A GENERAL EQUILIBRIUM MODEL OF NON-COMPETITIVE INTERDEPENDENCE

By

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ABSTRACT

The problem dealt with in this paper is that of generalizing the standard, general competitive equilibrium model to allow for several monopolists, producers who can affect relative prices. The traditional approach to problems of monopoly interaction, or non-competitive interdependence, has been to make assumptions inconsistent with rationality or perfect information. An example is the Cournot model of duopoly, where producers assume zero output reactions of their rivals to changes in their outputs even though the actual reactions are not zero. The approach adopted in this paper is to construct a general equilibrium model of monopoly which is consistent with rationality and perfect information. We show that between any pair of interacting monopolists, one and only one monopolist exhibits a reaction function while the other simply picks a point on the function. For both the case of uncontrolled monopoly and monopoly behavior under Federal anti-monopoly policy, we derive the rational firm's reaction function, demonstrate the existence of equilibrium, and characterize the resulting solutions. Results of preliminary tests of each of these models are very encouraging.

INTRODUCTION

This paper deals with the problem of deriving equilibrium quantities and prices within a general equilibrium system containing several monopolists, producers who can affect relative prices.

Since in a general equilibrium setting all relative prices are a function of all outputs, a monopolist's output decision will affect all the relative prices in the system. Hence, an individual monopolist's output change will, in general, render the remaining producers' previous outputs non-optimal with respect to the new set of prices generated by the monopolist's new output. The other producers, in recalculating their respective optimal outputs, generate a set of output reactions and again change relative prices. A monopolist who is aware of the relative price effects imposed on him and on the other producers will rationally take the other producers' reactions into account when calculating his profit-maximizing output. Similarly, other monopolistic producers,[1/] realizing that their reactions are being taken into account, will determine their reactions accordingly.

The task of this paper is to describe the rational form of the reaction functions, demonstrate the existence of a general equilibrium given these rational reaction functions, assuming they are known to all producers, and characterize the resulting equilibria.

Our model assumes perfect information regarding reaction functions, and prohibitively high transactions costs of making multi-lateral or collusive agreements. The traditional models of economic conflict attempting to describe such a world break down under the assumption of rationality and perfect knowledge of reaction functions. In the standard duopoly model of Cournot, each duopolist makes output decisions based on faulty information concerning the reaction behavior of his rival. The zero output reactions assumed in this model does not describe the true reactions that occur within the model. In the standard duopoly model of Stackelberg we will see that the selected reaction functions are irrational when there is perfect information concerning output reactions.

In Section I, we specify our general model and show that a necessary condition for the existence of solutions to all monopoly interaction problems featuring rational strategy selection and perfect information regarding reactions is that between any two, interacting monopolists, one and only one of them exhibits a reaction function. In the case of m' ($m' \geq 3$) interacting firms, a solution implies a recursive set of reaction functions for m'-1 firms. We specify a competitive process to determine which firm among a set of interacting firms is able to establish a reaction function over all of the other firms in the set.

In Section II, we derive the rational reaction functions and prove a theorem on the existence of a general equilibrium in a special case of the general model. We call this the "uncontrolled monopoly" case.

In Section III, we introduce government anti-trust policy. We show that such policy changes the form of the rational reaction functions and alters the characteristics of the controlled monopoly solution. We construct an equilibrium in a linear special case, contrast it to the m firm generalization of the Stackelberg duopoly model, and derive its equilibrium size distribution of firms and concentration ratios. The outputs of the three smallest firms are seen to be equal while the larger firms are each twice as large as the next smallest firm. And the concentration ratio of an industry asymptotically <u>increases</u> as the number of firms grows and the industry output approaches a competitive level. The asymptote is where $(2^t-1)/2^t$ is the market share of the top t firms in an industry.

In Section IV, we note the rough empirical accuracy of our theories of monopoly for the U. S. experience before and after federal anti-monopoly laws. A basic result of this analysis is that U. S. anti-monopoly laws, by altering the form of rational functions, have converted a world with highly inefficient monopolies into a world in which an industry with only a few firms produces an output which is very close to a competitive level.

I. THE GENERAL MODEL

A. The Environment

The model we will employ is a private-property, non-competitive, general equilibrium model containing k individuals, n commodities, and m firms. We give one, and only one, individual, called the "lawyer," the ability to write commitment contracts for others.

A commitment is defined as an enforceable promise of an individual to react in a specified way to the actions of another individual regardless of the costs incurred by the commitment-maker in carrying out the stated reaction. Commitments are effective when, and only when, they are written by the lawyer.

We shall assume: (1) it is prohibitively costly for firms to collude; (2) all firms are profit-maximizers; (3) all firms are aware of the effects of their output decisions on the profits and outputs of the other firms; and (4) there is no technical interdependence among firms.[2/]

The output allocation set is denoted

$$x = (x_1, x_2, \ldots, x_m) \ , \ x_f \geq 0, \text{ for all } f=1,\ldots,m, \qquad (1)$$

where x_f is the f^{th} firm's output vector. We will denote the set of output vectors of all firms except firm f as

$$x_{-f} = (x_1, x_2, \ldots, x_{f-1}, x_{f+1}, \ldots, x_m) \ . \qquad (2)$$

Each firm, f, can produce all commodities and has a profit function,

$$\pi_f(x) = \pi_f(x_f; x_{-f}), \quad f=1,\ldots,m, \qquad (3)$$

which summarizes firm f's technology, output demand conditions, and factor costs. We assume that each firm employs some "specific factors," factors for which that firm is the least-cost employer. Of course, $\pi_f(0;x_{-f}) = 0$.

A group of monopolists is said to be "interacting" when the output of one monopolist in the group affects the output choice of another monopolist in the group. We assume that the group of all m monopolists is interacting.

B. The Necessity of a strategy maker and a strategy taker

We define a monopolist, i, as a strategy-maker (or maker) relative to j when i exhibits an output reaction function,[3/]

$$x_i = h_i(x_j) \ , \tag{4}$$

from which firm j, called a strategy-taker (or taker) relative to i choose their respective outputs.

Theorem: Necessary for a solution to a problem of monopoly interaction with perfect information concerning the strategies of others is that between any pair of interacting monopolists, there exists one strategy-maker and one strategy-taker.

To demonstrate this proposition, consider any finite, two-person, non-cooperative game with perfect information. Let

$\text{action}_i \backslash \text{action}_j$	1	2	. . .	N
1	$(\pi_i,\pi_j)_{11}$	$(\pi_i,\pi_j)_{12}$	. . .	$(\pi_i,\pi_j)_{1N}$
2	$(\pi_i,\pi_j)_{21}$	$(\pi_i,\pi_j)_{22}$	. . .	$(\pi_i,\pi_j)_{2N}$
⋮	⋮	⋮		⋮
M	$(\pi_i,\pi_j)_{M1}$	$(\pi_i,\pi_j)_{M2}$		$(\pi_i,\pi_j)_{MN}$

be the payoff matrix for a two-person game, where i has M possible output sets and j has N possible output sets. Under perfect information each player's strategy set must be expanded beyond the simple actions used in describing the payoff matrix. Strategies expressing actions contingent on the other player's actions must be permitted. Hence, we must add additional strategies to each player corresponding to that player's possible reactions to his opponent's known strategy. The additional possible strategies and their payoffs are described below where, for example, 1/2 for i means that i will react with his action 1 to j's action 2.

strategy$_i$ \ strategy$_j$	1	2	. . . .	N	N+1	. . .	N^M
1	$(\pi_i,\pi_j)_{11}$	.	. . .	$(\pi_i,\pi_j)_{1N}$	$(\pi_i,\pi_j)_{11}$	. . . .	$(\pi_i,\pi_j)_{1N}$
2	.			.	.		.
.	.			.	.		.
.	.			.	.		.
.							
M	$(\pi_i,\pi_j)_{M1}$	.	. . .	$(\pi_i,\pi_j)_{MN}$	$(\pi_i,\pi_j)_{M2}$	. . . .	$(\pi_i,\pi_j)_{MN-1}$
M+1	$(\pi_i,\pi_j)_{11}$	$(\pi_i,\pi_j)_{12}$	. . . .	$(\pi_i,\pi_j)_{2N}$			
.	.	.		.			
.	.	.		.			
.	.	.		.			
M^N	$(\pi_i,\pi_j)_{M1}$	$(\pi_i,\pi_j)_{M2}$	. . . .	$(\pi_i,\pi_j)_{(M-1)N}$			

where

M+1 = 1/1, 1/2,...1/N-1, 2/N,

M^N = M/1, M/2,...M/N-1, M-1/N,

N+1 = 1/1, 1/2,...1/M-1, 2/M, and

N^M = N/1, N/2,...N/M-1, N-1/M .

The expanded payoffs are obtained directly from the original matrix, except that it is impossible to fill in a fully expanded payoff matrix because the southeast elements of the hypothetical matrix are undefined. This is because when <u>both</u> players are playing strategies which are contingent on the other's play, no action is taken so that no payoffs exist. Hence, these elements, and the hypothetical matrix cannot be defined.

The only possibility is to eliminate either <u>all</u> of the additional rows or <u>all</u> of the additional columns of the expanded matrix. This results in a payoff matrix <u>either</u> in the form,

$$
\text{(a)}\quad
\begin{array}{c|ccccc}
\text{action}_i \backslash \text{strategy}_j & 1 & \cdot & \cdot & \cdot & N^M \\
\hline
1 & (\pi_i,\pi_j)_{11} & \cdot & \cdot & \cdot & (\pi_i,\pi_j)_{1N} \\
\vdots & & & & & \\
M & (\pi_i,\pi_j)_{M1} & \cdot & \cdot & \cdot & (\pi_i,\pi_j)_{M(N+1)}
\end{array}
$$

or in the form

$$
\text{(b)}\quad
\begin{array}{c|ccc}
\text{strategy}_i \backslash \text{action}_j & 1 & \cdots\cdots & N \\
\hline
1 & (\pi_i,\pi_j)_{11} & \cdots\cdots & (\pi_i,\pi_j) \\
\vdots & \vdots & & \vdots \\
M^N & (\pi_i,\pi_j)_{M1} & \cdots\cdots & (\pi_i,\pi_j)_{(M-1)N}
\end{array}
\quad .
$$

It also results in an immediate solution. If j is the individual whose feasible strategies are not expanded (case (b) above), j picks the action that maximizes π_j given i's strategy; and i chooses the strategy which maximizes π_i given j's rational action under each of i's possible strategies. What this means is that one player, player i, has the exclusive rights to exhibiting a reaction function (summarized by his solution strategy). That is, between two interdependent players, one and only one player is the strategy-maker. The other, of course is the strategy-taker.

It is straightforward to extend this result to the case of a continuous action space.

C. Maker-Taker relationships over the entire set of firms

Equation (4) above is a cross section for given outputs of i's takers and given reaction functions of his makers. A like relation holds for each of the other firms which serve as a maker. It is possible that there are no such other firms. That is, while there must be at least one maker in a group of m interacting monopolists, this maker's optimal strategy may be to prevent interactions between the remaining firms. In this case, firm i is the maker and the other firms are takers from i. This must be the case, as we shall see in Section II, when monopolies are unregulated and there exists a "punishment output" from firm i.

In the case that monopolies are constrained by a certain anti-trust law, then all of the monopolies are interacting. This is shown in Section III.

For now, assume there is a maker besides individual i. And assume that j, a taker of i, is a maker with respect to k. Then k is a taker with respect to i. This is because when k selects an output and thus determines j's output according to j's reaction function, he is also determining i's output according to the reaction function i presents to j. If k were committed to a certain reaction to i, he would similarly be committed to a certain reaction to j. But k is a taker of j so an inconsistency would arise. Hence, if i is a maker to j, and j is a maker to j, and j is a maker to k, i is a maker to k. The maker relation is transitive.

Summarizing, there is a binary relation, call it $\mathcal{M}$, over the set of all mutually interacting firms, S = (1,2,...,m'), such that:

(a) either $i \mathcal{M} j$ or $j \mathcal{M} i$ for all $i,j \in S$,

(b) $i \mathcal{M} m$ and $j \mathcal{M} k$ implies $i \mathcal{M} k$ for all $i,j,k \in S$.

Since a non-reflexive, complete, transitive, binary relation over a finite set determines a unique maximal element in the set, there is one member of S, say 1, such that $1 \mathcal{M} j$ for all $j \in S$, $j \neq 1$. We call this firm the "primary maker." Since the same exercise can be repeated for the firms in $S' = (2,\ldots,m')$, there exists a firm, say 2, such that $2 \mathcal{M} j$ for all $j \in S'$, $j \neq 2$. We call this firm the "secondary" maker.

This exercise continues onto the last firm m', who is not a maker with respect to anyone. We call this firm the "pure taker."

We thus obtain a hierarchy of makers, with the primary maker a maker to everyone, exhibiting a reaction function,

$$x_1 = x_1(x_2,x_3,\ldots,x_{m'}) ,$$

while the secondary maker exhibits the reaction function,

$$x_2 = x_2(x_3,\ldots,x_{m'})$$

and the third firm exhibits

$$x_3 = x_3(x_4,\ldots,x_{m'}) ,$$

and so on up to the $m'-1^{\underline{st}}$ firm's simple reaction function.

Thus, the solution output is easily constructed once rational reaction functions are formed. The pure taker selects an output that maximizes his profit (assuming for now the existence of such an output) given all of these reaction functions. The $m'-1^{\underline{st}}$ firm then follows his established reaction function, which gives the two outputs, necessary for the $m'-2^{\underline{nd}}$ firm to determine his output. This process continues until the primary maker's output is determined. We shall derive the rational reaction functions for a special case in Section III.

D. Determination of the Strategy-Maker

To be the strategy-maker and exhibit such a reaction function, the firm must be able to make effective commitments.[4/] And a commitment is effective only if it is underwritten by the commitment contract writer, the lawyer. The services of the lawyer are obtained via competitive bidding by the firms. Each firm submits m-1 bids, each bid representing the amount the firm is willing to pay to be maker instead of a specified, alternative maker. A winning bidder, if one exists, is a firm whose bid against his least preferred alternative maker is no less than the maximum of the bids against him. The reason a winning bidder must bid as if the worst possible alternative is the actual alternative is that the lawyer is free to choose the bidder's alternative maker and will rationally choose an alternative which will maximize the bid of his most preferred winning bidder. The winning bidder, however, does not generally pay his bid to the lawyer; he need only match the second highest bid. Our auction is unusual in that the bidders have different payoffs; and therefore, different bids depending on who would otherwise win the auction and on what he would do as the winning bidder.

While our lawyer is a "deus ex machina" imposed upon the model to attain a solution, in the real world, we do sometimes observe such individuals. Moreover, we do observe firms in the real world devoting real resources to making commitments and communicating their validity.[5/] In our model we have summarized these activities into bids for the services of a lawyer. This serves the purpose of abstracting from the socially unnecessary resource drains which firms in the real world create by competing with other firms for prior commitments.

Section II contains, for the uncontrolled monopoly case, a specification of the cost of being a maker, a derivation of the identity of the maker, a characterization of general equilibrium solutions, and a proof of the existence of equilibrium solutions under some additional restrictions. Section III contains a similar analysis for a controlled monopoly case.

II. GENERAL EQUILIBRIUM WITH UNCONTROLLED MONOPOLY

A. The Existence of Punishment Outputs

We shall assume until Section III that for each firm, there exists an output vector which implies negative profits of any positive output vector to each of the other firms regardless of the outputs of the remaining firm. Such a set of outputs will be called a punishment set of outputs. More formally,

(a.1) For each i, there exists an x_i, say x_1', such that

$$\pi_j(x_j;\ x_1 \ldots x_{j-1}, x_{j+1}, \ldots x_i', \ldots x_m) < 0 \quad \text{for all } x_{-i} \text{ with } x_j > 0 \text{ and all } j \neq i.$$

Consider a pair of firms, i and j, where i is the strategy-maker and j is the strategy-taker. Firm i can induce firm j to produce any specified output that does not generate negative profits to j by making an effective commitment to produce a punishment set of outputs when j produces any output not equal to the specified output. Firm j, as a profit-maximizing firm facing i's output reaction function, will produce the specified output as it yields j its highest profits.

Since firm i has a punishment set of outputs, it can make effective commitments with respect to every firm to produce its specified output. Faced with firm i's commitment, each of the remaining m-1 firms will rationally choose to produce their respective profit-maximizing outputs, the outputs specified by firm i.

More formally, let x_i^i and x_{-i}^i be solution values to the problem,

$$\max_x \, [\pi_i(x) - C_i(x)] \text{ subject to } \pi_f \geq 0 \text{ for all } f \neq i, \qquad (5)$$

where $x_f = 0$ if $\pi_f(x_f) < 0$ for all $x_f > 0$, and where C_i is the cost to i of becoming maker. Thus, x_f^i is the output of the f^{th} firm which maximizes the net maker profit of firm i subject to the non-negativity of profits to any taker. The rational reaction function for firm i is then:

$$x_i = h_i(x_{-i}), \text{ such that} \tag{6}$$
$$x_i^i = h_i(x_{-i}^i), \text{ and}$$
$$x_i' = h_i(x_{-i}), \text{ all } x_{-i} \neq x_{-i}^i \quad .$$

The commitment made by i guarantees that i will produce x_i' when firm f deviates from producing x_f^i even if it implies lower profits to i than some alternative values of x_i given $x_f \neq x_f^i$. Such apparently irrational behavior by firm i is rational by virtue of our assumption of profit-maximizing behavior of all firms. This implies that firm f will produce x_f^i in equilibrium rather than an alternative output.

A strategy-taker, any firm $j \neq i$, faces the problem:

$$\max \pi_j(x_j;x_{-j}) \text{ subject to}$$
$$x_{-j} = x_{-j}^i \text{ if } x_j = x_j^i, \text{ and } x_k = x_k^i \text{ , all } k \neq j,i; \tag{7}$$
$$x_{-j} = (x_1,\ldots, x_i',\ldots,x_m) \text{ otherwise.}$$

This leads the j^{th} firm, knowing the rational responses of the other takers, to choose $x_j = x_j^i$. We have assumed this holds even if $\pi_j(x_j^i) = 0$, and $x_j^i > 0$. That is, the taker will choose to produce the maker's optimal output choice even though his profits there are zero and he has an equally profitable possibility (e.g., quitting business).

The m-1 takers are obviously non-interacting; the only response that counts when a taker changes his output is the response of the maker when punishment outputs exist and monopolists are uncontrolled.

Note that disregarding the cost of becoming a maker, no firm is ever worse off by being the strategy-maker as opposed to being a strategy-taker. This is because an individual firm can always do as well by choosing its own output as having it chosen by another. Hence, each firm will have a non-negative bid for the lawyer's services regardless of whom he is bidding against.

B. <u>The two-firm case</u>.

Consider two firms, i and j. The amount firm j is willing to offer to the lawyer equals the difference between j's profit as a maker and j's profit as a taker. Since j's profit as a taker depends on i's choice of outputs as a maker, the cost to i of being the maker, which is the cost of just beating j's bid, is a function of the x that i would choose as maker. Hence, we can write:

$$C_i(x) = \pi_j(x^j) - \pi_j(x) \ , \tag{8}$$

and using (5), describe firm i's <u>maximum maker profit</u> as:

$$\pi_i^M = \max_x \ [\pi_i(x) - (\pi_j(x^j) - \pi_j(x))] \text{ subject to } \pi_j(x) \geq 0, \tag{9}$$

where $\pi_j(x^j)$ is the value of π_j implied by the solution to:

$$\max_x \ [\pi_j(x) - (\pi_i(x^i) - \pi_i(x))] \text{ subject to } \pi_i(x) \geq 0, \tag{10}$$

where $\pi_i(x^i)$ is the solution value of π_i implied by (9).

Solutions to (9) and (10), if they exist, yield explicit values of $\pi_i(x^i)$, $\pi_j(x^j)$, $\pi_i(x^j)$, and $\pi_j(x^i)$ from which we obtain the value of each firm's bid. These values are interpreted as i's and j's operating profit as a maker, and i's and j's operating profit as a taker, respectively. Note that i's operating profit as a taker is his profit under j's optimal output choice.

Noting that x^i is independent of $\pi_j(x^j)$, we see from (9) that firm i is maximizing its joint-profits with firm j. Similarly, from (10), firm j is maximizing its joint-profits with firm i. If we assume that the joint-profit maximizing output is unique, then the same output vector will be chosen regardless of which firm is the strategy-maker. Hence, each firm's bid for the rights to be maker would equal zero since its profit as a maker is the same as its profit as a taker.[6/] In this case, the final determination of the strategy-maker is arbitrary.

It is now easy to show that the solution to a Cournot duopoly problem with rational reactions and no government regulation implies a joint profit maximizing total output and an arbitrary distribution of outputs between the duopolists.

But these results do not extend beyond two monopolists, for with m firms, $m \geq 3$, there are m-1 competing bids with which a prospective strategy-maker must contend. Since there is only the single source of the underwriting service for which the firms are bidding, any prospective maker, firm i, need only be concerned with the highest of its rivals' bids. This highest rival bid reflects the explicit cost to i of becoming the maker. Thus,

$$C_i(x) = \max_{f \neq i} (\pi_f(x^f) - \pi_f(x)) \; . \tag{11}$$

The m-1 opposing bidders are each rationally assuming that firm i will be the strategy-maker if they are not. The resulting bid of each firm then measures how much a firm is willing to pay to be maker instead of being a taker of i's reaction function. We can now describe firm i's maximum maker profit, π_i^M, as:

$$\pi_i^M = \max_x \; [\pi_i(x) - \max_{f \neq i} (\pi_f(x^f) - \pi_f(x))] \text{ subject to } \pi_f \geq 0 \; , \tag{12}$$

where $\pi_f(x^f)$ is the implied profit to firm f when f is solving for its maximum maker profit.

Firm i's alternative maker is that firm which will be the maker if i is not. Firm i's bid when j is his alternative maker, the difference between i's profit as maker and i's profit as taker of j, is

$$\pi_i(x^i) - \pi_i(x^j) = B_{ij}(x^j) , \tag{13}$$

where j is i's alternative maker.

By computing maximum maker operating profit for all m firms, if these profits exist, and taker profit in a similar fashion, we can compute each firm's bids from the explicit values of maker and taker profits.

D. Characterizing an m-Firm Equilibrium

Distinguishing features of the m-firm case ($m \geq 3$) are that at the solution there is an equality between the bids of some of the takers and that the solution is not a joint-profit maximum.

At any choice of output allocation set of the maker, i, there is either a distinct individual determining i's legal fee - i.e., an unique f, solving (11), or there is a tie bid between some of the takers. Suppose there is an unique maximum in (11). Since the maker is responsive only to changes in the bid of the single highest-bidding taker, say, j, the maker and this taker will adopt a joint-profit maximizing relationship as in the two-firm case. Therefore, if the output choice of i is, in fact, a solution, it also corresponds to a joint-profit maximum between i and j. If the joint-profit maximizing output is unique, j's bid against i is zero. Since the remaining bids are non-negative, such an output choice is unattainable because the alternative maker's zero bid is then not higher than the other takers.

If the joint-profit maximizing output of i and j is non-unique, the same result obtains. Suppose that joint-profits between i and j are maximum and therefore equal at both x^i and x^j. Then the difference between i's maker profit at x^i and x^j,

$$\pi_i(x^i) - [\pi_j(x^j) - \pi_j(x^i)] - \pi_i(x^j) + [\pi_j(x^j) - \pi_j(x^j)]$$

$$= \pi_i(x^i) + \pi_j(x^i) - [\pi_j(x^j) + \pi_i(x^j)] = 0 \ .$$

Thus, i is indifferent between x^i and x^j which implies that i's bid is zero. Again, an inconsistency results as the other bids are non-negative. Thus, the equilibrium solution is inconsistent with the existence of an unique maximum in the alternative bids.

Therefore the optimal output choice of the strategy-maker occurs where there exists a tie in the maximum bids of some of the takers.

A solution occurring at this point does <u>not</u> correspond to a joint-profit maximum. Although the maker is responsive to a change in the bid of any one of his several maximum-bidding-takers, he is not concerned with the <u>sum</u> of their bids, which i required for joint-profit maximization. Thus, the solution is not a joint-profit maximum. In case there is an m-way tie with all firm's bids equal to zero at the solution, we have a separate joint-profit maximum between the maker and each taker; however, this is still not an m-firm joint-profit maximum.[7/]

In both the two and m-firm cases, since the solution maker need only match the highest-bidding rival firm(s), the amount going to the lawyer will, at most, equal the value of the second highest bid over all firms. But, whereas the lawyer's fee is always zero in the two-firm-case, it may be positive in the m-firm-case.

E. A Theorem on the Existence of Equilibrium

We will be working in Euclidean space, R^y; the dimensionality y of the space equals the number of commodities (n) times the number of firms (m), or nm.

Let X, a subset of R^y, equal the feasible output set. An element, x, of this feasible output set is a y-dimensional vector of outputs of each commodity by each firm.

From the above discussion, for each firm there is a profit function, $\pi_f(x)$, f=1,...,m; defined on X. Similarly defined on X is:

(d.1) the i^{th} firm's maker profit,

$$\pi_i(x) - \max_f [\pi_f(x^f) - \pi_f(x)] \quad f=1,\ldots,m;\ f \neq i, \tag{14}$$

where x^i is the value of x yielding maximum maker profit, π_i^M , and

(d.2) the i^{th} firm's bid function, given that f, who produces output x, is the alternative maker

$$B_{if}(x) = [\pi_i(x^i) - \pi_i(x)] \ ,\ f=1,\ldots,m;\ f \neq i; \tag{15}$$

An <u>equilibrium</u> is a feasible output vector, $x^* \in X$, such that x^* maximizes the maker profit of some firm f and f is a winning bidder.

We now make the following assumptions:

(a.2) X is a non-empty, compact, convex set.

(a.3) $\pi_f(x)$ is a continuous, real-valued function, f=1,...,m.

(a.4) For any f and any given $(\pi_1^M,\ldots,\pi_{f-1}^M,\pi_{f+1}^M,\ldots,\pi_m^M)$, there is at most one value of x^f. (This is slightly weaker than the strict convexity of the set $X_i = \{x_i: \ \pi(x_i) \leq \pi\}$ for all π.)

<u>Theorem</u>: Given assumptions (a.1) - (a.4), there exists an equilibrium.

The proof will consist of two parts. Part 1 will prove that there exists a set of outputs $x^1,x^2,\ldots,x^m$. That is, for any i, there exists maximum maker profit, π_i^M, with consistent values of π_f^M, for all $f \neq i$. Part 2 will prove that there is always at least one firm which is a winning bidder, i.e., one firm whose maximum bid against his alternative makers exceeds the maximum of the bids against him.

<u>Proof</u>:

<u>Part 1</u>.

First we show that for given values of maker profit of other firms, firm i has a maximum maker profit. To do this we will employ the well-known theorem in analysis that a continuous, real-valued function defined over a closed and bounded set attains a maximum at some point in the set. Let

$$g_i(x) = \max_f \, [\pi_f(x^f) - \pi_f(x)] \quad f=1,\ldots,m;\ f \neq i, \quad i=1,\ldots,m. \tag{16}$$

That is, $g_i(x)$ is the function describing the maximum bids against i for each point in X selected by i. Since $\pi_i(x)$ is continuous by (a.2) and the sum of two continuous functions is continuous, firm i's maker profit in (14) is continuous if $g_i(x)$ is continuous.

<u>Lemma</u>: The function $g_i(x)$ is continous.

At any point in X, and any i, there is either (a) an unique maximum bid in (16), or (b) there is an equality between the highest two, or more bids in (16).

(a) If there is an unique maximum in (16) at some point in X, then since each bid function, $B_{fi}(x)$, is continuous (the difference of two continous functions is continuous), (16) is continuous at such points in X.

(b) Let x_s be a point in X where $B_{ji}(x_s) = B_{ki}(x_s) = g_i(x_s)$, $j \neq k$. Suppose, for any $\delta > 0$, there is an $\varepsilon < \delta$, $\varepsilon > 0$, such that

$$g_i(x_s-\varepsilon) = B_{ji}(x_s-\varepsilon) > B_{ki}(x_s-\varepsilon) \text{ , and} \tag{17}$$

$$g_i(x_s+\varepsilon) = B_{ji}(x_s+\varepsilon) \leq B_{ki}(x_s+\varepsilon) \text{ , } j,\ k=1,\ldots,m; \quad j\neq k,\ j,k\neq i.$$

It is obvious that since each bid function is continuous and equal at x_s, $g_i(x_s)$ is continuous at x_s.

In the case where the second relation in (17) does not hold, the bid of j is a maximal bid over the entire δ-neighborhood so the continuity of $g(x_s)$ follows from the continuity of $B_{ji}(x_s)$.

In the only remaining case, where only the first relation in (17) does not hold, $B_{ji}(x) \equiv B_{ki}(x)$ about an δ-neighborhood of x_s, then either bid is maximal in that neighborhood. Since the bid functions are continuous at all points in X, then $g_i(x)$, i=1,...,m, is continuous over all of X.

It follows from the lemma and the well-known theorem in analysis stated above that π_i^M and x^i exist for any set of values, $\{x^f\}$, $f \neq i$.

When $m \geq 3$ one firm's optimal maker output vector depends upon the optimal maker output vectors of other firms. This leads to the question of whether the optimal output vectors of the various firms are mutually consistent. Proving this establishes the existence of a set of output vectors, $(x^1,\ldots,x^m)$, such that, for each f, x^f yields maximum maker profit to firm f for the specified x^i of all $i \neq f$, $i,f = 1,\ldots,m$. Consider m feasible sets of nm outputs, representing an economy-wide output vector arbitrarily selected by each f, or $(x^{10},\ldots,x^{m0})$. Given the values of $(x^{20},\ldots,x^{m0})$, the value of x maximizing firm 1's maker profit, x^{11}, is calculated. Using x^{11} and $x^{30},\ldots,x^{m0}$, the value of x maximizing firm 2's maximum maker profit, x^{21}, is calculated. Continuing in this manner, the output set, $(x^{11},\ldots,x^{m1})$ is attained. Then given $x^{21},\ldots,x^{m1}$ the value of x maximizing firm 1's maker profit is re-calculated and labelled x^{12}. And the process continues as above.

The transformation

$$(x^{10},\ldots,x^{m0}) \to (x^{11},x^{21},\ldots,x^{m1}) \to ,\ldots, \to (x^{1r},\ldots,x^{mr}) \to ,\ldots, \tag{18}$$

then is a chain of transformations from a set of output sets, X^m, into itself. To show that there exists a consistent set of maximum maker profit over all m firms, it is sufficient to show that there exists a set of outputs, $(x^1,\ldots,x^m)$, which remains unchanged over the complete transformation (18). By the Brouwer fixed point theorem, the set, $(x^1,\ldots,x^m)$ exists if X^m is a compact, convex set, and the complete transformation (18) is continuous.

By assumption (a.1) - and the fact that the intersection of closed, bounded, and convex sets is itself closed, bounded, and convex - we know that X^m is closed, bounded, and convex. Since each transformation in (18) is a calculation of some firm's profits, it is sufficient to show the continuity of (18) by showing the continuity of x^i as a function of

$$x^{-i} = (x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^m) \tag{19}$$

for any i, i=1,...,m. Suppose the function,

$$x^i = x^i(x^{-i}), \tag{20}$$

is not continuous at some $\bar{x}^{-i}$. This implies that there is an infinite sequence, $\{x^{-i}\}$, approaching say, $\bar{x}^{-i}$, such that

$$x^i(\bar{x}^{-i}) \neq \lim_{x^{-i}\to\bar{x}^{-i}} \{x^i(x^{-i})\}.$$

(The existence of this limit is implied by the boundedness assumption in (a.2) and the Weierstrass Theorem.) Since each firm's maker profit is a continuous function of $(x^1,\ldots,x^m)$, there is also an infinite sequence,

$$\{\pi^M_{-i}\} = \{\pi^M_1,\ldots,\pi^M_{i-1},\pi^M_{i+1},\ldots,\pi^M_m\} \tag{21}$$

which approaches say, $\bar{\pi}^M_{-i}$, such that

$$x^i(\bar{\pi}^M_{-i}) \neq \lim_{\pi^M_{-i}\to\bar{\pi}^M_{-i}} \{x^i(\pi^M_{-i})\}.$$

Now the uniqueness of $x^i(\bar{\pi}^M_{-i})$ expressed in (a.4) implies that there is a $\delta(\varepsilon), \delta > 0$, such that for any $x^i \in X$ not in an ε-neighborhood of $\bar{x}^i$,

$$\pi^i(\bar{x}^i, \bar{\pi}^M_{-i}) - \pi^i(x^i, \bar{\pi}^M_{-i}) > \delta(\varepsilon), \tag{22}$$

where π^i is firm i's maker profit. Then, from the linear manner in which π^M_{-i} enters i's maker profit function (14), there is an $\omega > 0$ such that for all π^M_{-i} satisfying $|\pi^M_{-i} - \bar{\pi}^M_{-i}| < \omega$, and for all $x^i \in X$ not in an ε-neighborhood of $\bar{x}^i$,

$$\pi^i(\bar{x}^i, \pi^M_{-i}) - \pi^i(x^i, \pi^M_{-i}) > \delta(\varepsilon). \tag{23}$$

Consider the ε-neighborhood of

$$\lim_{\pi^M_{-i} \to \bar{\pi}^M_{-i}} x^i ,$$

and select an ε sufficiently small that the intersection of this neighborhood and the ε-neighborhood of $\bar{x}^i$ is empty. If the x^i in the former neighborhood are indeed profit maximizing, for all x^i in that neighborhood,

$$\pi^i(\bar{x}^i, \pi^M_{-i}) - \pi^i(x^i, \pi^M_{-i}) < 0, \tag{24}$$

for all $\pi^M_{-i} \to \bar{\pi}^M_{-i}$ generating the ε-neighborhood of

$$\lim_{\pi^M_{-i} \to \bar{\pi}^M_{-i}} x^i .$$

This is a direct contradiction of the immediately preceding inequality (23).

Hence, x^i is a continuous function of x^{-i} and likewise the transformation (18) is continuous. This is sufficient for the Brouwer fixed point theorem to apply; and therefore, the set $(x^1, \ldots, x^m)$ exists.

Part 2.

We shall now prove that -- given the array of maximum maker profits in (14), and therefore an array of bids against all alternative makers described in (15) -- a winning bidder exists. Consider the matrix B, representing the bids of each firm against the others, with zeroes along the main diagonal:

$$B = \begin{bmatrix} 0 & B_{12} & B_{13} & \cdots & B_{1m} \\ B_{21} & 0 & B_{23} & \cdots & B_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & B_{m3} & & 0 \end{bmatrix} . \tag{25}$$

From the definition of a solution maker, i is a solution maker if $\max_j B_{ij} \geq \max_k B_{ki}$, that is, if the maximum bid by i exceeds the maximum of the bids against i. In B, i is a solution maker if the maximum of the elements in the i^{th} row exceeds the maximum of the elements in the i^{th} column. Let B_{qr} be a maximal element of B. Then, $B_{qr} \geq \max_k B_{kq}$ so that q is a maker. Hence, there is always a winning bidder.

III. ANTI-MONOPOLY POLICY AND MULTIPLE-MONOPOLY

The results obtained in Section I are based on a decentralized model of monopoly decision-making without government intervention. However, in order to arrive at a more realistic calculation of monopoly behavior, we now introduce government participation in the economy in the form of federal anti-monopoly policy. Since most anti-monopoly policy is concerned with economic performance within an industry, we will confine our analysis to intra-industry behavior.

A. The Revised Reaction Functions

On the basis of existing anti-monopoly laws, it is reasonable to assume that if any firm expands its output in reaction to increases in output of its competitors - either present firms or entering firms - that firm would be subject to prosecution under the law for its "predatory practices."[8/] Thus, whenever a firm increases its output for a given level of industry demand we shall assume that government policy prohibits another firm from increasing its output. (We are assuming that detection of violators and enforcement of the law are carried out at zero cost.) This restriction upon the output reactions of firms precludes the use of punishment strategies.

In the absence of punishment strategies, a firm may try to induce the production of some desired industry output by "rewarding" other firms for their outputs; that is, decreasing its output if the other firm(s) decrease its (their) output(s) up to the desired industry output. We shall assume that such behavior will be viewed as collusion by the government policy-makers, and likewise be prohibited. Therefore, the effect of our anti-monopoly policy on a firm's choice of reaction function is to limit these choices to a class of non-increasing functions.

With the elimination of punishment and reward outputs, no firm is able to force the production of any output allocation set on the other firms. Since there is no single firm determining the outputs of all other firms with $m \geq 3$, some of the firms which were previously takers, and did not exhibit a reaction function, are now free to react to the outputs of all other takers. We are thus back to the model of IC, in which a hierarchy of makers appears. Firm i, $1 < i < m$, selects its reaction function in light of the reactions of firms' 1 through i-1 to the outputs of firms i through m. That is, the reaction functions of preceding makers are parameters in any given firms' choice of a reaction function. Firm i is a maker with respect to i+k but is a taker with respect to i-j ($k=1,\ldots,m-i$; $j=1,\ldots,i-1$).

Firm m exhibits no output reaction function but chooses its profit-maximizing output subject to the reaction functions of the other firms and is a pure taker.

Since the restricted reaction functions are such that all partial derivatives of the functions are non-positive, firm j, $j < m$, faces two alternatives given the output choice of firm k, $k > j$. If j exhibits a negative reaction to k's output, the other makers must keep their outputs constant since any non-zero reaction by these firms would be positive with respect to the output variation of either j or k, and thereby be disallowed. If j chooses a zero reaction function, then others may react negatively to k's output choice. Comparing the results of these two alternatives, firm j rationally decides upon the zero reaction function. The reasoning is as follows: If k increases his output, it is not rational for j to decrease his output since then (a) the outputs of the other firms could not decrease and the result would be to encourage the expansion of the aggregate output of his competitors and (b) the reaction would yield no benefit to j in terms of a different solution output for him because j's output level for given outputs of others may be chosen independently of these outputs. Similarly if k reduces his output, j will not rationally increase his output and thereby discourage k from decreasing his output in the first place. Firm j could produce the higher output by simply committing itself to produce the output regardless of k's reaction and thereby induce k to produce a lower output than he would under a negative reaction by k. When firm j exhibits a constant output reaction function with respect to k, the same argument holds with respect to the rational reaction functions of the other firms. Thus, each firm's rational reaction function is a constant output reaction function.

B. Equilibrium with no competition for hierarchal position.

Using this result we now illustrate equilibrium with m firms producing a homogeneous output in which each firm's hierarchal position is exogenously given. This amounts to a generalization of the Stackelberg duopoly model.[9/] Consider m firms, each producing a homogeneous output. We assume a hierarchy of makers, with firm 1 being the primary maker, firm 2 the secondary maker, and firm m the pure taker. These positions are, in the analysis of this section, imposed upon the model and not the result of any bidding process. Industry demand is assumed to be linear and of the form:

$$p = a-b\sum_{i=1}^{m} x_i \tag{26}$$

where p is the price of the industry's output and a and b are positive constants. Marginal costs are assumed to be constant and identical for each firm so that

$$C_f = cx_f + d_f \, , \tag{27}$$

where c and d_f are positive constants and, to assure positive outputs, $c < a$. The condition for profit maximization for each firm is

$$p - x_f b \sum_{i=1}^{m} \frac{dx_i}{dx_f} - c = 0 \text{ , or} \tag{28}$$

$$x_f = \frac{a-c-b\sum_{i\neq f} x_i}{b\left(2+\sum_{i\neq f} \frac{dx_i}{dx_f}\right)} = \frac{p - c}{b\left(1+\sum_{i\neq f} \frac{dx_i}{dx_f}\right)} \tag{29}$$

Since all reaction functions are constant output reaction functions, for each firm j,

$$\frac{dx_i}{dx_j} = 0 \quad \text{for all } i < j.$$

This yields a profit-maximizing expression for m of:

$$x_m = \frac{a-c-b\sum_{i=1}^{m-1} x_i}{2b} = \frac{p-c}{b} \quad . \tag{30}$$

Since firm m-1 is a taker with respect to 1,..., m-2, and the latter exhibit constant output reaction functions,

$$\sum_{i=1}^{m-2} \frac{dx_i}{dx_{m-1}} = 0 \; .$$

And from (30), we know $\frac{dx_m}{dx_{m-1}} = -\frac{1}{2}$. Thus, using (29),

$$x_{m-1} = \frac{a-c-b \sum_{i\neq 1} x_i}{3b/2} = \frac{p-c}{b/2} \tag{31}$$

To obtain m-2's profit=maximizing condition, we have to calculate j's and k's rational responses to a change in x_i. From (31) we know

$$\frac{dx_{m-1}}{dx_{m-2}} = -\, 2/3$$

However, when m-1 rationally adjusts its output in response to a change in m-2's reaction function (in this case, a constant output), m-1 is also altering the reaction function (in this case, a constant output) it exhibits to m. Hence, both of the reaction functions shown to m change when m-2 changes his reaction function. Thus, since

$$\frac{\partial x_m}{\partial x_{m-2}} = -\,1/2, \text{ and } \frac{dx_{m-1}}{dx_{m-2}}\;\frac{dx_m}{dx_{m-1}} = (-2/3)(-1/2) = 1/3 \;,$$

$$\frac{dx_m}{dx_{m-2}} = \frac{\partial x_m}{\partial x_{m-2}} + \frac{dx_m}{dx_{m-1}} \cdot \frac{dx_{m-1}}{dx_{m-2}} = -1/6$$. Hence, again using (29),

$$x_{m-2} = \frac{a-c-b\sum_{i\neq m-2} x_i}{7b/6} = \frac{p-c}{b/6} \quad . \tag{32}$$

Similarly, we find that

$$x_{m-3} = \frac{a-c-b\sum_{i\neq m-3} x_i}{43b/42} = \frac{p-c}{b/42} \tag{33}$$

$$x_{m-4} = \frac{a-c-b\sum_{i\neq m-4} x_i}{1807/1806} = \frac{p-c}{b/1806}$$

$$x_{m-5} = \frac{a-c-b\sum_{i\neq m-5} x_i}{b\ 3{,}263{,}443/3{,}263{,}442} = \frac{p-c}{b/3{,}263{,}442}$$

.
.
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The resulting distribution of outputs among firms from firm m to firm m-6, indexing firm m's output to unity, is

1; 2; 6; 42; 1806; 3,263,442; 10,649,994,950,808; ... (34)

This pattern satisfies the formula:

$$x_f - x_{f-1} = x_{f-1}^2 \quad . \tag{35}$$

(An inductive proof of this formula for all i, i=1,...,m can be provided; it follows essentially the same outline as the proof of the output distribution formula in the next subsection, where competitive bidding for hierarchial positions is permitted.)

Firm size thus increases very rapidly with the firm's position in the hierarchy. A simple exponential expansion of x with f would have $x_f - x_{f-1}$ approximated by x_{f-1} instead of x^2_{f-1}. Indeed, our rate of increase is so fast that there is no differentiable function which increases this fast over the entire interval.[10/] Altering the analysis to permit non linearities, decreasing marginal costs, different marginal costs for different firms, or slopes of industry demand curves which increase with industry output would appear to only reinforce the dramatic increase of the firm's output with its rank in the industry. We shall see in Section IV that such a rapid expansion of output with respect to rank in the industry is wholly unrealistic for all industries in the U. S. for which data is available. This leads to the conclusion that either (a) for all industries in the U. S., the member firms have substantially rising long run marginal costs or the slopes of industry demand curves substantially fall as industry output rises, (b) that anti-trust laws are severely misrepresented in our analysis, or (c) that the lack of competition for hierarchal positions in our generalized Stackelberg model is inappropriate. Our previous discussion suggests that the last of these possible conclusions is true.

C. Equilibrium with competition for positions in the hierarchy

The introduction of competition in making commitments and thus competitive bidding for each position in the hierarchy substantially alters the analysis in the model above. Such competition increases the costs of all firms except, of course, the costs of the pure taker. Since all firms have the same variable cost functions, the variable maker profit of each firm is, in equilibrium, equal to the simple variable profit of the pure taker. It also follows from the equality of variable costs between our firms that every firm is indifferent to any position in the hierarchy so that all bids for each position in the hierarchy are identical. One's position in the hierarchy is determined by the priority of his committment. Hence, the first auction is for the position of primary maker, the second for the secondary maker, etc. In the first auction, there are m bidders, in the second there are m-1, and so on until, finally in the m-1st auction, there are only two bidders. Let us see how these auctions alter the rationally chosen outputs from those selected in the model with no competition for hierarchal position.

The pure taker obviously has the same output choice function as in the model with no competitive bidding for hierarchal positions, so again

$$x_m^* = \frac{a-c-b\sum_{i\neq 1}^{m-1} x_i}{2b} = \frac{p-c}{b}, \tag{36}$$

where x_i^* is the solution output of firm i in the present model. But the output of the m-1st firm is now sensitive to a bid of the $m^{\underline{th}}$ firm for his hierarchal position. The bid of m for position m-1 equals

$$B_{m,m-1}(x_{m-1}) = \max[0, K_{m,m-1}-\pi_m(x_{m-1})] ,$$

where $K_{m,m-1}$ is the operating profit m would make if he were the $m\text{-}1^{\underline{st}}$ maker. Firm m-1 therefore selects an x_{m-1} which maximizes

$$\pi_{m-1}^M = \pi_{m-1} - \max[0, K_{m,m-1}-\pi_m(x_{m-1})] \text{ subject to (36) and the given values, } x_1,\ldots,x_{m-2} . \tag{37}$$

If the solution x_{m-1} were such that $B_{m,m-1}(x_{m-1}) > 0$, then the output that satisfies (37) would be a <u>joint</u> profit maximum subject to (36) and $x_1,\ldots x_{m-2}$. This joint profit is $(p-c)(x_{m-1}+x_m)-d_m-d_{m-1}$, which, as above, reaches its maximum at $x_{m-1}+x_m = \frac{p-c}{b}$. But then, from (36), x^*_{m-1} would equal zero. Since variable profits exceed zero at some positive outputs, firm m-1 would do better as the pure taker. Hence, $B_{m,m-1}(x^*_{m-1})=0$. But if so, the variable profits are equal for both firms m and m-1. Therefore,

$$x^*_{m-1} = x^*_m \tag{38}$$

and using (36),

$$x^*_{m-1} + x^*_m = \frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b/2} = 2\left(\frac{p-c}{b}\right) \quad . \tag{39}$$

This solution may be constructed by starting firm m-1 at its output in the previous model, an output which maximizes his operating profit and exceeds x_m, and then make him pay m's bid to be maker to a lawyer. It is then obvious from (37) that it pays m-1 to reduce his output in order to reduce m-1's bid against him. This occurs until $x_{m-1} = x_m$, at which point the bids become zero. It then no longer pays m-1 to reduce his output for there is no further reduction in m's bid that is possible. To compute m-2's optimal output, we need the profit of firm's m and m-1 as a function of x_{m-2}. Using (39), and computing, from now on, profits as variable profits (profits net of d_i), we have

$$\pi_{m-1} = \pi_m = \left[a-c-b\sum_{i=1}^{m-2} x_i - b\left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b/2}\right)\right]\left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b}\right) = \frac{\left(a-c-b\sum_{i=1}^{m-2} x_i\right)^2}{9b} . \tag{40}$$

Firm m-2's maker profits can now be written, using (39),

$$\pi^M_{m-2} = \left[a-c-b\sum_{i=1}^{m-2} x_i - b\left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b/2}\right)\right] x_{m-2} - \max[0, K_{m,m-2} - \pi_m(x_{m-2})] \ . \qquad (41)$$

Assume that $K_{m,m-2}-\pi_m(x^*_{m-2}) \geq 0$. Then, using (40) and (41),

$$\pi^M_{m-2} = \left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{8}\right) x_{m-2} + \frac{\left(a-c-b\sum_{i=1}^{m-2} x_i\right)^2}{9b} - K_{m,m-2} \ .$$

Maximizing this profit, we find

$$x^*_{m-2} = \frac{a-c-b\sum_{i=1}^{m-3} x_i}{4b} \ , \text{ and} \qquad (42)$$

$$\pi_{m-2}(x^*_{m-2}) = \frac{\left(a-c-b\sum_{i=1}^{m-3} x_i\right)^2}{16b} \qquad (43)$$

Substituting x^*_{m-2} into (4),

$$x^*_{m-2} = x^*_{m-1} = x^*_m \qquad (44)$$

The assumption that $K_{m,m-2}-\pi_m(x^*_{m-2}) \geq 0$ is satisfied, for at x^*_{m-2} the bid of m (and of m-1) is zero. And m-2 obviously would reduce his profit by contracting his output so as to make $K_{m,m-2}-\pi_m(x_{m-2})$ negative because (42) and (43) show that even if m-2 could profit from negative bids, he would still maintain a zero-bid solution. Since this holds for any firm, if the output solution to the problem,

$$\max_{x_i} \ [(p-c)x_i - \max_j(K_{ji}-\pi_j(x_i))] \ , \ i < m, \qquad (45)$$

exceeds x_m, then, because bids are positive in such a solution, this output solution is also the solution to the general problem:

$$\max_{x_i} \left[(p-c)x_i - \max_j \left[0, K_{ji} - \pi_j(x_i)\right]\right] . \tag{46}$$

Performing the maximization in (45) for firm m-3, using (42), (43) and (44), we maximize

$$\left[\left(a-c-b\sum_{i=1}^{m-3} x_i\right)x_{m-3} - 3/4\left(a-c-b\sum_{i=1}^{m-3} x_i\right)x_{m-3} + \frac{\left(a-c-b\sum_{i=1}^{m-3} x_i\right)^2}{16b}\right]$$

with respect to x_{m-3}. The solution output can be written:

$$x^*_{m-3} = \frac{a-c-b\sum_{i=1}^{m-4} x_i}{3b} = \frac{a-c-b\sum_{i=1}^{m-3} x_i}{2b} . \tag{47}$$

This output is twice the output of firm m-2 and thus is also a solution to (46). We also find, using (42), (44) and (47) that variable profits for each firm are:

$$\pi^*_m = \pi^*_{m-1} = \pi^*_{m-2} = \pi^{M*}_{m-3} = \frac{\left(a-c-b\sum_{i=1}^{m-4} x_i\right)^2}{36b} . \tag{48}$$

Thus variable maker profit to the m-4th firm is

$$\pi^M_{m-4}(x_{m-4}) = a-c-b\sum_{i=1}^{m-4} x_i - \frac{5}{6}\left(a-c-b\sum_{i=1}^{m-4} x_i\right)x_{m-4} + \frac{a-c-b\sum_{i=1}^{m-4} x_i}{36b} .$$

Maximizing this with respect to x_{m-4}, we find

$$x^*_{m-4} = \frac{a-c-b\sum_{i=1}^{m-5} x_i}{5b/2} = \frac{a-c-b\sum_{i=1}^{m-4} x_i}{3b/2} . \tag{49}$$

This is twice the output of firm m-3 and four times the outputs of firms m, m-1, and m-2. We also find that

$$\pi^*_m = \pi^*_{m-1} = \pi^*_{m-2} = \pi^{M*}_{m-3} = M^{M*}_{m-4} = \frac{a-c-b\sum_{i=1}^{m-5} x_i}{100b} \quad .$$

These profits form the bids for the $m-5^{\underline{th}}$ position in the hierarchy, and the procedure continues until we reach the top position. The distribution of outputs, moving on to m-5, m-6 and m-7 and again indexing the output of the $m^{\underline{th}}$ firm to unity, is easily seen to be

1, 1, 1, 2, 4, 8, 16, 32.

The obvious generalization is that

$$x^*_m = x^*_{m-1} \text{ and } x^*_{m-2-i} = x_m \cdot 2^i ,\; i = 0,1,\ldots,\, m-3 \;. \tag{50}$$

To prove that this generalization is, in fact, the solution distribution of firms, we provide an inductive proof. In particular, we shall prove that if the hypothesized distribution holds for $i = r$, i.e., if $x_{m-2-i}/x_{m-1-i} = 2$ for any i such that $0 \leq i \leq r$, then it holds for $i = r + 1$, i.e., $x_{m-3-r}/x_{m-2-r} = 2$. To do this, we first note that variable profit to the $m-3-r^{th}$ firm is

$$\pi_{m-3-r} = \left(a-c-b\sum_{i=1}^{m} x_i\right) x_{m-3-r} + \frac{\left(a-c-b\sum_{i=1}^{m} x_i\right)^2}{b} \;.$$

By hypothesis, for the firms from m to m-2-r we have:

$$x_{m-2-r} = \frac{a-c-b\sum_{i=1}^{m-3-r} x_i}{b(2^r+1)/2^{r-1}} \tag{51}$$

and

$$\sum_{i=m-2-r}^{m} x_i = \frac{a-c-b\sum_{i=1}^{m-3-r} x_i}{b} \left(\frac{2^{r+1}+1}{2^{r+1}+2}\right) , \quad r \geq 0 \tag{52}$$

Using (52),

$$\pi_{m-3-r} = \left(a-c-b\sum_{i=1}^{m-3-r} x_i - b\sum_{i=m-2-r}^{m} x_i\right) x_{m-3-r} + \frac{\left(a-c-b\sum_{i=1}^{m-3-r} x_i - b\sum_{i=m-2-r}^{m} x_i\right)^2}{b}$$

$$= \frac{1}{2^{r+1}+2}\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right) x_{m-3-r} + \frac{\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right)^2}{(2^{r+1}+2)^2 b}$$

Maximizing this expression with respect to x_{m-3-r},

$$0 = \frac{1}{2^{r+1}+2}\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right) - \frac{bx_{m-3-r}}{2^{r+1}+2} - \frac{2\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right)}{(2^{r+1}+2)^2}$$

$$= \frac{2^{r+1}\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right)}{(2^{r+1}+2)^2} - \frac{bx_{m-3-r}}{2^{r+1}+2}$$

$$x_{m-3-r} = \frac{a-c-b\sum_{i=1}^{m-3-r} x_i}{b\left(\frac{2^r+1}{2^r}\right)} .$$

Using (51), we see that

$$\frac{x_{m-3-r}}{x_{m-2-r}} = \frac{(2^r+1)/2^{r-1}}{(2^r+1)/2^r} = 2 .$$

This establishes the theorem.

There is a rather remarkable corollary concerning the "concentration ratio" of our industries. It is that the t-firm concentration ratio, the share of the top t firms in the industry ($t < m-2$), increases with the number of firms in the industry. From the above theorem, the total output of the top t firms in the industry can be written:

$$K \sum_{i=m-2-t}^{m-3} 2^i = K2^{m-3-t} \sum_{i=1}^{t} 2^i = K2^{m-2-t}(2^t-1) ,$$

where $t \leq m-3$ and K is some positive number. The total output of the m-2-t firms from firm m-2 to firm t-1 is then given by

$$K\sum_{i=0}^{m-3-t} 2^i = (2^{m-2-t}-1)K ,$$

while the total output of firms m and m-1 is, of course, 2K. Hence, the output share of the top t firms in the industry, $t \leq m-3$, is given by:

$$S_t = \frac{2^{m-2-t}(2^t-1)}{2^{m-2-t}(2^t-1) + 2^{m-2-t}-1+2} = \frac{2^t-1}{2^t + \frac{1}{2^{m-2-t}}} . \qquad (53)$$

Thus we see that as the number of firms in the industry expands and thus the output becomes more competitive, the concentration ratio, S_t, for any t ($t \leq m-3$) <u>increases</u>.

This increase, however, is very slight once the number of firms in the industry becomes at all significant. For example, if $m \geq 10$, then the percentage error in using $\frac{2^{t-1}}{2^t}$ as an estimate of S_t is always less than one quarter of one percent.

From (36), the equilibrium mark-up in our model is bx_m. Under pure monopoly, the mark-up would be $b\sum_{i=1}^{m} x_i$. This is the same as the uncontrolled monopoly mark-up since the rational maker in this industry model could not do better than he could by producing an output such that $p < c$ whenever any other firm produced a positive output. Given the distribution of output among firms in our controlled monopoly solution, the equilibrium mark-up relative to the pure monopoly mark-up is therefore given by

$$\frac{x_m}{\sum_{i=1}^{m} x_i} = \frac{1}{2^{m-2}+1} \quad .$$

So with, say, 10 firms in the industry, the equilibrium mark-up is less than $\frac{1}{4}$ of 1% of the pure monopoly mark-up. This seems safe enough to ignore for policy purposes, especially since some positive mark-up is necessary to guarantee the presence of producers in an industry. The result speaks for the powerful efficiency of the simple anti-trust policy outlined above.

IV. PRELIMINARY EVIDENCE

The rational reaction function-perfect information approach to non-competitive interdependence can be tested by attempting to verify empirically the implications of the above two models. We use the U. S. experience since we are somewhat less ignorant of it than of the experiences of other countries.

According to most accounts, no substantial monopolies other than government-granted and natural, local monopolies appeared before the Civil War. After that war, the communications-transportation revolution and the emergence of the corporate form of organization apparently opened up new opportunities for large scale organizations and thus private monopolies operating in nation-wide markets. In this environment, industrial giants grew in several industries, each coming to dominate his industry by using unprofitable price-cutting as a weapon against smaller firms in order to keep them "in line." These "robber barons" were, in our terms, simply rational makers over a set of takers and their "cutthroat competition" was merely their application of punishment outputs to deviant firms.

The development of anti-trust policy in response to the obvious inefficiencies in this system took several decades and has operated, as we have suggested, to remove collusion (as well as mergers with the purpose of raising prices) and cutthroat competition. This policy implies, as we have pointed out, a hierarchy of makers in which each of the makers presents the industry with a fixed output for a given level of market demand and industry costs. Empirically, this means that the larger firms in an industry can be expected to commit themselves to an announced share of the market and retain this share regardless of the peculiar economics of individual firms. That large firms in the U. S. determine their outputs in this way rather than computing their own demand and supply curves is obvious for certain firms and has been claimed as a fairly general description by numerous institutionalist authors in spite of the expected deluge of criticism by economists.

Further evidence for the controlled monopoly model was obtained from observations on relative firm sizes within selected U. S. industries. Our hypothesis, from equation (50), implies that

$$\log x_{m-2-i} = K_1 + (\log 2)\, i \ , \quad i = 0, i, \ldots, m-3 \tag{54}$$

for each industry. The hypothesis relating market share to rank which we find in the literature (Simon (5)) is

$$\log x_{m-i} = K_2 + b \log i \ , \quad b < 1 \ . \tag{55}$$

This hypothesis, which has no theoretical rationale, is clearly contrary to ours in that ours has firm size increasing more than in proportion to a firm's rank in the industry (i) while (55) has firm size increasing less than in proportion to the firm's rank. Finally we tested the size distribution of firms implied by the generalized Stackelberg model. As was shown in footnote (10), when this size distribution can be approximated, it is approximately:

$$x_i = K_3\, i\, x_i - K_3 \qquad , K_3 > 0 \ . \tag{56}$$

We obtained our data from Standard and Poor's "Compustat" tape for 1971, which has data on all of the relatively large U. S. companies within industries disaggregated to the four-digit industry level.[11/] This data was used to generate least-squares fits of the three hypotheses.[12/] The generalized Stackelberg model did not generate a negative constant term for any of our 41 industries. So we immediately rejected that hypothesis. The regressions for (55) produced coefficients less than unity in only three industries. In each of these industries (cement, roof and wallboard, and savings and loans) it appeared that we had erred in considering the markets for their product a national rather than a local market. It is not surprising that (55) fits better than (54) for local industries as it is well-known (cf. Simon (5)) that city sizes follow a distribution such as (55).

For the remaining thirty-eight industries, the fit in (54) was better (higher R^2) than that in (55) in over 90% of the cases and the average estimated percentage excess of each firm's size over the next smallest firm's size (the average of the antilogs of the estimated coefficients in (54)) was 86% compared to the theoretical value of 100%. To us, these results amount to fairly strong preliminary evidence in favor of our theory. The regression results on (54) are described in Table 1.

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Table 1: Fit of Equation (54)

Industry	Coeff.	$S_{Coeff.}$	$e^{Coeff.}$	D.W.	R^2
gold mining	.68	.1	1.98	1.57	.85
coal	1.13	.21	3.10	1.66	.90
housing construction	.39	.04	1.48	2.37	.93
packaged foods	.26	.03	1.30	1.13	.88
dairies	.64	.13	1.90	1.78	.83
canned foods	.33	.02	1.39	1.55	.96
animal foods	1.14	.16	3.13	1.70	.95
biscuits	1.22	.67	3.39	3.00	.80
confectionary	.69	.09	2.00	2.06	.91
brewers	.27	.01	1.31	1.81	.97
distillers	.51	.03	1.67	2.53	.98
soft drinks	.69	.09	2.00	2.06	.91
tobacco	.47	.10	1.60	1.73	.83
forest products	.49	.04	1.63	1.67	.95
mobile homes	.33	.04	1.39	2.11	.90
home furnishings	.27	.02	1.31	1.85	.94
paper	.35	.04	1.42	1.28	
books	.49	.05	1.63	1.75	.95
drugs-ethical	.22	.02	1.25	1.21	.93
drugs-proprietary	.57	.07	1.77	.70	.91
medical & hospital supply	.38	.03	1.46	2.08	.94
soap	.99	.18	2.69	1.57	.91
cosmetics	.30	.01	1.35	2.45	.99
paint	.60	.11	1.83	1.74	.86
tires & rubber goods	.31	.02	1.36	0.89	.92
plastics	.33	.04	1.39	1.77	.91
shoes	.38	.02	1.46	1.81	.96
concrete, gypsum * plastic	.37	.08	1.45	1.98	.83
aluminum	.60	.21	1.83	2.29	.80
motor vehicles	1.01	.22	2.75	2.14	.91
photographic	1.07	.07	2.92	2.38	.98
watches	.65	.08	1.92	1.93	.96
musical instruments, parts	.97	.23	2.64	2.27	.86
games	.41	.05	1.51	1.85	.94
radio-TV broadcasters	.34	.03	1.40	2.91	.95
wholesale foods	.52	.06	1.68	1.78	.93
retail lumber yards	.96	.23	2.61	2.37	.90
motion pictures	.55	.06	1.73	1.37	.91
average			1.86		

FOOTNOTES

* The authors benefited substantially from the comments of Louis Makowski on an earlier draft of this paper.

1. These relative price effects are not theoretically interesting with respect to competitive firms. Since individual competitive firms have no effect on relative prices, they do not consider relative price effects when choosing their outputs.

2. Since the purpose of this paper is to generalize the standard competitive model to allow for multiple monopoly, we want to build a general equilibrium model which is void of other problems not handled by the standard competitive model. Thus, we have assumed away problems of externalities and collective goods. We have assumed individual profit-maximizing behavior and prohibitive transactions costs of collusive activity in order to achieve a spirit of decentralized decision-making.

3. Actually, it is a correspondence, not a function.

4. Although the idea of commitments is not novel to the area of conflict resolution (see, for example, Boulding (1) and Schelling (4)), its economic rationale and use in economic theory is rare. A recent exception is contained in Thompson (6). Here, commitments characterize an equilibrium distribution of property between countries through each country's commitment to protect, with all of its resources, any part of its capital stock from foreign aggression.

5. Examples of communicating commitments are: (1) when a producer of diamonds or gold stockpiles his commodity beyond the quantity rationalized by unpredicted demand fluctuations. Thus, the producer is exhibiting his ability and desire (by incurring higher production costs) to flood the market in the event of entry or increased production by his competitors: (2) strikes and lockouts when there exist only small differences in contract negotiations. This establishes the union's, or the firm's, willingness to engage in irrational (non-profit-maximizing) behavior when its desires are not met: and (3) when nations establish commitments by placing the power to make certain decisions beyond their control; for example, a "doomsday machine" which can destroy the whole world, but once turned on, cannot be turned off.

6. If the joint-profit maximizing output is non-unique, the two firms' bids will be equal, but positive. For example, at x^i, let $\pi_i(x^i) = 50$, $\pi_j(x^i) = 40$, and at x^j, let $\pi_j(x^j) = 60$, $\pi_i(x^j) = 30$. Notice that both i's and j's bid will equal 20. Since the bids are equal, the selection of strategy-maker is still arbitrary.

7. An m-firm joint-profit maximum would mean that the maker's marginal profit (assuming differentiability) equals the <u>sum</u> of the other firms' marginal profits. In our case, the marginal profit of the maker is equal to each firm's marginal profit. Let there be an m-firm tie where m = 3. If marginal profit to i, the maker, equals \$1 (thus, marginal profit equals \$1 a piece to the takers), then marginal joint-profit equals minus \$1, rather than 0.

8. This interpretation assumes that the courts can distinguish between increases in output due to efficiency reasons, and increases for predatory or punishment reasons. Certain anti-trust cases, such as the U. S. Steel Case, 1921, lend support to this assumption. The anti-trust policy in this paper is based on past interpretations of the Sherman Anti-Trust Act. For further discussion of this law and its judicial interpretations, see Neale (7).

9. In the Stackelberg model of duopoly (see Intriligator (2)), one firm, the "leader," believes that his rival will behave as a Cournot duopolist. That is, the rival firm, called the "follower," assumes the leader will exhibit a constant output; and thus the follower makes his rational output choice based on this assumption about his rival's output. The leader then selects his output subject to the follower's rational output choice-function. If the two duopolists behave as each believes, the result is a Stackelberg equilibrium. In our controlled monopoly model, the pure taker acts as a Stackelberg follower by choosing his output subject to constant-output reaction functions. The pure maker behaves as a Stackelberg leader since he chooses his reaction function (output) subject only to the profit-maximizing behavior of the other firms. Firms 2,...,m-1 introduce into the model additional maker/taker relationships not described in previous models of which these authors are aware. Nevertheless, our controlled monopoly model with its (constrained) reaction functions and added maker/taker relationships generates what can be interpreted as a generalized Stackelberg model. For, as derived above, adding more firms to our model merely

creates a hierarchy of makers, or partial Stackelberg leaders.

10. This can be seen by noting from (42) that a differentiable function $x(f)$ admitting x_f at each f would have to have the property that at

$$f = f^i + \phi_i,\ 0<\phi_i<1,\ i, f^i=1,2,\dots,m-1,\ \frac{dx(f)}{df} = x^2(f^i)\ .$$

Maintaining this rate of expansion over the whole interval (not just at $i=1,\dots,m-1$), this equality would hold for all f, $1 \leq f \leq m-1$. Then

$$df = \frac{dx}{x^2(f^i)}\ .$$

Antidifferentiating, we have

$$f + K = \frac{1}{x(f^i)}\ .$$

Taking $x(1)= 1$ and $x(3) = 6$, we have

$$1 + \phi_1 + K = 1; \qquad K = -\phi_1$$

$$3 + \phi_3 + K = \frac{1}{6}; \qquad K = -2\tfrac{5}{6} + \phi_3\ .$$

These equations are inconsistent since $\phi_i<1$. Hence, the differentiable approximation cannot hold everywhere in our interval. As is obvious from (42), $x(f^i)$ expands so rapidly with f there are intervals of f over which it cannot remain finite (e.g., the interval [1, 3]).

11. We only included an industry when it (1) included 4 or more companies (for statistical reasons), (2) had a firm producing over 50 million dollars of sales (to avoid the exclusion of large producers due to their being a subsidiary of a diversified firm), (3) sold its product in a national

market (to avoid local monopoly effects and interactions with firms in foreign markets), (4) sold its product to economic agents which are not substantially larger than itself (to avoid including industries in which some of the outputs are produced by vertically integrated firms, which would not be counted as part of the industry), (5) and marketed a relatively homogeneous commodity. This is a highly subjective selection of industries, but we know of no better way to provide a fair test of the hypothesis with so much of the data obviously irrelevant.

12. We had data on both current sales and assets as measures of size, assets being perhaps better than current sales as a measure of future sales. We ran regressions for both measures of size and chose the measure for each hypothesis that yielded Durbin-Watson statistics closest to 2. The rationale here is that we wanted to be as generous as we could to each hypothesis regarding which measure of size would conform the best to the curvature assumptions of the hypothesis.

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