ENHANCED BANZHAF POWER INDEX AND ITS MATHEMATICAL PROPERTIES

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Abstract: The main objective of the paper is to make the well-known Banzhaf-Coleman power index more adequate to the practice of coalition formation. To achieve this, a new power index, which is denoted by BS, is introduced and studied. In the second section of the paper, going after introduction, some game theoretic denotations and definitions necessary for understanding are given. In the third section, properties of coalitions in proper simple games are studied. The BS power index is introduced in the fourth section. It is demonstrated how properties of this index make possible to obtain some intrinsic properties of players (such, for example, as being a blocking player). An axiomatic characterization of the BS power index is given in the fifth section (Theorems 5.1 and 5.2). Problems of axiom independence are considered in the sixth section. It is proved that axioms for the Banzhaf-Coleman power index given by Dubey and Shapley (Theorem 6.1) as well as axioms for the BS power index defined in this paper are independent (Theorems 6.2 and 6.3).

1. Introduction

Different indices have been suggested for measuring political power. The most popular of them are the Banzhaf-Coleman (Banzhaf, 1965; 1968; Coleman, 1971) and Shapley-Shubik (Shapley and Shubik, 1954) indices. The first of them emerged from the legislature practice while the second one was introduced as a specialization of the so-called "Shapley value" (Shapley, 1953), which is one of the most useful tools in the game theory. These indices have been extensively used for practical purposes in the U.S. and abroad (cf. Dubey and Shapley, 1979). Thus, in the 1960s the U.S. Supreme Court handed down a series of "one person one vote" decisions, setting forth new standards of constitutional fairness for systems of electoral representation at the state and local levels. As a result, many existing voting systems had to be revised or at least re-examined. It gave to the proponents or opponents of the proposed reform frequent occasion to invoke political power indices. Multiple calculations using real data were carried out on computers and presented as evidence in the courtroom or at legislative hearings. The main ideas underlying the game-theoretic approach to power estimation found wide legal acceptance (Dubey and Shapley, 1979). It is necessary to remark that the use of the game theory to study voting systems can be traced back to the introduction of simple games by von Neumann and Morgenstern (von Neumann and Morgenstern,1944).

It has been demonstrated that different power indexes reflect specific conditions in the voting body (Straffin, 1977). If all members have the equal probabilities for voting for or against the discussed measure and this probability is selected from the uniform distribution on [0,1], then the political power of individual members is estimated by the Shapley-Shubik power index. Otherwise, when all members are voting independently, that is, the voting probabilities are selected from the uniform distribution on [0,1], then the political power of individual members is estimated by the Banzhaf-Coleman power index.

Usage of the Banzhaf-Coleman power index as a measure of political power presupposes that the influence of any swinger on a coalition C is independent of other swingers. However, in real life, it is also important if this swinger i is unique for C or not. In the first case to become winning, the coalition has to use i and thus, i has a great power in C. In the second case when c has other swingers making a winning coalition out of C, importance of i for C diminishes because i can be substituted by another swinger. Consequently, the power of i with respect to C has to be shared with other swingers of C. Thus, we come to the concept of the enhanced Banzhaf power index, which is denoted BS and studied in this paper. This makes possible to obtain more realistic means for measuring political power.

In the second section of the paper, going after introduction, some game theoretic denotations and definitions necessary for understanding are given. In the third section, properties of coalitions in proper simple games are studied. The BS power index is introduced in the fourth section. It is compared to the Banzhaf-Coleman and Shapley-Shubik indices. It is demonstrated how characteristics of the BS power index make possible to obtain some intrinsic properties of players (such, for example, as being a blocking player). An axiomatic characterization of the BS power index is given in the fifth section.

Problems of axiom independence are considered in the sixth section. It is proved that axioms for the Banzhaf-Coleman power index given by Dubey and Shapley (1979) as well as axioms for the BS power index, which are introduced in this paper, are independent. Thus, the given sets of axioms are minimal in the sense that any their subset does not characterize the corresponding power index.

2. Game Theoretic Preliminaries

Let N be a set. A game on N is a real-valued function v that is defined on the subsets of N and is vanishing on the empty set. According to (Bourbaki, 1968; Burgin, 1990) any function is precisely represented by a triad. Thus, a game G is a triad (N, v, L) where $N = \mathrm{N(G)}$ is the set of players of the game G and L that is called the scale of G is the codomain of the function v. The range of the function v is a subset of L. The subsets of N are called coalitions. $\varnothing$ denotes the void coalition in any game. $C_o(N)$ denotes the set of all coalitions and $C(N)$ denotes the set of all non-void coalitions. If A and B are coalitions from $C_o(N)$ and $B \supseteq A$, then A is called a subcoalition of B and B is called a supercoalition of A .

Two games $G = (N, v, L)$ and $H = (M, u, K)$ are called equal if $N = M$, $L = K$, and $v = u$.

A game G is called:

a) superadditive or proper if

$$v(C \cup D) \geq v(C) + v(D) \text{ for any two coalitions } C \text{ and } D \text{ when } C \cap D = \varnothing ; \quad (1)$$

b) simple if v assumes only the values 0 and 1, is not identically 0, and obeys the condition of monotonicity:

$$\text{for any two coalitions } C \text{ and } D ,\ v(C) \geq v(D) \text{ whenever } C \supseteq D . \quad (2)$$

For proper games, condition (2) is equivalent to the condition:

$$v(C) + v(N \setminus C) \leq 1 \text{ for any coalition } C . \quad (3)$$

If X is a set, then 2^X denotes the set of all subsets of X and $|X|$ denotes the number of elements in X. Thus, we assume that $|N| = n$. Then $|C_o(N)| = 2^n$. The set 2^N usually coincides with the set $C_o(G)$ of all potential coalitions in any game G that has N as the set of players.

For games, the union and intersection operations are defined:

the union $G \vee H$ of two games G and H is the game defined by the function $v(G \vee H) = \max \{v(G), v(H)\}$; the intersection (or the meet) $G \wedge H$ of two games G and H is the game defined by the function $v(G \wedge H) = \min \{ v(G) , v(H) \}$.

For simple games, their union is obtained taking the union of the sets of their winning coalitions while intersection is obtained taking the meet of the sets of their winning coalitions.

Operations of union and meet are associative. So, we denote by $\vee_i^m G_i$ the union of m games and by $\wedge_i^m G_i$ the intersection of m games. By $\Sigma_i^m k_i$ we denote the sum of m numbers.

3. Coalitions in Simple Games

In what follows, we consider only proper simple games and call them also voting games.

Definition 3.1. *A coalition C is called winning (losing) if v(C) is equal to* 1 (*to* 0).

W(G) denotes the set of all winning coalitions in the game G. If the game G is fixed, we write simply **W**.

L(G) denotes the set of all losing coalitions in the game G. If the game G is fixed, we write simply **L**.

Definition 3.2. *A player i is called a dictator in a game G, if* $\{ i \}$ *is a winning coalition in G.*

As the name presupposes, a dictator can dictate his/her decision to all other players in the corresponding game. Any dictator in a game is unique.

Definition 3.3. *A player i is called a blocking player in a game G, if any winning coalition in G contains i.*

As the name presupposes, a blocking player can prevent any decision-making by deserting a winning coalition.

Any dictator in a game G is its blocking player. However, a game may have several blocking players while a dictator is unique. In this case, the game does not have a dictator.

Example 3.1. Let N consists of five players: α, β, δ, γ, and ε. The player α has 4 votes, the player β has 5 votes, each of the players δ, γ, and ε has 3 votes. To take an action (make a decision), it is necessary to have 10 votes. In this game, both players α and β are blocking players.

A possible generalization of the concept of a blocking player is the concept of a blocking coalition (Dubey and Shapley, 1979).

Definition 3.4. *A coalition C is called blocking in a game G, if any winning coalition in G contains some element i from C.*

Remark 3.1. For simple games, this definition is equivalent to the definition given by Dubey and Shapley (1979).

Remark 3.2. Any winning coalition in a simple game G is a blocking coalition. However, this is not true for games that are not simple. Besides, as example 3.1 demonstrates, there are blocking coalitions in simple games that are not winning.

Lemma 3.1. *If C is a blocking coalition and* $B \supseteq C$, *then B is a blocking coalition.*

Remark 3.3. In a proper game, the intersection of two winning coalitions is not empty. However, the intersection of two blocking coalitions may be empty, as is demonstrated by

Example 3.1 because any blocking player forms a blocking coalition. Consequently, the complement of a blocking coalition may be a blocking coalition. For winning coalitions, this is not true (Dubey and Shapley, 1979).

Another generalization of the concept of a blocking player is the concept of a glued blocking coalition.

Definition 3.5. *A coalition C is called a glued blocking coalition in a game G , if any winning coalition in G contains C.*

Lemma 3.2. *If C is a glued blocking coalition and $C \supseteq D$, then D is a glued blocking coalition.*

Definition 3.6. *A winning coalition C is called critical if there is such i from C that $C \setminus \{ i \}$ is a losing coalition.*

Informally, a winning coalition C is critical if it may become a losing coalition by losing only one member.

$\mathbf{W}_{\mathbf{C}}(G)$ denotes the set of all critical coalitions in the game G. If the game G is fixed, we write simply $\mathbf{W}_{\mathbf{C}}$.

Definition 3.7. *A winning coalition C is called minimal critical if it is critical and does not contain proper critical subcoalitions.*

$\mathbf{W}_{\mathbf{MC}}(G)$ denotes the set of all critical coalitions in the game G . If the game G is fixed, we write simply $\mathbf{W}_{\mathbf{MC}}$.

Definition 3.8. *A losing coalition C is called subcritical if there is such i from N that $C \cup \{ i \}$ is a winning coalition.*

Informally, a losing coalition C is subcritical if it may become a winning coalition by gaining only one member. In such a way, a subcritical coalition may become only a critical winning coalition.

$\mathbf{L}_{\mathbf{C}}(G)$ denotes the set of all subcritical coalitions in the game G . If the game G is fixed, we write simply $\mathbf{L}_{\mathbf{C}}$.

Definition 3.9. *A losing coalition C is called minimal subcritical if it is subcritical and does not contain proper subcritical subcoalitions.*

$\mathbf{L}_{\mathbf{MC}}(G)$ denotes the set of all subcritical coalitions in the game G . If the game G is fixed, we write simply $\mathbf{L}_{\mathbf{MC}}$.

Proposition 3.1. *The set $\mathbf{W}_{\mathbf{C}}(G)$ ($\mathbf{L}_{\mathbf{C}}(G)$ or $\mathbf{W}_{\mathbf{MC}}(G)$) uniquely defines a simple game G.*

That is, two simple games on N are equal if they have the same sets of critical (subcritical or minimal critical) coalitions.

Proof. Any simple game G is determined by the set $\mathbf{W}(G)$ of all its winning coalitions. In its turn, $\mathbf{W}(G)$ is determined by the set $\mathbf{W}_C(G)$ and thus, by $\mathbf{W}_{MC}(G)$ because $\mathbf{W}(G) = \{ A \; ; \exists B \in \mathbf{W}_{MC}(G) \; (B \subseteq A) \}$.

At the same time, if two games on N have equal sets of critical coalitions, then their sets of subcritical coalitions are also equal. Thus, if we suppose that there are two different games $G1$ and $G2$ that have equal sets $\mathbf{L}_C(G1)$ and $\mathbf{L}_C(G2)$, then they have different sets $\mathbf{W}_C(G1)$ and $\mathbf{W}_C(G2)$. It is possible to assume also that $\mathbf{W}_C(G1) \subseteq \mathbf{W}_C(G2)$ because the set $\mathbf{L}_C(G)$ is obtained from the set $\mathbf{W}_C(G)$ by deleting consequently one member from each of its coalitions.

Both sets $\mathbf{W}_C(G1)$ and $\mathbf{W}_C(G2)$ are closed with respect to supercoalitions. Consequently, at least one coalition $A \in \mathbf{W}_C(G2) \setminus \mathbf{W}_C(G1)$ has to be subcritical in $\mathbf{L}_C(G1)$. However, $\mathbf{L}_C(G1) = \mathbf{L}_C(G2)$ and $\mathbf{L}_C(G2) \cap \mathbf{W}_C(G2) = \varnothing$. It means that such cannot exist and $\mathbf{W}_C(G1) = \mathbf{W}_C(G2)$. As it is demonstrated, it implies equality of the games $G1$ and $G2$.

Proposition is proved.

Remark 3.4. The set $\mathbf{L}_{MC}(G)$ does not define the game G uniquely, as is demonstrated by the following example.

Example 3.2. Let us consider two voting games $G1$ and $G2$. Each of them has five players: α, β, γ, δ, and ε. They are defined by their sets of winning coalitions: $\mathbf{W}(G1) = \{(\alpha, \beta, \varepsilon), (\gamma, \delta, \varepsilon), (\alpha, \beta, \gamma, \varepsilon), (\alpha, \beta, \delta, \varepsilon), (\alpha, \gamma, \delta, \varepsilon), (\beta, \gamma, \delta, \varepsilon), (\alpha, \beta, \gamma, \delta, \varepsilon)\}$ and $\mathbf{W}(G2) = \mathbf{W}(G1) \cup \{ (\alpha, \beta, \gamma, \delta) \}$. Then $\mathbf{L}_C(G1) = \{(\alpha, \beta), (\gamma, \delta), (\alpha, \varepsilon), (\beta, \varepsilon), (\gamma, \varepsilon), (\delta, \varepsilon), (\alpha, \gamma, \varepsilon), (\alpha, \delta, \varepsilon), (\alpha, \gamma, \varepsilon), (\beta, \gamma, \varepsilon), (\alpha, \beta, \gamma, \delta), (\beta, \gamma, \varepsilon), (\beta, \delta, \varepsilon), (\alpha, \delta, \varepsilon), (\beta, \delta, \varepsilon), (\alpha, \beta, \gamma), (\alpha, \beta, \delta), (\alpha, \gamma, \delta), (\beta, \gamma, \delta)\}$ and $\mathbf{L}_C(G2) = \{ (\alpha, \beta), (\gamma, \delta), (\alpha, \varepsilon), (\beta, \varepsilon), (\gamma, \varepsilon), (\delta, \varepsilon), (\alpha, \gamma, \varepsilon), (\alpha, \delta, \varepsilon), (\alpha, \gamma, \varepsilon), (\beta, \gamma, \varepsilon), (\beta, \gamma, \varepsilon), (\beta, \delta, \varepsilon), (\alpha, \delta, \varepsilon), (\beta, \delta, \varepsilon), (\alpha, \beta, \gamma), (\alpha, \beta, \delta), (\alpha, \gamma, \delta), (\beta, \gamma, \delta) \}$. Thus, $\mathbf{L}_{MC}(G1) = \mathbf{L}_{MC}(G2) = \{ (\alpha, \beta), (\gamma, \delta), (\alpha, \varepsilon), (\beta, \varepsilon), (\gamma, \varepsilon), (\delta, \varepsilon) \}$ while the game $G1$ is not equal to the game $G2$.

Let us assume that $|N| = n > 1$.

Theorem 3.1. *The number* 2^{n-1} *is the exact upper bound for the number of subcritical coalitions, i.e.,* $| \mathbf{L}_C | \leq 2^{n-1}$ *and there is a simple game G on N for which the equality holds.*

Proof. We will show that there is a one-to-one mapping h: $\mathbf{L_C} \rightarrow C_o(N) \setminus \mathbf{L_C}$. As $|C_o(N)| = 2^n$, we have that $2|\mathbf{L_C}| \leq |C_o(N)| = 2^n$. Consequently, $|\mathbf{L_C}| \leq 2^{n-1}$.

To do this, let us consider a mapping w: $\mathbf{L_C} \rightarrow \mathbf{W_C}$ that is defined as follows:

For any $C \in \mathbf{L_C}$, let us put $w(C) = C \cup \{i\}$ where i is some swinger for C . If this mapping is one-to-one, then the first statement of the theorem is proved.

Let us assume that this mapping w is not one-to-one. Then there are, at least, two coalitions C_1 and C_2 that have the same image D in $\mathbf{W_C}$. That is $w(C_1) = w(C_2) = D$. We show how to change the mapping w to another mapping w_1 in such a way that C_1 and C_2 will have different images. By the definition of w, $w(C_1) = C_1 \cup \{i\} = w(C_2) = C_2 \cup \{j\} = D$ for some i and j from N.

Let us take the coalition $A = C_1 \cap C_2$. If $A \notin \mathbf{L_C}$, then we put $w_1(C_1) = A$. When $A \in \mathbf{L_C}$, there is some k from N such that $A \cup \{i\} = B \in \mathbf{W}$. As $k \notin D$ and $i, j \notin B$, we have $B \cup D \neq B$ and $B \cup D \neq D$. For all other elements from N but C_2, we put $w_1(C_1) = w(C_2)$. Repeating this process, we either get for an image of C_2 some E that does not belong to or $\mathbf{L_C}$ or $\mathbf{M_C}$. In both cases, E is not an image of any other coalition from $\mathbf{L_C}$. Thus, we decrease the number of coincidences in the image of $\mathbf{L_C}$.

Repeating this process, we build the necessary one-to-one mapping h: $\mathbf{L_C} \rightarrow C_o(N) \setminus \mathbf{L_C}$. This concludes the proof of the first part of the theorem.

Thus, 2^{n-1} is a bound for number of subcritical coalitions in N . To demonstrate that this is the exact bound, let us consider the following situation.

If a game G on N has one dictator (who is denoted, for example, by 1), then all coalitions in this game are divided into two equal classes: one of them consists of all coalitions without the player 1 and the second consists of all coalitions containing the player 1. The first class is the class of all subcritical coalitions and the second class is the class of all winning coalitions. As each of these classes contains 2^{n-1} elements, theorem is proved.

Remark 3.5. The number of all losing coalitions in a game G is usually larger and may be as large as $2^n - 1$. This is just the case when there is only one winning coalition, N.

4. An Enhanced Banzhaf Power Index

Let G be a game with the set N containing n players.

Definition 4.1 (Banzhaf, 1965; 1968; Coleman, 1971). *A player i is called a swinger for a losing coalition C, if $C \cup \{ i \} \in$* **W**.

In this case, transition from the coalition C to $C \cup \{ i \}$ is called a swing of i and is denoted by the pair $(C, C \cup \{ i \})$ (as in (Dubey and Shapley, 1979)) or by the triple (C, i, D) where $D = C \cup \{ i \}$.

Definition 4.2. *A player i is called a downswinger for a winning coalition B, if $B \setminus \{ i \} \in$* **L**.

In this case, transition from the coalition B to $B \setminus \{ i \}$ is called a downswing of i and is denoted by the pair $(B, B \setminus \{ i \})$ or by the triple (B, i, E) where $E = B \setminus \{ i \}$.

Lemma 4.1. *A player i is a swinger for a losing coalition C if and only if i is a downswinger for a winning coalition $C \cup \{ i \}$.*

Corollary 4.1. *For any player i from N, the number of his swings with respect to losing coalitions is equal to the number of his downswings with respect to winning coalitions.*

Lemma 4.2. a) *A losing coalition C has a swinger if and only if it is subcritical.*

b) *A winning coalition C has a downswinger if and only if it is critical.*

Lemma 4.3. *Any blocking player is a swinger.*

Definition 4.3. *A player i is called a dummy in a game G if it is not a swinger for any losing coalition C in* $\mathbf{L}(G)$.

Lemma 4.4. *The following statements are equivalent:*

a) *A player i is a dummy in a game G.*

b) *A player i is not a downswinger for any winning coalition A in* $\mathbf{W}(G)$.

c) *A player i is not a member any critical coalition A from* $\mathbf{W}_C(G)$.

Definition 4.4. *Swings (downswings) with respect to a minimal subcritical (critical) coalition are called minimal.*

Lemma 4.5. a) *C is a minimal winning coalition if and only if any player of C is a swinger for it.*

b) *C is a minimal subcritical coalition if and only if there is a winning coalition B containing C such that any player of B is its swinger.*

Definition 4.5. *The Banzhaf-Coleman power index of a player i in the game G is equal to the quantity of coalitions in N for which i is a swinger.*

We denote the Banzhaf-Coleman power index of a player i by $BC_i(G)$, but when the game G is fixed, we write simply BC(i).

Remark 4.1. It is possible to define as the number of swings of i in G.

Remark 4.2. This is an actual (or "raw" in the terminology of (Dubey and Shapley, 1979)) index introduced by Banzhaf (1965; 1968) as a measure for political power. Coleman (1971) introduced two indices proportional to the original Banzhaf power index: $C_1(i) = BC_i(G)/S(G)$ and $C_2(i) = BC_i(G)/2^{n-1}$ where $S(G)$ denotes the number of all swings in G.

From Corollary 4.1, it follows that the Banzhaf-Coleman power index is symmetric. Namely, the result is the same whether we define this index by the number of swings or by the number of down swings of a player.

Definition 4.6. *The weight $w_C(i)$ of a player i in a coalition C is equal to* 0 *when i is not a swinger for C and is equal to* $1/k$ *where k is the number of all swingers for C when i is a swinger for C.*

Remark 4.3. It is necessary to remark that in our terminology only losing coalitions may have swingers, but not winning coalitions.

The weight of a player reflects the following property of coalition formation. If in a coalition C one swinger i may be changed for another swinger j, then it is natural to suppose that these players divide power with respect to this coalition. When there are k swingers, then they have equal power with respect to the coalition C. Taking 1 as the whole power that is demanded to transform C into a winning coalition, we come to the definition of the weight.

Definition 4.7. *The enhanced Banzhaf power index of i in the game G is*

$$BS_i(G) = \sum_{C \subseteq N} w_C(i)$$

When the game G is fixed, we write simply BS(i).

Proposition 4.1. a) $BS(i) \leq BC(i)$.

b) $BS(i) = BC(i)$ *if and only if for any coalition C,* $w_C(i)$ *is either equal to zero or to* 1.

Voting games have different types of players: dictators, blocking players, swingers, dummies, etc. Let us look how power indices characterize these types. It is known (Banzhaf, 1965; Coleman, 1971; Shapley and Shubik, 1954) that a player is a dummy (a swinger) if and only if his corresponding power index is equal (is not equal) to zero. The same is true for the enhanced Banzhaf power index. For dictators and blocking players, characteristic properties of their power indices are more complex.

Proposition 4.2. *The following statements are equivalent:*

a) *Some player i is a blocking player in a game G.*

b) $BS(i) = BC(i) = |\mathbf{W}| = |\mathbf{W}_C|$.

Really, if i is a blocking player in a game G, then this player is a member of all winning coalitions in this game. Besides, any different swings of i bring this player to different winning coalitions. This implies the equality $BC(i) = |\mathbf{W}|$.

In addition to this, any blocking player i cannot be changed for any other swinger j of the same subcritical coalition. Consequently, we have $BS(i) = BC(i) = |\mathbf{W}|$.

As deleting a blocking player from a coalition makes this coalition a losing one, it follows that $|\mathbf{W}| = |\mathbf{W}_C|$.

Besides, the equality $BC(i) = |\mathbf{W}|$ implies that all winning coalitions contain i, and consequently, is a blocking player.

Proposition is proved.

Let A be a coalition and $\mathbf{L}_C(A, G)$ denotes the set of all subcritical coalitions in the game G that contain A.

Proposition 4.3. The following statements are equivalent:

a) *D is a glued blocking coalition in a game G.*

b) *Any player i from D is a blocking player in a game G.*

c) $\mathbf{L}_C(G) = \mathbf{L}_C(D, G) \cup (\cup_{i\in D} \mathbf{L}_C(D_i, G))$ *where* $D_i = D \setminus \{i\}$.

d) *For any player i from C, we have* $BS(i) = BC(i) = |\mathbf{W}| = |\mathbf{W}_C|$.

Theorem 3.1 makes possible to prove the following result.

Proposition 4.4. *The following statements are equivalent:*

a) *Some player i is a dictator in a game G.*

b) $BS(i) = BC(i) = 2^{n-1}$.

c) $|\mathbf{L}| = |\mathbf{L}_C| = 2^{n-1}$.

d) $|\mathbf{W}| = |\mathbf{W}_C| = 2^{n-1}$.

e) $|\mathbf{W}| = |\mathbf{W}_C| = |\mathbf{L}| = |\mathbf{L}_C|$.

Really, if i is a dictator in a game G, then all coalitions in this game are divided into two equal classes: of all subcritical coalitions and of all winning coalitions. As each of these classes contains 2^{n-1} elements. Thus, $|\mathbf{L}| = |\mathbf{L_C}| = 2^{n-1}$, $|\mathbf{W}| = |\mathbf{W_C}| = 2^{n-1}$, and BS($i$) = BC($i$) = 2^{n-1} because the dictator i is a single swinger in this game.

Let us suppose that BS(i) = BC(i) = 2^{n-1}. Then i is a unique swinger for all subcritical coalitions because the number of these coalitions is less or equal to 2^{n-1} by Theorem 3.1. Thus, i is a blocking player in the game G. Then, by Proposition 4.2.b, the number of all winning coalitions is equal to2^{n-1}. Consequently, the empty coalition $\varnothing$ is subcritical and $\{i\}$ is a winning coalition. It means that i is a dictator in the game G.

Each of the conditions $|\mathbf{L}| = |\mathbf{L_C}| = 2^{n-1}$ and $|\mathbf{W}| = |\mathbf{W_C}| = 2^{n-1}$ implies that the empty coalition $\varnothing$ is subcritical and consequently, i is a dictator in the game G.

Finally, when $|\mathbf{W}| = |\mathbf{L}| = |\mathbf{L_C}|$ or $|\mathbf{W_C}| = |\mathbf{L}| = |\mathbf{L_C}|$, we have $|\mathbf{W}| = |\mathbf{W_C}| = 2^{n-1}$ or what is equivalent, $|\mathbf{L}| = |\mathbf{L_C}| = 2^{n-1}$.

Proposition 4.3 is proved.

Proposition 4.5. *If i is dummy in G from* **E**(*N*), *then* BS(i) = BC(i) = 0.

For the BS power index, we introduce the vector index BS(G) = (BS(1), BS(2), ... , BS(n)) of the game G and the distribution of power indices **DBC** = { BC(1) : BC(2) : ... : BC(n) }, **DBS** = { BS(1) : BS(2) : ... : BS(n) }, and **DSS** = {SS(1) : SS(2) : ... : SS(n) }.

Let us consider some examples comparing the new indices with the original Banzhaf-Coleman and Shapley-Shubik power indices.

Example 4.1. Let N consists of five players: α, β, δ, γ, and ε. The player α has 10 votes, the player β has 2 votes, each of the players δ, γ, and ε has 1 vote. To take an action (make a decision), it is necessary to have 12 votes. In this game, political power indices have the following values: BS(α) = BC(α) = 12, while BC(β) = 4, BS(β) = 2, BC(δ) = BC(γ) = BC(ε) = 3, and BS(δ) = BS(γ) = BS(ε) = 2/3. We see that those players that are swingers for the same critical coalition share political power and consequently, the values of their BS indices are less than the values of their BC indices. It is interesting that the differences between the values of indices in the case of BS index become greater that in the case of the BC index.

Example 4.2. Let us consider a 9-person tricameral assembly that has been popular in the game theoretic literature (Shapley and Shubik, 1954; Shapley, 1977). The voting body of this game

G consists of three chambers: A, B, and C. The first chamber A has a single player α. The second chamber B has three players β1, β2 and β3. The third chamber D has five players δ1, δ2, δ3, δ4, and δ5. All players in each chamber have the same number of votes (e.g., one vote). To take an action (make a decision), it is necessary to have a majority of votes in all three chambers. In this game, the Banzhaf-Coleman power index has the following values: BC(α) = 64 , BC(βi) = 32 , and BC(δj) = 24 for all i = 1, 2, 3 and all j = 1, 2, 3, 4, 5. At the same time, BS power index has the following values: BS(α) = 64 , BS(βi) = 16 , and BS(δj) = 8 for all i = 1, 2, 3 and all j = 1, 2, 3, 4, 5.

To compare these values with the Shapley-Shubik index, let us take the distribution of power according to these indices. Then **DBC** = {64:32:32:32:24:24:24:24:24} and **DBS** = {64:16:16:16:8:8:8:8:8}. At the same time, for the Shapley-Shubik index, we have (Shapley, 1977) **DSS** = {64:18:18:18:10:10:10:10:10}. We can see that the BS and SS power indices give very close results in this case.

The idea of probabilistic interpretation of power indices (Dubey and Shapley, 1979; Straffin, (1977) implies normalization of indices making their sum equal to 1. In the normalized form, we have **NDSS** = {32/84 : 9/84 : 9/84 : 9/84 : 5/84 : 5/84 : 5/84 : 5/84 : 5/84} and **NDBS** = {32/76 : 8/76: 8/76 : 8/76 : 4/76 : 4/76 : 4/76 : 4/76 : 4/76 } . Thus, the difference between the Shapley-Shubik and BS indices becomes even less.

However, if we calculate the combined indices for separate chambers, we find that the SS index has the highest correspondence to each other values for political power for separate chambers. Really, for the BC index we have: (64 : 96 : 120); for the BS index we have: (64 : 48 : 40); and for the SS index we have: (64 : 54 : 50). This is more realistic, assuming that these chambers are independent. At the same time, the BS index is very close to the SS index in values for separate chambers.

Taking the normalized forms, (32/76 : 24/76 : 20/76) for the BS index and (32/84 : 27/84 : 25/84) for the SS index, we obtain even more similarity. For example, calculating the difference between values for the second chamber for both cases, we see that it is less than 0.01. Actually, (27/84) - (24/76) = 3/432 < 1/140.

Remark 4.4. An important property of the Banzhaf-Coleman power index is that its values are natural numbers. This is not true for the BS index, as is demonstrated by Example 4.1. As we will see, some other properties of these two indices are also different.

5. Axioms for the Enhanced Banzhaf Power Index

Before starting our search for axioms for the BS power index, let us look what properties it has in common with the initial BC power index. One of such properties, which are included into the set of axioms for the BC index (Dubey and Shapley, 1979), is the Dubey lattice axiom:

for any games G and H from $E(N)$, $f_i(G \wedge H) + f_i(G \vee H) = f_i(G) + f_i(H)$.

Here: $G \vee H$ denotes the union of two games G and H; $G \wedge H$ denotes the intersection of two games G and H.

This axiom is true for many power indices (Straffin, 1980) including the BC power index (Dubey and Shapley, 1979) and SS power index (Dubey, 1975). However, the following example demonstrates that the BS power index does not satisfy the Dubey lattice axiom.

Example 5.1. Let us consider two voting games $G1$ and $G2$. Each of them has three players: α, β, and γ. They are defined by their winning coalitions: $\mathbf{W}(G1) = \{ (\alpha, \beta), (\alpha, \beta, \gamma) \}$ and $\mathbf{W}(G1) = \{ (\alpha, \gamma), (\alpha, \beta, \gamma) \}$. Then by definition (cf., (Dubey and Shapley, 1979)), $\mathbf{W}(G1 \vee G2) = \mathbf{W}(G1) \vee \mathbf{W}(G2)$ and $\mathbf{W}(G1 \wedge G2) = \mathbf{W}(G1) \wedge \mathbf{W}(G2)$. Consequently, we have $\mathbf{W}(G1 \vee G2) = \{ (\alpha, \beta), (\alpha, \gamma), (\alpha, \beta, \gamma) \}$ and $\mathbf{W}(G1 \wedge G2) = \{ (\alpha, \beta, \gamma) \}$.

Easy calculations show that $BS_\alpha(G1) = 2$, $BS_\alpha(G2) = 2$, $BS_\alpha(G1 \vee G2) = 3$, $BS_\alpha(G1 \wedge G2) = 1$, and $BS_\beta(G1) = 2$, $BS_\beta(G2) = 1$, $BS_\beta(G1 \vee G2) = 1.5$, $BS_\beta(G1 \wedge G2) = 1$. Thus, $BS_\alpha(G1) + BS_\alpha(G2) = 4 = BS_\alpha(G1 \vee G2) + BS_\alpha(G1 \wedge G2) = 4$, while $BS_\beta(G1) + BS_\beta(G2) = 3 \neq BS_{\beta G1 \vee G2}(G1 \vee G2) + BS_\beta(G1 \wedge G2) = 2.5$. It means that the Dubey lattice axiom is not valid for the enhanced Banzhaf power index.

Thus, to obtain an axiomatic representation for BS index, we need to introduce additional structures. Consequently, we define the i-completion iG of a game G, and two types of game groups that are called B-sets and CB-sets.

Definition 5.1. *A game H is called one-generated if there is some coalition $D \subseteq N$ such that*

$$v(A) = \begin{cases} 1 & \text{if } A \supseteq D; \\ 0 & \textit{otherwise}. \end{cases}$$

Remark 5.1. Such games form a basis of the linear space of all games on N. The dimension of this space is equal to $2^n - 1$.

Definition 5.2. a) *The coalition D is called the generator of the game H.*

b) *All subcritical coalitions C of the game H are called roots of this game.*

Lemma 5.1. a) *The generator of a one-generated game H is its single critical coalition.*

b) *All roots of a one-generated game H form the set of minimal subcritical coalitions of H.*

Example 5.2. Let us consider a voting game G with three players: α, β, and γ, and two winning coalitions: $\mathbf{W}(G) = \{ (\alpha, \beta), (\alpha, \beta, \gamma) \}$. Then by definition, the coalition (α, β) is the generator and (α) and {β } are the roots of the game G.

Definition 5.3. *A system of games* $\mathbf{H} = \{H_t;\ t\in I\}$ *is called a* B-*set if all* H_t *are one generated and have a common root.*

Lemma 5.2. *A common root of a non-trivial, i.e., having more then one element, B-set is unique.*

Proof. Let us assume that games H_1 and H_2 belong a B-set $\mathbf{H} = \{ H_t;\ t\in I\}$, C_1 is a root of H_1, D_i is the generator of the game H_i, (i = 1,2), and $C \neq C_1$ is a common root of H_1 and H_2 . From a definition of a root, it follows that for some winning coalitions D_1 and D_2 we have $D_1 = C\cup\{i\}$ and $D_2 = C\cup\{j\}$ for some i and j from N. Besides, there is some k in N such that $D_1 = C_1\cup\{k\}$because $C_1 \neq C = D_1 \setminus \{i\}$.

Let us suppose that C_1 is also a common root of H_1 and H_2 . Then there is some h in N such that $D_2 = C_1\cup\{h\}$. As $D_1 = C \cup\{i\}$, D_1 contains i . At the same time, $D_1 = C_1\cup\{k\}$ and $k \neq i$. Consequently, C_1 contains i . As $D_1 = C\cup\{i\}$, C does not contain i , otherwise $D_1 = C$. Consequently, D_2 does not contain i because $D_2 = C\cup\{j\}$ and $j\neq i$.

However, we have $D_2 = C_1\cup\{h\}$ and C_1 contains i. It means that D_2 contains C_1 and thus, contains i . This contradiction shows that C_1 cannot be a common root of H_1 and H_2 .

A common root of a B-set **H** is a common root of each pair of games that belong to **H**. Thus, **H** has only one common root.

Lemma is proved.

Definition 5.4. *The union* $H = \vee_{t\in I}\ H_t$ of a B-*set* $\mathbf{H} = \{H_t;\ t\in I\}$ *is called a bush.*

Definition 5.5. *The common root C of all games* H_t *is called the root of the bush H* .

Proposition 5.1. *If $H = \vee_{t \in I} H_t$ is a bush and i is a swinger of its root C, then the set* $\mathbf{L}_{\mathrm{Mc}i}$ *of minimal subcritical coalitions for which i is a swinger consists of a single coalition C, which is the root of H.*

Really any bush has a single minimal subcritical coalition.

Corollary 5.1. *If $H = \vee_{t \in I} H_t$ is a bush, then any element of N has at most one minimal swing in the game H.*

Proposition 5.2. *A game H is a bush if and only if all its minimal critical coalitions are one-element extensions of one and the same subcritical coalition.*

Definition 5.6. *The i-completion iH of the union of games $H = \vee_{t \in I} H_t$ is the game G for which* $\mathbf{W}(G) = \mathbf{W}(H) \cup \{ D \in 2^N;\ D = A \setminus \{i\},\ A \in \mathbf{W}(H),\ D \notin \cup_{t \in I} \mathbf{L}_{\mathrm{MC}}(H_t) \}$.

Example 5.3. Let us consider two one-generated voting games $G1$ and $G2$. Each of them has five players: α, β, γ, δ, and ε. They are defined by their generators: $G1$ by (α, β) and $G1$ by (α, γ). $G1$ has two roots: (α) and (β). $G2$ also has two roots: (α) and (γ). Their unique common root is (α). Then by definition of the union of games (cf., (Dubey and Shapley, 1979)), we have $\mathbf{W}(G1 \vee G2) = \mathbf{W}(G1) \vee \mathbf{W}(G1)$ = { (α, β) , (α, γ) , (α, β, γ) , (α, β, δ) , (α, γ, δ) , (α, β, γ, δ) , (α, β, ε) , (α, γ, ε) , (α, β, γ, ε) , (α, β, δ, ε) , (α, γ, δ, ε) , (α, β, γ, δ, ε) } and $\mathbf{L}_{\mathrm{C}}(G1 \vee G2)$ = { (α) , (β), (γ) , (β, γ) , (β, δ) , (γ, δ) , (β, γ, δ) , (β, ε) , (γ, ε) , (β, γ, ε) , (β, δ, ε) , (γ, δ, ε) , (β, γ, δ, ε), (α, δ) , (α, ε) , (α, δ, ε) } . Here, β is a swinger for the following coalitions: (α), (α, δ) , (α, ε) , and (α, δ, ε) .

At the same time, by the definition, the β-completion $\beta(G1 \vee G2)$ of the union of games $G1$ and $G2$ has the following set of winning coalitions $\mathbf{W}(\beta(G1 \vee G2))$ = { (α, β) , (α, γ) , (α, β, γ) , (α, β, δ) , (α, γ, δ) , (α, β, γ, δ) , (α, β, ε) , (α, γ, ε) , (α, β, γ, ε) , (α, β, δ, ε) , (α, γ, δ, ε) , (α, β, γ, δ, ε), (α, δ) , (α, ε) , (α, δ, ε) }. At the same time, $\mathbf{L}_{\mathrm{C}}(\beta(G1 \vee G2))$ = {(α) , (β), (γ) , (β, γ) , (β, δ) , (γ, δ) , (β, γ, δ) , (β, ε) , (γ, ε) , (β, γ, ε) , (β, δ, ε) , (γ, δ, ε) , (β, γ, δ, ε), (δ) , (ε) , (δ, ε) }. Thus, β becomes a swinger only for one coalition (α), which is the common root for games $G1$ and $G2$.

This property is a stable regularity for *i*-completions as it is demonstrated by the following result.

Proposition 5.3. *If $H = \vee_{j \in I} H_j$ is a bush, then any element i of N has at most one swing in iH and this swing is a minimal one.*

Informally, the operation i-completion of the union of games $H = \vee_{t \in I} H_t$ excludes all swings of the player i that are not minimal.

Corollary 5.2. *If* $H = \vee_{j \in I} H_j$ *is a bush, then* $BC_i(iH) \leq 1$.

Corollary 5.3. *If* $H = \vee_1^m H_j$ *and all* H_j *are such bushes that i is a swinger for their roots, then* $BC_{iH}(i) = m$ *, i.e., the number of all swings of i in iH is equal to the number of bushes.*

Really, by Proposition 5.3, in each bush H_j , i has at most one swing, while by initial condition on H_j this swing actually exists.

Corollary 5.4. *If* $H = \vee_1^m H_t$ *and all* H_t *are such bushes that i is a swinger for their roots, then* $OBS_H(i) = m$ *, i.e., the number of all minimal swings of i in H is equal to the number of bushes.*

Definition 5.7. *A system of games* $\mathbf{H} = \{H_t\,;\ t \in I\}$ *on N is called a* CBi-*set if all* H_t *are bushes with different roots and* $i(\vee_1^m H_t) = \vee_1^m H_t$.

Definition 5.8. *A system of games* $\mathbf{H} = \{H_t;\ t \in I\}$ *on N is called a* CB-*set if it is a* CBi-*set for all* $i \in N$.

Let $G = (N, v, \{1, 0\})$ be an arbitrary simple game.

Definition 5.9. *The game* $\mu G = F = (M, u, \{1, 0\})$ *is called the compactification of the game G if* $M = (\cup_{A \in \mathbf{W}_C(G)} A) \setminus (\cap_{A \in \mathbf{W}_C(G)} A)$ *and* $\mathbf{W}_C(F) = \{ B \setminus (\cap_{A \in \mathbf{W}_C(G)} A);\ B \in \mathbf{W}_C(G)\}$.

From the definition, it follows that G defines μG uniquely and there is an epimorphism of G onto μG.

Example 5.4. Let us consider the following proper simple game G with five players: α, β, γ, δ, and ε. It is defined by its set of winning coalitions $\mathbf{W}(G) = \{(\alpha, \beta, \varepsilon), (\gamma, \delta, \varepsilon), (\alpha, \beta, \gamma, \varepsilon), (\alpha, \beta, \delta, \varepsilon), (\alpha, \gamma, \delta, \varepsilon), (\beta, \gamma, \delta, \varepsilon), (\alpha, \beta, \gamma, \delta, \varepsilon)\}$. Then $\mathbf{W}_C(G) = \{(\alpha, \beta, \varepsilon), (\gamma, \delta, \varepsilon)\}$, $\cup_{A \in \mathbf{W}_C(G)} A = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, $\cap_{A \in \mathbf{W}_C(G)} A = \{\varepsilon\}$, the set M of the players from the game μG is equal to $\{\alpha, \beta, \gamma, \delta\}$, and $\mathbf{W}_C(\mu G) = \{(\alpha, \beta), (\gamma, \delta)\}$. In this case, μG is not a proper game.

Here are some properties of compactifications that will be useful in what follows.

Lemma 5.3. $\cap_{A \in \mathbf{W}_C(G)} A = \cap_{A \in \mathbf{W}(G)} A$.

Lemma 5.4. $\mathbf{W}_C(\mu G) = \{ B \cap M;\ B \in \mathbf{W}_C(G)\}$.

Lemma 5.5. $\mathbf{W}(\mu G) = \{ B \cap M;\ B \in \mathbf{W}(G)\}$.

Lemma 5.6. *If G is a one generated game, then* μG *has only one player (for example, a),* $\mathbf{L}_C(\mu G) = \{\varnothing\}$, $\mathbf{W}(\mu G) = \mathbf{W}_C(\mu G) = \{(a)\}$.

Lemma 5.7. If $H = \vee_1^m H_j$ *is a bush, then* μH has m players (*let it be players* 1,2, … , m), is also a *bush, and* $\mathbf{L}_C(\mu G) = \{\varnothing\}$, $\mathbf{W}_C(\mu G) = \{(1), (2), \dots, (m)\}$.

At first, we give axiomatic characterization of BS power index for all simple games, and then for all proper simple games.

Let E_s be the set of all simple games with finite number of players and $\delta(i, X)$ be a characteristic function of membership, that is the function in which the first argument is an element, the second argument is a set, and which is defined in the following way:

$$\delta(i, X) = \begin{cases} 1 & \text{when } i \in X; \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 5.1. *There is a unique function* $f: E_s \rightarrow R^n$ *that satisfies the following axioms:*

BS 1: *For any simple game G with N players,* $F(G) = \Sigma_{i \in N} f_i(G) = |\mathbf{L}_C(G)|$.

BS 2: *For any simple game G with N players and any permutation* π *of* N, $f_{\pi(i)}(\pi G) = f_i(G)$.

BS 3: $f_i(\vee_1^k G_p) = \Sigma_1^k f_i(\mu G_p) \cdot \delta(i, N(\mu G_p))$ *for any* CB-*set* $\mathbf{G} = \{G_p; p = 1, \dots, k\}$.

Moreover, $f(G) = \mathrm{BS}(G)$ *for all G from* E_s. *In other words, axioms* BS 1-3 *give an axiomatic characterization for the enhanced Banzhaf-Coleman power index on* E_s.

Proof. The proof consists of two parts. First, we show that for any bush H the function $f(\mu H)$ is uniquely defined by axioms BS 1 and BS 2 , and this function coincides with the Banzhaf-Coleman power index for μH.

Second, we demonstrate that any simple game is unique union of a BC-set. Then, according to axiom BS 3, we come to the conclusion that the function f, which we have constructed in such a way, is defined uniquely and is equal to the BC power index.

Thus, we begin with a consideration of an arbitrary bush H. If it is the simplest case, then it is a one-generated game. By Lemma 5.6, μG has only one player (for example, a), $\mathbf{L}_C(\mu G) = \{\varnothing\}$, $\mathbf{W}(\mu G) = \mathbf{W}_C(\mu G) = \{(a)\}$. Consequently, by axiom BS 1, $f_a(\mu_w G) = 1$.

If the bush is non-trivial, i.e., $H = \vee_l^m H_j$, then according to the definition of a bush, each H_j is a one-generated game. By Lemma 5.9, μH has m players $\{1,2, \dots, m\}$, is also a bush, and $\mathbf{L}_C(\mu G) = \{\varnothing\}$, $\mathbf{W}_C(\mu G) = \{(1), (2), \dots, (m)\}$. Consequently, by axiom BS 2, $f_i(\mu G) = f_j(\mu G)$ for any two elements i, j from the set $\{1,2, \dots, m\}$. At the same time, by axiom BS 1, $\Sigma_1^m f_i(\mu G_p) = 1$. Thus, $f_i(\mu_w G)$ is uniquely defined and is equal to $1/m$ for all elements i from the set $\{1,2, \dots, m\}$.

To prove the second part of the theorem, let us consider an arbitrary simple game $G = (N, v, \{1, 0\})$ and the set of subcritical coalitions $\mathbf{L_C}(G)$ in G. Then for each $D \in \mathbf{L_C}(G)$, we take all its swingers $S(D) = \{i, j, \dots, t\}$ and build the bush $H = \vee_{j \in I} H_j$ where each $H_j = G_{D_i}$ is a one-generated game with the generator $D_i' = D \cup \{i\}$.

Taking all such bushes, we obtain the set $\{H_D ; D \in \mathbf{L_C}(G)\}$ where H_D is defined by a coalition $D \in \mathbf{L_C}(G)$. From the construction of this set, it follows that $G = \vee_{D \in \mathbf{Lc}(G)} H_D$. This representation is unique for the game G because all bushes have different roots.

Let us prove that if $G = \vee_{D \in \mathbf{Lc}(G)} H_D$, then $i(\vee_{D \in \mathbf{Lc}(G)} H_D) = \vee_{D \in \mathbf{Lc}(G)} H_D$ for any $i \in N$. Operation of i-completion changes the set of subcritical coalitions $\mathbf{L_C}(G)$ making some of them winning coalitions. Let us take such $K \in \mathbf{L_C}(G)$ that in the process of i-completion becomes a winning coalition. As $K \in \mathbf{L_C}(G)$, then by Definition 3.8, there is some coalition $A \in \mathbf{W}(G)$, for which the equality $A = K \cup \{i\}$ holds and $K \notin \cup_{D \in \mathbf{LC}(G)} \mathbf{L_{MC}}(H_D)$. However, K is a minimal subcritical coalition in H_K because $K \in \mathbf{L_C}(G)$ and is the root of the bush H_K. Consequently, the coalition K cannot be changed by i-completion. As K has been taken arbitrarily from $\mathbf{L_C}(G)$, we come to the conclusion that $i(\vee_{D \in \mathbf{LC}(G)} H_D) = \vee_{D \in \mathbf{LC}(G)} H_D$. Thus, by Definition 5.7, $\{H_D ; D \in \mathbf{L_C}(G)\}$ is a BCi-set. As the player i is chosen arbitrarily from the set N, Definition 5.8 implies that $\{H_D ; D \in \mathbf{L_C}(G)\}$ is a BC-set and this representation of the game G is unique.

This concludes the proof of the second part of Theorem 5.1, and as consequence, the whole proof.

Theorem 5.1 characterizes the BS power index on the set of all simple games. It does not give such characterization for the set of all proper simple games because μG is not a proper game in a general case. Thus, to find a similar axiomatic characterization for the smaller set of all proper simple games, we have to change axiom BS 3. This becomes possible when we introduce a weak compactification of a game. .

Let $G = (N, v, \{1, 0\})$ be an arbitrary simple game.

Definition 5.10. *The game $\mu_w G = K = (Q, w, \{1, 0\})$ is called a weak compactification of the game G* if $Q = (\cup_{A \in \mathbf{Wc}(G)} A) \setminus X$, $\mathbf{W_C}(F) = \{B \setminus X;\ B \in \mathbf{W_C}(G)\}$, *and X consists of all elements from $(\cap_{A \in \mathbf{Wc}(G)} A)$ but one.*

From the definition, it follows that in contrast to μG , G does not define $\mu_w G$ uniquely. However, there is always an epimorphism of G onto $\mu_w G$.

Example 5.5. Let us consider the following proper simple game G with five players: α, β, γ, δ, and ε. It is defined by its set of winning coalitions $\mathbf{W}(G) = \{ (\alpha, \beta, \varepsilon), (\alpha, \gamma, \varepsilon), (\alpha, \delta, \varepsilon), (\alpha, \beta, \gamma, \varepsilon), (\alpha, \beta, \delta, \varepsilon), (\alpha, \gamma, \delta, \varepsilon), (\alpha, \beta, \gamma, \delta, \varepsilon) \}$. Then $\mathbf{W_C}(G) = \{ (\alpha, \beta, \varepsilon), (\alpha, \gamma, \varepsilon), (\alpha, \delta, \varepsilon) \}$, $\cup_{A \in \mathbf{W_C}(G)} A = \{\alpha, \beta, \gamma, \delta, \varepsilon \}$, $\cap_{A \in \mathbf{W_C}(G)} A = \{\alpha, \varepsilon \}$, the set M of the players from the game $\mu_w G$ is equal to $\{\alpha, \beta, \gamma, \delta \}$, and $\mathbf{W_C}(\mu_w G) = \{(\alpha, \beta), (\alpha, \gamma), (\alpha, \delta) \}$. In contrast to μG, $\mu_w G$ is a proper game.

However, it is also possible to take as the weak compactification $\mu_w G$ the game with the set of players $\{ \beta, \gamma, \delta, \varepsilon \}$, and $\mathbf{W_C}(\mu_w G) = \{(\beta, \varepsilon), (\gamma, \varepsilon), (\delta, \varepsilon) \}$.

Remark 5.2. To avoid ambiguity in the definition of the weak compactification $\mu_w G$, in what follows, we join to the set of players of the game μG such a player from $\cap_{A \in \mathbf{W_C}(G)} A$ that has the least number in this set and such a way we obtain the set of players of the game $\mu_w G$.

Here are some properties of compactifications that will be useful in what follows.

Lemma 5.8. *If G is a one generated game, then either G has a dictator (let it be a) and in this case, $\mu_w G$ has only one player a,* $\mathbf{L_C}(\mu_w G) = \{\varnothing \}$, $\mathbf{W}(\mu_w G) = \mathbf{W_C}(\mu_w G) = \{ (a) \}$, *or $\mu_w G$ has two players (let it be a and b) and in this case, $\mu_w G$ has only one player a,* $\mathbf{L_C}(\mu_w G) = \{(a) \}$, $\mathbf{W}(\mu_w G) = \mathbf{W_C}(\mu_w G) = \{ (a, b) \}$.

Lemma 5.9. *If $H = \vee_l^m H_l$ is a bush, then $\mu_w H$* has $m + 1$ players (let it be $1, 2, \dots, m, m + 1$), *is also a bush, and* $\mathbf{L_C}(\mu_w H) = \{C = (i), D ;$ *D is an arbitrary subset of the set* $\{1, 2, \dots, i-1, i+1, \dots, m, m+1\} \}$, *here C is the common root of the bush $\mu_w H$,* $\mathbf{W_C}(\mu_w H) = \{ D \cup \{i\} ;$ *D is an arbitrary subset of the set* $\{1, 2, \dots, i-1, i+1, \dots, m, m+1\} \}$.

Let E_{ps} be the set of all proper simple games with finite number of players.

Theorem 5.2. *There is a unique function f: $E_{ps} \rightarrow R^n$ that satisfies the following axioms:*

BS 1a: *For any proper simple game G with N players,* $F(G) = \Sigma_{i \in N} f_i(G) = | \mathbf{L_C}(G) |$.

BS 2a: *For any proper simple game G with N players and any permutation π of N,* $f_{\pi(i)}(\pi G) = f_i(G)$.

BS 3a: $f_i(\vee_1^k G_p) = \Sigma_1^k f_i(\mu_w G_p)\cdot\delta(i, N(\mu G_p))$ *for any such* CB-*set* $\mathbf{G} = \{ G_p; p = 1, \ldots, k \}$ *that* $\vee_1^k G_p$ *is a proper game.*

Moreover, $f(G) = \mathrm{BS}(G)$ *for all G from* $\mathbf{E}_{ps}$ *. In other words, axioms* BS 1a-3a *give an axiomatic characterization for the enhanced Banzhaf-Coleman power index on* $\mathbf{E}_{ps}$.

Proof. The proof, like in the case of Theorem 5.1, consists of two parts. At first, we show that for any bush H the function $f(\mu_w H)$ is uniquely defined by axioms BS 1a and BS 2a , and this function coincides with the Banzhaf-Coleman power index for $\mu_w H$.

Then we use the assertion that any simple game is the unique union of a BC-set as it is demonstrated in the proof of Theorem 5.1. This makes possible, according to axiom BS 3a, to come to the conclusion that the function f, which is determined by axioms BS 1a-3a, is defined uniquely and is equal to the BC power index.

Let H be a bush. If it is the simplest case, then it is a one-generated game. This game either has a dictator or not. In the first case (cf. Lemma 5.8), we come to the conclusion that $\mu_w G$ has only one player a, $\mathbf{L_C}(\mu_w G) = \{\varnothing\}$, and $\mathbf{W}(\mu_w G) = \mathbf{W_C}(\mu_w G) = \{ (a) \}$. Then by axiom BS 1a, we have $f_a(\mu_w G) = 1$.

In the second case (cf. Lemma 5.8), $\mu_w G = \{ (a, b), v, \{1, 0\} \}$, $v((a)) = v((b)) = 0$ and $v((a, b)) = 1$. Consequently, by axiom BS 2a, $f_a(\mu_w G) = f_b(\mu_w G)$, while by axiom BS 1a, $f_a(\mu_w G) + f_b(\mu_w G) = 2$. Thus, $f_a(\mu_w G) = 1$ and $f_b(\mu_w G) = 1$.

Now we can consider nontrivial bush $H = \vee_1^m H_j$. In it, according to the definition of a bush, each H_j is a one-generated game. By Lemma 5.9, we have $\mu_w H = (\{ 1,2, \ldots , m, m + 1\}, w, \{1,0\})$, $\mathbf{L_C}(\mu_w H) = \{C = (i), D ; D$ is an arbitrary subset of the set $\{1, 2, \ldots , i - 1, i + 1, \ldots , m, m + 1\} \}$, here C is the common root of the bush $\mu_w H$. $\mathbf{W_C}(\mu_w H) = \{ D \cup \{i\}$; D is an arbitrary subset of the set $\{1, 2, \ldots , i - 1, i + 1, \ldots , m, m + 1\} \}$.

Let G_D be a simple game generated by the coalition $D \cup \{i\}$. Then, $\{ \mu_w H$, G_D ; D is an arbitrary subset of the set $\{1, 2, \ldots , i - 1, i + 1, \ldots , m, m + 1\} \}$ is a BC-set. Let $\mathbf{K}$ be the set of all such coalitions D. Then applying axiom BS 3a to the element i , we have $f_i(\mu_w H) = \Sigma_{D\in\mathbf{K}} f_i(\mu_w G_D)$ because $i \notin N(\mu_w H)$ and $i \in N(\mu_w G_D)$ for all D from $\mathbf{K}$.

As it is demonstrated above for an arbitrary one-generated game, $f_i(\mu_w G_D) = 1$ for all D from $\mathbf{K}$. Consequently, $f_i(\mu_w H) = 2^m$.

Simultaneously, $|\mathbf{L_C}(\mu_w H)| = 2^m + 1$. As a result, we derive from these equalities and axiom BS 1a that

$$\Sigma_1^{\,i-1} f_j(\mu_w H) + \Sigma_{i+1}^{\,m-1} f_j(\mu_w H) = 1 \,. \qquad (1)$$

At the same time, by axiom BS 2a,

$$f_j(\mu_w H) = f_k(\mu_w H) \qquad (2)$$

for all elements j and k from the set $\{1, 2, \ldots, i-1, i+1, \ldots, m, m+1\}$.

Equalities (1) and (2) imply that $f_j(\mu_w H) = 1/m$ for all elements j from the set $\{1, 2, \ldots, i-1, i + 1, \ldots, m, m+1\}$.

This concludes the first part of the proof of Theorem 5.2 because values of the function f are determined uniquely and are equal to the corresponding values of the Banzhaf-Coleman power index.

As the second part of the proof of Theorem 5.2 is actually similar to second part of the proof of the Theorem 5.1, this also concludes the whole proof of Theorem 5.2.

Remark 5.3. The axiomatizations of the BS power index that are obtained in Theorems 5.1 and 5.2 are in some aspects similar to the axiomatization of Banzhaf-Coleman power index given by Dubey and Shapley (1979). Another axiomatization of Banzhaf-Coleman power index was obtained by Owen (1982). Based on linear and sequential compositions of games, the set of Owen's axioms B1-B5 defines the Banzhaf-Coleman power index uniquely up to the two other power indices: null and dictator indices. Adding up the axiom A2 from (Dubey and Shapley, 1979) to the axioms B1-B5 from (Owen, 1982) gives a complete characterization of the Banzhaf-Coleman power index on the set of simple games. It might be an interesting problem to find a similar characterization for the enhanced Banzhaf-Coleman power index.

6. Independence of Axioms for Power Indices

Having an axiomatic representation of some structure, a natural problem arises in mathematics whether the given axioms are independent. In other words, the question is if it is possible to find for one of the axioms its proof from the other axioms from the list. In such a way, one of the properties of the structure in question might be eliminated from the list of its axioms.

In some cases, important results may evolve from investigation of independence problems. The most luminous result was the discovery of non-Euclidean geometries in the process of attempts to prove the fifth postulate of Euclid basing on his first four axioms.

In our case, we come to an interesting question whether the axioms characterizing the BC, SS, and BS indices are independent. Here are some results in this direction.

As it is demonstrated in (Dubey, 1975) and (Dubey and Shapley, 1979), systems of axioms characterizing BC and SS indices coincide in all axioms but the second one, A2. Consequently, we have the following result.

Proposition 6.1. *Axiom* A2 *is independent of axioms* A1, A3, *and* A4.

What concerns the independence of axiom A1 about dummies, we see that in the characterization of BS index we do not need this axiom because it is deduced from axioms BS 1-BS 3. It is possible to suggest that the same is true for the BC and SS indices. However, it is not the case for the BC index as the following result demonstrates.

Proposition 6.2. *Axiom* A1 *is independent of the axioms* A2 - A4 *for the* BC *index.*

Proof. To make our exposition complete in itself, we give below axioms A1-A4 from (Dubey and Shapley, 1979). Let G be an arbitrary simple game with N players.

A 1: *If i is dummy in G, then* $f_i(G) = 0$.

A 2: $\sum_{i\in N} f_i(G) = S(G)$ *where* $S(G)$ *is the number of swings in the game* G.

A 3: *For any permutation* π *of* N, $f_{\pi(i)}(\pi G) = f_i(G)$.

A 4: *For any games* G *and* H *on* N, $f_i(G) + f_i(H) = f_i(G\vee H) + f_i(G\wedge H)$.

To prove the independence of A1 we consider the set of all simple games with three players and build a function on this set that satisfies axioms A2-A4 but does not satisfy axiom A1.

Let $N = \{a, b, c\}$. There are eleven proper simple games on this set: $G_\wedge$, G_{ab}, G_{ac}, G_{bc}, G_a, G_b, G_c, G^a, G^b, G^c, $G_\vee$ and seven improper games $G^c{}_{ab}$, $G^b{}_{ac}$, $G^a{}_{bc}$, G^{ab}, G^{ac}, G^{bc}, $G^\vee$. They are defined by the sets of their winning coalitions: $\mathbf{W}(G_\wedge) = \{(a, b, c)\}$, $\mathbf{W}(G_{ab}) = \{(a, b), (a, b, c)\}$, $\mathbf{W}(G_{ac}) = \{(a, c), (a, b, c)\}$, $\mathbf{W}(G_{bc}) = \{(b, c), (a, b, c)\}$, $\mathbf{W}(G_a) = \{(a, b), (a, c), (a, b, c)\}$, $\mathbf{W}(G_b) = \{(a, b), (b, c), (a, b, c)\}$, $\mathbf{W}(G_c) = \{(a, c), (b, c), (a, b, c)\}$, $\mathbf{W}(G^a) = \{(a), (a, b), (a, c), (a, b, c)\}$, $\mathbf{W}(G^b) = \{(b), (a, b), (b, c), (a, b, c)\}$, $\mathbf{W}(G^c) = \{(c), (a, c), (b, c), (a, b, c)\}$, $\mathbf{W}(G_\vee)$

$=\{(a, b), (a, c), (b, c), (a, b, c)\}$, and $\mathbf{W}(G^{c}_{ab}) = \{(c), (a, c), (b, c), (a, b), (a, b, c)\}$, $\mathbf{W}(G^{b}_{ac}) = \{(b), (a, b), (b, c), (a, c), (a, b, c)\}$, $\mathbf{W}(G^{a}_{bc}) = \{(a), (a, b), (a, c), (b, c), (a, b, c)\}$, $\mathbf{W}(G^{ac}) = \{(a), (c), (a, c), (b, c), (a, b), (a, b, c)\}$, $\mathbf{W}(G^{ab}) = \{(a), (b), (a, c), (b, c), (a, b), (a, b, c)\}$, $\mathbf{W}(G^{bc}) = \{(b), (c), (a, c), (b, c), (a, b), (a, b, c)\}$, $\mathbf{W}(G^{\vee}) = \{(a), (b), (c), (a, c), (b, c), (a, b), (a, b, c)\}$.

We begin with calculation of the values of the Banzhaf-Coleman power index for these games:

$BC_a(G_{\wedge}) = BC_b(G_{\wedge}) = BC_c(G_{\wedge}) = 1;$

$BC_a(G_{ab}) = BC_b(G_{ab}) = 2,\ BC_c(G_{ab}) = 0;$

$BC_a(G_b) = BC_c(G_b) = 1,\ BC_b(G_b) = 3;$

$BC_a(G_{\vee}) = BC_b(G_{\vee}) = BC_c(G_{\vee}) = 2;$

$BC_a(G^{\vee}) = BC_b(G^{\vee}) = BC_c(G^{\vee}) = 1;$

$BC_a(G^{ab}) = BC_b(G^{ab}) = 2,\ BC_c(G^{ab}) = 0;$

$BC_a(G^{a}) = 4,\ BC_b(G^{a}) = BC_c(G^{a}) = 0;$

$BC_a(G^{a}_{bc}) = 3,\ BC_b(G^{a}_{bc}) = BC_c(G^{a}_{bc}) = 1.$

For all other games on the set of three players, the values of the Banzhaf-Coleman power index are derived using axiom A3 from the given set of values because all games the values of the BC index of which are not in this list may be obtained from the considered set of games by permutations of the set $N = \{a, b, c\}$.

Here we define a function f that is different from the BC index:

$f_a(G_{\wedge}) = f_b(G_{\wedge}) = f_c(G_{\wedge}) = 1;$

$f_a(G_{ab}) = f_b(G_{ab}) = 1,\ f_c(G_{ab}) = 2;$

$f_a(G_b) = f_c(G_b) = 2,\ f_b(G_b) = 1;$

$f_a(G_{\vee}) = f_b(G_{\vee}) = f_c(G_{\vee}) = 2;$

$f_a(G^{\vee}) = f_b(G^{\vee}) = f_c(G^{\vee}) = 1;$

$f_a(G^{ab}) = f_b(G^{ab}) = 2,\ f_c(G^{ab}) = 0;$

$f_a(G^{a}) = 2,\ f_b(G^{a}) = f_c(G^{a}) = 1;$

$f_a(G^{a}_{bc}) = 3,\ f_b(G^{a}_{bc}) = f_c(G^{a}_{bc}) = 1.$

All other values of the function f may be obtained using permutations of the set N and axiom A3.

As the function f corresponds non-zero values to dummies in some games, it does not satisfy axiom A1.

Let us show that so defined function f satisfies axioms A2-A4 from (Dubey and Shapley, 1979). As for each game, the sums of the function f and BC index are the same and the BC index satisfies axiom A2, f also satisfies axiom A2.

Direct proof shows that so defined function f is stable with respect to all permutations of the set $N = \{a, b, c\}$ and thus, f satisfies axiom A3.

To test axiom A4, we need all identities with operations $\vee$ and $\wedge$ involving all games on the set $N = \{a, b, c\}$. Let x, y, z be different elements from the set $\{a, b, c\}$. Then we have the following identities:

$G_{xy} = G_{yx}$; $G^{xy} = G^{yx}$; $G^{z}_{xy} = G^{z}_{yx}$;

$G_{\wedge} \wedge G_x = G_{\wedge} \wedge G_{xy} = G_{\wedge} \wedge G^{x} = G_{\wedge} \wedge G^{xy} = G_{\wedge} \wedge G^{\vee} = G_{\wedge} \wedge G^{x}_{yz} = G_{xy} \wedge G_{yz} = G_{xy} \wedge G_z = G_{\wedge}$;

$G_{\wedge} \vee G_x = G_x$; $G_{\wedge} \vee G_{xy} = G_{xy}$; $G_{\wedge} \vee G^{x} = G^{x}$; $G_{\wedge} \vee G^{x}_{yz} = G^{x}_{yz}$;

$G_{xy} \vee G_{yz} = G_y$; $G_{xy} \vee G_x = G_x$; $G_{xy} \vee G^{x} = G_x \vee G^{x} = G^{x}$; $G_{xy} \vee G_z = G_x \vee G_y = G_{\vee}$;

$G_{xy} \wedge G^{x} = G_{xy}$; $G_{\vee} \wedge G^{x} = G_x \wedge G^{x} = G_x$; $G_{xy} \wedge G_x = G_x \wedge G_y = G_{xy}$; $G_{yz} \wedge G^{x} = G_{\wedge}$;

$G^{x}_{yz} \vee G^{x} = G^{x}_{yz} \vee G_{yz} = G^{x}_{yz} \vee G_x = G^{x}_{yz} \vee G_y = G^{x}_{yz}$; $G_{yz} \vee G^{x} = G_y \vee G^{x} = G^{x}_{yz}$;

$G^{x}_{yz} \wedge G^{x} = G^{x}$; $G^{x}_{yz} \wedge G_{yz} = G_{yz}$; $G^{x}_{yz} \wedge G_x = G_x$; $G^{x}_{yz} \wedge G_y = G_y$; $G^{x}_{yz} \wedge G^{y}_{xz} = G_{\vee}$;

$G^{xy} \wedge G^{yz} = G^{y}_{xz}$; $G^{xy} \wedge G^{x}_{yz} = G^{x}_{yz}$; $G^{xy} \wedge G^{z}_{xy} = G_{\vee}$; $G^{xy} \wedge G^{z} = G_z$;

$G^{x}_{yz} \vee G^{y}_{xz} = G^{xy} \vee G^{x}_{yz} = G^{xy} \vee G^{x} = G^{xy} \vee G_{yz} = G^{xy} \vee G_x = G^{xy} \vee G_y = G^{xy}$; $G^{x}_{yz} \vee G^{y}_{xz} = G^{xy}$;

$G^{\vee} \vee G^{xy} = G^{\vee} \vee G^{x}_{yz} = G^{\vee} \vee G^{x} = G^{\vee} \vee G_{yz} = G^{\vee} \vee G_x = G^{xy} \vee G^{z}_{xy} = G^{xy} \vee G^{z} =$
$G^{xy} \vee G^{yz} = G^{\vee}$;

$G^{\vee} \wedge G^{xy} = G^{xy}$; $G^{\vee} \wedge G^{x}_{yz} = G^{x}_{yz}$; $G^{\vee} \wedge G^{x} = G^{x}$; $G^{\vee} \wedge G_{yz} = G_{yz}$; $G^{\vee} \wedge G_x = G_x$.

Having these identities, we can check validity of the Dubey lattice axiom A4 for the function f. The results are given in the following table 1. It contains only non-trivial equalities implied by the written above identities and axiom A4. Trivial ones (such as $f_a(G^{a}) + f_a(G^{a}_{bc}) = f_a(G^{a}) + f_a(G^{a}_{bc})$) and those that can be obtained through a permutation of players are not included in the table.

An equality that is implied by axiom A4	**The corresponding values of the function *f***
$f_a(G_a) + f_a(G_b) = f_a(G_{\vee}) + f_a(G_{ab})$	$1 + 2 = 2 + 1$
$f_c(G_a) + f_c(G_b) = f_c(G_{\vee}) + f_c(G_{ab})$	$2 + 2 = 2 + 2$
$f_a(G_{ab}) + f_a(G_{bc}) = f_a(G_{\wedge}) + f_a(G_b)$	$1 + 2 = 1 + 2$
$f_b(G_{ab}) + f_b(G_{bc}) = f_b(G_{\wedge}) + f_b(G_b)$	$1 + 1 = 1 + 1$
$f_a(G^a) + f_a(G_{bc}) = f_a(G_{\wedge}) + f_a(G^a{}_{bc})$	$2 + 2 = 1 + 3$
$f_b(G^a) + f_b(G_{bc}) = f_b(G_{\wedge}) + f_b(G^a{}_{bc})$	$1 + 1 = 1 + 1$
$f_a(G^a) + f_a(G_{\vee}) = f_a(G_a) + f_a(G^a{}_{bc})$	$2 + 2 = 1 + 3$
$f_b(G^a) + f_b(G_{\vee}) = f_b(G_a) + f_b(G^a{}_{bc})$	$1 + 2 = 2 + 1$
$f_a(G_{ab}) + f_a(G_c) = f_a(G_{\wedge}) + f_a(G_{\vee})$	$1 + 2 = 1 + 2$
$f_c(G_{ab}) + f_c(G_c) = f_c(G_{\wedge}) + f_c(G_{\vee})$	$2 + 1 = 1 + 2$
$f_a(G^{ab}) + f_a(G^c) = f_a(G_c) + f_a(G^{\cdot})$	$2 + 1 = 2 + 1$
$f_c(G^{ab}) + f_c(G^c) = f_c(G_c) + f_c(G^{\cdot})$	$0 + 2 = 1 + 1$
$f_a(G^{ab}) + f_a(G^{bc}) = f_a(G^b{}_{ac}) + f_a(G^{\cdot})$	$2 + 0 = 1 + 1$
$f_c(G^{ab}) + f_c(G^{bc}) = f_c(G^b{}_{ac}) + f_c(G^{\vee})$	$0 + 2 = 1 + 1$
$f_a(G^a{}_{bc}) + f_a(G^b{}_{ac}) = f_a(G_{\vee}) + f_a(G^{ab})$	$3 + 1 = 2 + 2$
$f_c(G^a{}_{bc}) + f_c(G^b{}_{ac}) = f_c(G_{\vee}) + f_c(G^{ab})$	$1 + 1 = 2 + 0$
$f_a(G^{ab}) + f_a(G^c{}_{ab}) = f_a(G_{\vee}) + f_a(G^{\cdot})$	$2 + 1 = 2 + 1$
$f_c(G^{ab}) + f_c(G^c{}_{ab}) = f_c(G_{\vee}) + f_c(G^{\cdot})$	$0 + 3 = 2 + 1$

Table 1. Equalities for the function *f* that are induced by the axiom A4.

The results from the table demonstrate validity of the axiom A4 for the function f and thus, they complete the proof of Proposition 6.2.

The proof of Proposition 6.2 makes it possible to obtain the following result.

Proposition 6.3. *Axiom* A4 *is independent of axioms* A1 – A3 *for the BC index.*

Proof. To prove independence of A1, we consider the same set of all simple games with three players as in the proof of Proposition 6.2 and build such a function on this set that satisfies axioms A1-A3 from (Dubey and Shapley, 1979), but does not satisfy axiom A4. However, we define this function in a different way.

Let all denotations be the same as in the proof of Proposition 6.2 and G be an arbitrary simple game on the set $N = \{a, b, c\}$. We define a function g that is different from the BC index as follows:

$$g(G) = \begin{cases} f(G_x) & \text{when } G = G_x \text{ for all } x = a, b, c; \\ \mathrm{BC}(G) & \text{otherwise}. \end{cases}$$

Here f is the function from the proof of Proposition 6.2.

As both the BC power index and the function $f(G)$ satisfy axioms A2 and A3, the same is true for the function $g(G)$.

Besides, everything is fine with dummies, i.e., axiom A1 is also satisfied by the function $g(G)$ because the BC index satisfies axiom A1 and the game G_a on which $g(G)$ differs from the BC index does not have dummies.

At the same time, axiom A4 is not valid for the function $g(G)$ because it is equal to the BC index on all but one game. For example, the first and second equalities from the table 1 are not true for the function $g(G)$.

Proposition 6.3 is proved.

Proposition 6.4. *Axiom* A3 *is independent of axioms* A1, A2, A4 *for the* BC *index.*

Proof. To prove independence of A1, we consider the set of all simple games with two players and build such a function on this set that satisfies axioms A1, A2, A4 from (Dubey and Shapley, 1979), but does not satisfy axiom A3.

Let $N = \{ a, b \}$. There are four simple games on this set: $G_\wedge$, G_a, G_b, and $G_\vee$. They are defined by the sets of their winning coalitions: $\mathbf{W}(G_\wedge) = \{(a, b)\}$, $\mathbf{W}(G_a) = \{(a), (a, b)\}$, $\mathbf{W}(G_b) = \{(b), (a, b)\}$, and $\mathbf{W}(G_\vee) = \{(a), (b), (a, b)\}$.

At first, we calculate the values of the BC power index for these games:

$BC_a(G_\wedge) = BC_b(G_\wedge) = 1$;

$BC_a(G_a) = 2$, $BC_b(G_a) = 0$;

$BC_a(G_b) = 0$, $BC_b(G_b) = 2$;

$BC_a(G_\vee) = BC_b(G_\vee) = 1$.

There are only two non-trivial identities involving these games and lattice operations:

$G_a \wedge G_b = G_\wedge$ and $G_a \vee G_b = G_\vee$.

Thus, we define a function *f* that is different from the BC index as follows:

$f_a(G_\wedge) = 2$, $f_b(G_\wedge) = 0$;

$f_a(G_a) = 2$, $f_b(G_a) = 0$;

$f_a(G_b) = 0$, $f_b(G_b) = 2$;

$f_a(G_\vee) = 0$, $f_b(G_\vee) = 2$.

Direct test shows that this function satisfies axioms A1, A2, A4 from (Dubey and Shapley, 1979), but does not satisfy axiom A3 because a permutation of the set $\{ a, b \}$ does not change the games $G_\wedge$ and $G_\vee$, but interchanges players a and b.

Proposition 6.4 is proved.

Propositions 6.1-6.4 imply the following result.

Theorem 6.1. *Axioms* A1-A4 *from* (Dubey and Shapley, 1979) *characterizing the* BC *power index are independent.*

Remark 6.1. The set of all simple games on a three-element set is the least possible one for proving independence of axiom A4. For simple games on a two-element set, it is possible to characterize the BC power index only by axioms A1 - A3. Really (cf. proof of the Proposition 6.4), the values of the function *f* for the games $G_\cdot$ and $G_\vee$ are defined uniquely by axioms A2 and A3, while the values of the function *f* for the games G_a and G_b are defined uniquely by axioms A1 and A2.

Now, let us look what is the situation with the axioms for the BS power index. While axioms for the BC power index given by Dubey and Shapley (1979) involve only games with the same number of players, axioms for the BS power index (and namely, axiom BS 3) are related to games with different numbers of players. This implies some complications in the case of BS power index. However, we have the following result.

Proposition 6.5. *Axiom* BS 1 *is independent of axioms* BS 2 *and* BS 3 *for the* BS *index.*

Proof. To prove independence of BS 1, we consider the set of all simple games with two players and build such a function on this set that satisfies axioms BS 2 and BS 3 but not BS 1.

At first, we calculate the values of the BS power index for these games:

$BS_a(G_\wedge) = BS_b(G_\wedge) = 1;$

$BS_a(G_a) = 2, BS_b(G_a) = 0;$

$BS_a(G_b) = 0, \ BS_b(G_b) = 2;$

$BS_a(G_\vee) = BS_b(G_\vee) = ½ .$

To define a function f that is different from BC index, we consider a game $G(a)$ on an one-element set $\{ a \}$. In this game, $BS_a(G(a)) = 1$. We change this value for the function $f(G)$ and put $f_a(G_\wedge) = 2$ to be able to exclude axiom BS 1 from the properties of the function $f(G)$. As a result, this function does not satisfy axiom BS 1 and in a trivial way satisfies axioms BS 2 and BS 3 because these axioms do not apply to this function.

To get a non-trivial example we apply axioms BS 2 and BS 3 to find values of the function $f(G)$ for all games on the set $\{ a, b \}$. As a result, we have:

$f_a(G_\wedge) = f_b(G_\wedge) = 4;$

$f_a(G_a) = 4, \ f_b(G_a) = 0 \ ;$

$f_a(G_b) = \ 0, \ f_b(G_b) = 4;$

$f_a(G_\vee) = \ f_b(G_\vee) = 1.$

A direct test shows that this function satisfies axioms BS 2 and BS 3 but not BS 1.

Proposition 6.5 is proved.

Proposition 6.6. *Axiom* BS 2 *is independent of axioms* BS 1 *and* BS 3 *for the* BS *index.*

Proof. We consider the same set of all simple games with two players. However, a function on this set, which satisfies axioms BS 1 and BS 3 but not BS 2, is built in a different way. We define a function g as follows:

$g_a(G_\wedge) = g_b(G_\wedge) = 1;$

$g_a(G_a) = 2, \ g_b(G_a) = 0 \ ;$

$g_a(G_b) = \ 0, \ g_b(G_b) = 2;$

$g_a(G_\vee) = ¼$, $g_b(G_\vee) = ¾$.

A direct test shows that this function satisfies axioms BS 1 and BS 3 but not BS 2 because transposition of the players a and b does not change the game $G_\vee$ but changes the values of the function $g(G)$. This contradicts axiom BS 2.

Proposition 6.6 is proved.

Proposition 6.7. *Axiom* BS 3 *is independent of axioms* BS 1 *and* BS 2 *for* BS *index.*

Proof. To prove this independence, we consider the same set of all simple games with three players as in the proof of Proposition 6.2 and build such a function on this set that satisfies axioms BS 1 and BS 2 but not BS 3.

Taking $N = \{a, b, c\}$ as the set of players we have eighteen simple games: $G_\wedge, G_{ab}, G_{ac}, G_{bc}, G_a, G_b, G_c, G^a, G^b, G^c, G_\vee, G^c_{ab}, G^b_{ac}, G^a_{bc}, G^{ab}, G^{ac}, G^{bc}$, and $G^\vee$.

We begin with the calculation of the values of the BS power index for these games: we have

$BS_a(G_\wedge) = BS_b(G_\wedge) = BS_c(G_\wedge) = 1$;

$BS_a(G_{ab}) = BS_b(G_{ab}) = 2,\ BS_c(G_{ab}) = 0$;

$BS_a(G_b) = BS_c(G_b) = 1/2,\ BS_b(G_b) = 3$;

$BC_a(G_\vee) = BS_b(G_\vee) = BS_c(G_\vee) = 1$;

$BS_a(G^\vee) = BS_b(G^\vee) = BS_c(G^\vee) = 1/3$;

$BS_a(G^{ab}) = BS_b(G^{ab}) = 1,\ BS_c(G^{ab}) = 0$;

$BS_a(G^a) = 4,\ BS_b(G^a) = BS_c(G^a) = 0$;

$BS_a(G^a_{bc}) = 3,\ BS_b(G^a_{bc}) = BS_c(G^a_{bc}) = 1$.

For all other games on the set of three players, the values of BS power index are derived from the given set of values by means of axiom BS 2 because all games where BS index values not in this list may be obtained from those in the list by permutations of the set $N = \{a, b, c\}$.

We define a function g that is different from BC index as follows. If $G \neq G^x$ for some $x = a, b, c$, then $g(G) = BS(G)$. For the game G^x, we put $g_x(G^x) = 2$, $g_y(G^x) = g_z(G^x) = 1$ where $x, y, z \in \{a, b, c\}$ and $x \neq y, x \neq z, y \neq z$.

The BS power index and the function $f(G)$ satisfy axioms BS 1 and BS 2. Besides, both these axioms hold for the function g on the games G_x. Consequently, the same is true for the function $g(G)$.

At the same time, the game G^a is equal to the union $G_\wedge \vee G_{ab} \vee G_{ac} \vee G^a$ of the one-generated games $G_\wedge, G_{ab}, G_{ac}$, and G^a that form a BC set. If g satisfies axiom BS 3, then we would have the equality $g_a(G^a) = g_a(\mu G_\wedge) + g_a(\mu G_{ab}) + g_a(\mu G_{ac}) + g_a(\mu G^a)$. At the same time, $g_a(\mu G_\wedge) = g_a(\mu G_{ab}) = g_a(\mu G_{ac}) = g_a(\mu G^a) = 1$. Thus, by the definition of g the equality is not true and consequently, axiom BS 3 is not valid for g.

Proposition 6.7 is proved.

Propositions 6.5-6.7 imply the following result.

Theorem 6.2. *Axioms* BS 1 - BS 3 *characterizing the* BS *power index on the set of all simple games are independent.*

Similar methods make it possible to obtain the following result.

Theorem 6.3. *Axioms* BS 1a - BS 3a *characterizing the* BS *power index on the set of all proper simple games are independent.*

7. Conclusion

Thus, we have introduced a new power index, which is denoted by BS. This index allows achieving better reflection of political power in voting bodies. Properties of this index reveal essential features of players in voting games. An axiomatic characterization of the BS power index is given and we have proved that these axioms are independent.

In addition we have proved that axioms A1 – A4 for the Banzhaf-Coleman power index, given by Dubey and Shapley (1979), are independent. It is also interesting to consider a similar problem of independence for the axiomatization of the Banzhaf-Coleman power index that was obtained by Owen (1982) as well as for the axioms for the Shapley-Shubik power index, given by Dubey (1975).

Another problem connected with political power indices has a practical issue. It might be useful to find under what conditions one or the other power index is more relevant. Some comparison of Banzhaf-Coleman and Shapley-Shubik in this aspect will be found in (Straffin, 1977).

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