

INTERTEMPORAL INSURANCE¹

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1 Introduction

This paper applies techniques of intertemporal finance to insurance markets. The essential concepts employed are the Harrison and Kreps [1979] characterization of asset prices as martingales and the use of sequential trading to achieve dynamic spanning as formulated by Kreps [1982]. General equilibrium treatments of insurance markets and financial markets share common roots in Arrow and Debreu's notion of a contingent commodity, developed in the 1950's. Since that time, however, the theories have gone their separate ways. Competitive insurance theory has seen little further development, the literature on insurance choosing instead to focus on problems of asymmetric information (moral hazard or adverse selection) as potential sources of market failure. The theory of financial markets, in contrast, moved from the static model of Arrow [1953] to a fully developed intertemporal setting, clarifying how dynamic trading of assets can facilitate the efficient bearing of risk as information unfolds over time (see Radner [1972]).

A key contribution of the theory of intertemporal finance is the demonstration that, provided "uncertainty resolves nicely," markets can be completed with far fewer securities than states of nature. With insurance, the specter of incomplete markets appears in the simplest of settings. Consider, for example, an exchange economy with n consumers and two dates: 0 (today) and T (tomorrow). Assume that all consumers have the same endowments and the same preferences and all face the same risks. Specifically, suppose that each consumer has endowment $Y > 0$ of today's commodity and an independent and identically distributed risk of an accident tomorrow with probability $p \in (0, 1)$. If no accident occurs, consumer i will have an endowment Y of tomorrow's goods; but, in the event of an accident, her endowment falls to $Y - L \geq 0$. In this setting, the uncertain state of the world in this economy can be represented by a sequence of n zeros and ones, say

$$\omega = 10011010 \cdots 0110,$$

where a 1 in the i^{th} place indicates that consumer i has an accident and a 0 that she does not. The sample space Ω is the collection of all such sequences, a total of 2^n possibilities. The number of "natural" insurance contracts, promises to deliver a unit of consumption in the event of an accident to a particular consumer, total only n : one insurance contract for each consumer. Even if we add a riskless asset, we have only $n + 1$ "securities" to cover 2^n contingencies. Thus, even in the case of two consumers, markets

will necessarily be incomplete with three instruments asked to deal with four possible contingencies. As the number of consumers increases, the “incompleteness gap” will widen. Insurance markets, it seems, are necessarily incomplete.

Fortunately, this conclusion is unwarranted, an artifact of a static formulation. As we will show, once we allow for intertemporal trading of insurance contracts, markets can be completed with far fewer assets: one contract per consumer plus a riskfree asset. Just as in Kreps [1982], this conclusion relies on a hypothesis that uncertainty resolves “nicely.” What justifies this hypothesis in our model is an assumption that accident arrivals are governed by a counting process, a characterization which captures the intuition that, on a sufficiently fine time scale, accidents are few and far between. Returning to the simple example described above, what we are doing in effect is to “spread” the terminal date T over many time periods, keeping the total number of accidents (essentially) the same. If time periods last only a nanosecond, at most dates there will be no accident and at no date (to a reasonable approximation) will there be more than one. These are the assumptions underlying use of the familiar Poisson process to model accidents. Counting processes generalize the Poisson process to allow for “hazard rates” which vary over time, vary across individuals, or depend on past history while retaining the property that markets can be completed with one insurance contract per customer plus a riskfree asset. As an important byproduct, it is then possible to price a wide variety of insurance contracts under a wide variety of accident generating mechanisms, suggesting a role for insurance far broader than traditional insurance theory would seem to imply.

Our ultimate goal is to develop this theory of intertemporal insurance in a continuous time setting. However, in this paper we treat only discrete time. Section 2 applies “standard finance,” as represented by Dothan [1990], Duffie [1988], or Huang and Litzenberger [1988], to insurance markets. We begin with a formal description of a discrete time counting process, a special type of event tree. Using a counting process as the basic source of uncertainty, we then describe an intertemporal exchange economy on the event tree generated by the accident process and characterize the Walrasian equilibrium involving trade in Arrow-Debreu-Radner (ADR) time-event contingent commodities on this tree. The corresponding ADR time-event contingent prices allow us to price insurance contracts, which are **not** standard ADR contracts, as redundant securities. A transformation of the underlying probability measure gives an alternative characterization of insurance

contract prices plus accumulated payouts as martingales in the fashion of Harrison-Kreps [1979]. With this standard Walrasian equilibrium in place, we then “remove” the ADR contingent commodities, leaving only the insurance contracts to deal with risks. A direct application of the usual dynamic spanning argument (as presented, for example, in Huang and Litzenberger [1988]) demonstrates that these insurance contracts suffice “generically” for dynamic spanning.

In Section 3 we give a detailed illustration of this theory applied to a simple setting with two consumers. While dynamic spanning and martingale pricing may be unfamiliar to most readers, this example should demonstrate that the implications of the theory developed here are straightforward and reasonable. In this “Edgeworth box” economy, consumers tend to buy and hold the same insurance portfolio over time whatever the realization of the accident history, and the martingale pricing formula generates pricing rules for insurance assets with surprising ease.

A concluding section points to future directions for this research.

2 The model

We confine attention to a model of pure exchange in discrete time with a single commodity available for consumption at each date. We begin with a representation of the underlying source of uncertainty as a discrete time, multivariate counting process.

2.1 Characterizing the accident process

Let $\mathcal{T} := \{0, 1, \dots, T\}$ represent the time set. Accidents happen between dates, and the information that an accident has occurred between dates $t - 1$ and t is known to all traders prior to trade at date t . As a shorthand convention, an accident happening between dates $t - 1$ and t is said to happen “at date t .” Accidents at any date can be classified into one of a finite number of types indexed by $J := \{0, \dots, K\}$ with $j = 0$ signifying “no accident.” All uncertainty in the economy is captured entirely by the probability space (Ω, \mathcal{F}, P) with state space $\Omega := J^T$, σ -algebra $\mathcal{F} = 2^\Omega$, and probability measure P . Throughout this paper, we assume that $P(\omega) > 0$ for all $\omega \in \Omega$.

Figure 1 illustrates for the case $T = K = 2$ which contains nine sample points. The point ω_{20} , for example, corresponds to the realization

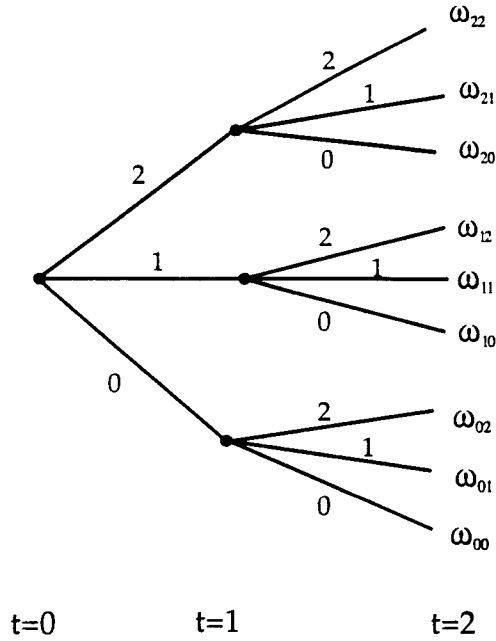


Figure 1: The sample space.

- accident of type 2 at date 1;
- no accident at date 2.

For $j > 0$, let

$$N_j: T \times \Omega \rightarrow \mathbf{Z}_+$$

represent the stochastic process which counts the number of accidents of type j : i.e., $N_j(t, \omega)$ is the number of accidents of type j which have occurred up to date t . N_j is nondecreasing, and $N_j(0, \omega) = 0$ for all $\omega \in \Omega$. The \mathbf{Z}_+^K -valued stochastic process defined by

$$N(t, \omega) := (N_1(t, \omega), \dots, N_K(t, \omega))$$

is called a discrete time, K -variate counting process. Figure 2, which imitates a similar figure for a continuous time counting process in Brémaud¹, illustrates a typical realization for an economy with two accident types:

- accidents of type 1 occur at dates 1 and 4;
- accidents of type 2 occur at dates 3 and 5; and

¹See Brémaud [1981], p. 20.

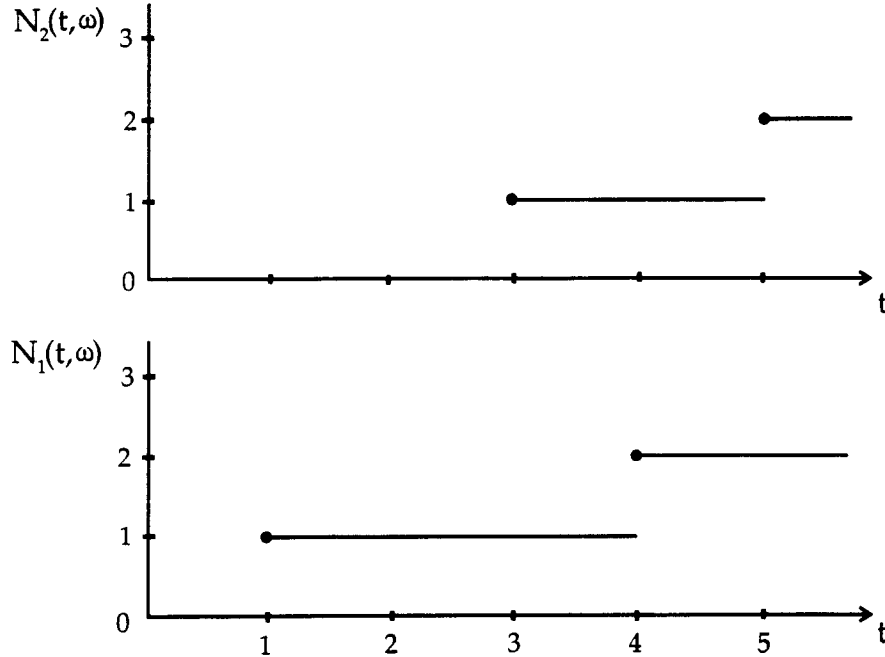


Figure 2: A realization of the accident process.

- there is no accident at date 2.

Note that there is never more than one accident of any type at any given date.

In the usual way, the stochastic process N generates a filtration² \mathcal{F} , a non-decreasing sequence of σ -algebras

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_T = \mathcal{F}$$

with the property that \mathcal{F}_t is the coarsest σ -algebra with respect to which the random variable

$$N(t): \Omega \rightarrow \mathbf{Z}_+$$

is measurable. Let f_t denote the partition of Ω which generates the σ -algebra \mathcal{F}_t . By definition of the accident process, $f_0 = \{\Omega\}$ and $f_T = \{\{\omega\} \mid \omega \in \Omega\}$, corresponding to the absence of information at date $t = 0$ and complete information at date T . In Figure 1, the partition f_1 at date 1 consists of the three sets

$$a_{10} := \{\omega_{00}, \omega_{01}, \omega_{02}\}, \quad a_{11} := \{\omega_{10}, \omega_{11}, \omega_{12}\}, \quad a_{12} := \{\omega_{20}, \omega_{21}, \omega_{22}\}$$

²As is common in the literature, we use the same notation for the σ -algebra \mathcal{F} and the filtration \mathcal{F} . The proper interpretation should always be clear from the context.

corresponding to the events

- no accident at date 1;
- accident of type 1 at date 1;
- accident of type 2 at date 2

respectively.

2.2 Describing the economy

Consumers in this economy are indexed by the finite set $I = \{1, \dots, n\}$. For each consumer $i \in I$, a consumption process x_i is a function

$$x_i: \mathcal{T} \times \Omega \rightarrow \mathbf{R}$$

and an endowment process w_i a function

$$w_i: \mathcal{T} \times \Omega \rightarrow \mathbf{R}.$$

adapted to the filtration \mathcal{F} generated by the accident process N . Let L denote this vector space, the set of all functions

$$x: \mathcal{T} \times \Omega \rightarrow \mathbf{R}$$

such that $x^{-1}(G) \in \mathcal{F}_t$ for every Borel subset G of \mathbf{R} and for every $t \in \mathcal{T}$. Equivalently, processes adapted to \mathcal{F} are constant on the sets $\{t\} \times a_t$.

Contingent commodities provide a convenient way to represent these consumption or endowment processes. The (t, a_t) -contingent commodity, representing one unit of consumption in event $a_t \in f_t$ at date t , is represented by the indicator function $1(t, a_t): \mathcal{T} \times \Omega \rightarrow \mathbf{R}$ defined by³

$$1(t, a_t)(t', \omega') = \begin{cases} 1 & \text{if } t' = t \text{ and } \omega' \in a_t; \\ 0 & \text{otherwise.} \end{cases}$$

Using these contingent commodities as a basis, a consumption process $x_i \in L$ has the representation

$$x_i = \sum_{t=0}^T \sum_{a_t \in f_t} x_i(t, a_t) 1(t, a_t)$$

³We write $1(t, a_t)$ rather than the more usual $1_{(t, a_t)}$ for typographical convenience.

and an endowment process the representation

$$w_i = \sum_{t=0}^T \sum_{a_t \in f_t} w_i(t, a_t) 1(t, a_t).$$

Similarly, consumption at date t can be written

$$x_i(t) = \sum_{a_t \in f_t} x_i(t, a_t) 1(t, a_t)$$

and endowment at date t as

$$w_i(t) = \sum_{a_t \in f_t} w_i(t, a_t) 1(t, a_t).$$

Letting

$$L_+ := \{x \in L \mid x(t, a_t) \geq 0 \forall t \in \mathcal{T} \ \& \ a_t \in f_t\}$$

represent the nonnegative orthant of L , assume that each consumer $i \in I$ has consumption set $X_i = L_+$, endowment $w_i \in L_+$, and preference relation \succeq_i on $X_i \times X_i$ which is a complete preordering and strongly monotonic. Walrasian prices are given by a linear functional $p: L \rightarrow \mathbf{R}$ with representation

$$p(x) = \pi \cdot x = \sum_{t \in \mathcal{T}} \sum_{a_t \in f_t} \pi(t, a_t) x(t, a_t)$$

where $\pi(t, a_t)$ is the price of a (t, a_t) -contingent commodity. Just as for consumption processes x , we can also view $\pi \in L_+$ as a stochastic process $\pi: \mathcal{T} \times \Omega \rightarrow \mathbf{R}$ adapted to the filtration \mathcal{F} . With respect to the Walrasian price system π , consumer i has budget set⁴

$$\beta_i(\pi) := \{x_i \in X_i \mid \pi \cdot x_i \leq \pi \cdot w_i\}$$

and demand set

$$\phi_i(\pi) := \{x_i \in X_i \mid \beta_i(\pi) \cap P_i(x_i) = \emptyset\}$$

where

$$P_i(x_i) := \{x'_i \in X_i \mid x'_i \succ_i x_i\}$$

is the strict preference set of consumer i . An allocation $x: I \rightarrow L_+$ is **feasible** if

$$\sum_{i \in I} x_i = \sum_{i \in I} w_i.$$

A **Walrasian equilibrium** for this exchange economy consists of a feasible allocation x and a price system π such that $x_i \in \phi_i(\pi)$ for all $i \in I$.

⁴The notation adopted here is that of Ellickson [1993].

2.3 The pricing of insurance

The Walrasian equilibrium described above requires a large number of contingent commodities, one such commodity for each date-event (t, a_t) . Suppose that, in addition to this complete set of ADR contingent commodities, we introduce a set of redundant assets, one for each type of accident $j \neq 0$. Associated with each security $j \neq 0$ is a dividend process $d_j \in L_+$ where $d_j(t, a_t)$ represents the payout of insurance policy j at date-event (t, a_t) , measured in units of the consumption good at (t, a_t) . We assume that $d_j(0, \Omega) = 0$ for all $j \neq 0$. As for ADR prices and consumption processes, each dividend process d_j can be viewed as a stochastic process adapted to \mathcal{F} , reflecting the fact that payouts on an insurance contract must be based only on accidents which have already happened and not on those which are yet to come. In addition to these insurance policies, we also assume there exists a riskfree asset available at each date-event (t, a_t) which costs one unit of the consumption good at (t, a_t) and returns $d_0(t, a_{t+1}) = 1 + r(t, a_t)$ of the consumption good at each successor event $a_{t+1} \subset a_t$, $a_{t+1} \in f_{t+1}$. We will refer to $r(t, a_{t+1})$ as the riskfree rate.

From now on, we adopt the price normalization⁵ $\pi(0) = 1$ so that the Walrasian price functional has the representation

$$\pi \cdot x = x(0) + \sum_{t=1}^T \sum_{a_t \in f_t} \pi(t, a_t) x(t, a_t).$$

Given the ADR prices π , define for each event $a_t \in f_t$ and $a_{t+1} \in f_{t+1}$, where $a_{t+1} \subset a_t$, the **martingale conditional probability**

$$Q(a_{t+1} | a_t) := \frac{\pi(t+1, a_{t+1})}{\sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1})}. \quad (1)$$

Assuming that prices of all insurance assets are ex dividend, we define for each $t < T$ the price process S_j for asset j according to the relation

$$\pi(t, a_t) S_j(t, a_t) = \sum_{s=t+1}^T \sum_{\substack{a_s \in f_s \\ a_s \subset a_t}} \pi(s, a_s) d_j(s, a_s),$$

⁵Because the initial and terminal information partitions have a special structure,

$$f_0 = \{\Omega\} \quad \text{and} \quad f_T = \{\{\omega\} | \omega \in \Omega\},$$

it will often be convenient to abuse notation slightly, writing $\pi(0)$ in place of $\pi(0, \Omega)$ and $x(0)$ in place of $x(0, \Omega)$.

the Arrow-Debreu valuation of the dividend stream following the date-event (t, a_t) . At terminal nodes, $S_j(T, \omega) = 0$. Note that $S_j \in L_+$ so that security prices can also be viewed as nonnegative stochastic processes adapted to the accident filtration \mathcal{F} . As with payouts, the security price $S_j(t, a_t)$ is measured in units of the (t, a_t) -consumption good.

As a simple consequence of the tree structure of the filtration,

$$\pi(t, a_t)S_j(t, a_t) = \sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1}) [S_j(t+1, a_{t+1}) + d_j(t+1, a_{t+1})]. \quad (2)$$

where the sum is over events a_{t+1} belonging to the partition f_{t+1} and contained in a_t . In the special case of the riskfree asset, which costs one unit of the consumption good at (t, a_t) and pays $1+r(t, a_t)$ at each of the immediate successor nodes, the above condition specializes to

$$\pi(t, a_t) = \sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1})(1 + r(t, a_t))$$

or, equivalently,

$$1 + r(t, a_t) = \frac{\pi(t, a_t)}{\sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1})} \quad (3)$$

for all $t < T$. Using the definitions of the martingale conditional probability and the riskfree rate, equation (2) can now be written

$$S_j(t, a_t) = \frac{\sum_{a_{t+1} \subset a_t} [S_j(t+1, a_{t+1}) + d_j(t+1, a_{t+1})] Q(a_{t+1} | a_t)}{1 + r(t, a_t)}.$$

Letting $r(t)$ denote the interest rate at date t ,

$$r(t) := \sum_{a_t \in f_t} r(t, a_t) 1(t, a_t),$$

and $S_j(t)$ the price of the j^{th} asset at date t ,

$$S_j(t) := \sum_{a_t \in f_t} S_j(t, a_t) 1(t, a_t),$$

we have

$$S_j(t) = \frac{E_Q[S_j(t+1) + d_j(t+1) | \mathcal{F}_t]}{1 + r(t)} \quad (4)$$

where $E_Q[\cdot | \mathcal{F}_t]$ denotes conditional expectation relative to the filtration \mathcal{F} under the martingale measure Q .

Using the riskfree rate to discount insurance asset prices and their payouts, define

$$S_j^*(t) := \frac{S_j(t)}{\prod_{s=0}^{t-1} (1 + r(s))}$$

and

$$d_j^*(t) := \frac{d_j(t)}{\prod_{s=0}^{t-1} (1 + r(s))}.$$

Define the cumulative discounted dividend process for security j as

$$D_j^*(t) = \sum_{s=0}^t d_j^*(s).$$

From equation (4) it follows that

$$\begin{aligned} S_j^*(t) + D_j^*(t) &= \frac{E_Q[S_j(t+1) + d_j(t+1) | \mathcal{F}_t]}{\prod_{s=0}^t (1 + r(s))} + D_j^*(t) \\ &= E_Q[S_j^*(t+1) + D_j^*(t) + d_j^*(t+1) | \mathcal{F}_t] \\ &= E_Q[S_j^*(t+1) + D_j^*(t+1) | \mathcal{F}_t]; \end{aligned} \quad (5)$$

i.e., the sum of the discounted insurance asset price and its discounted cumulative payout is a martingale with respect to the martingale measure Q .

2.4 Dynamic spanning

Although the machinery is rather elaborate, the basic idea behind dynamic spanning is quite simple. In an event tree context, the key to dynamic spanning is the **index** of the filtration, the maximum number of branches leaving any node of the event tree: the number of accident types plus one in our model. As shown by Kreps [1982], the number of securities required for dynamic spanning is no greater than the index of the filtration. Thus, in addition to the riskfree asset, one insurance contract for each type of accident is all that is necessary for dynamic spanning.

Let α represent an arbitrary event $a_{t-1} \in f_{t-1}$ and

$$B_t(\alpha) = \{a_t \in a_{t-1} | a_t \in f_t\} = \{\beta_0, \beta_1, \dots, \beta_K\}$$

the collection of its immediate successors (see Figure 3). Let

$$\theta_i(t, \alpha) = (\theta_i^0(t, \alpha), \theta_i^1(t, \alpha), \dots, \theta_i^K(t, \alpha))^T$$

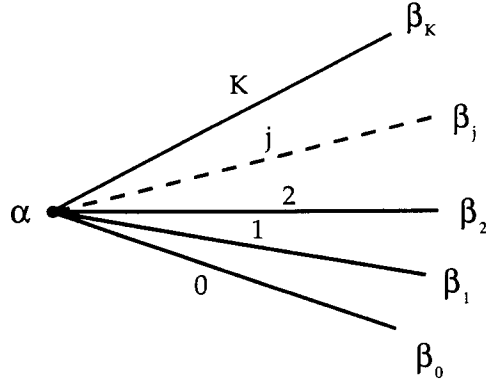


Figure 3: Event α and its successors

represent the portfolio of securities purchased by consumer i at date-event $(t - 1, \alpha)$ and

$$\theta_i(t + 1, \beta) = (\theta_i^0(t + 1, \beta), \theta_i^1(t + 1, \beta), \dots, \theta_i^K(t + 1, \beta))^T$$

the portfolio acquired at date-event (t, β) , $\beta \in B_t(\alpha)$ where T denotes transpose. Note that, by definition, the **trading process** θ_i is not only adapted to the filtration, but also **predictable**: i.e., for each t , $\theta_i(t)$ is measurable with respect to \mathcal{F}_{t-1} . Requiring predictability captures the economically natural restriction that a consumer must buy insurance prior to acquiring knowledge whether the insured event will occur. Finally, let

$$\Delta_w x_i(t, \beta) := x_i(t, \beta) - w_i(t, \beta)$$

represent the net trade of consumer i at date-event (t, β) . For each $\beta \in B_t(\alpha)$, budget balance in the spot market at date-event (t, β) requires

$$\Delta_w x_i(t, \beta) = \sum_{j \in J} \theta_i^j(t, \alpha) [S_j(t, \beta) + d_j(t, \beta)] - \sum_{j \in J} \theta_i^j(t + 1, \beta) S_j(t, \beta).$$

which, letting

$$c_i(t, \alpha, \beta) := \Delta_w x_i(t, \beta) + \sum_{j \in J} \theta_i^j(t + 1, \beta) S_j(t, \beta),$$

can be written

$$\sum_{j \in J} \theta_i^j(t, \alpha) [S_j(t, \beta) + d_j(t, \beta)] = c_i(t, \alpha, \beta). \quad (6)$$

For each event α , there are $K + 1$ such equations, one for each of the successor events $\beta \in B_t(\alpha)$, and $K + 1$ unknowns, the portfolio

$$\theta_i(t, \alpha) = (\theta_i^0(t, \alpha), \theta_i^1(t, \alpha), \dots, \theta_i^K(t, \alpha)).$$

For each date-event (t, α) , define the $K + 1$ by $K + 1$ matrices

$$S(t, \alpha) = \begin{pmatrix} S_0(t, \beta_0) & S_1(t, \beta_0) & \dots & S_K(t, \beta_0) \\ S_0(t, \beta_1) & S_1(t, \beta_1) & \dots & S_K(t, \beta_1) \\ \vdots & \vdots & \ddots & \vdots \\ S_0(t, \beta_K) & S_1(t, \beta_K) & \dots & S_K(t, \beta_K) \end{pmatrix}$$

and

$$D(t, \alpha) = \begin{pmatrix} d_0(t, \beta_0) & d_1(t, \beta_0) & \dots & d_K(t, \beta_0) \\ d_0(t, \beta_1) & d_1(t, \beta_1) & \dots & d_K(t, \beta_1) \\ \vdots & \vdots & \ddots & \vdots \\ d_0(t, \beta_K) & d_1(t, \beta_K) & \dots & d_K(t, \beta_K) \end{pmatrix}.$$

The system of equations of type (6) at date-event (t, α) then takes the form

$$[S(t, \alpha) + D(t, \alpha)]\theta_i(t, \alpha) = c_i(t, \alpha) \quad (7)$$

where

$$c_i(t, \alpha) = (c_i(t, \alpha, \beta_0), c_i(t, \alpha, \beta_1), \dots, c_i(t, \alpha, \beta_K))^T.$$

An ADR equilibrium allocation is said to be **dynamically spanned** by the set of securities J provided there is a portfolio $\theta_i(t, \alpha)$ which solves equation system (7) for every date-event (t, α) in the filtration \mathcal{F} and every consumer $i \in I$.

If T is finite and the matrix $S(t, \alpha) + D(t, \alpha)$ is invertible for every date-event (t, α) , then finding a set of dynamically spanning portfolio trades is straightforward. Using system (7), we solve first for the portfolios purchased at date $T - 1$ and then work back recursively to the initial holdings at date 0. Assuming that securities are priced ex dividend, for each security j we have $S_j(T, \omega) = 0$ for all $\omega \in \Omega$. Consequently, for $t = T$ and $\alpha \in f_{T-1}$, system (7) reduces to

$$D(T, \alpha)\theta_i(T, \alpha) = c_i(T, \alpha).$$

Since consumers will carry no portfolio holdings beyond date T , $c_i(T, \alpha, \beta)$ is simply the net trade $\Delta_w x_i(T, \beta)$ at the successor node β . Therefore, provided the dividend matrix $D(T, \alpha)$ is invertible, the equation system can

be solved for a unique portfolio $\theta_i(T, \alpha)$ for each $\alpha \in f_{T-1}$ and each consumer $i \in I$.

Moving back a date, consider an event $\alpha \in f_{T-2}$. From the computations at date T , the required portfolios $\theta_i(T, \beta)$ are known for each successor event $\beta \in B_t(\alpha)$ and consequently $c_i(T-1, \alpha)$ can be computed. Therefore, provided that the matrix $S(T-1, \alpha) + D(T-1, \alpha)$ is invertible, system (7) can be solved for a unique portfolio $\theta_i(T-1, \alpha)$ for each $\alpha \in f_{T-2}$.

Continuing recursively in this fashion, portfolios $\theta_i(t, \alpha)$ can be computed for each consumer $i \in I$ and date-event (t, α) until finally we reach $t = 0$. Since the initial partition is trivial, $f_0 = \{\Omega\}$, we can write $\theta_i(1, \Omega) = \theta_i(1)$. Assuming that consumers have no initial endowments of securities, $\theta_i(0) = 0$. System (7) then reduces to the requirement $c_i(0, \alpha) = 0$ or

$$\Delta_w x_i(0) + \sum_{j \in J} \theta_i^j(1) S_j(0) = 0 \quad (8)$$

which requires the initial purchase of assets at date 0 to offset the net trade at date 0.

3 A Cobb-Douglas illustration

In this section we illustrate our model of intertemporal insurance with a simple example close in spirit to traditional models of insurance markets: consumers face an accident process with hazard rates which are independent of the past history of the accident process and constant over time. We begin by deriving the general equilibrium results: the Arrow-Debreu-Radner prices and the equilibrium net trades in contingent commodities. Two insurance regimes are then considered, one providing short-term insurance on next period's events and the other providing long-term contracts paying a unit of the consumption good every time an accident occurs in the future.

Assume that consumer $i \in I$ has von-Neumann Morgenstern utility

$$u_i(x_i) = \ln x_i(0) + \sum_{t=1}^T \delta^t \sum_{a_t \in f_t} P(a_t) \ln x_i(t, a_t) \quad (9)$$

where $\delta \in [0, 1)$ and $P(a_t)$ is the probability of event $a_t \in f_t$. Let

$$w(t, a_t) := \sum_{i \in I} w_i(t, a_t)$$

denote the aggregate endowment of the (t, a_t) -contingent commodity and

$$w(0) := \sum_{i \in I} w_i(0)$$

the corresponding endowment at date 0. Using the normalization $\pi(0) = 1$, ADR market clearing prices are given by

$$\pi(t, a_t) = \frac{\delta^t w(0) P(a_t)}{w(t, a_t)}. \quad (10)$$

Specializing to the case of two consumers, assume that in each period,

- no accident occurs with probability 1/4;
- an accident occurs to consumer one with probability 1/2;
- an accident occurs to consumer two with probability 1/4.

Thus, we take the set of accident types to be $J = \{0, 1, 2\}$ where

$j = 0$ means “there is no accident;”

$j = 1$ means “an accident happens to consumer 1;” and

$j = 2$ means “an accident happens to consumer 2;”

and hazard rates are constant over time: i.e., if $a_t \in f_t$, $a_{t+1} \in f_{t+1}$, and $a_{t+1} \subset a_t$, then

$$P(a_{t+1} | a_t) = \begin{cases} 1/4 & \text{if } j = 0 \text{ at date } t + 1; \\ 1/2 & \text{if } j = 1 \text{ at date } t + 1; \\ 1/4 & \text{if } j = 2 \text{ at date } t + 1. \end{cases}$$

Assume each consumer is endowed with Y units of the single commodity at each node gross of any loss to accidents and that an accident at date t results in the total loss of the endowment at that date. Formally,

$$w_i(t, a_t) = Y - Y \Delta N_i(t, a_t) = \begin{cases} 0 & \text{if } j = i \text{ at date } t; \\ Y & \text{if } j \neq i \text{ at date } t. \end{cases}$$

Under these assumptions, aggregate endowment $w(0) = 2Y$ and aggregate endowment

$$w(t, a_t) = \begin{cases} 2Y & \text{if } j = 0 \text{ at date } t; \\ Y & \text{if } j \neq 0 \text{ at date } t. \end{cases}$$

From equation (10), ADR prices are

$$\pi(t, a_t) = \begin{cases} \delta^t P(a_t) & \text{if } j = 0 \text{ at date } t; \\ 2\delta^t P(a_t) & \text{if } j \neq 0 \text{ at date } t. \end{cases}$$

Equilibrium wealth for consumer i is

$$\begin{aligned} \pi \cdot w_i &= w_i(0) + \sum_{t=1}^T \sum_{a_t \in f_t} \pi(t, a_t) w_i(t, a_t) \\ &= Y + 2Y \sum_{t=1}^T \delta^t \sum_{a_t \in f_t} \frac{P(a_t) w_i(t, a_t)}{w(t, a_t)} \\ &= Y + 2Y \sum_{t=1}^T \delta^t \sum_{a_{t-1} \in f_{t-1}} \sum_{a_t \subset a_{t-1}} \frac{P(a_t) w_i(t, a_t)}{w(t, a_t)} \\ &= Y + 2Y \sum_{t=1}^T \delta^t \sum_{a_{t-1} \in f_{t-1}} P(a_{t-1}) \sum_{a_t \subset a_{t-1}} P(a_t | a_{t-1}) \frac{w_i(t, a_t)}{w(t, a_t)}. \end{aligned}$$

For consumer 1,

$$\sum_{a_t \subset a_{t-1}} P(a_t | a_{t-1}) \frac{w_i(t, a_t)}{w(t, a_t)} = \frac{1}{4} \cdot \frac{Y}{2Y} + \frac{1}{2} \cdot \frac{0}{Y} + \frac{1}{4} \cdot \frac{Y}{Y} = \frac{3}{8}$$

while for consumer 2

$$\sum_{a_t \subset a_{t-1}} P(a_t | a_{t-1}) \frac{w_i(t, a_t)}{w(t, a_t)} = \frac{1}{4} \cdot \frac{Y}{2Y} + \frac{1}{2} \cdot \frac{Y}{Y} + \frac{1}{4} \cdot \frac{0}{Y} = \frac{5}{8}$$

Therefore,

$$\pi \cdot w_1 = Y + \frac{3Y}{4} \sum_{t=1}^{\infty} \delta^t = Y + \frac{3Y}{4} \frac{\delta(1 - \delta^T)}{1 - \delta}$$

and

$$\pi \cdot w_2 = Y + \frac{5Y}{4} \sum_{t=1}^{\infty} \delta^t = Y + \frac{5Y}{4} \frac{\delta(1 - \delta^T)}{1 - \delta}$$

Consumer 1, the more accident prone, has as a consequence lower wealth in equilibrium.

Equation (1) gives as the martingale conditional probabilities for $a_t \in f_t$, $a_{t+1} \in f_{t+1}$, and $a_{t+1} \subset a_t$:

$$Q(a_{t+1} | a_t) = \begin{cases} 1/7 & \text{if } j = 0 \text{ at date } t + 1; \\ 4/7 & \text{if } j = 1 \text{ at date } t + 1; \\ 2/7 & \text{if } j = 2 \text{ at date } t + 1. \end{cases}$$

Table 1: Equilibrium trades and net trades: $T = 2$

j	$x_1(t, a_{tj})$	$x_2(t, a_{tj})$	$\Delta_w x_1(t, a_{tj})$	$\Delta_w x_2(t, a_{tj})$
0	1,148	1,456	-154	154
1	574	728	574	-574
2	574	728	-728	728

Table 2: Equilibrium trades and net trades: $T = \infty$

j	$x_1(t, a_{tj})$	$x_2(t, a_{tj})$	$\Delta_w x_1(t, a_{tj})$	$\Delta_w x_2(t, a_{tj})$
0	1,116	1,488	-186	186
1	558	744	558	-558
2	558	744	-744	744

Thus, the “risk neutral” martingale conditional probabilities compensate for risk aversion in the economy by increasing the risk of accidents relative to the actuarial conditional probabilities $P(a_{t+1} | a_t)$.

Equation (3) yields

$$1 + r(t, a_t) = \begin{cases} 4/7\delta & \text{if } j = 0 \text{ at date } t; \\ 8/7\delta & \text{if } j \neq 0 \text{ at date } t. \end{cases}$$

For computational convenience, we set the discount factor $\delta = 4/7$ so that the riskfree rate becomes

$$r(t, a_t) = \begin{cases} 0 & \text{if there is no accident at date } t; \\ 1 & \text{otherwise.} \end{cases}$$

From now on we assume $Y = 1,302$, chosen to yield integer results for security prices and net trades in the computations which follow. We consider two horizons: a finite horizon, with $T = 2$, and a horizon which is infinite. In the finite horizon case, equilibrium wealth for the two consumers becomes

$$\pi \cdot w_1 = \left(\frac{82}{49}\right) 1,302 \approx 2,179 \quad \text{and} \quad \pi \cdot w_2 = \left(\frac{104}{49}\right) 1,302 \approx 2,763$$

while for the infinite horizon

$$\pi \cdot w_1 = 2,604 \quad \text{and} \quad \pi \cdot w_2 = 3,472.$$

Equilibrium allocations in this example have a very simple structure: $x_i(t, a_t)$ depends only on the accident type j at date t which functions as a “state variable.” Letting $a_{tj} \in f_t$ denote an event in which an accident of type j occurs, Table 1 shows for each possible event the equilibrium trades and net trades for $T = 2$ and Table 2 the corresponding trades and net trades for $T = \infty$. For example, in the case $T = 2$ consumer 1 consumes 1,148 if there has been no accident at date-event (t, a_t) and 574 if an accident has happened to either consumer 1 or consumer 2. Note that gross trades depend only on the “macro risk” in the economy, i.e., whether the total endowment is Y or $2Y$, while net trades also depend on who has the accident.

We consider how these ADR equilibria are implemented under two insurance regimes, one offering short-term insurance to consumers and the other offering long-term contracts.

3.1 Short-term insurance

We know that to achieve dynamic spanning, it is necessary to offer a separate insurance contract for each type of accident. In the first regime we consider, insurance contracts are short-term: one unit of insurance issued at date t on accidents of type j at date $t + 1$ returns a payout at $t + 1$ of

$$d_j(t + 1, a_{t+1}) = \begin{cases} 1 & \text{if an accident of type } j \text{ occurs at date } t + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Because security prices are ex dividend and this contract is short-term, it is worthless at date $t + 1$: $S_j(t + 1) = 0$. According to the fundamental equation (4) of asset pricing, the price of this asset depends both on the riskfree rate and on who is being insured. For consumer 1,

$$S_1(t, a_t) = \frac{1}{1 + r(t, a_t)} \left[\frac{1}{7}(0) + \frac{4}{7}(1) + \frac{2}{7}(0) \right]$$

and so

$$S_1(t, a_t) = \begin{cases} 4/7 & \text{if there were no accidents at date } t; \\ 2/7 & \text{otherwise;} \end{cases}$$

Similarly, for consumer 2,

$$S_2(t, a_t) = \begin{cases} 2/7 & \text{if there were no accidents at date } t; \\ 1/7 & \text{otherwise.} \end{cases}$$

Equation (6), the budget balancing condition in the spot-market at date-event (t, a_t) , must be modified slightly to account for the fact that the insurance securities are short-term. At date t there are two securities of type j in existence: the “old” contracts issued at $t - 1$ and paying off at t and the “new” contracts issued at t and paying off at $t + 1$. Since contracts issued yesterday are worthless today (i.e., their ex dividend price is zero), we reserve $S_j(t, a_t)$ to represent the price of contracts issued at date t . With this modification, equation system (7) becomes

$$D(t, \alpha)\theta_i(t, \alpha) = c_i(t, \alpha)$$

where

$$c_i(t, \alpha) = (c_i(t, \alpha, \beta_0), c_i(t, \alpha, \beta_1), \dots, c_i(t, \alpha, \beta_K))^T$$

and

$$c_i(t, \alpha, \beta) := \Delta_w x_i(t, \beta) + \sum_{j \in J} \theta_i^j(t + 1, \beta) S_j(t, \beta)$$

as before.

Suppose first that the horizon $T = 2$ so that the event tree of Figure 1 applies. For the date 1 node $\alpha = a_{10}$ at which no accident has happened,

$$D(2, a_{10}) = \begin{pmatrix} 1 + r(1, a_{10}) & d_1(2, \omega_{00}) & d_2(2, \omega_{00}) \\ 1 + r(1, a_{10}) & d_1(2, \omega_{01}) & d_2(2, \omega_{01}) \\ 1 + r(1, a_{10}) & d_1(2, \omega_{02}) & d_2(2, \omega_{01}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and so for consumer 1 the modified equation system (7) becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} \theta_1^0(2, a_{10}) \\ \theta_1^1(2, a_{10}) \\ \theta_1^2(2, a_{10}) \end{bmatrix} = \begin{bmatrix} -154 \\ 574 \\ -728 \end{bmatrix}$$

where the right hand side is simply the vector of net trades for consumer 1 at the terminal nodes $\{\omega_{00}, \omega_{01}, \omega_{02}\}$. Since the matrix $D(2, a_{10})$ is invertible, this equation system has as its unique solution the portfolio

$$\theta_1(2, a_{10}) = \begin{bmatrix} -154 \\ 728 \\ -574 \end{bmatrix}.$$

For nodes $\alpha = a_{11}$ or $\alpha = a_{12}$ at which an accident has happened to consumers 1 or 2, the equation system becomes

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{bmatrix} \theta_1^0(2, \alpha) \\ \theta_1^1(2, \alpha) \\ \theta_1^2(2, \alpha) \end{bmatrix} = \begin{bmatrix} -154 \\ 574 \\ -728 \end{bmatrix}$$

where the shift of the leading column of the dividend matrix from 1's to 2's reflects the higher riskfree rate when an accident occurs. Solving gives the portfolios

$$\theta_1(2, a_{11}) = \theta_1(2, a_{12}) = \begin{bmatrix} -77 \\ 728 \\ -574 \end{bmatrix}.$$

Shifting back one date and using the information generated by the first step of the recursion, at the initial date-event $(0, a_0)$ the equation system becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} \theta_1^0(1) \\ \theta_1^1(1) \\ \theta_1^2(1) \end{bmatrix} = \begin{bmatrix} -56 \\ 623 \\ -679 \end{bmatrix},$$

yielding the portfolio

$$\theta_1(1) = \begin{bmatrix} -56 \\ 679 \\ -623 \end{bmatrix}.$$

Since

$$-\Delta_w x_1(0) = 154 = \sum_{j=0}^2 \theta_1^j(1) S_j(0),$$

equation (8) is satisfied: the initial purchase of assets by consumer 1 at date 0 is exactly offset by her negative net trade at date 0.

Consumer 2 holds precisely the opposite portfolio from consumer 1,

$$\theta_2(t, a_t) = -\theta_1(t, a_t) \quad \text{for all } (t, a_t),$$

as can be easily verified. Each consumer purchases insurance on herself at each date and sells insurance to the other consumer with the portfolio adjusting over time in response to accident history.

In the infinite horizon case, the trading portfolios are stationary:

$$\theta_1(t, a_t) = \begin{bmatrix} 0 \\ 651 \\ -651 \end{bmatrix} \quad \text{and} \quad \theta_2(t, a_t) = \begin{bmatrix} 0 \\ -651 \\ 651 \end{bmatrix}$$

for every date-event (t, a_t) with each consumer buying 651 units of insurance on herself and selling the same amount to the other consumer. The riskless asset is not held in equilibrium.

Verifying that (modified) equation system (7) is satisfied is straightforward. At a date-event (t, α) at which there has been no accident, we have for consumer 1

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 651 \\ -651 \end{bmatrix} = \begin{bmatrix} 0 \\ 651 \\ -651 \end{bmatrix},$$

while, if there has been an accident,

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 651 \\ -651 \end{bmatrix} = \begin{bmatrix} 0 \\ 651 \\ -651 \end{bmatrix}.$$

Finally, since

$$-\Delta_w x_1(0) = 186 = \sum_{j=0}^2 \theta_1^j(1) S_j(0),$$

the initial purchase of assets by consumer 1 at date 0 is again exactly offset by her negative net trade at date 0.

It is also easy to verify that the martingale equation (5) is satisfied for these short-term insurance contracts. For contract j issued at date t , the discounted price

$$S_j^*(t) = \frac{S_j(t)}{\prod_{s=0}^{t-1} (1+r(s))}$$

while $S_j^*(t+1) = 0$. Because the contracts are short term, there is no accumulation of dividends over time and hence

$$D_j^*(t) = 0 \quad \text{and} \quad D_j^*(t+1) = \frac{d_j(t+1)}{\prod_{s=0}^t (1+r(s))}.$$

Thus, equation (5) reduces to

$$S_j(t) = \frac{1}{1+r(t)} E_Q[d_j(t+1) \mid \mathcal{F}_t]$$

which is equivalent to equation (4).

3.2 Long-term insurance

Although short-term insurance contracts substitute perfectly for ADR contingent contracts, they seem only slightly more realistic than their ADR counterparts. Taking a step closer to reality, we now consider long-term

insurance contracts. Specifically, an insurance policy on an accident of type j is a long-term obligation which, at a price $S_j(t, a_t)$ at date-event (t, a_t) , returns one unit of the consumption good at each subsequent date-event at which an accident of type j occurs.

To make matters more interesting, we also replace the riskfree asset with a bond which pays one unit of the consumption good at every date-event (t, a_t) from time one forward. For $a_t \in f_t$ let $\beta_j \in f_{t+1}$ be the immediate successor event in which an accident of type j occurs. From equation (4), the price of the bond at (t, a_t) for $t < T$ is

$$\begin{aligned} S_0(t, a_t) &= \frac{\frac{1}{7}[S_0(t+1, \beta_0) + 1] + \frac{4}{7}[S_0(t+1, \beta_1) + 1] + \frac{2}{7}[S_0(t+1, \beta_2) + 1]}{1 + r(t, a_t)} \\ &= \frac{1 + \frac{1}{7}S_0(t+1, \beta_0) + \frac{4}{7}S_0(t+1, \beta_1) + \frac{2}{7}S_0(t+1, \beta_2)}{1 + r(t, a_t)}. \end{aligned}$$

Letting $\alpha_j \in f_t$ be an event in which an accident of type j occurs, we conclude that

$$S_0(t, \alpha_0) = 2S_0(t, \alpha_1) = 2S_0(t, \alpha_2)$$

and similarly

$$S_0(t+1, \beta_0) = 2S_0(t+1, \beta_1) = 2S_0(t+1, \beta_2).$$

Substitution gives

$$S_0(t, a_t) = \frac{1}{1 + r(t, a_t)} \left[1 + \frac{4}{7}S_0(t+1, \beta_0) \right].$$

If the horizon $T = 2$, then $S_0(2, \omega) = 0$ for all $\omega \in \Omega$. Letting $a_{1j} \in f_1$ if an accident of type j occurs at date t , we conclude that

$$S_0(1, a_{10}) = 1 \quad \text{and} \quad S_0(1, a_{11}) = S_0(1, a_{12}) = 1/2$$

and hence

$$S_0(0) = 1 + \left[\frac{1}{7} + \frac{2}{7} + \frac{1}{7} \right] = \frac{11}{7}.$$

When the horizon $T = \infty$, we can use the fact that the price of the bond will clearly be stationary to conclude that

$$S_0(t, \alpha_0) = 1 + \frac{4}{7}S_0(t, \alpha_0)$$

and hence that

$$S_0(t, a_t) = \begin{cases} 7/3 & \text{if there is no accident at date } t; \\ 7/6 & \text{if there is an accident at date } t. \end{cases}$$

Turning now to the insurance contracts, equation (4) implies that for $t < T$

$$\begin{aligned} S_1(t, a_t) &= \frac{\frac{1}{7}S_0(t+1, \beta_0) + \frac{4}{7}[S_0(t+1, \beta_1) + 1] + \frac{2}{7}S_0(t+1, \beta_2)}{1 + r(t, a_t)} \\ &= \frac{\frac{4}{7} + \frac{1}{7}S_0(t+1, \beta_0) + \frac{4}{7}S_0(t+1, \beta_1) + \frac{2}{7}S_0(t+1, \beta_2)}{1 + r(t, a_t)} \end{aligned}$$

and

$$\begin{aligned} S_2(t, a_t) &= \frac{\frac{1}{7}S_0(t+1, \beta_0) + \frac{4}{7}S_0(t+1, \beta_1) + \frac{2}{7}[S_0(t+1, \beta_2) + 1]}{1 + r(t, a_t)} \\ &= \frac{\frac{2}{7} + \frac{1}{7}S_0(t+1, \beta_0) + \frac{4}{7}S_0(t+1, \beta_1) + \frac{2}{7}S_0(t+1, \beta_2)}{1 + r(t, a_t)} \end{aligned}$$

Using the same notation as for the bond, we conclude that

$$S_j(t, \alpha_0) = 2S_j(t, \alpha_1) = 2S_j(t, \alpha_2)$$

and similarly

$$S_j(t+1, \beta_0) = 2S_j(t+1, \beta_1) = 2S_j(t+1, \beta_2).$$

Substitution gives

$$S_1(t, a_t) = \frac{1}{1 + r(t, a_t)} \left[\frac{4}{7} + \frac{4}{7}S_0(t+1, \beta_0) \right]$$

and

$$S_2(t, a_t) = \frac{1}{1 + r(t, a_t)} \left[\frac{2}{7} + \frac{4}{7}S_0(t+1, \beta_0) \right].$$

If the horizon $T = 2$, we use the fact that $S_j(2) = 0$ for $j = 1, 2$ to conclude that

$$\begin{aligned} S_1(1, a_{10}) &= \frac{4}{7} & \text{and} & & S_1(1, a_{11}) &= S_1(1, a_{12}) &= \frac{2}{7} \\ S_2(1, a_{10}) &= \frac{2}{7} & \text{and} & & S_2(1, a_{11}) &= S_2(1, a_{12}) &= \frac{1}{7} \end{aligned}$$

and hence

$$S_1(0) = \frac{44}{49} \quad \text{and} \quad S_2(0) = \frac{22}{49}.$$

When $T = \infty$, we can use stationary of the insurance asset prices to conclude that

$$S_1(t, \alpha_0) = \frac{4}{7} + \frac{4}{7}S_1(t, \alpha_0)$$

and

$$S_2(t, \alpha_0) = \frac{2}{7} + \frac{4}{7}S_1(t, \alpha_0)$$

and consequently

$$S_1(t, a_t) = \begin{cases} 4/3 & \text{if there is no accident at date } t; \\ 2/3 & \text{if there is an accident at date } t \end{cases}$$

and

$$S_2(t, a_t) = \begin{cases} 2/3 & \text{if there is no accident at date } t; \\ 1/3 & \text{if there is an accident at date } t. \end{cases}$$

Because the insurance contracts are long-term, equation (6), the budget balancing condition in the spot-market at date-event (t, a_t) , now requires no modification: $S_j(t, a_t)$ is the price at which an insurance contract of type j can be bought or sold.

Assuming first a horizon $T = 2$ and applying equation system (7) to the date 1 node $\alpha = a_{10}$ at which no accident has happened, we have for consumer 1

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} \theta_1^0(2, a_{10}) \\ \theta_1^1(2, a_{10}) \\ \theta_1^2(2, a_{10}) \end{bmatrix} = \begin{bmatrix} -154 \\ 574 \\ -728 \end{bmatrix}$$

which has as its solution

$$\theta_1(2, a_{10}) = \begin{bmatrix} -154 \\ 728 \\ -574 \end{bmatrix},$$

just as in the case of short-term insurance.

For nodes $\alpha = a_{11}$ or $\alpha = a_{12}$ at which an accident has happened, the equation system becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} \theta_1^0(2, \alpha) \\ \theta_1^1(2, \alpha) \\ \theta_1^2(2, \alpha) \end{bmatrix} = \begin{bmatrix} -154 \\ 574 \\ -728 \end{bmatrix}$$

with solution

$$\theta_1(2, a_{11}) = \theta_1(2, a_{12}) = \begin{bmatrix} -154 \\ 728 \\ -574 \end{bmatrix}.$$

At date 0, the matrix

$$S(0) + D(0) = \begin{pmatrix} 1 & 4/7 & 2/7 \\ 1/2 & 2/7 & 1/7 \\ 1/2 & 2/7 & 1/7 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4/7 & 2/7 \\ 3/2 & 9/7 & 1/7 \\ 3/2 & 2/7 & 8/7 \end{pmatrix},$$

so equation system (7) becomes

$$\begin{pmatrix} 2 & 4/7 & 2/7 \\ 3/2 & 9/7 & 1/7 \\ 3/2 & 2/7 & 8/7 \end{pmatrix} \begin{bmatrix} \theta_1^0(1) \\ \theta_1^1(1) \\ \theta_1^2(1) \end{bmatrix} = \begin{bmatrix} -56 \\ 623 \\ -679 \end{bmatrix},$$

yielding the portfolio

$$\theta_1(1) = \begin{bmatrix} -154 \\ 728 \\ -574 \end{bmatrix}.$$

Since

$$-\Delta_w x_1(0) = 154 = \sum_{j=0}^2 \theta_1^j(1) S_j(0),$$

equation (8) is satisfied: the initial purchase of assets by consumer 1 at date 0 is exactly offset by her negative net trade at date 0.

Once again, consumer 2 holds precisely the opposite portfolio from consumer 1,

$$\theta_2(t, a_t) = -\theta_1(t, a_t) \quad \text{for all } (t, a_t).$$

As in the case of short-term insurance, each consumer purchases insurance on herself and sells insurance to the other. However, in contrast to the market with short-term insurance, here consumers buy and hold the same portfolio at all dates regardless of the accident history.

In the infinite horizon case, the trading portfolios are also stationary:

$$\theta_1(t, a_t) = \begin{bmatrix} -186 \\ 744 \\ -558 \end{bmatrix} \quad \text{and} \quad \theta_2(t, a_t) = \begin{bmatrix} 186 \\ -744 \\ 558 \end{bmatrix}$$

for every date-event (t, a_t) with each consumer buying insurance on herself and selling insurance to the other consumer. Consumer 1 also goes short

186 units of the bond while consumer 2 goes long. For any date-event (t, α) , equation (7) for consumer 1 becomes

$$\begin{pmatrix} 10/3 & 4/3 & 2/3 \\ 13/6 & 5/3 & 1/3 \\ 13/6 & 2/3 & 4/3 \end{pmatrix} \begin{bmatrix} -186 \\ 744 \\ -558 \end{bmatrix} = \begin{bmatrix} 0 \\ 651 \\ -651 \end{bmatrix}.$$

Since

$$-\Delta_w x_1(0) = 186 = \sum_{j=0}^2 \theta_1^j(1) S_j(0),$$

the initial purchase of assets by consumer 1 at date 0 is again exactly offset by her negative net trade at date 0.

Verifying the martingale equation (5) for these long-term insurance contracts is straightforward. Suppose that we arrive at date-event (t, a_t) having experienced m accidents, $0 \leq m < t$. Then the discounted price for security j at dates t and $t + 1$ respectively is

$$S_j^*(t) = \frac{S_j(t)}{2^m} \quad \text{and} \quad S_j^*(t+1) = \frac{S_j(t+1)}{2^m(1+r(t))}$$

while the cumulative dividend for security j at date $t + 1$ is

$$D_j^*(t+1) = D_j^*(t) + \frac{d_j(t+1)}{2^m(1+r(t))}.$$

Thus, equation (5) becomes

$$S_j(t) = \frac{1}{1+r(t)} E_Q[S_j(t+1) + d_j(t+1) \mid \mathcal{F}_t]$$

which is the same as equation (4).

3.3 Earthquakes and the law of large numbers

What happens as the number of consumers increases? We consider two cases. In the first, which we call the earthquake case, the number of accident types remains the same; in the second, the number of accident types increases in direct proportion to the number of consumers.

In case 1, suppose there are now n consumers but only three types of accident: $J = \{0, 1, 2\}$ as before. For the sake of interpretation, imagine a world consisting of two regions, either of which can experience an earthquake

at date t . There is never more than one earthquake at a given date, and there may be none. The set of consumers is divided into two equal subsets: I_1 who reside in region 1 and I_2 who reside in region 2. For each $i \in I := I_1 \cup I_2$, preferences are once again represented by equation (8). However, the endowment of consumer $i \in I$ is now given by

$$w_i(t, a_t) = \begin{cases} 0 & \text{if } j \neq 0 \text{ and } i \in I_j \text{ at date } t; \\ Y & \text{otherwise.} \end{cases}$$

In other words, if an earthquake strikes at date t and consumer i resides in the region in which it strikes, the consumer loses her entire endowment at that date; otherwise, she is unaffected. It is easy to see that in this case our results remain essentially unchanged: in particular, ADR prices, equilibrium net trades, and security prices (short or long term) remain the same.

In case 2, suppose once again there are two types of consumer with equal numbers of each type, but accidents happen only to individuals. As in our two-person economy, assume there is a chance $1/2$ that an accident happens to some consumer of the first type and a chance $1/4$ that an accident happens to some consumer of the second type. Assuming consumers of the same type share equally in the risk to their type, let $p_1(t)$ represent the probability⁶ that an accident happens to any specific consumer of type 1 at date t and $p_2(t)$ the corresponding probability that an accident happens to any specific consumer of type 2 at date t . Since we assume all consumers of a given type are equally at risk,

$$p_j(t) = \begin{cases} 1/n & \text{if } j = 1; \\ 1/2n & \text{if } j = 2. \end{cases}$$

The probability there is no accident at date t is

$$p_0(t) = 1 - \frac{n}{2} \frac{1}{n} - \frac{n}{2} \frac{1}{2n} = \frac{1}{4}$$

as before.

In contrast to the case of earthquakes, in this case increasing n does alter equilibrium trades and prices. In particular, from equation (10) we conclude that the ADR price associated with date-event (t, a_t) is

$$\pi(t, a_t) = \begin{cases} \delta^t P(a_t) & \text{if there is no accident at date } t; \\ \frac{n}{n-1} \delta^t P(a_t) & \text{if there is an accident at date } t. \end{cases}$$

⁶ $p_j(t)$ is really the conditional probability $P(a_t | a_{t-1})$ that the event “an accident happens to consumer i at date t ” occurs conditional on event a_{t-1} , but we are exploiting independence to simplify notation. A similar remark applies to the martingale probabilities $q_j(t)$ defined below.

Letting $q_j(t)$ denote the risk neutral or martingale probability that an accident happens to a consumer of type j at date t and $q_0(t)$ the martingale probability there is no accident at all, it follows from equation (1) that

$$q_0(t) = \frac{1}{1 + 3n/(n-1)}$$

while the ratios

$$\frac{q_1(t)}{p_1(t)} = \frac{q_2(t)}{p_2(t)} = \frac{4n/(n-1)}{1 + 3n/(n-1)}.$$

As $n \rightarrow \infty$, the martingale probability

$$q_0(t) \rightarrow p_0(t) = 1/4$$

and the ratios

$$\frac{q_1(t)}{p_1(t)} \rightarrow 1 \quad \text{and} \quad \frac{q_2(t)}{p_2(t)} \rightarrow 1.$$

Thus, we obtain a result similar in spirit to those associated with an appeal to the law of large numbers in the traditional insurance literature: as the number of consumers increases, the risk neutral (martingale) probabilities converge to the true, actuarial accident probabilities or, stated in more familiar terms, the pooling of risk lowers the risk premia charged to individual consumers.

4 Conclusion

As will be apparent to those familiar with the finance literature, the research reported here only begins to tap the potential for applying the tools of intertemporal finance to insurance markets. Insurance contracts are clearly more complex than the simple instruments captured here, typically insuring a variety of types of accident over varying periods of time with options to renew and the like. All such contracts are “redundant assets” in this setting and, as such, can be priced using martingale measure. Insurance contracts also typically pay out in real rather than nominal terms, a distinction we have not addressed in the single commodity version of the model presented here.

Tools developed in finance for analyzing the structure of risk and return apply directly to assets such as ours. In particular, if we employ a Dobb-Meyer decomposition of realized return for each insurance contract into a

predictable part and an innovation, a CAPM-like relationship emerges linking excess returns for each security to the covariance between the likelihood ratio process,

$$z(t) = E_P \left[\frac{Q}{P} \mid \mathcal{F}_t \right]$$

and the innovation component of the security. Development of some version of a mutual fund theorem would also be appealing in which consumers buy insurance on themselves and invest in a market portfolio of insurance contracts on others.

As the theoretical discussion clearly shows, there is no reason to assume hazard rates are independent of the past history of the process and constant or that the effect of an accident is confined to the date at which it occurs. When one medical problem strikes, it may announce the increased chance of other problems arising. And an accident today may put a worker out of commission for months or years to come.

Finally, perhaps the top priority in this research agenda is extension of the results to continuous time. The assumptions underlying the counting process, that at most one accident happens at each date, are only an approximation in discrete time. As our “earthquake example” illustrates, a suitable interpretation of accident type takes much of the sting out of this assumption: any one earthquake can affect a large number of people, but earthquakes are discrete events. Nevertheless, the clearest justification for our hypothesis comes in continuous time. Much of the formalism of this paper is aimed at making the transition from discrete to continuous time as effortless as possible.

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