COMPETITIVE PRICING OF PERSONS

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A characterization of perfectly competitive equilibrium called the no-surplus condition developed in Ostroy [1980, 1981], Makowski [1980] and Artzner and Ostroy [1981] is extended in this paper to a fairly broad class of nonatomic economies.

Of principal concern are the connections between the no-surplus condition and

(i) Fréchet differentiability

(ii) Walrasian equilibrium

(iii) sufficient conditions for its realization

(iv) cooperative game-theoretic approaches to perfect competition

(v) the marginal productivity theory of distribution.

Basic to the mathematical exposition is the notion of a direct market in which individuals "sell themselves" directly and an indirect market in which individuals sell various commodities. The mathematical concept of a linear operator is used to establish an equivalence between direct and indirect markets and its adjoint establishes an equivalence between the competitive pricing of persons and commodities.

The term "direct market" appears in Shapley and Shubik [1969] and the construction used below differs from theirs only in permitting a continuum rather than a finite number of individuals/commodities.

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I. PRELIMINARIES

The set of agents is the unit interval, denoted by A, along with its Borel subsets, $\mathcal{A}$, and Lebesgue measure, λ.

It will be useful to regard the measure space of agents as elements of a normed linear space. To this end define for each $E \in \mathcal{A}$ the <u>characteristic function</u> $\chi_E : A \to \mathbb{R}$ where

$$\chi_E(a) = \begin{cases} 1 & \text{if } a \in E \\ 0 & \text{if } a \in A\backslash E. \end{cases}$$

Let

$$\chi = \{\chi_E : E \in A\}.$$

This provides a one-to-one correspondence between elements of χ and $\mathcal{A}$.

To embed χ in a linear space, let $\pi = \{E_i\}$ be a finite set of pairwise disjoint elements of $\mathcal{A}$ whose union is A. Regarded as $\{\chi_{E_i}\}$, a linear combination of the elements of the partition π is an $x : A \to \mathbb{R}$ defined by

$$x = \Sigma\alpha_i\chi_{E_i} ,$$

where α_i are scalars. Let X be the set of all such elements and X_+ the positive convex cone formed when all α_i are restricted to be nonnegative. Thus, X_+ is the smallest convex cone containing χ.

The norm of $x \in X$ is given by

$$|x| = \Sigma|\alpha_i|\lambda(E_i) .$$

Therefore,

$$|\chi_E| = \lambda(E) \quad .$$

(The closure of X is $L^1(A, \mathcal{A}, \lambda)$, the space of Lebesgue integrable functions on A.)

The commodity space will be denoted by Y, also a linear space with norm denoted by $\|\cdot\|$. A <u>norm</u> on Y is a nonnegative function such that

(a) $\|y\| = 0$ only when $y = \underset{\sim}{0}$, the zero element of Y,

(b) absolutely homogeneous: $\|\alpha y\| = |\alpha| \; \|y\|$,

(c) subadditive: $\|y + y'\| \leq \|y\| + \|y'\|$.

A linear transformation, or <u>linear operator</u>, from X to Y is a mapping T that is homogeneous ($\alpha Tx = T\alpha x$) and additive ($T(x+x') = Tx + Tx'$). A linear operator is <u>bounded</u> if there is a scalar $\beta > 0$ such that for all $x \in X$

$$\|Tx\| \leq \beta|x|.$$

When the range of T includes its domain define the <u>identity operator</u> by $T = I$, where $Ix = x$.

Let $\ell(\cdot)$ be a <u>linear functional</u> on X, i.e., a linear operator from X to $\mathbb{R}$. A linear functional ℓ is bounded if there is a β' such that for all $x \in X$, $|\ell(x)| \leq \beta'|x|$. It is well-known that any bounded linear functional on X can be represented by a $p \in L^\infty(A, \mathcal{A}, \lambda)$ -- i.e., for all $x \in X$

$$\ell(x) = \int_A p(a)x(a)d\lambda$$

The norm of p is given by

$$|p|_\infty = \sup_{|x| \leq 1} \frac{|px|}{|x|} .$$

Let $q(\cdot)$ be a linear functional on Y. The set of all bounded linear functionals on Y is denoted by Y^*. The norm of $q \varepsilon Y^*$ is

$$||q||_* = \sup_{||y|| \leq 1} \frac{|q(y)|}{||y||} .$$

Below $q(y)$ will be written as qy just as $\ell(x)$ is written as px.

The space X was introduced to embed the set of agents into a normed linear space. An element $p \varepsilon L^\infty$ is a price vector for the set of agents, where $p(a)$ is the price of agent $a \varepsilon A$. Similarly, q is a price vector for commodities. The goal is to establish a relation between the pricing of commodities and the pricing of persons, i.e., between q and p.

Assume T is bounded. Since $T : X \to Y$ and $q \varepsilon Y^*$, $q(Tx)$ is a linear functional on Y. If $\ell(x)$ is the scalar defined by $(qT)x$, $\ell(x)$ is a linear functional on X. Thus, there is a $p \varepsilon L^\infty$ such that

$$\ell(x) = (qT)x = px.$$

Holding T fixed, $\ell(\cdot) = p$ varies with q. Denote this functional dependence of p on q by the <u>adjoint operator</u> $T^* : Y^* \to L^\infty$ where T^* is defined by the condition that for all x

$$(T^*q)x = (qT)x.$$

Thus, T^* is a mapping from the space of prices for commodities to the space of prices for persons.

II. DIRECT AND INDIRECT MARKETS

Let Y_+ be a positive, convex cone in Y, i.e., $y, y' \varepsilon Y_+$ and $\alpha \geq 0$ imply $(\alpha y + y') \varepsilon Y_+$. Just as $\mathbb{R}^2_+$ is used to denote the relevant commodity space when there are two inputs, Y_+ plays the same role for the commodity space Y.

The function $g : Y_+ \to \mathbb{R}_+$ is a production function from the relevant space of inputs to scalar outputs. It is assumed that for all $\alpha \geq 0$ and $y, y' \varepsilon Y_+$,

(g.1): (positively homogeneous) $g(\alpha y) = \alpha g(y)$,

(g.2): (superadditive) $g(y + y') \geq g(y) + g(y')$,

(g.3): (Lipschitz) There exists $\gamma > 0$ such that

$$|g(y) - g(y')| \leq \gamma\|y-y'\| .$$

REMARK 1: A bounded linear functional is both Lipschitz and positively homogeneous on Y_+. Therefore g differs from a bounded linear functional only be permitting superadditivity rather than additivity. The norm $\|\cdot\|$ is both positively homogeneous and Lipschitz. Therefore g differs from $\|\cdot\|$ by being superadditive rather than subadditive and by the absence of the requirement that $g(y) = 0$ implies $y = \underset{\sim}{0}$. (g.1) and (g.2) imply g is concave.

Let T be a linear operator from X to Y such that

(T.1): $T[\chi] \subset Y_+$

(T.2): T is bounded.

Note that by putting

$$W(E) = T\chi_E ,$$

$W : A \to Y_+$ defines a Y_+-valued measure on A. $W(E)$ describes the initial endowment of inputs held by E.

(T.2) implies that

W is countably additive : if $\{E_m\}$ is a sequence of pairwise disjoint sets in A whose union is E,

$$\lim_k \|W(E) - W(\bigcup_{m=1}^{m=k} E_m)\| = 0; \text{ and,}$$

W is nonatomic : if $\|W(E)\| \neq 0$, there is an

$E' \subset E$, $E' \in A$ such that $0 \neq \|W(E')\| \neq \|W(E)\|$

REMARK 2: If W is countably additive and nonatomic there may be no <u>bounded</u> T such that $W(E) = T\chi_E$. To illustrate suppose $Y = \mathbb{R}$. By the Radon-Nikodym Theorem there is an $x \in L^1(A, \mathbf{A}, \lambda)$ such that $W(E) = \int x\chi_E d\lambda$ but there is no guarantee that $x \in L^\infty(A, A, \lambda)$ as is necessary if T is to be bounded. Thus, (T.2) imposes the requirement that <u>per capita</u> endowments, $\|T\chi_E\|/|\chi_E|$, be bounded. This does not appear to be a serious restriction. It is not, however, imposed in the Aumann [1964] or Vind [1964] formulation of nonatomic economies.

It is useful to distinguish among the following three subsets of the commodity space,

$$T[\chi] \subset T[X_+] \subset Y_+ .$$

$T[\chi]$ is the set of inputs that are actually available to the agents. It is the range of the vector measure $W(E) = T\chi_E$. The convex cone $T[X_+]$ may be regarded as the relevant "subspace" for $T[\chi]$ on which certain mathematical concepts may be defined. Finally, there is Y_+ which may be interpreted as the space of conceivably available, rather than actually available, inputs.

Taking Y_+ as given, define an <u>indirect market</u> by the pair (g,T). Let g_T be the function g restricted to the subset $T[X_+]$ of its domain. To every indirect market (g,T) there corresponds a <u>direct market</u> (f,I) where $f : X_+ \to \mathbb{R}_+$, I is the identity operator on X, and for all $x \in X_+$,

$$f(x) = f(Ix) = g(Tx).$$

f is simply g_T defined on its underlying domain X_+. In an indirect market E sells commodities $T\chi_E$ whereas in the corresponding direct market the members of E sell themselves directly.

The connections between direct and indirect markets are described in Figure 1.

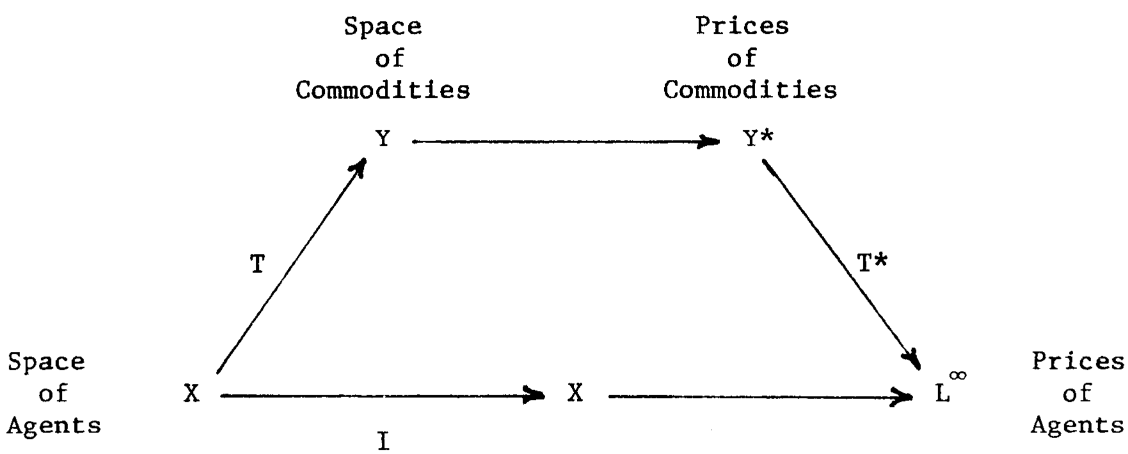

Figure 1

The upper route goes from the space of agents, X, via T to the space of commodity inputs, to the space of prices for those inputs and then, via the adjoint T*, to the prices of agents. This is the path through the indirect market (g,T). The direct market (f,I) takes the more direct lower route.

III. WALRASIAN AND NO-SURPLUS DEFINITIONS OF COMPETITIVE EQUILIBRIUM

The pair (g,T) defines a simple economy in which agents supply inputs, inelastically, to produce output. A distribution of the total output is an additive measure $\mu : \mathcal{A} \to \mathbb{R}_+$ such that

$$\mu(A) = g(T\chi_A)$$

$$\mu(E \cup E') = \mu(E) + \mu(E'), \text{ whenever } E \cap E' = \emptyset$$

These restrictions on μ are assumed throughout.

Equilibrium in (g,T) amounts to a definition of what constitutes an equilibrium distribution. In this section the Walrasian and no-surplus definitions are given.

A Walrasian equilibrium (WE) for (g,T) is a pair (μ,q) where $q \in Y^*$ such that

WE.1: $\quad q T\chi_E = \mu(E),\ E \in \mathcal{A}$

WE.2: $\quad g(T\chi_A) - qT\chi_A \geq g(y) - qy, \quad y \in Y_+$

(WE.2) says that taking WE prices, q, as given, profit-maximizing demands for inputs equal their supply, $T\chi_A$. (WE.1) says that output, assumed to have a price of unity, is distributed according to the WE prices of inputs.

It follows from the homogeneity of g that (WE.2) is equivalent to

WE.2a: $\quad qT\chi_A = g(T\chi_A)$

WE.2b: $\quad qy \geq g(y), \quad y \in Y_+$

This, in turn, is equivalent to the condition that q belongs to the subdifferential of g on Y_+ at $T\chi_A$ defined by

$$\partial g(T\chi_A) = \{q \in Y^* : g(T\chi_A) - q(T\chi_A - y) \geq g(y),\ y \in Y_+\}.$$

Since the condition (WE.1) describing μ is derived from q it suffices to define a WE for (g,T) as

WE_q: $\quad q \in \partial g(T\chi_A)$.

Thus, a WE is defined as a weak derivative (subderivative of a concave function) of g at $T\chi_A$.

A definition of WE entirely in terms of μ may also be obtained. If $q \in \partial g(T\chi_A)$, then by definition of the adjoint T^* and the definition of the direct market (f,I) associated with (g,T),

$$T^*q \in \partial f(\chi_A) = \{p \in L^\infty : f(\chi_A) - p(\chi_A - x) \geq f(x),\quad x \in X_+\}.$$

<u>If</u> q <u>represents WE prices for commodities in the indirect market</u> (g,T), T^*q <u>represents WE prices for persons in the corresponding direct market</u>.

Putting $\mu(E) = p\chi_E$, where $p \in \partial f(\chi_A)$, it is easily verified that the distribution of output defined by p satisfies for all $\pi = \{E_i\}$ and $\alpha_i \geq 0$,

WE_μ: $\quad \Sigma\alpha_i\mu(E_i) \geq f(\Sigma\alpha_i\chi_{E_i})$.

Therefore, if $q \in \partial g(T\chi_A)$ there is a μ satisfying WE_μ. The converse is

THEOREM 1: <u>If</u> μ <u>satisfies</u> WE_μ <u>there is a</u> $p \in \partial f(\chi_A)$ <u>such that</u> $\mu(E) = p\chi_E$ <u>and a</u> $q \in T^{*-1}p$ <u>such that</u> $q \in \partial g(T\chi_A)$.

To prove the Theorem it must be established that

LEMMA: $\partial g(T\chi_A) \neq \emptyset$.

PROOF: Let $\bar{y} = T\chi_A$ and $\bar{\tau} = g(\bar{y})$. Define

$$B_g = \{(\tau,y) : g(y) \geq \tau, \ (\tau,y) \in \mathbb{R} \times Y_+\},$$

$$B = \{(\tau',y') = (\bar{\tau},\bar{y}) + \alpha[(\tau,y) - (\bar{\tau},\bar{y})]: \alpha \geq 0, \ (\tau,y) \in B_g\}.$$

By (g.1-2) B_g is a convex cone in $\mathbb{R} \times Y_+$ and by construction B is a convex cone in $\mathbb{R} \times Y$. Setting $\alpha = 1$ reveals that $B_g \subset B$ and setting $\alpha = 0$ reveals that $(\bar{\tau},\bar{y})$ is a boundary point of B. (According to Dunford and Schwartz [1957, 451-452], B would be defined as the cone with vertex $(\bar{\tau},\bar{y})$ generated by B_g.)

1. It will be demonstrated that

$$(\bar{\tau}+\gamma,\bar{y}) \notin \text{cl } B,$$

where cl $\equiv$ closure and γ is the parameter defining the Lipschitz condition, (g.3). It suffices to show that for any $y \in Y_+$ and $\alpha \geq 0$ such that

$$\|[\bar{y} + \alpha(y-\bar{y})] - \bar{y}\| = \|\alpha(y - \bar{y})\| \leq \delta < 1,$$

there is the inequality,

$$|[\bar{\tau} + \alpha(g(y)-g(\bar{y}))] - (\bar{\tau} + \gamma)| = |g(\alpha y) - g(\alpha\bar{y}) - \gamma| > \frac{\gamma(1-\delta)}{2} .$$

This follows from the Lipschitz condition,

$$|g(\alpha y) - g(\alpha\bar{y})| \leq \gamma \|\alpha(y-\bar{y})\| \leq \gamma\delta.$$

2. As a consequence of 1., there is by the Separation Theorem (Dunford and Schwartz, 417-418) an $(\xi,q) \in \mathbb{R} \times Y^*$, $(\xi,q) \neq (0,\underset{\sim}{0})$, separating the point $(\bar{\tau}+\gamma,\bar{y})$ from cl B. Thus, for all $y \in Y_+$ and $\alpha \geq 0$,

$$\xi(\bar{\tau}+\gamma) + q\bar{y} > \beta = \sup_{\substack{\alpha \geq 0 \\ (\tau,y) \in B_g}} \xi(\bar{\tau}+\alpha[\tau-\bar{\tau}]) + q(\bar{y} + \alpha[y-\bar{y}])$$

$$\geq \xi\bar{\tau} + q\bar{y}.$$

3. To show that $\beta = \xi\bar{\tau} + q\bar{y}$, suppose the contrary that a $(\tau,y) \in B_g$ exists such that $\xi\tau - qy > \xi\bar{\tau} + q\bar{y}$, or

$$\xi(\tau-\bar{\tau}) - q(y-\bar{y}) > 0.$$

Since B is a cone $(\bar{\tau} + \alpha[\tau-\bar{\tau}], \bar{y} + \alpha[y-\bar{y}]) \in B$ for all $\alpha \geq 0$. But the above inequality would make

$$\xi(\bar{\tau}+\alpha[\tau-\bar{\tau}]) + q(\bar{y}+\alpha[y-\bar{y}])$$

arbitrarily large when α is arbitrarily large, contradicting the existence of the sup = β.

4. To show that $\beta = 0$, note that $g(\underset{\sim}{0}) = 0$ and therefore $\beta \geq 0$. If β were positive, homogeneity, (g.1), would make it unbounded. By the inequality in 2., $\xi(\bar{\tau}+\gamma) + q\bar{y} > \xi\bar{\tau} + q\bar{y} = 0$. Thus, $\xi > 0$.

5. Setting $q' = -\xi^{-1}q$, it follows from 3. and 4. that

$$q'T\chi_A = q'\bar{y} = \bar{\tau} = g(\bar{y})$$

$$q'y \geq g(y)$$

The inequality is obtained by setting $\alpha = 1$. Thus, $q' \in \partial g(T\chi_A)$.

PROOF OF THEOREM 1: From the assumptions that g is Lipschitz and T is bounded,

$$\infty > \sup_{|\chi_E|>0} \left\{ \frac{g(T\chi_A) - g(T\chi_{A\setminus E})}{|\chi_E|} = \frac{f(\chi_A) - f(\chi_{A\setminus E})}{\lambda(E)} \right\}.$$

From the hypothesis of the Theorem, $\mu(A\setminus E) \geq f(\chi_{A\setminus E})$. Since $\mu(A) = f(\chi_A)$,

$$\infty > \frac{f(\chi_A) - f(\chi_{A\setminus E})}{\lambda(E)} \geq \frac{\mu(A) - \mu(A\setminus E)}{\lambda(E)} = \frac{\mu(E)}{\lambda(E)} \geq 0.$$

The Radon-Nikodym Theorem (Halmos [1950, 128]) along with the above inequality implies the existence of a $p \in L^\infty$ such that

$$\mu(E) = p\chi_E.$$

Therefore, $p\chi_A = f(\chi_A)$ and $p(\Sigma\alpha_i\chi_{E_i}) \geq f(\Sigma\alpha_i\chi_{E_i})$, or

$$p \in \partial f(\chi_A).$$

Define

$$\partial g_T(T\chi_A) = \{q'\varepsilon Y^* : g(T\chi_A) - q'T(\chi_A - x) \geq g(Tx), \quad x \in X_+\}.$$

By construction if $p \in \partial f(\chi_A)$ and $q' \in T^{*-1}p$, then $q' \in \partial g_T(T\chi_A)$. This follows from

$$q'(Tx) = (T^*q')x = px.$$

For any $q' \in \partial g_T(T\chi_A)$ there exists $q \in \partial g(T\chi_A)$ such that for all $y \in T[X]$,

$$qy = qy'.$$

Combining these results

$$qT\chi_E = q'T\chi_E = (T^*q')\chi_E = p\chi_E = \mu(E).$$

Therefore, the hypothesis on μ implies a p leading to a $q \varepsilon \partial g(T\chi_A)$ that is a WE for (g,T).

The Walrasian definition of equilibrium has individuals responding passively to announced prices rather than bargaining for more favorable terms; and, prices are obtained for all commodities in Y_+, not only those actually available in $T[\chi]$. In the no-surplus approach individuals are relied upon to bargain vigorously for all they can obtain; and, the definition is based entirely on the properties of g on $T[\chi]$. Therefore, it may be described via the corresponding direct market.

Relative to the distribution μ, E contributes

$$\left.\begin{array}{l}\text{a positive surplus}\\ \text{no surplus}\\ \text{a negative surplus}\end{array}\right\} \text{ if } f(\chi_A) - f(\chi_{A\setminus E}) \left\{\begin{array}{l} > \mu(E)\\ = \mu(E)\\ < \mu(E)\end{array}\right.$$

The quantity $f(\chi_A) - f(\chi_{A\setminus E})$ is the addition to total output attributable to E -- the marginal product of E. When this is greater (less) than what it receives, $\mu(E)$, E contributes a positive (negative) surplus to $A\setminus E$. When the amount it receives is exactly equal to the output it adds E is contributing no surplus, i.e., E is extracting all the surplus it contributes.

It would be both unreasonably demanding and inconsistent with the notion of perfect competition to require of an equilibrium μ that it exhibit no surplus for every <u>group</u> E. The smallest amount of superadditivity of f on X_+ would preclude such a distribution. Further, the term "perfectly competitive equilibrium" describes a relation between <u>individuals</u> and the rest of the economy rather than between arbitrary groups E and $A\setminus E$.

Let $\{E_n\}$ be an arbitrary sequence of elements in $\mathcal{A}$ such that $\lambda(E_n) \searrow 0$. Thus $\{\lambda(E_n)\}$ is tending to the scale of an individual agent. The direct market (f,I) exhibits <u>no-surplus</u> (NS) if there exists a distribution μ such that for all $\{E_n\}$,

$$\text{NS:} \qquad \lim \frac{|f(\chi_A) - f(\chi_{A\setminus E_n}) - \mu(E_n)|}{\lambda(E_n)} = 0 .$$

When μ is NS the smaller the group, and therefore the closer it is to the scale of an individual, the more nearly it is extracting all of the <u>per capita</u> surplus attributable to it.

Just as a WE is equivalent to a subderivative of g on Y_+ at $T\chi_A$, it will be shown that a NS distribution is equivalent to a <u>Fréchet</u> derivative of g_T at $T\chi_A$. This is defined as

$$Dg_T(T\chi_A) = \{q \in Y^* : \lim_{\substack{x_n \in X_+ \\ \|T(\chi_A - x_n)\| \to 0}} \frac{|g(T\chi_A) - g(Tx_n) - qT(\chi_A - x_n)|}{\|T(\chi_A - x_n)\|} = 0\}.$$

It is more convenient to work with the corresponding direct market in which the Fréchet derivative of f at χ_A is

$$Df(\chi_A) = \{p \in L^\infty : \lim_{\substack{x_n \in X_+ \\ |\chi_A - x_n| \to 0}} \frac{|f(\chi_A) - f(x_n) - p(\chi_A - x_n)|}{|\chi_A - x_n|} = 0\}.$$

From the definition of f and the assumption that T is bounded,

$$q \in Dg_T(T\chi_A) \quad \text{iff} \quad T^*q \in Df(\chi_A).$$

Because $T[X_+]$ need not be dense in Y_+ or $T\chi_A$ need not be in the relative interior of Y_+, $Dg_T(T\chi_A)$ may contain more than one element. When translated to the direct market, however, this redundancy disappears -- i.e., if $q, q' \in Dg_T(T\chi_A)$, then $|T^*(q-q')|_\infty = 0$.

REMARK 3: The <u>Gâteaux</u> derivative of f at χ_A denoted $\nabla f(\chi_A)$, may be defined by the condition that $\partial f(\chi_A)$ is a singleton. Therefore, a <u>necessary</u> condition for $Dg_T(T\chi_A)$ is that

$$T^*[\partial g_T(T\chi_A)] = \nabla f(\chi_A).$$

If $p = Df(\chi_A)$ it follows immediately that (f,I) exhibits an NS distribution. Simply substitute $\chi_{A\setminus E_n}$ for x_n and $p(\chi_A - \chi_{A\setminus E_n}) = p\chi_{E_n} = \mu(E_n)$ into the definition of $Df(\chi_A)$. Along with a demonstration of the converse, the differentiability properties of concave functions may be further exploited to yield an equivalent definition of perfectly competitive equilibrium in terms of perfectly elastic demand/supply schedules facing individuals. Related equivalences between NS and definitions of perfectly elastic demand/supply appear in Ostroy [1980], Makowski [1980], and Artzner and Ostroy [1981].

Let $p' \in \partial f(\chi_{A\setminus E})$. Therefore,

$$f(\chi_{A\setminus E}) - p'\chi_{A\setminus E} \geq f(x) - p'x, \quad x \in X_+ .$$

Even though x may contain strictly positive quantities of all inputs in A, at the prices p' profits are maximized when no inputs in E are hired. Clearly, the values of $p'(a)$, $a \in E$, must be sufficiently high to make their employment unprofitable, whereas $p'(a)$, $a \in A\backslash E$, must be such as to permit their full employment.

In (g,T) or its associated (f,I) individuals would accept any nonnegative price for their inputs if they could not do any better. (Reservation demands are nil.) In terms of prices, the boundary of bargains between E and $A\backslash E$ for the prices paid to the members of E lies in the interval $[0, p'(a)]$, $a \in E$, where $p' \in \partial f(\chi_{A\backslash E})$. _If_ $p' \in \{\partial f(\chi_A) \cap \partial f(\chi_{A\backslash E})\}$, E may be said to have extracted the most favorable prices it could have obtained. At prices any higher than $p'(a)$, $a \in E$, demand for the inputs owned by E would be zero while at p' they are fully employed. Thus E may be said to face perfectly elastic demands for the inputs it supplies at prices p'.

Of course, not all _groups_ can be expected to face such prices. This would require that $p' \in \{\partial f(\chi_A) \bigcap_{E \in A} \partial f(\chi_{A\backslash E})\}$ which would imply that f is the linear function $f(x) = p'x$. What is required is that each _individual_ face perfectly elastic demands. This will obtain and $p \in \partial f(\chi_A)$ will be said to be _perfectly determinate_ (PD) if for any $\{E_n\}$ and $p^n \in \partial f(\chi_{A\backslash E_n})$,

$$\text{PD:} \qquad \lim |p^n - p|_\infty = 0 .$$

Note the convergence of the prices of persons with respect to the L^∞ norm. Convergence with respect to any less demanding norm such as the L^1 would be too weak to capture the notion of perfectly competitive equilibrium among individuals. This is illustrated in Figure 2.

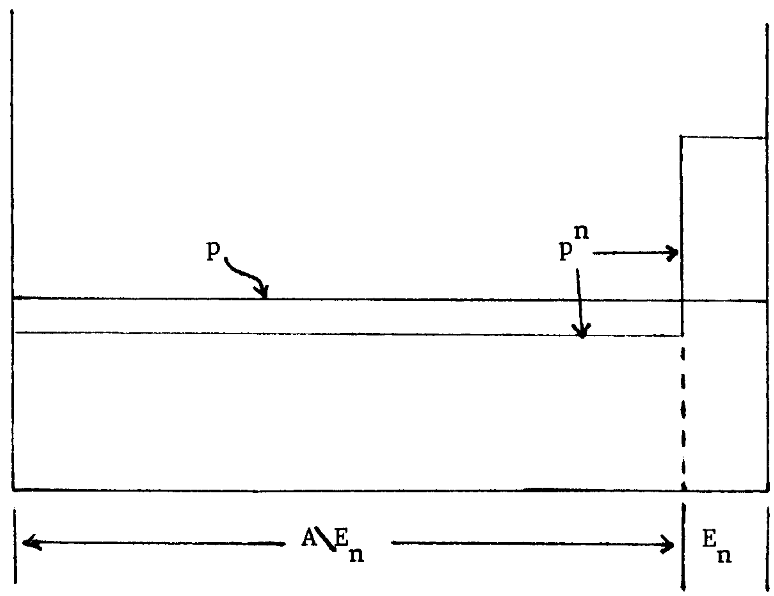

Figure 2

Assume that $\int|p-p^n|d\lambda \searrow 0$. This is obviously a necessary condition for PD but it is not sufficient because it does not preclude the possibility that in $\{p^n\}$ the prices of the persons in E_n remain much higher than they are in p no matter how small is $\lambda(E_n)$. (See example 1, below.)

The summary characterization of the NS definition of equilibrium is

THEOREM 2: _The following are equivalent_:

(i) (f,I) _exhibits an_ NS _distribution_,

(ii) $p = Df(\chi_A)$,

(iii) p is PD.

The following steps prove Theorem 2.

1. (iii) $\rightarrow$ (i) : Let μ be defined by $\mu(E) = p\chi_E$. When p is PD, $p \in \partial f(\chi_A)$ and therefore

$$\frac{f(\chi_A) - f(\chi_{A\setminus E_n}) - p(\chi_A - \chi_{A\setminus E_n})}{|\chi_A - \chi_{A\setminus E_n}|} = \frac{f(\chi_A) - f(\chi_{A\setminus E_n}) - \mu(E_n)}{\lambda(E_n)} \geq 0. \quad (1)$$

Similarly, when $p^n \varepsilon\ \partial f(\chi_{A\setminus E_n})$,

$$\frac{f(\chi_{A\setminus E_n}) - f(\chi_A) - p^n(\chi_{A\setminus E_n} - \chi_A)}{|\chi_A - \chi_{A\setminus E_n}|} \geq 0. \qquad (2)$$

Adding (1) and (2),

$$0 \leq \frac{(p^n-p)(\chi_A-\chi_{A\setminus E_n})}{|\chi_A - \chi_{A\setminus E_n}|} \leq \frac{|p_n-p|_\infty\ \lambda(E_n)}{\lambda(E_n)}. \qquad (3)$$

From PD, $\lim\ |p^n-p|_\infty = 0$. Since (3) dominates (1), its limit is zero. Thus, μ is NS.

2. (ii) $\rightarrow$ (iii) : A theorem of Smulian [1940] (see also Yamamuro [1974, p. 81]) for homogeneous, subadditive functions adapted here to homogeneous, superadditive functions on X_+ is: <u>If</u> $p = Df(\chi_A)$, <u>then for any</u> $\{p^n\}$ <u>such that</u> (a) $p^n\chi_A \rightarrow f(\chi_A)$ <u>and</u> (b) $p^n x \geq f(x)$, $x \varepsilon X_+$, $\lim\ |p^n - p|_\infty = 0$. If $p^n \varepsilon\ \partial f(\chi_{A\setminus E_n})$, p^n fulfills (b) and since $\{p^n\}$ is known to be weak-star convergent to p it fulfills (a). Therefore, p is PD.

3. (i) $\rightarrow$ (ii): If M is a subset of X_+ containing χ_A define

$$Df(\chi_A;M) = \{p \ \varepsilon\ L^\infty : \lim_{\substack{x_n \varepsilon M \\ |\chi_A - x_n| \to 0}} \frac{|f(\chi_A) - f(x_n) - p(\chi_A - x_n)|}{|\chi_A - x_n|} = 0\}.$$

3a. From the Lemma to Theorem 1 applied to f there is a $p\ \varepsilon\ \partial f(\chi_A)$. To show that $p = Df(\chi_A;\chi)$ it suffices to show that $\mu(E) = p\chi_E$ where μ is NS. For any $\{E_n\}$,

$$\lim \frac{f(\chi_A) - f(\chi_{A\setminus E_n}) - p\chi_{E_n}}{\lambda(E_n)} \geq 0.$$

The definition of μ as NS says that when $p\chi_{E_n}$ is replaced by $\mu(E_n)$ the limit is zero. Therefore,

$$\lim \frac{\mu(E_n) - p\chi_{E_n}}{\lambda(E_n)} \geq 0.$$

Since $\{E_n\}$ is arbitrary and $\mu(A) = p\chi_A$, $\mu(E) = p\chi_E$.

Note that $p = Df(\chi_A;\chi)$ implies by substituting $p\chi_A$ for $f(\chi_A)$,

$$\lim \frac{|p\chi_{A\setminus E_n} - f(\chi_{A\setminus E_n})|}{\lambda(E_n)} = 0. \qquad (4)$$

3b. Let $\pi = \{E_i\}$, $i = 1, \ldots, n$.

Define s^π: $[0,1]^n \to \mathbb{R}_+$ and $e = (\alpha_1,\ldots,\alpha_n) \in [0,1]^n$ as

$$s^\pi(e) = p(\Sigma\alpha_i w(E_i)) - f(\Sigma\alpha_i w(E_i)) \geq 0.$$

For fixed χ_{E_i}, $i = 1, \ldots, n$, p and f are functions on $[0,1]^n$; p is linear and therefore convex and f is concave and therefore $-f$ is convex. s^π, being the sum of two convex functions, is also convex.

Let $e^0 = (1,1,\ldots,1)$, $e_j = (0,\ldots,0,1,0,\ldots,0)$ and $e^i = e^0 - e_i$. Then $\{e^0, e^1, \ldots, e^n\}$ are the extreme points of the convex set

$$\Delta^n = \{e \in [0,1]^n : e = (\alpha_1,\ldots,\alpha_n), \ \Sigma(1 - \alpha_i) \leq 1\}.$$

The maximum of a convex function on a convex set occurs at an extreme point. (Rockafeller [1970, p. 344]). Therefore, since $s^\pi(e) \geq s^\pi(e^0) = 0$,

$$\max_{e\in\Delta} s^\pi(e) = \max_{1\leq i\leq n} s^\pi(e^i) = \max_{1\leq i\leq n} p(\chi_{A\setminus E_i}) - f(\chi_{A\setminus E_i}).$$

Let $\{x_n = \Sigma\alpha_{i(n)}\chi_{E_{i(n)}}\}$ be a sequence in $<\chi>$ converging to χ_A such that $\lambda(E_{i(n)}) = n^{-1}\lambda(A) = n^{-1}$ and

$$\lim \frac{|\chi_A - x_n|}{n^{-1}} = \lim \frac{|\Sigma(1-\alpha_{i(n)})|}{n^{-1}} \neq 0.$$

Letting $\pi(n) = \{E_{i(n)}\}$, $s^{\pi(n)}(e^{i(n)}) = p\chi_{A\setminus E_{i(n)}} - f(\chi_{A\setminus E_{i(n)}})$; and for some $e \in \Delta^n$, $s^{\pi(n)}(e) = px_n - f(x_n)$. Therefore,

$$\lim \max_{1\leq i(n)\leq n} \frac{p\chi_{A\setminus E_{i(n)}} - f(\chi_{A\setminus E_{i(n)}})}{n^{-1}} \geq \lim \frac{px_n - f(x_n)}{|\chi_A - x^n|} \geq 0 .$$

By (4), this limit is zero.

3c. $p = Df(\chi_A)$: Let X_+^α be the subset of X_+ consisting of elements $x = \Sigma\alpha_i\chi_{E_i}$ such that $\lambda(E_i) = n^{-1}$, $i = 1, \ldots, n$, and $\alpha_i \in [0,\alpha]$, $\alpha \geq 1$. The argument in 3b. demonstrates that $p = Df(\chi_A;X_+^1)$ and it follows immediately that for any $\alpha \geq 1$, $p = Df(\chi_A;X_+^\alpha)$.

For any $\delta_1 > 0$ there is an α such that if $x' \in X_+$ there is an $x \in X_+^\alpha$ for which $|x' - x| < \delta_1$. For any $\alpha \geq 1$ and $\delta_2 > 0$ there is a δ_3 such that

$$\frac{f(\chi_A) - f(x) - p(\chi_A - x)}{|\chi_A - x|} < \delta_2,$$

whenever $|\chi_A - x| < \delta_3$. This follows from $p = Df(\chi_A;X_+^\alpha)$.

Therefore, if $x' \in X_+$ and $|\chi_A - x'| < \delta_3$,

$$\frac{f(\chi_A) - f(x') - p(\chi_A - x')}{|\chi_A - x'|} < \delta_2 + \delta_1(|p|_\infty + \gamma\beta),$$

where γ is the Lipschitz parameter for (g.3) and β is the bound for T, i.e., $|g(Tx) - g(Tx')| \leq \gamma \|T(x-x')\| \leq \gamma\beta |x-x'| = \gamma\beta\delta_1$. Therefore, $p = Df(\chi_A)$.

IV. NO-SURPLUS: SUFFICIENT CONDITIONS

Since the direct market is derived from some indirect market (g,T) it is the properties of g and T upon which the existence of an NS distribution depends. In Remark 1 it was observed that g differs from a linear function on Y_+ only by permitting superadditivity. Since all linear functions are Fréchet differentiable, the failure of NS can be imputed to the superadditivity of g -- more precisely, the superadditivity of g_T. The purpose of this section is to examine the rather substantial restrictions on the supperadditivity of g_T imposed by the requirement that $Dg_T(T\chi_A)$, and therefore an NS distribution, exists.

From Theorem 2, (g,T) is NS iff for all $\{E_n\}$, $q^n \varepsilon \partial g(T\chi_{A\setminus E_n})$ and $q \varepsilon \partial g(T\chi_A)$,

$$\lim|T^*(q^n-q)|_\infty = 0 .$$

Two quite separate conditions are subsumed by this. The first is,

$$T^*[\partial g(T\chi_A)] = \nabla f(\chi_A).$$

The second is,

$$\{T^* q^n\} \text{ contains a Cauchy subsequence.}$$

They are each necessary and jointly sufficient for the desired conclusion.

Granting $\nabla f(\chi_A)$ (see Remark 6, below), differentiability will depend on the properties of T^*. T^* is said to be <u>compact</u> if it maps bounded sets in Y^* onto sets in L^∞ whose closures are compact. The nonemptiness of $\partial g(y)$ for all y in a Y_+ - neighborhood of $T\chi_A$ (Lemma 1) implies that there is a bounded set $B \subset L^{\infty}$ such that

$$B \cap \partial g(T\chi_{A \setminus E_n}) \neq \emptyset .$$

Therefore,

PROPOSITION 1: <u>If</u> $T^*[\partial g(T\chi_A)] = \nabla f(\chi_A)$, <u>and</u> T^* <u>is compact</u>, (g,T) <u>exhibits NS.</u>

The restrictions on the superaddivity of g_T when T^* is compact will be examined. Translating the problem from the indirect market (g,T) to its associated direct market (f,I), superadditivity implies that for any $\pi = \{E_i\}$

$$\sigma(\pi) = f(\chi_A) - \sum_{E_i \in \pi} f(\chi_{E_i}) \geq 0.$$

The difference, $\sigma(\pi)$, is a measure of the gains in output from the sharing of inputs. In general, the smaller is each $\lambda(E_i)$ the larger the value of $\sigma(\pi)$.

Say that (f,I) exhibits <u>population constant returns</u> (PCR) if for any $\delta > 0$ and $\delta' > 0$ there exists a $\pi = \{E_i\}$ such that

PCR.1: $\lambda(E_i) < \delta$, all i,

PCR.2: $\sigma(\pi) < \delta'$.

PCR is similar to the <u>commodity</u> constant returns of g, assumption (g.1), in that both permit maximum efficiency to be achieved at arbitrarily small scale. However, PCR is stronger than commodity constant returns because it requires that these small scale productive units draw their (small scale) inputs from <u>entirely disjoint sets of factor owners</u>. Thus, PCR implies that the economy as a whole exhibits no gains associated with the phenomenon of specialization and division of labor. PCR is also weaker than commodity constant returns since it does not require that a doubling of persons doubles outputs -- only that there is some way always to divide any group of factor owners into disjoint halves without damage to total productivity.

THEOREM 3: *If* T^* *is compact, the direct market* (f,I) *associated with* (g,T) *exhibits* PCR.

The following steps prove Theorem 3.

1. A basic result in operator theory is that *T^* is compact iff T is compact*. (See for example, Kantorovich and Akilov [1964, p. 309].)

2. The compactness of T is closely associated with its finite dimensionality. An operator T is finite-dimensional if $T[X]$ is a finite-dimensional subspace of Y. Of course this represents no restriction if Y is itself finite-dimensional. It is well-known that bounded operators having finite-dimensional range are compact. Further, T is compact iff there exists a sequence of finite-dimensional operators $\{T^k\}$ such that

$$\lim \sup_{|x| \leq 1} ||Tx - T^k x|| = 0.$$

(See Diestel and Uhl [1977, p. 69].) Thus the compactness of T, an essential sufficient condition for NS, implies that T is "virtually" finite-dimensional. The implications of this are carried one step further to yield the restrictions on the superadditivity of g_T.

3. Suppose T is finite-dimensional. Let $\hat{T}$ be defined by $\hat{T}\chi_E = (T\chi_E, \lambda(E))$.

Then $\hat{T}[\chi]$ is the range of a finite-dimensional, nonatomic vector measure. By Lyapunov's Theorem (Diestel and Uhl [1977, p. 264]), if $<\hat{T}[\chi]>$ is the convex hull of $\hat{T}[\chi]$, then

$$\hat{T}[\chi] = <\hat{T}[\chi]>.$$

Therefore, for any integer n there exists $\pi = \{E_i\}$, $i = 1, \ldots, n$, such that

$$\hat{T}\chi_{E_i} = (n^{-1}T\chi_A, n^{-1}\lambda(A)).$$

Each E_i comprises one-n^{th} of the population and owns one-n^{th} of the total inputs. By the homogeneity of g,

$$g(T\chi_{E_i}) = n^{-1}g(T\chi_A).$$

4. To complete the demonstration of Theorem 3, let $n^{-1} < \delta$ and k be such that $\sup_{|x|\leq 1} ||Tx - T^k x|| \leq \delta'$.

REMARK 4 (Thick Markets): An operator T is <u>representable</u> if there is a Bochner integrable function $w : A \to Y$ such that for all $x \in X$,

$$Tx = \int x(a)w(a)d\lambda.$$

If T is representable the Y-valued measure W defined by $W(E) = T\chi_E$ has w as its <u>Radon-Nikodym derivative</u>,

$$W(E) = \int\chi_E w d\lambda = \int_E w d\lambda.$$

When the measure W describing holdings of commodities has a Radon-Nikoydm derivative markets will be said to be <u>thick</u>. To justify this term suppose Y is a set of real-valued functions on C, the index set of commodities (e.g., C might be the integers or a subset of $\mathbb{R}$). Denote by $W_c(A)$ the aggregate quantity of commodity $c \varepsilon C$ available in the market. If W has a Radon-Nikodym derivative, then $W_c(A) > 0$ only if there is a set E with $\lambda(E) > 0$ such that

$$W_c(E) = \int_E w_c(a)d\lambda > 0,$$

i.e., only if there is a non-null set of agents all of whom have positive quantities of commodity c.

If T is compact it is known to be representable. (See Diestel and Uhl, [1977, p. 68]). Therefore compactness of T implies that markets are thick. Remark 2 shows that the converse does not hold.

The model Aumann [1964] used to prove the core equivalence theorem may be described as having introduced thick markets into general equilibrium theory. Bewley's [1973] generalization displays the power of the thick markets hypothesis by showing that Aumann's conclusions for finite-dimensional spaces can be extended. Indeed, it is possible to show that Aumann's result holds for any (Banach) commodity space as long as markets are thick. Such markets also exhibit PCR. (See Remarks 6 and 9.)

REMARK 5 (Thin Markets): Define markets to be thin if they are not thick. Since a compact operator necessarily implies that $T\chi_E = W(E)$ defines a W with a Radon-Nikodym derivative, for a market to be thin, T cannot be compact.

The canonical example of thin markets is the identity operator, I. I is bounded ($|Ix| = |x|$) and therefore $W(E) = I\chi_E = \chi_E$ defines a countably additive, nonatomic measure; but there can be no $w : A \to L^1(A,\mathcal{A},\lambda)$ such that $I\chi_E = \int_E w d\lambda$. To demonstrate, let $c \in C = A$. The only E having positive endowment of c is $E = \{a\}$ where $a = c$. Therefore,

$$W_c(A) = \int w_c(a)d\lambda = \int_E w_c(a)d\lambda = 0;$$

but $W_c(A) = 1$.

The identity operator would be appropriate to the depiction of an economy with commodity heterogeneity, the opposite of thick markets. This heterogeneity need not be incompatible with perfect competition.

Let T be a bounded operator from X to Y and U a compact operator from Y to Y'. Y' may, for example, be a subspace of Y. Then $T' = UT$ is from X to Y' and it is known that T' is compact. (See Kantorovich [1964, 309].) Define $g' : Y' \to \mathbb{R}_+$ to be a function on the same space as the range of T'. To demonstrate that (g,T) exhibits an NS distribution, assuming that $T^*[\partial g(T\chi_A)] = \nabla f(\chi_A)$, it suffices to impose the added assumption on g that there is a g' such that

$$g(Tx) = g'(T'x).$$

Therefore the indirect market (g,T) is equivalent to (g',T') and step 1. of Theorem 3 can be applied to Proposition 1 to show that (g,T) exhibits NS.

The added assumption on g increases the substitution possibilities among commodities. Therefore, physical differences among the heterogeneous commodities supplied by individuals, for example when $T = I$, overstates their economic individuality. This point may be given further emphasis by noting that although markets in (g,T) may be thin, if (g,T) is equivalent to (g',T') where T' is compact, then (g,T) is equivalent to an indirect market in which markets are thick.

An examplar of this version of thin markets is Mas-Colell's [1975] model of perfectly competitive product differentiation.

REMARK 6 (Perfectly Competitive Environments): Sufficient conditions for NS described above have focused on the particular properties required of T or g. Define Y as a perfectly competitive environment if for all, or "almost all," (g,T) satisfying (g.1-3) and (T.1-2) an NS distribution exists. Since g is concave and NS is equivalent to $Dg_T(T\chi_A)$, a space Y such that convex functions are almost always Fréchet differentiable constitutes a perfectly competitive environment. Asplund [1968] classifies Y as a strong differentability space if every continuous convex function is Fréchet differentiable on a dense G_δ set of its domain. Therefore, a strong differentiability space is a perfectly competitive environment. Asplund's results imply, for example, that for $1 < r < \infty$, $L^r(C,\mathcal{C},\rho)$ is a strong differentiability space. [$L^r(C,\mathcal{C},\rho)$ is the r^{th}-integrable functions on the σ-finite measure space $(C,\mathcal{C},\rho)$.]

A weak differentiability space is one in which continuous convex functions are Gâteaux differentiable on a dense G_δ set. L^1 is known to be a weak, but not a strong, differentiability space.

There appears to be a close connection between the differentiability properties of convex functions on Y and the PCR property of g_T. Call Y a Lyapunov convexity space if for every bounded operator $T : X \to Y$,

$$(*) \qquad \text{cl } T[\chi] = \text{cl } \langle T[\chi]\rangle.$$

It is from (*) that the PCR property of Theorem 3 is obtained. Precisely the same spaces Asplund finds to be strongly differentiable can be shown, via a result of Uhl [1969], to be Lyapunov convexity spaces. (Uhl proves that if W is a Y-valued measure with finite total variation and Y is either reflexive or a separable dual space, the range of W, $T[\chi]$, satisfies (*). Asplund [1968] proves that these are strong differentiability spaces.)

The identity operator shows that L^1 is not a Lyapunov convexity space. There is no element in $\mathrm{cl}\ I[\chi]$ equal to $1/2\chi_A \in \langle I[\chi]\rangle$.

REMARK 7 (Euler's Theorem): The relation between perfectly competitive environments (strong differentiability spaces) and PCR (Lyapunov convexity spaces) is reminiscent of Euler's Theorem for homogeneous functions. Let $e : \mathbb{R}^k_+ \to \mathbb{R}_+$ be a production function and $\nabla e(\cdot)$ its vector of partial derivatives. When inputs are paid their marginal products, output is exactly exhausted by payments to inputs $t = (t_1,\ldots,t_k)$ iff

$$\nabla e(t)t = e(t). \tag{1}$$

Similarity to the NS condition

$$Dg_T(T\chi_A)T\chi_A = g(T\chi_A), \tag{2}$$

is evident.

Just as (1) might hold for an arbitrarily given e at <u>some</u> t, so (2) might hold for arbitrary g at <u>some</u> $T\chi_A$. However, to guarantee (1) for all t it is necessary to have commodity constant returns; and, to guarantee (2) for all $T\chi_A$ it may be necessary to have PCR.

EXAMPLE 1: Since L^1 is not a strong differentiability space and the identity operator does not satisfy (*), an example that is neither NS nor PCR can be found. Let $(g,T) = (f,I)$, where for $x \varepsilon L^1_+(A,\mathcal{A},\lambda)$

$$g(Tx) = f(x) = (\int [x(a)]^\beta \, d\lambda)^{1/\beta}, \quad 0 < \beta < 1 .$$

It is readily verified that (g,T) satisfies (g.1-3) and (T.1-2). Further,

$$\chi_A = \nabla f(\chi_A),$$

and therefore $(\mu,p) = (\lambda,\chi_A)$ is the unique WE. Thus, one of the necessary conditions for NS holds. However, for any disjoint, non-null E and E',

$$f(\chi_{E \cup E'}) > f(\chi_E) + f(\chi_{E'}).$$

The market does not exhibit PCR.

To demonstrate that (f,I) is not NS, part (iii) of Theorem 2 is used. It may be verified that $p^n \varepsilon \partial f(\chi_{A\setminus E_n})$ where,

$$p^n(a) = \begin{cases} [\lambda(A\setminus E_n)]^{1/\beta - 1}, & a \varepsilon A \; E_n \\ 1/\beta, & a \varepsilon E_n. \end{cases}$$

Letting $p = \chi_A = \nabla f(\chi_A)$,

$$\lim |p^n - p|_\infty = 1/\beta - 1 \neq 0.$$

Therefore p is not PD. (See Figure 2.)

V. INDIRECT MARKETS WITH DIRECT INPUTS

The construction in this section provides the bridge to carry over the above results to economies with many outputs and ordinal preferences.

Let $(h, (T,I))$ be an <u>indirect market with direct inputs</u> where T and I are defined as above and T satisfies (T.1-2). The production function $h : Y_+ \times X_+ \to \mathbb{R}_+$ is assumed to satisfy (h.1), positively homogeneous; (h.2), superadditive; and (h.3), Lipschitz on its domain just as g satisfies (g.1-3). The Lipschitz condition is that for all $y, y' \in Y_+$ and $x, x' \in X_+$ there is a γ' such that

$$|h(y,x) - h(y',x')| \leq \gamma'(\|y-y'\| + |x-x'|).$$

Although production possibilities are described by the single function h, it is useful to regard $h(\cdot, I\chi_E) = h(\cdot, \chi_E)$ as the production function available to E so that there may be as many production functions as there are distinct groups. Thus, $h(T\chi_E, I\chi_E)$ is the output E could produce when its endowment of "raw material" inputs is $T\chi_E$ and its endowment of "entrepreneurial" inputs is, by definition, $I\chi_E$.

Let $r \in L^\infty$ and interpret $r(a)$ as the <u>rent</u> assigned to one unit of entrepreneurial input $a \in A$. An element of $\partial h(T\chi_A, I\chi_A)$ is a pair $(q,r) \in Y^* \times L^\infty$ such that

$$qT\chi_A + rI\chi_A = h(T\chi_A, I\chi_A)$$

$$qy + rx \geq h(y,x), \quad (y,x) \in Y_+ \times X_+.$$

If $(q,r) \in \partial h(T\chi_A, I\chi_A)$ and μ is a distribution of the total output such that $\mu(E) = qT\chi_E + rI\chi_E$, then in a manner entirely analogous to the definition for (g,T), $(\mu,(q,r))$ is a WE for $(h,(T,I))$.

Similarly, μ is an NS distribution for $(h,(T,I))$ if for every $\{E_n\}$,

$$\lim \frac{|h(T\chi_A, I\chi_A) - h(T\chi_{A\setminus E_n}, I\chi_{A\setminus E_n}) - \mu(E_n)|}{\lambda(E_n)} = 0.$$

Letting h_T be the restriction of h to the subset of its domain which is the graph of T,

$$\{(y,x) : (y,x) = (Tx, Ix),\ x \in X_+\},$$

define,

$$Dh_T(T\chi_A, I\chi_A) = \{(q,r) : \lim_{\substack{x_n \in X_+ \\ |\chi_A - x_n| \to 0}} \frac{|h(T\chi_A, I\chi_A) - h(Tx_n, Ix_n) - [qT(\chi_A - x_n) + rI(\chi_A - x_n)]|}{\|T(\chi_A - x_n)\| + |I(\chi_A - x_n)|} = 0.$$

An extension of Theorem 2 characterizing an NS distribution for (g,T) yields the result that

$$\mu \text{ is NS iff } \mu(E) = qT\chi_E + rI\chi_E, \text{ where } (q,r) \in Dh_T(T\chi_A, I\chi_A).$$

The difference between a $(q,r) \in \partial h(T\chi_A, I\chi_A)$ and a (q,r) which also belongs to $Dh_T(T\chi_A, I\chi_A)$ is the difference between Walrasian and perfectly competitive pricing. Applied to r, it is the difference between rents such that if they are taken as given along with q, hiring all the inputs represents an aggregate, profit-maximizing production plan, and rents such that each agent is extracting all the surplus or marginal product contributed by its entrepreneurial input. This is more fully elaborated in Makowski [1980].

REMARK 8: Suppose there are only a finite number of raw material inputs (T is finite-dimensional) and $\partial h(T\chi_A, I\chi_A)$ is unique. Then the existence of a perfectly competitive equilibrium reduces to the existence of perfectly competitive rents. However, since I is not a compact operator (see Remark 5), the competitiveness of r is dubious unless there is a compact U such that $h(y,x) = h'(y,x')$ whenever $x' = UIx = Ux$ -- i.e., unless there are substantial substitution possibilities among the entrepreneurial inputs. When U is compact and T is finite-dimensional, entrepreneurial inputs are sufficiently redundant to guarantee PCR.

The direct market (f,I) corresponding to $(h,(T,I))$ is defined by

$$f(\chi_E) = h(T\chi_E, I\chi_E).$$

Let (T^*,I^*) be the adjoint of (T,I), where $(T^*,I^*) : Y^* \times L^\infty \to L^\infty$ and $(T^*,I^*)(q,r) = p$ is defined by the equality that for all $x \in X$,

$$(T^*q)x + (I^*r)x = q(Tx) + r(Ix) = px.$$

Thus, if q represents the prices of raw material inputs and r the prices of entrepreneurial inputs, then

$$p = T^*q + I^*r$$

represents the prices of persons in the corresponding direct market.

The representation of an ordinal <u>exchange</u> economy by an indirect market with direct inputs will impose an added restriction on h. For all $\alpha > 0$ and $(y,x) \in Y_+ \times X_+$, another assumption on h is

(h.4): (zero homogeneity in entrepreneurial inputs) $h(y,\alpha x) = h(y,x)$.

With (h.1-3) additions to entrepreneurial inputs allow access to another technology that may be, in terms of raw material inputs, more productive. This is also permitted with (h.4) but the productivity of these additional entrepreneurial inputs is limited to those changes that do not take place by positive scalar multiplication.

As a consequence,

PROPOSITION 2: If h satisfies (h.1-4) and $(q,r) \in \partial h(y,x)$, then $rx = 0$.

PROOF: By (h.1), $qy + rx = h(y,x)$ and $qy' + rx' \geq h(y',x')$, $(y',x') \in Y_+ \times X_+$. When $\delta > 0$ and $x' = \alpha x$, $qy + \alpha rx \geq h(y,\alpha x) = h(y,x)$ which can only be satisfied if $rx = 0$.

Therefore, if $x = \chi_A$ and $(q,r) \in \partial h(y,x)$ then r must be the null function. The implication is that if $(\mu,(q,r))$ is a WE when h satisfies (h.4), the total value of output is entirely accounted for by the values given to the raw material inputs -- i.e., $qT\chi_A = h(T\chi_A, I\chi_A)$ and $\mu(E) = qT\chi_E$. Even though r is null, the existence of an NS distribution will depend in part on whether or not $\lim |r^n|_\infty = 0$, where $(q^n,r^n) \in \partial h(T\chi_{A\setminus E_n}, I\chi_{A\setminus E_n})$. The interpretation is that if $\lim |r^n|_\infty = 0$, the preferences of any small group of agents are not very different from the preferences exhibited by others. There is, of course, no guarantee of this. For example, locational differences among commodities and persons might preclude it. If it is satisfied, the existence of an NS distribution would depend on whether or not $\lim |T^*(q^n - q)|_\infty = 0$, precisely the condition for an NS distribution in (g,T).

Not treated in this paper are ordinal economies with production, but the framework of this section and Makowski's results for finite economies suggest that such an extension may be straightforward.

VI. ORDINAL EXCHANGE ECONOMIES

The following description of an ordinal exchange economy is based on Vind [1964].

It was observed above that by setting $W(E) = T\chi_E$, W defines a countably additive, nonatomic, Y_+-valued measure which will be referred to as the <u>initial allocation</u>. A <u>final allocation</u>, or simply <u>allocation</u>, is a countably additive $Z : \mathcal{A} \to Y_+$. Unlike W, Z is not required to come from a bounded linear operator on X. An allocation is <u>feasible</u> if

$$Z(A) = W(A) = T\chi_A.$$

It is simply a rearrangement among agents of the initial allocation.

Let $\mathcal{Z}$ be the set of allocations, feasible or not. For $Z, Z' \in \mathcal{Z}$, Z agrees with Z' on E if for all $E' \in \mathcal{A}$, $E' \subset E$, $Z'(E') = Z(E')$.

Preferences are defined by a mapping $S : \mathcal{Z} \times \mathcal{A} \to 2^{Y+}$. $S(Z,E)$ is the set of total resources that could be distributed to E in such a way that the members could obtain an allocation unanimously preferred by them to Z. $S(Z,E)$ is derived from some underlying preference relation, $\succ_E$, defined on $\mathcal{Z} \times \mathcal{Z}$. Whatever these preferences may be, attention is confined to the conclusion that if there is some $Z' \in \mathcal{Z}$ such that (i) $Z' \succ_E Z$, (ii) Z' agrees with Z on $A \setminus E$, and (iii) $y = Z'(E)$, then $y \in S(Z,E)$.

An ordinal exchange economy is defined by the pair (S,T). T is subject to the above restrictions (T.1-2) and

(T.3): $\|T\chi_E\| \neq 0$ whenever $\lambda(E) \neq 0$.

Non-null sets of agents have non-null initial endowments.

For $(Z,E) \in \mathcal{Z} \times \mathcal{A}$, assumptions on S are:

(S.1): (Absence of External Effects) If Z agrees with Z' on E, $S(Z,E) = S(Z',E)$.

(S.2): (Countable Additivity) If $\{E_m\}$ is a sequence of pairwise disjoint set in $\mathcal{A}$ whose union is E, then $y \in S(Z,E)$ iff there exists $y_k \in S(Z, \bigcup_{m=1}^{m=k} E_m)$ such that $\lim \|y-y_k\| = 0$.

(S.3): $S(Z,E)$ is Y_+-open

(S.4): (Local Non-satiation) $Z(E) \in \text{cl } S(Z,E)$

(S.5): (Monotonicity) $y \in \text{cl } S(Z,E)$ implies $y + \{Y_+ \backslash \{0\}\} \subset S(Z,E)$.

(S.6): $S(Z,E)$ is convex.

Denote by $\partial S(Z,E)$ the Y_+-boundary of $S(Z,E)$. (S.3-4) imply that if $Z(E) \in \partial S(Z,E)$, then $Z(E) \notin S(Z,E)$; otherwise, Z would be preferred to itself by E.

A representation of an ordinal exchange economy by a real-valued function appears in Shafer and Sonnenschein [1975] where it is used in the proof of existence of WE in finite economies. The construction employed here is a concave variant of the Minkowski function describing the distance between a point and a convex set. Shepard [1953] has characterized production technologies using this device.

Let S be a subset of Y_+ whose closure does not contain the origin and which satisfies (S.3), (S.5), and (S.6). Define the distance between

$y \in Y_+$ and S by

$$d(y;S) = \begin{cases} 0, & \text{if } \alpha y \notin S, \text{ all } \alpha > 0 \\ \sup\{\alpha^{-1} : \alpha > 0, \alpha y \in S\}, & \text{otherwise.} \end{cases}$$

The function $d(y;S)$ has the following properties.

$$\text{(d.1):} \quad \left.\begin{array}{l} d(y;S) > 1 \\ d(y;S) = 1 \\ d(y;S) < 1 \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} y \in S \\ y \in \partial S \\ y \notin \text{cl } S \end{array}\right.$$

For $\alpha > 0$, its various homogeneity properties are

$$\text{(d.2):} \quad \begin{array}{l} d(\alpha y;\alpha S) = d(y;S) \\ d(\alpha y;S) = \alpha d(y;S) \\ d(y;\alpha S) = \alpha^{-1} d(y;S) \end{array}$$

It is superadditive,

$$\text{(d.3):} \quad d(y+y';S) \geq d(y;S) + d(y';S).$$

(d.3) follows from the well-known result that (a) $(\alpha+\alpha')S = \alpha S + \alpha' S$ when $\alpha, \alpha' \geq 0$ and S is convex (Rockafeller [1970, p. 17]) and (b) the positive homogeneity of $d(\cdot;S)$.

It will be assumed throughout the remainder of this section that Z is a feasible allocation such that

$$\|Z(E)\| \neq 0 \quad \text{whenever} \quad \lambda(E) \neq 0.$$

By (S.3-6) this implies that $\underset{\sim}{0} \notin \text{cl } S(Z,E)$ whenever $\lambda(E) \neq 0$.

For $\pi = \{E_i\}$, $\lambda(E_i) > 0$, and $\alpha_i > 0$, let

$$d^Z(y,\Sigma\alpha_i\chi_{E_i}) = d(y;\Sigma\alpha_i S(Z,E_i)).$$

Because $d(\cdot;\cdot)$ is a homogeneous of degree zero so is d^Z. To obtain a positively homogeneous and also superadditive function, define

$$h^Z(y,x) = d^Z(y,x)|x|$$

where $x = \Sigma\alpha_i \chi_{E_i}$. Thus, an allocation in an ordinal exchange economy yields a function of the form associated with an indirect market with direct inputs. By the homogeneity of degree minus one of $d(y;\cdot)$, it exhibits property (h.4).

Assume that h^Z also satisfies the Lipschitz condition (h.3). This is an implicit restriction on Z and the preference mapping S. As a consequence of (h.1-4) for h^Z, there exists $q \in Y^*$ such that

$$qT\chi_A = h^Z(T\chi_A, I\chi_A),$$

$$qy \geq h^Z(y,x), \qquad (y,x) \in Y_+ \times X_+,$$

which is to say that $(q,\underset{\sim}{0}) \in \partial h^Z(T\chi_A, I\chi_A)$. (This follows from the Lemma to Theorem 1 and Proposition 2.)

A feasible allocation is <u>Pareto-optimal</u> when $T\chi_A = Z(A) \notin S(Z,A)$. By (S.4), $Z(A) \in \text{cl } S(Z,A)$. Therefore, Z is Pareto-optimal iff $T\chi_A \in \partial S(Z,A)$. By (S.4) and (S.5) and the maintained assumption on Z, if $Z(A) \in \partial S(Z,A)$ then $Z(E) \in \partial S(Z,E)$ whenever $\lambda(E) \neq 0$. By property (d.1) of d, $d(Z(E);S(Z,E)) = 1$. Therefore,

$$h^Z(Z(E), I\chi_E) = d(Z(E);S(Z,E))|\chi_E| = \lambda(E).$$

The summary statement is that Z <u>is Pareto-optimal iff the values of</u> h^Z <u>coincide with Lebesgue measure on the subset</u> $(Z(E),\chi_E)$ <u>of its domain</u>.

If Z is Pareto-optimal and $(q,\underset{\sim}{0}) \in \partial h^Z(T\chi_A, I\chi_A)$ it is easily verified that

$$qZ(E) = h^Z(Z(E), I\chi_E) = \lambda(E).$$

This is equivalent to the condition that the hyperplane defined by q is supporting for $Z(E) \in \partial S(Z,E)$ -- i.e.,

$$\inf q[S(Z,E) - Z(E)] = 0 ,$$

when q is normalized so that $qZ(E) = \lambda(E)$.

The allocation Z is a WE for (S,T) if there exists a $q \in Y^*$, $\|q\|_* \neq 0$, such that for all $E \in A$, $\lambda(E) > 0$,

$$\inf q[S(Z,E) - T\chi_E] = 0, \qquad (1)$$

$$y \in S(Z,E) \text{ implies } qy > qT\chi_E \qquad (2)$$

Under the above assumptions, (2) is superfluous. (T.3) and (S.5) are known to imply $qT\chi_E > 0$ whenever $\lambda(E) > 0$ and from this one may conclude (2) from (1).

Since $qT\chi_A > 0$ when q is the commodity prices associated with the WE Z, it may be assumed that $qT\chi_A = \lambda(A) = h^Z(T\chi_A, I\chi_A)$. Therefore, the WE pair (q,Z) satisfies

$$qT\chi_E = qZ(E) = h^Z(Z(E), I\chi_E) = \lambda(E) .$$

These remarks are summarized in

PROPOSITION 3: (q,Z) <u>is a</u> WE <u>with</u> $qT\chi_A = \lambda(A)$ <u>iff</u> $(q,\underset{\sim}{0}) \in \partial h^Z(Z(E), I\chi_E)$ <u>and</u> $qT\chi_E = \lambda(E)$.

The purely ordinal version of NS for the economy (S,T) is a feasible allocation Z such that for any $\{E_n\}$,

$$\lim \frac{d(T\chi_{A\backslash E_n}; S(Z, A\backslash E_n))}{\lambda(E_n)} = 0 .$$

An NS allocation is evidently Pareto-optimal. Representing Z by h^Z, the Pareto-optimal allocation Z is NS iff the distribution $\mu = \lambda$ is NS : for any $\{E_n\}$,

$$\lim \frac{|h^Z(T\chi_A, I\chi_A) - h^Z(T\chi_{A\setminus E_n}, I\chi_{A\setminus E_n}) - \lambda(E_n)|}{\lambda(E_n)} = 0$$

This is equivalent to the existence of $Dh^Z_T(T\chi_A, I\chi_A)$.

Let $f^Z(x) = h^Z(Tx, Ix)$ be the direct market derived from h^Z which is in turn derived from Z and (S,T). The following result shows that precisely the same differences in the differentiability properties that distinguished WE and NS distributions in Section III apply to WE and NS allocations in ordinal exchange economies.

PROPOSITION 4: <u>Let</u> h^Z <u>satisfy</u> (h.1-4) <u>and let</u> f^Z <u>be the direct market derived from</u> h^Z.

(i) Z <u>is a WE iff</u> $\chi_A \in \partial f^Z(\chi_A)$

(ii) Z <u>is NS iff</u> $\chi_A = Df^Z(\chi_A)$

PROOF: Part (ii) follows from inspection of the definition of NS and Theorem 2.

For (i), suppose Z is a WE allocation and its associated price vector q is normalized so that $qT\chi_A = h^Z(T\chi_A, I\chi_A) = f^Z(\chi_A)$. Then $(q, \underset{\sim}{0}) \in \partial h^Z(T\chi_A, I\chi_A)$ and therefore $T^*q \in \partial f^Z(\chi_A)$. For any E,

$$T^*q\chi_E = qT\chi_E = qZ(E) = h^Z(Z(E), I\chi_E) = \lambda(E).$$

But this can only be satisfied if $T^*q = \chi_A$.

Conversely, suppose $\chi_A \in \partial f^Z(\chi_A)$. Applying the Lemma to Theorem 1, the hypotheses on h^Z ensure that $\partial h^Z(T\chi_A, I\chi_A) \neq \emptyset$. From Proposition 2, if $(q,r) \in \partial h^Z(T\chi_A, I\chi_A)$ then $r = \underset{\sim}{0}$. Applying the argument used in the proof of Theorem 1, if $\chi_A \in \partial f^Z(\chi_A)$ there is a $q \in Y^*$ such that $T^*q = \chi_A$ and $(q,\underset{\sim}{0}) \in \partial h^Z(T\chi_A, I\chi_A)$. Since $T\chi_A = Z(A)$ and Z is Pareto-optimal,

$$T^*q\chi_E = qT\chi_E = \lambda(E) = h^Z(Z(E), I\chi_E) \ .$$

It follows from $(q,\underset{\sim}{0}) \in \partial h^Z(T\chi_A, I\chi_A)$ that $qZ(E) \geq h^Z(Z(E), I\chi_E)$ and therefore $q(Z(E) - T\chi_E) \geq 0$; but this implies

$$q(Z(E) - T\chi_E) = 0. \tag{1'}$$

If $y \in S(Z,E)$ and $\lambda(E) > 0$, then by property (d.1), $d(y;S(Z,E)) > 1$ and therefore,

$$qy \geq h^Z(y, I\chi_E) > h^Z(Z(E), I\chi_E) = qT\chi_E \ . \tag{2'}$$

Conditions (1') and (2') imply conditions (1) and (2) defining Z as a WE.

VII. THE NO-SURPLUS CONDITION AND THE COOPERATIVE THEORY OF GAMES

Two solution concepts from the cooperative theory of games that have been linked to WE are the core (Debreu and Scarf [1963], Aumann [1964], Vind [1964], Hildenbrand [1974]), and the (Shapley) value (Aumann and Shapley [1974], Aumann [1975], Champsaur [1975], Mas-Colell [1977]). By demonstrating an equivalence between the core and WE or the value and WE, game theory provides a rationalization of its competitiveness. It is the purpose of this section to provide an alternative interpretation. Instead of using core or value equivalence to justify perfect competition, the NS definition of perfect competition will be shown to imply core and value equivalence (but not conversely).

Solution concepts in cooperative game theory with transferable utility are based on the game-theoretic characteristic function, a mapping $v : A \to \mathbb{R}$. Using the h^Z construction comparisons between NS and the core established below have immediate extensions to games with non-transferable utility; but this construction cannot be used for the value and therefore the connections, if any, must be demonstrated by other means. (See Geanakoplos [1978] and Mas-Colell [1978].)

A restriction on v, due to Shapley [1967], called a balanced game has been shown to capture much of the essential structure of general equilibrium models. It is defined by the condition that for any $\{E_i\}$, not necessarily a partition, and $\alpha_i \geq 0$,

$$\Sigma\alpha_i v(E_i) \leq v(A) \quad \text{whenever } \Sigma\alpha_i \chi_{E_i} \leq \chi_A .$$

Every indirect market yields a v according to

$$v(E) = g(T\chi_E) = f(\chi_E) .$$

Further, the homogeneity and superadditivity properties of g imply that v is balanced. Throughout this section v is assumed to be derived from a (g,T).

Let μ be a distribution of the total output: $\mu(A) = v(A) = g(T\chi_A)$. The following restrictions on μ describe various notions of an equilibrium distribution.

NNS: (Non-negative Surplus) For any $\{E_n\}$, $\lim \frac{v(A) - v(A\backslash E_n) - \mu(E_n)}{\lambda(E_n)} \geq 0.$

CORE: For all $E \in \mathcal{A}$, $\mu(E) \geq v(E)$.

WE: For all $\pi = \{E_i\}$ and $\alpha_i \geq 0$, $\Sigma\alpha_i\mu(E_i) \geq f(\Sigma\alpha_i\chi_{E_i})$.

NS: For any $\{E_n\}$, $\lim \frac{v(A) - v(A\backslash E_n) - \mu(E_n)}{\lambda(E_n)} = 0.$

For the definition of the value, Kannai's asymptotic approach [1966] is followed. A $\pi = \{E_i\}$ defines a game-theoretic characteristic function v_π on $\mathcal{A}_\pi$, where $\mathcal{A}_\pi$ is the set of elements of $\mathcal{A}$ formed by unions of elements in π and v_π is the restriction of v to $\mathcal{A}_\pi$. The value for v_π is demonstrated by Shapley [1953] to be

$$\mu_\pi(E_i) = \sum_{E\in\mathcal{A}_\pi} \frac{(n-|E|)!(|E|-1)!}{n!}[v(E) - v(E\backslash E_i)],$$

where $|E|$ is the cardinality of E as a element of $\mathcal{A}_\pi$ and $|A| = n$. To extend μ_π to a measure on $\mathcal{A}$, let

$$\mu_\pi(E) = \lambda(E)\lambda(E_i)^{-1}\mu_\pi(E_i), \qquad E \subset E_i .$$

VALUE: μ is the (asymptotic) value if for all sequences $\pi(n) = \{E_i^n\}$, where

$$\max_i \{\lambda(E_i^n)\} \searrow 0, \quad \text{if} \quad \mu_{\pi(n)} \quad \text{is the value for} \quad v_{\pi(n)},$$

$$\lim \mu_{\pi(n)}(E) = \mu(E).$$

The NNS condition represents an obvious weakening of NS. Since $v(A) - v(A\backslash E) \geq \mu(A) - \mu(A\backslash E) = \mu(E)$ when μ is in the core, the core condition implies that there is NNS for all groups not only small ones. In fact, the core and NNS conditions are equivalent, provided $v(E) = 0$ whenever $\lambda(E) = 0$, as is the case when v is derived from (g,T).

The definition of μ as a WE is justified by Theorem 1. It makes transparent the well-known result that a WE is in the core. The definition of NS is identical to that given above.

According to the core (value) criterion the game v is perfectly competitive if the set of distributions in the core (value) coincide with the WE distributions.

$$\{\mu : \mu(E) = p\chi_E, \quad p \in T^*[\partial g_T(T\chi_A)]\}.$$

Since the value is unique, a necessary condition for value equivalence is

$$T^*[\partial g_T(T\chi_A)] = \nabla f(\chi_A).$$

It follows from Kannai's results that if the range of T is finite-dimensional and $\nabla f(\chi_A)$ exists, the value coincides with the unique WE. This conclusion can be extended to operators whose range is not finite-dimensional.

PROPOSITION 5: <u>If T has separable range in Y and $T^*[\partial g_T(T\chi_A)] = \nabla f(\chi_A)$, there is value equivalence.</u>

PROOF: Let $T^k : X \to Y$ be a finite-dimensional operator such that $T^k\chi_A = T\chi_A$; let v^k be the game defined by $v^k(E) = g(T^k\chi_E)$; and, $v^k_{\pi(n)}$ the game defined by the restriction of v^k to the elements of $A_{\pi(n)}$.

By hypothesis $T^*q = \nabla f(\chi_A)$ if $q \in \partial g(T\chi_A)$. If $q \in \partial g(T\chi_A)$ then $q \in \partial g_{T^k}(T^k\chi_A)$ and therefore $T^*q = \nabla f^k(\chi_A)$, where f^k is defined by $f^k(x) = g(T^kx)$, $x \in X_+$. Kannai's result is that for fixed k, if $\mu^k_{\pi(n)}$ is the value for $v^k_{\pi(n)}$ and $T^*q = \nabla f^k(\chi_A)$ then μ^k defined by $\mu^k(E) = qT^k\chi_E$ is the value for v^k, i.e.,

$$\lim_n |\mu^k_{\pi(n)}(E) - qT^k\chi_E| = 0. \tag{1}$$

Because T is separably valued, there is a sequence $\{T^k\}$ such that for any $x \in X$, $\lim ||Tx - T^kx|| = 0$. Therefore, letting μ be defined by $\mu(E) = qT\chi_E$ where $T^*q = \nabla f(\chi_A)$,

$$\lim_k \; |\mu^k(E) - \mu(E)| = \lim |qT^k\chi_E - qT\chi_E| = 0 \tag{2}$$

Further, $\{T^k\}$ may be chosen such that for $k = n$ the range of T^k tains the convex cone spanned by $\{T\chi_E\}$, $E \in A_{\pi(n)}$. Thus, $v^n_{\pi(n)}$ coincides with $v_{\pi(n)}$ which implies that their respective values $\mu^n_{\pi(n)}$ and $\mu_{\pi(n)}$ coincide. Combining this with (1) and (2),

$$\lim |\mu_{\pi(n)}(E) - qT\chi_E| = 0.$$

Therefore μ such that $\mu(E) = qT\chi_E$ is the (asymptotic) value of the game defined by (g,T).

The relation between NS and core and value equivalence is summarized in

PROPOSITION 6: Assume that T has separable range.

(i) If v exhibits NS there is core and value equivalence.

(ii) Neither core nor value equivalence implies that v exhibits NS.

(iii) If T is compact, then v exhibits NS iff there is value equivalence and this implies core equivalence.

PROOF: (i) If μ is NS then by Theorem 2 $\mu(E) = p\chi_E$ where $p = Df(\chi_A)$. Thus, $\nabla f(\chi_A) = Df(\chi_A)$ and by Proposition 5 there is value equivalence. If μ is in the core it is NNS. But if v contains an NS distribution every NNS distribution is NS.

(ii) Example 1 in Section IV satisfies the hypotheses of Proposition 5 and therefore exhibits value equivalence but it is not NS. If T is finite-dimensional, Lyapunov's Theorem shows that $T[\chi] = \langle T[\chi] \rangle$ (see step 3. in the proof of Theorem 3) and it is well-known that this suffices for core equivalence. However if $T^*[\partial g_T(T\chi_A)] \neq \nabla f(\chi_A) = \emptyset$, then $Df(\chi_A) = \emptyset$ and by Theorem 2 there is no NS distribution.

(iii) By Proposition 1 and step 1 in the proof of Theorem 3 T^* is compact and $Df(\chi_A)$ exists. Repeat the argument of part (i).

REMARK 9: From part (iii) of Proposition 6, it might appear that value equivalence is closer than core equivalence to the NS characterization of perfectly competitive equilibrium. In fact, it is more nearly the opposite.

A perfectly competitive environment was defined in Remark 6 as any space such that $Dg_T(T\chi_A)$ exists for "most" g and T. Applying the same criterion to value equivalence would lead to the conclusion that L^1 was a perfectly competitive environment since a concave, Lipschitz function is Gâteaux differentiable on a dense G_δ set of its domain. L^1 is, in Asplund's terminology, a weak differentiability space but not a strong differentiability space.

Core equivalence makes no direct demands on differentiability as does value equivalence (Gâteaux) or NS (Fréchet). What is required of an environment for it to be classified as perfectly competitive according to the core criterion (applied to bounded operators) is that Y be a Lyapunov convexity space. (See Remark 6.) From condition (*) defining such a space, it may be shown that if μ is in the core of (g,T) and $\partial g(T\chi_A) \neq \emptyset$, there is a $q \in \partial g(T\chi_A)$ such that $\mu(E) = qT\chi_E$, i.e., there is core equivalence. L^1 is not a core equivalence environment. In Example 1, it may be verified that the core is larger than the unique WE. If (?) strong differentiability and Lyapunov convexity spaces coincide the classification of perfectly competitive environments according to the core equivalence criterion would be similar to the NS classification.

VIII. THE NO-SURPLUS CONDITION AS A REFORMULATION OF THE MARGINAL PRODUCTIVITY THEORY OF DISTRIBUTION

The theory of value is concerned with the pricing of commodities under perfectly competitive conditions. The connection between the pricing of commodities and the pricing of persons is made through the marginal productivity theory (MPT) of distribution. According to MPT there is a basic asymmetry between the pricing of commodities and the pricing of persons. Factor incomes are determined, and therefore persons are "priced," by the theory of commodity prices, but the theory of commodity prices cannot be derived from MPT. Three well-known properties of MPT make this clear: (i) It is a theory of the demand for factors, not their supply; (ii) It takes prices of products as given in the determination of demand for factors; (iii) Like the cost-of-production approach to the theory of value, it cannot be applied to the determination of values in an exchange economy.

With the NS approach there is complete symmetry. In mathematical terms this symmetry is based on the duality between the operator T and its adjoint T*. Unlike MPT, <u>the NS theory of distribution implies the competitive pricing of commodities</u>.

This difference belies the obvious parallels between MPT and the NS approach listed below.

MPT	NS
marginal product of a factor	marginal product of a person
product-exhaustion with respect to factors	no-surplus condition
constant returns to factors	population constant returns

MPT is based on the marginal product of a factor, an argument of a production function. Essential to its logical validity is the concept of product-exhaustion: after each factor is paid the marginal product of the last unit employed the sum of the payments exactly exhausts the quantity of output produced. Constant returns to factors is necessary to guarantee product-exhaustion.

NS is based on the marginal product of a person. To extract all the surplus, a concept that is well-defined even for ordinal exchange economies, is the same as obtaining one's marginal product. The NS condition is precisely product-exhaustion with respect to persons. Population constant returns seems to be a necessary condition for an environment to be perfectly competitive. (See Theorem 3 and Remarks 6 and 7.)

Treatment of "labor" illustrates that the marginal product of a factor and the marginal product of a person can coincide. The last, infinitesimal unit of labor and an individual worker have traditionally been regarded as identical. Indeed, it is difficult to see how marginal analysis would apply unless buyers and sellers operated on a small scale. Why would a person supplying a large-scale quantity allow an infinitesimal marginal product to determine the price received for all units?

"Capital" has been treated differently, as a factor disembodied from the person supplying it. One of the issues in capital theory is whether or not its marginal product <u>can</u> be defined. With the MPT perspective, an inability to define its marginal product would imply that a capitalist's income need not be determined by the same principles as those governing the income of a supplier of labor. This amibugity can only arise because of the asymmetry in MPT in which a person's price, or factor income, is determined by the marginal product of the input supplied.

The NS approach establishes a direct connection between the prices of persons and the prices of commodities without any explicit reliance on the marginal products of disembodied factors. Whether or not meaning can be given to the marginal product of an aggregate called "capital," the NS characterization of perfectly competitive equilibrium says that if there is perfect competition the same theory of income determination must apply to all suppliers of inputs. Any differences are attributable to the presence of monopoly power.

Both MPT and WE are sensitive to increasing returns to factors, depending upon non-increasing returns for their existence. What they fail to provide is a sensitivity to increasing returns to persons, in which the sum of the marginal products of persons exceeds the total product. This may be illustrated for an economy with a finite number of persons. If $\pi = \{E_i\}$ and $MP(E_i) = f(\chi_A) - f(\chi_{A\setminus E_i})$, the homogeneity and superadditivity of f imply

$$\Sigma MP(E_i) \geq f(\chi_A).$$

Since equality is exceptional there is the rather obvious conclusion that economies with large-scale suppliers are typically not perfectly competitive.

In the nonatomic direct market (f, I) assume that for any $\{E_n\}$ such that $E_n = \{a\}$,

$$\lim \frac{f(\chi_A) - f(\chi_{A\setminus E_n})}{\lambda(E_n)} = p(a) = MP(a).$$

If, as above, f is homogeneous and superadditive,

$$p\chi_A = \int MP(a)d\lambda \geq f(\chi_A).$$

There is equality iff f is Fréchet differentiable at χ_A.

Of course, these inequalities would apply if f exhibited increasing returns and that is exactly the point. For perfectly competitive equilibrium to fail to exist, it does not matter whether the source of the inequality is increasing returns to factors or constant factor returns without differentiability with respect to persons.

The importance of differentiability was recognized by the early marginalists. Fixed-proportions production functions were known to create difficulties for MPT. But this absence of what might be called micro-differentiability is probably not, by itself, an important source of imperfect competition. For example, Houthakker [1955] showed that if firms in an industry do not have the same fixed-proportions technologies the aggregate production function of the industry will exhibit macro-differentiability.

What the early marginalists may not have fully recognized is that when the number of complementary factors/commodities is sufficiently large -- large enough so that the economy does not exhibit PCR -- then even if there is micro-differentiability and constant returns to factors there may be increasing returns to persons. This would preclude the MPT-NS approach to perfectly competitive price determination. Perhaps it is this formulation of increasing returns through commodity heterogeneity that underlies Chamberlin's [1962] vision of monopolistic competition.

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