(\mu,w)).$$

Furthermore, $\forall w$, $MP(\cdot,w;\cdot)$ *is continuous in the sense that for any sequence* $\{(\mu_n,v)\} \subset \Delta$ *such that* $\mu_n \to \mu$, $\lim MP(\mu_n,w;v) = \xi(\mu,w) + v(y(\mu,w))$. (N.B.: (μ,v) *need not be in* Δ.)

PROOF: Let $(\mu,v) \in \Delta$. Rewrite $MP(\mu,w;v)$ as

$$\text{(a)} \quad MP(\mu,w;v) = \lim_{t\to 0_+} \frac{g(\mu+t(\delta_w-\delta_v);\mu) - g(\mu) + [g(\mu)-g(\mu-t\delta_v)]}{t}$$

$$= \lim_{t\to 0_+} \frac{g(\mu+t(\delta_w-\delta_v);\mu) - g(\mu)}{t} - g'(\mu;-\delta_v)$$

Note that

$$\text{(b)} \quad g'(\mu;\delta_w-\delta_v) = \lim_{t\to 0_+} \left[\int \frac{[z(y(\mu+t(\delta_w-\delta_v)),z)-z(y(\mu,z))]d\mu(z)}{t} + \frac{tw(y(\mu+t(\delta_w-\delta_v),w))}{t} - \frac{tv(y(\mu+t(\delta_w-\delta_v),v)}{t}\right]$$

By inspection of the definition of $g(\mu + t(\delta_w-\delta_v);\mu)$ given above and (b), we have using (R.5)

$$\text{(c)} \quad g'(\mu;\delta_w-\delta_v) = \lim_{t\to 0_+} \frac{g(\mu+t(\delta_w-\delta_v);\mu) - g(\mu)}{t} + w(y(\mu,w)) - v(y(\mu,w)).$$

Similarly, suppose $v \in \operatorname{supp} \mu$ were to announce the characteristic w. Then v's utility can be written as

$$u(f(\mu,w),v) = MP(\mu,w;v) - H(\mu,w),$$

where $MP(\mu,w;v)$ is v's marginal product when he/she announces w.

The marginal productivity reward principle has a built-in dominant strategy property.

LEMMA 3: *If* $v \in \operatorname{supp} \mu$, *then*

$$\max_{w} MP(\mu,w;v) = MP(\mu,v;v) = MP(\mu,v).$$

PROOF: It is evident from the definition that for all $t > 0$,

$$g(\mu - t\delta_v + t\delta_w;\mu) \leq g(\mu),$$

i.e., the *total* gains from trade cannot possibly be increased through misrepresentation. Therefore,

$$MP(\mu,w;v) = \lim_{t \to 0_+} \frac{g(\mu - t\delta_v + t\delta_w;\mu) - g(\mu - t\delta_v)}{t}$$

$$\leq \lim_{t \to 0_+} \frac{g(\mu) - g(\mu - t\delta_v)}{t} = MP(\mu,v). ||$$

Say that $H: \Omega \to R$ is a *lump sum* function if there is an $h: M\#[V] \to R$ such that

$$h(\mu) = H(\mu,w).$$

This might be better termed an anonymous lump sum function in contrast with the lump sum function described for finite agent models (see Section 5). In the latter, the lump sum is invariant to the individual's characteristics

REMARK 5: The lump-sum property of H is intended to reflect the condition that individuals cannot influence that part of their reward. However, if H were not μ-continuous, the condition $H(\mu,v) = H(\mu,w) = h(\mu)$ would say that while singleton subsets of agents cannot influence H, sets of arbitrarily small size could have a substantial impact. This would be contrary to the spirit of the nonatomic model as an idealization of a large but finite model, and to the asymptotic interpretation given above to DS mechanisms. However, since $u(f(\mu,w),v) = \xi(\mu,w) + v(y(\mu,w)) - h(\mu)$, the regularity of f establishes the μ-continuity of u, ξ and v, hence h must also be μ-continuous.

4B. The Marginal Product Reward Principle as a Necessary Condition

There remains the converse, that to achieve the DS property a regular PO_Y mechanism must be specified as in Theorem 1. Based on the preparations given above and those to follow, we shall show that Holmstrom's [1979] demonstration of necessity for the finite agent model can be "lifted" to the nonatomic case.

For the sufficient conditions on $DSPO_Y$ to become necessary it is well-known that V must exhibit a certain amount of variety. A simple method of insuring enough variety is to assume that

V *is a convex set:* $\forall v,w \in V$, $\forall\ \alpha \in (0,1)$, there is a $v_\alpha \in V$ such that

$$\forall y,\ v_\alpha(y) = \alpha v(y) + (1-\alpha)w(y).$$

The role of convexity will be to ensure that for any $v,w \in V$ such that $v(y(\mu,w)) > -\infty$ and $w(y(\mu,v)) > -\infty$, i.e., the y-optimal allocation to w is feasible for v and vice-versa, the environment will be sufficiently rich to include many other characteristics "between" v and w. In Holmstrom's terminology, this is an example of an environment that is where because μ is fixed we may write $\psi(\beta;\alpha) = MP(\mu,v_\beta;v_\alpha)$ and $k(\beta) = H(\mu,v_\beta)$.

For continuous mechanisms the DS property applies to all agents $w \in V$, not just $w \in \text{supp}\ \mu$. Proof: Let $\mu_n \to \mu$ and let $\{(\mu_n,w)\} \subset \Delta$. Then if f satisfies DS, $u(f(\mu_n,w),w) \geq u(f(\mu_n,z,w))$ for all $z \in V$. Hence, taking limits, $u(f(\mu,w),w) \geq u(f(\mu,z),w)$. So from the hypothesis that f is a regular DR mechanism for all $\alpha,\beta \in [0,1]$,

(a) $\alpha \in \arg_\beta\max\ \psi(\beta,\alpha) + k(\beta)$.

Further, Lemma 3 implies for any v_α (not just for $v_\alpha \in \text{supp}\ \mu$),

$$\max_\beta MP(\mu,v_\beta;v_\alpha) = MP(\mu,v_\alpha;v_\alpha).$$

Proof: Suppose the contrary, that $MP(\mu,v_\beta;v_\alpha) > MP(\mu,v_\alpha;v_\alpha)$. Then let $\mu_n \to \mu$ with $\{(\mu_n,v_\alpha)\} \subset \Delta$. The continuity of $MP(\cdot,v_\beta;\cdot)$ on Δ (recall Lemma 2) implies for sufficiently large n $MP(\mu_n,v_\beta;v_\alpha) > MP(\mu_n,v_\alpha,v_\alpha)$, contradicting Lemma 3. Hence, for all $\alpha,\beta \in [0,1]$

(b) $\alpha \in \arg_\beta\max\ \psi(\beta,\alpha)$

From the definition of v_α, $v_\alpha(\cdot) = \alpha v(\cdot) + (1-\alpha)w(\cdot)$. Hence, differentiating ψ with respect to α,

$$\frac{\partial\psi(\beta,\alpha)}{\partial\alpha} = v(y(\mu,v_\beta)) - w(y(\mu,v_\beta))$$

To show

(c) $\sup_{\alpha,\beta} \left|\frac{\partial\psi(\beta,\alpha)}{\partial\alpha}\right| < \infty$,

suppose the contrary. Then there is a sequence $\beta_k \in [0,1]$ such that $v(y(\mu,v_{\beta_k})) - w(y(\mu,v_{\beta_k})) \to -\infty$ or $+\infty$. We show a contradiction for the one lets $\epsilon \to 0$. The details are left to interested readers.

Having established (a)-(c), now apply the following basic result proved in Holmstrom [1979],

LEMMA: *Let* ψ: $[0,1] \times [0,1] \to R$ *and* k: $[0,1] \to R$ satisfy (a), (b) *and* (c), *then* k *is constant.*

Therefore, there is an h such that $h(\mu) = H(\mu,v) = H(\mu,w)$, as was to be demonstrated.||

In some settings, such as models of exchange economies, we must deal with the fact that individual characteristics include, besides variations in tastes, variations in what is individually feasible. The following assumption, by providing for sufficient variation in what is individually feasible in V, allows for a more complete converse to Theorem 1.

V *is feasibly connected:* $\forall\mu\forall v,w$ there exists z such that

(1) z could have delivered $y(\mu,v)$ or $y(\mu,w)$: $y(\mu,v), y(\mu,w) \in Y_z$,

(2) v and w could have delivered $y(\mu,z) \in Y_v \cap Y_w$.

In the above, $y(\mu,\cdot)$ is the y-optimal allocation in f.

To illustrate feasible connectedness consider a two-commodity-plus-money exchange economy in which $\forall v$, $v(0,0) > -\infty$, i.e., it is individually feasible for any agent not to trade. Suppose $y(\mu,v) = (1,-1)$ and $y(\mu,w) = (-1,1)$. This assumption requires, for example, that there is a z which can feasibly make the trade (1,-1) or (-1,1) but is called upon in an optimal allocation to deliver $y(\mu,z) = (0,0)$.

REMARK 6: A weaker assumption would suffice. It is enough to postulate that for $\forall\mu$ $\forall v \in$ supp μ $\forall w$, there exists a finite sequence $(z_o, z_1, \ldots, z_n)$

A finite agent economy can be described by a rather special measure μ with finite support where,

$$\mu = \Sigma\alpha_{v_i}\delta_{v_i} \quad \text{and } \alpha_{v_i} = 1 \text{ for all } v_i \in \text{supp } \mu.$$

Let $\mu^i = \sum_{j\neq i} \alpha_{v_j}\delta_{v_j} = \sum_{j\neq i}\delta_{v_j}$ be the economy without v_i. Thus, if $v_i \in \text{supp } \mu$, $\mu = (\mu^i, v_i)$.

An allocation for μ is a $f_i(\mu^i,v_i) = (y_i(\mu^i,v_i),\ m_i(\mu^i,v_i)) \in R^\ell \times R$, $i = 1,\ldots,n$. Here, $f_i(\mu^i,v_i)$ plays the role of $f(\mu,v)$ above. In contrast with nonatomic models where, as it were $\mu^i = \mu$ and therefore changes in any one agent's characteristics do not change μ, here changes in v_i necessarily change μ, although they do not change μ^i.

The DS property of f is given by

$$u(f_i(\mu^i,v_i),v_i \geq u(f_i(\mu^i,w_i),v_i)$$

for all i, μ^i, $v_i \in \text{supp } \mu$ and w_i where $w \in V$.

A y-optimal mechanism (PO_Y) is one that achieves

$$g(\mu) = \max\{\Sigma v_i(y_i(\mu^i,v_i)); \quad v_i \in \text{supp } \mu\}.$$

Vickrey [1961], Clarke [1971], Groves [1973] and Groves and Loeb [1975], among others, have demonstrated that in a y-optimal mechanism $f = (y,m)$, if money payments are given by

$$(*) \qquad m_i(\mu^i,v_i) = \sum_{\substack{j\neq i \\ v_j \in \text{supp}\mu}} v_j(y_i(\mu^i,v_i)) - k_i(\mu^i),$$

then f will be DS. Conversely, Green and Laffont [1977], Walker [1978] and Holmstrom [1979] have shown that if f is DS and PO_Y then $m_i(\mu^i,v_i)$ must be given by the formula (*)

differs from $g(\mu)$ only by a lump sum, clearly any mechanism satisfying (*) can be rewritten as (**) and conversely. However, it is in the form (**), rather than (*), that the formula for $DSPO_Y$ mechanisms is suitable for extension from the finite agent to the nonatomic case. In (**) the RHS terms are both the same order of magnitude as that of an individual agent, *even for nonatomic models*. In the nonatomic case, the MP of an agent is the derivative (i.e., the rate of change) of the total gains from trade with respect to that agent. (Note: Unlike the continuum model, only the existence of a y-optimal mechanism is required for MP_i to be well-defined in the finite case; compare this with the rather delicate considerations in Section 3 above to prove the existence of MP's in the continuum.)

6. Existence Theorems for DSPO and DSPOIR Mechanisms

6.A A Possibility Theorem for DSPO Mechanisms

Recall that a DSPO mechanism is a $DSPO_Y$ mechanism in which the sum of money transfers, $\int m(\mu,v)d\mu(v)$, is always zero. If that sum were positive, the allocation of the money commodity would not be feasible for the participants in the economy and the balance would have to be made up by some outside authority; or if it were negative, the sum would represent the departure from full utility maximization and Pareto-optimality. While the results for $DRPO_Y$ mechanisms in nonatomic models completely parallel the finite agent mechanism results (the literature concentrates on DR, rather than the more general DS mechanisms), the situation for DRPO is quite the opposite. Instead of the impossibility results for DRPO cited above for finite agent models, there is always possibility -- even for DSPO mechanisms.

Let

$u(f(\mu,w),v) = MP(\mu,w) - h(\mu)$. To be DSPO, it must satisfy

$$\int u(f(\mu,v))d\mu(v) = \int MP(\mu,v)d\mu(v) - h(\mu)\int d\mu(v) = g(\mu),$$

or

$$\sigma(\mu) = \int MP(\mu,v)d\mu(v) - g(\mu) = h(\mu)\int d\mu(v).$$

Thus, $h(\mu) = \sigma(\mu)/\bar{\mu}$. ||

REMARK 7: Obviously, the conclusions of Theorem 3 also hold for DRPO mechanisms, without the assumption that V is feasibly connected.

REMARK 8: To interpret our existence result as idealizing the case in large but finite economies, we need

$$m(\mu,v) = MP(m,v) - v(y(\mu,v)) - \sigma(\mu)/\bar{\mu}$$

to be μ-continuous (recall the discussion at the end of Section 3A and also Remark 5). We already know MP and v are μ-continuous, and $\bar{\mu}$ certainly is also. Thus since $g(\mu) = \int v(y(\mu,v))d\mu(v)$ is μ-continuous, we see that σ and, hence, m are μ-continuous.

Thus, using the interpretation of our model as an idealization of a large but finite economy given at the end of Section 3A, Theorem 3 implies that DSPO can be approximately achieved in large but finite economies. Indeed, for sufficiently large but finite economies, the approximation can be made arbitrarily close to exact budget balancing -- using the equal division of the "deficit" sharing rule. Of course, in large finite economies this rule implies only an approximate lump sum tax.

We want to emphasize that this Groves approximation result holds for any class of (smooth) large finite economies -- not just for costless collective good models, where approximation results have been previously

Attention is confined to environments satisfying the following conditions:

(E.1) (Non-negative surplus) $\forall\mu,\ \sigma(\mu) \geq 0$.

(E.2) (Characteristics are benign) $\forall\mu\ \forall w,\ \ MP(\mu,w) \geq 0$.

(E.3) (Existence of "dummies") $\forall\mu\ \exists\ v^{o}(\mu)$ such that $MP(\mu,v^{o}(\mu)) = 0$.

Were we to formulate more explicitly a particular model of an economy with private or public goods of the kinds referred to above, assumptions (E.1) and (E.2) could be derived as conclusions. Here we simply assert that these conditions do not go beyond conventional restrictions. The nonnegative surplus condition (E.1) is fulfilled whenever the g function is homogeneous (in which case $\sigma = 0$, see Section 7). It is also a necessary condition for the game in characteristic function form associated with μ to have a nonempty core. Given (E.1), the hypothesis that individual characteristics are benign (E.2) would be satisfied "on average" whenever g could be normalized so that $g(\mu) \geq 0$. In fact, nonatomic models typically satisfy (E.2) for all characteristics, not just on average.

(E.3) postulates the existence of individuals having no effect on the gains from trade. For example, in a private good exchange economy with convex preferences if $p(\mu)$ were the efficiency price vector corresponding to the y-optimal allocation in the population μ, then $v^{o}(\mu)$ could be taken to be those preferences for which the hyperplane $\{x \in R^{\ell}: p(\mu)x = 0\}$ is tangent to the indifference curve of v^{o} passing through the origin of R^{ℓ}; with public goods, v^{o} would be the tastes of someone entirely indifferent to public goods and who, furthermore, has no resources that contribute toward their production.

PROOF: From Theorem 3, if f is DSPO and $\sigma = 0$, then $u(f(\mu,v),v) = MP(\mu,v)$. But by (E.2), $MP(\mu,v) \geq 0$, so f satisfies IR.

To demonstrate the DSPOIR implies $\sigma = 0$, suppose the contrary. Then, by (E.1) there is a μ' such that $\sigma(\mu') > 0$. Let $v^o = v^o(\mu')$ (recall E.3), and let $\mu = \mu' + t\delta_{v^o}$. Notice by the continuity of $\sigma(\cdot)$ (see Remark 8) and the continuity of $MP(\cdot,v^o)$, as $t \to 0$, $\sigma(\mu)/\bar{\mu} \to \sigma(\mu')/\bar{\mu}' > 0$ and $MP(\mu,v^o) \to MP(\mu',v^o) = 0$. Hence, $\exists t > 0$ such that $MP(\mu,v^o) - \sigma(\mu)/\bar{\mu} < 0$. But by Theorem 3, DSPO implies $u(f(\mu,v^o),v^o) = MP(\mu,v^o) - \sigma(\mu)/\bar{\mu}$. The RHS, we have verified is negative for some μ, contradicting IR. ||

7. The Adding-Up Problem Revisited

We shall conclude this investigation into dominant strategy mechanisms in nonatomic economies by pointing out the connections with the century old adding-up problem in the marginal productivity theory of distribution. Think of an agent's characteristics not as a utility function on a net trade possibility set but as a labor input in the production of a commodity whose technology is given by g. Regard μ as a vector of total labor inputs with the property that each individual is endowed with one unit of labor type, v. Thus, since $\mu = \Sigma\alpha_v\delta_v$, α_v is interpreted as the mass of individuals having "labor input" v.

To more closely conform to traditional marginal productivity notation, let us define

$$\frac{\partial g(\mu)}{\partial v} = MP(\mu,v).$$

The marginal productivity theory of distribution postulated that competition would lead a supplier of input v to receive $\partial g(\mu)/\partial v$ for each infinitesimal quantity supplied. Since each individual agent in a nonatomic model

We are led to an obvious question: What is the difference between mechanisms which can reward individuals with their MP's directly without any further lump sum modifications and those which require such modifications? Note that this is a question about the properties of economies in which $\sigma(\mu) = 0$ versus those in which $\sigma(\mu) \neq 0$.

We want to emphasize that *as it stands the theory of dominant strategy mechanisms*, i.e., the theory having to do with $DSPO_Y$ and DSPO mechanisms in models with quasi-linear preferences, *is remarkably unforthcoming on the free-rider problem*. To repeat, for $DSPO_Y$ mechanisms a single characterization holds for models with public and/or private goods and finite or continuum of agents: reward individuals with their MP's and possibly a lump sum. For DSPO mechanisms, finite agent models yield impossibility theorems with or without public goods while nonatomic models yield possibility theorems in either case. So, according to these results, the summary conclusion is that (1) if there is a demand revelation problem it does not occur in economies with large numbers of agents and (2) in finite agent economies demand revelation problems do occur, but indiscriminately among those containing private or public goods.

Continuing to assume quasi-linearity, if we take DSPOIR as the definition of an efficient incentive-compatible mechanism, the overall conclusions are more satisfying. Here the traditional approach to the adding-up problem -- modified to the marginal products of persons, not commodities -- comes into its own. Theorem 4, above, in its finite analog (see Theorem 3 in Makowski and Ostroy [1986]) combine to show that DSPOIR mechanisms can be constructed in nonatomic or finite agent environments if and only if $\sigma = 0$.

produce $g(\mu)$, then y will also be chosen for the population $\lambda\mu$ and this will produce total utility $g(\lambda\mu) = \lambda g(\mu)$.) However, when we consider models with public goods and non-negligible costs (in terms of private goods), the g function fails to be linearly homogeneous. Two identical populations μ, each producing the same optimal quantities of public goods with the same resources and producing total utility $2g(\mu)$ could, simply by combining while halving the *per capita* resources contributed, maintain the same total quantity of public goods and therefore produce a total gains from trade for the population 2μ such that $g(2\mu) > 2g(\mu)$. In such a case we have $\sigma(\mu) > 0$ but, by Theorem 4, this is incompatible with DSPOIR.

The summary conclusions for the DSPOIR definition of incentive compatibility is that since such mechanisms require adding-up of marginal products in the traditional sense that σ must equal zero, incentive compatibility will typically fail to exist (a) whenever there is a finite number of agents (because of indivisibilities) or (b) whenever there is a public good with non-negligible costs of production (because of failure of linear homogeneity). This Euler's Theorem explanation appears to confirm economic intuition (see Samuelson [1955]).

Kleinberg, N. [1980]: "Fair Allocations and Equal Income," *Journal of Economic Theory*, 23, 189-200.

Laffont, J.-J. and E. Maskin [1980]: "A Differential Approach to Dominant Strategy Mechanisms," *Econometrica*, 48(6), 1507-1520.

Makowski, L. and J. Ostroy [1986]: "Vickrey-Clarke-Groves Mechanisms and Perfect Competition," forthcoming in *Journal of Economic Theory*.

Mas-Colell, A. [1983]: "On the Second Welfare Theorem for Anonymous Net Trades in Exchange Economies with Many Agents," forthcoming in *Information, Incentives and Economic Mechanisms, Essays in Honor of Leonid Hurwicz*, eds. T. Groves, R. Radner and S. Recter, Univ. of Minnesota Press.

McLennan, A. [1981]: Ph.D. Thesis, Princeton University.

Mitsui, T. [1983]: "Asymptotic Efficiency of the Pivotal Mechanism with General Project Space," *Journal of Economic Theory*, 31, 318-331.

Ostroy, J. [1984]: "A Reformulation of the Marginal Productivity Theory of Distribution," *Econometrica*, 52(3), 599-630.

Rob, R. [1982]: "Asymptotic Efficiency of the Demand-Revealing Mechanism," *Journal of Economic Theory*, 28, 207-220.

Roberts, D.J. and A. Postelwaite [1976]: "The Incentives for Price-Taking Behavior in Large Exchange Economies," *Econometrica*, 44, 115-127.

Roberts, J. [1976]: "The Incentives for Correct Revelation of Preferences and the Number of Consumers," *Journal of Public Economics*, 6, 359-374.

Samuelson, P. [1955]: "Diagrammatic Exposition of a Theory of Public Expenditure," *Review of Economics and Statistics*, 350-356.

Schmeidler, D. and K. Vind [1972]: "Fair Net Trades," *Econometrica*, 40(4), 637-642.

Stigler, G. [1946]: *Production and Distribution Theories*. Macmillan.