HETEROGENEITY AND DISTRIBUTION IN THE COMMONS

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Working Paper No. 643
Department of Economics
University of California, Los Angeles
December 1991

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Abstract

This paper analyzes the problem of the commons in a model in which the individual users are heterogeneous. Explicit treatment of heterogeneity leads to five results, some that differ from those of the previous literature: 1) The free-access equilibrium Pareto dominates the equilibrium in which the commons is prohibited. 2) The optimum level of employment in the commons can be supported by a taxes or subsidies. 3) Examples exist where private ownership leads to a higher level of income to the free factor than that in the free-access equilibrium. 4) Under reasonable conditions, the income of the variable factor that remains employed in the commons will be higher in the private ownership equilibrium. 5) A non-discriminatory proportional reduction of the use of the commons can be inferior to the free-access equilibrium.

I. Introduction

This paper addresses the welfare and distributional implications of regulating an activity that produces an externality, but where the externality is restricted to those individuals that engage in the activity. Thus, it treats the problem of the *commons*, defined as a resource in which a group of individuals have a common right of access, but for which each individual's use has an impact on the others. Examples include the extraction of oil from a common pool; the use of facilities subject to congestion, such as roads, bridges or airports; and the exploitation of such biological resources as fisheries. This paper extends previous treatments of this problem by explicitly treating implications of heterogeneity in the users of the commons.[1]

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[1]The explicit treatment of heterogeneity in the users of the commons in this paper is motivated, in part, by two recent legal cases: *Los Angeles Department of Water and Power v. Manhart*, 435 U.S. 702 (1978). and *International Union, United Automobile, Aerospace and Agricultural Implement Workers of America, UAW, Et. Al. v. Johnson Controls, Inc.* No. 89-1215 (1991). In *Manhart*, the Court decided that employers could not use gender in determining the cost of annuities. Even if a class shared a characteristic such as gender and a second characteristic that was statistical such as long expected life, the Court held that under Title VII of the Civil Rights Act of 1964 it was not valid to use the first characteristic (gender) with regard to the class since it was not possible to determine a priori which member of the class shared the second characteristic (long expected life). Heterogeneity of women as a class was central to the logic in *Manhart*, and this case was central to the outcome in *Johnson Controls*. In that case the company had instituted a

In the commons the output of each individual is a function of his or her own efforts, but each person's marginal product depends in a declining manner, given a negative externality, on the aggregate input of all individuals. Typically, individuals are assumed to be homogeneous or it is assumed that inputs are such that they can be quantified in equivalence or efficiency units.[2] These assumptions are relaxed in this paper. The *free-access equilibrium* is one in which individuals do not face any restriction on their use of the commons. It results in the common property resource being used to excess because no individual takes into account the impact of his or her efforts on the productivity of others, the so-called "tragedy of the commons."[3] It is commonly accepted that it is efficient to reduce the level of exploitation.[4] Further, it has been shown that private ownership of the commons will redistribute income away from the variable factor.[5]

A feature of this paper is that it explicitly considers the implications of *heterogeneity* among the users of the commons. This paper analyzes this problem of the common property resource in the context of a model in which individuals are heterogeneous in their ability to exploit the commons. It treats the welfare implications of regulating an activity that produces an externality,

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policy that barred all women capable of bearing children from jobs involving exposure to lead at levels that exceeded certain standards. Employees affected by this policy filed a class action suit claiming that the policy constituted sex discrimination in excluding women from certain jobs. The Supreme Court held that a policy that excluded women with childbearing capacity from lead exposed jobs creates a classification based on gender and thus explicitly discriminates against women on the basis of their sex . The company policy explicitly classifies employees on the basis of potential for pregnancy and thus must be regarded in the same light as explicit sex discrimination. This paper is not an essay on these cases. The main proposition that applies to the issues in these cases is the one that states that of banning the use of the hazardous technology is Pareto inferior to the free access equilibrium. Properly viewed, this proposition is self evident. In fact, Judge Posner invoked it implicitly in his dissent at the 7th Circuit. He stated: "The plaintiffs would have won a Pyrrhic victory if as a result of their winning this suit Johnson Controls shut down its battery operation, or if, as has happened with so many products formerly manufactured in the country, production shifted overseas. . . ." 886 F.2d 871, 907. The formal model is similar to the model we used to study vaccines and other excludable externalities. See Brito, Sheshinski, and Intriligator (1991).

[2] See Weitzman p. 226 (1974).

[3] See Hardin (1968), Hardin and Baden, eds. (1977), and Ostrom (1991).

[4] See Stiglitz (1986) p. 216.

[5] See Weitzman *op.cit.*

but where the externality is restricted to those individuals that engage in the activity. The externality treated here is a function of *inputs* in the production process as opposed to *output* externalities.[6] It differs from Diamond and Levin who have treated heterogeneity in a similar context, as well as Weitzman, in that it assumes that an individual's marginal product is independent of his/her own level of effort, but the amount of effort has an upper bound.[7] This is similar to the Mirrlees tax model, where an individual's marginal product depends only on the skill level and is independent of effort.[8] An example is an airport with planes queuing to land. Small planes occupy landing slots just like large planes, the externality being caused by congestion which is independent of the size of the plane.[9] Productivity per plane is a function of plane capacity and is limited.

An implication of heterogeneity is that the free-access equilibrium Pareto dominates the equilibrium in which the commons is prohibited or shut down even in the case in which the commons involves the use of a risky technology. Furthermore, if the difference between the average productivity of the users and the productivity of the marginal user is sufficiently large, a non-discriminatory proportional reduction of the use of the commons is inferior to the free-access equilibrium. Thus, if all individuals are reduced in the use of the commons by a proportional amount then total output will fall. The two allocations, however, are not Pareto comparable since

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6 The model, thus, would not apply to a dynamic model of fisheries where the externality is caused by the overexploitation of the common property resource. If the fish density in a fishery is reduced, it costs more to catch a fish. Thus, the cost will generally not depend on the number of fishermen at any on time, but rather on the history of exploitation of the resource. In a steady state, however, the analysis would hold. See Appendix. Dasgupta and Heal (1979) p. 144 consider the case where both kinds of externalities exist. See also Clark (1990).

7 See Diamond (1973) and Levin (1985)

8 See Mirrlees (1971)

9 Since small planes are slower, often do not have a copilot, and have less sophisticated instrumentation, it can be argued that they impose a greater burden on the Instrument Landing System (ILS) than a larger plane.

the output of some of the less-productive individuals will be lower in the free-access equilibrium.[10]

Heterogeneity may result in an increase in the income of the variable factor in the private ownership equilibrium. In this case, however, the income of the more productive individuals is increased at the expense of the less productive individuals. It can be shown that under reasonable conditions, the income of the variable factor that remains employed in that sector will be higher in the private ownership equilibrium.

II. The Model

Assume that a single homogeneous good y can be produced in two sectors. In the first sector, the commons, the technology is risky and the marginal product of individuals is a(n), where n is an unobservable parameter characterizing both individual productivity and susceptibility to risk in the commons. It will be assumed that net productivity in the commons is a decreasing function of n.[11] In the second sector the technology is safe, and the marginal product of all individuals is the wage w. Individuals are distinguished by their marginal product, a(n) where $a'(n) < 0$. Individuals of the same type have a constant marginal (and average) product when employed in the commons and this marginal product varies across classes. The distribution function of this unobservable characteristic, F(n) is non-degenerate over $[0, \bar{n}]$, and it has a corresponding density function, f(n). Individuals supply one unit of labor, and they have a pretax income equal to their marginal product. The technology that uses the commons involves a level of risk which, for simplicity, is assumed to be proportional to the number of individuals engaged

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[10] Consider the marginal user of the commons. His or her output in the commons or in the alternative technology must be equal under free access. If there is a reduction in his or her access to the commons, the time he or she has to reallocate is no less productive and the time allocated to the commons is now more productive due to the decrease in the externality. That individual is unambiguously better off, so the allocations cannot be Pareto comparable.

[11] The use of distributions over an unobservable characteristic follows Mirrlees (1971). An alternative interpretation of the model is that the parameter is observable but there exist legal restrictions against the use of such information, e.g. employers can not use gender in determining the cost of annuities, as in the *Manhart* case discussed in footnote 1.

in that activity. Individuals also differ as to their susceptibility to this risk, as measured by the unobservable parameter n.

The risk probability, p(n,x), increases with n, so $\frac{\partial p(n,x)}{\partial n} \geq 0$, where x is the number of individuals employed in the commons. Individuals with high n are more susceptible to the hazard, where the loss associated with the risk, $\alpha[a(n)]$, is assumed to be proportional to the marginal product a(n), and $\frac{d\alpha[a(n)]}{dn} < 0$. In the context of the aircraft example, a large plane may, because of better instrumentation, face a lower probability of delay in landing, however, because of the greater number of passengers the expected cost of a delay may be greater.

The expected product of an individual employed in the commons is

$$(1) \quad a(n)[1 - p(n,x)] + \{a(n) - \alpha[a(n)]\}p(n,x) = a(n) - \alpha[a(n)]p(n,x) \equiv a(n) - c(n,x)$$

where c(n,x) is the expected loss, given as the probability p(n,x) times the loss $\alpha[a(n)]$. The expected product is assumed to decrease with n, so $\frac{\partial}{\partial n}[a(n) - c(n,x)] < 0$. Individuals are treated here as risk neutral, so an n-individual will choose to be employed in the commons if and only if his or her return is higher than the wage in the safe sector,

$$(2) \qquad a(n) - c(n,x) > w\ .$$

Thus, those individuals with 'low' n will tend to be employed in the commons while those with 'high' n will tend to be employed in the safe sector. Letting $\hat{n}$ be the characteristic of the marginal individual employed in the commons, the number of individuals employed in the commons is given by

$$(3) \qquad x(\hat{n}) = \int_0^{\hat{n}} dF(z) = F(\hat{n}).^{12}$$

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12 This assumption is that the externality is a function of the number of individuals in the commons. Alternative assumptions are that the externality is a function of output or that the level

To find the free-access equilibrium, define the *excess return* as the difference of the return in the commons over the safe sector

(4) $$\psi(n,x) \equiv a(n) - c[n,x] - w, \quad n \in [0,\tilde{n}].$$

By assumption and the nature of the commons, the excess return decreases with greater n and greater x, that is the partial derivatives of $\psi(n,x)$ satisfy $\psi_1(n,x)$ and $\psi_2(n,x) < 0$. Assuming that $\psi(0,0) > 0$ and $\psi(\tilde{n},1) < 0$, these conditions guarantee a unique interior solution to the equation $\psi[n,x(n)] = 0$.

Let V(z) be the set of individuals that would not choose to work in the commons if the number of individuals employed in the commons is z

(5) $$V(z) = \{n \mid \psi(n,z) \leq 0\}.$$

Define ζ as the cumulative value of for n in V(z),

(6) $$\zeta = \int_{n \in V(z)} dF = \phi(z).$$

Proposition 1. There exists a *unique free-access equilibrium.*

A unique fixed point exists if $\zeta = \phi(z)$ is a continuous monotonic mapping of [0,1] to [0,1]. The assumption that. $\psi_2(n,x) < 0$ implies that V(z) is a subset of $V(z + \delta)$ for all $\delta > 0$, thus $\phi(z)$ is monotonic. The conditions that $\psi(0,0) > 0$ and $\psi(\tilde{n},1) < 0$ rule out the possibility that the fixed point is on the boundary. Let x* be the unique fixed point, it defines the equilibrium.•

The unique fixed point x* implicitly defines the *productivity level* n*, where $a(n^*) - c[n^*,x(n^*)] - w = 0$. This partitions the population into two groups. Those with high

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of the externality produced varies with individuals. These assumptions will be discussed in Section VI. There are no essential differences in the results.

productivity in the commons, $n \leq n^*$ are employed in the commons, and those with low productivity, $n > n^*$, are employed in the alternative technology. Thus n^*, the fixed point, provides a complete characterization of the *unique, free-access equilibrium.*

The free-access equilibrium, so characterized, however, does *not* maximize social welfare. *Social welfare*, W(n), is defined here as the sum of utilities

(7)
$$W(n) = \int_0^n \{a(z) - c[z,x(n)]\}dF(z) + \int_n^{\bar{n}} w dF(z),$$

where individuals in the interval [0, n] are employed in the commons and individuals in the interval $(n,\bar{n}]$ are in the safe sector. Adding and subtracting $\int_0^n w dF(z)$ yields

(8)
$$W(n) = \int_0^n \{a(z) - c[z,x(n)] - w\}dF(z) + w.$$

If the commons is banned then all individuals in the interval $[0, \bar{n}]$ are employed in the alternative sector, so social welfare in this case is

(9)
$$W(0) = \int_0^{\bar{n}} w dF(z) = w,$$

Proposition 2. $W(0) < W(n^*)$, so banning the use of the commons is Pareto inferior to the free-access equilibrium.

Proof

Subtracting equation (9) from equation (8) evaluated at n^* yields

(10)
$$W(n^*) - W(0) = \int_0^{n^*} \{a(z) - c[z,x(n)] - w\}dF(z) > 0,$$

since $a(n) - c[n,x(n)] > w$ for all $n < n^*$. Since the proof does not depend on the specific form of the welfare function, the result holds for all social welfare functions that respect individual preferences.[13]•

While this proposition that banning the use of the commons is inferior to the free-access equilibrium is reasonably straightforward, the formally equivalent proposition that banning the use of risky technology in circumstances where the danger is limited to those employed in that sector is Pareto inferior to the free-access equilibrium would be debated by some individuals. Indeed, such bans have been both suggested and implemented.[14] The proposition is stated here formally since it is necessary for the proof of subsequent propositions.

The first-order condition for an interior maximization of W(n) with respect to n yields

$$\frac{dW(n)}{dn} = a(n) - c[n,x(n)] - w - \int_0^n c_2[z,x(n)]x'(n)dF(z) = 0. \tag{11}$$

This condition has a solution, n^{**}, which yields the maximum welfare. Since $c_2[n,x(n)]x'(n) > 0$, it follows that $n^{**} < n^*$. Thus, at the social optimum fewer individuals are employed in the commons than were employed in the free-access equilibrium. It follows directly from Proposition 2 that the solution, n^{**}, which yields the maximum social welfare, is interior.

III. Subsidization and Taxation

The government will, in general, not have sufficient information to choose the unique social optimum n^{**} maximizing social welfare (satisfying (11)) since it generally cannot distinguish those individuals who have a "high" n from those who have a "low" n. Thus, it cannot identify which people are to be excluded from the use of the commons. Even if the

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13 We are grateful to Tim Besley for pointing this out to us in another context.

14 The argument usually involves a paternalistic view of the individual. For example the debate over breast implants suggests a view that the state is better able to judge the costs and benefits of such procedures than the woman involved. See Hilts (1991). For a discussion of legal prohibitions against contractual waivers of liability, see Landes and Posner (1987) pp. 280-284.

government *does* have sufficient information to choose the unique social optimum, it may be prohibited from implementing this solution on non-discriminatory grounds, due to law or custom. It may face the constraint that it must treat all individuals the same and thus not be able to discriminate in order to reach the efficient solution.

The government *can*, however, reach the efficient solution by providing a subsidy, s, to individuals in the safe technology and/or imposing a tax, t, on individuals in the commons, letting individuals voluntarily decide, on the basis of their own productivity and the government-established levels of tax and subsidy, whether or not to use the commons. The equilibrium *cutoff level*, $\hat{n}$, is defined as the level of cost below which individuals will work in the safe technology and above which they will work in the commons. This level satisfies the condition that the utility of individuals with productivity $\hat{n}$ be equal in the two technologies

$$(12) \qquad a(\hat{n}) - t - c(n,x) = w + s,$$

where tax t is levied on workers in the commons and subsidy s is provided to workers in the safe technology. This condition will be referred to below as the *self-selection constraint* as it determines which individuals will choose, given the government determined levels of tax and subsidy, whether or not to be employed in the risky technology.[15]

To attain the social optimum at n^{**} requires that both (11) and (12) be satisfied. Since $a(\hat{n}) - w = s + t > 0$ there always exists a subsidy s or a tax t that makes (11) and (12) hold jointly, satisfying (12) for either s = 0 or t = 0. If the combined tax/subsidy program is to be *revenue neutral*, where the tax collections equal subsidy payments, then this revenue constraint requires that

[15] See Stiglitz (1982) for a discussion and use of the self-selection constraint.

$$(13) \qquad tx[\hat{n}] - s[1 - x(\hat{n})] = 0.$$

Setting $\hat{n} = n^{**}$, equations (12) and (13), together with (11), determine a unique revenue neutral tax/subsidy combination (t^{**}, s^{**}) yielding n^{**}. Risk neutrality of individuals will imply that the optimal cutoff level under a tax/subsidy scheme be the full information cutoff level at n^{**} given by equation (11).

Proposition 3. If all individuals are risk neutral, then the government can obtain the socially optimum level of employment in the commons by subsidizing those individuals who are employed in the safe technology and taxing those individuals who are employed in the commons.

Proof

Social welfare is, in this case of subsidy s and tax t,

$$(14) \qquad W(n,s,t) = \int_0^n \{a(z) - c[z,x(n)] - t\}dF(z) + \int_n^{\bar{n}} (w + s)dF(z).$$

The revenue neutral constraint (13) can be used to simplify (14) as

$$(15) \qquad W(n,s,t) = \int_0^n \{a(z) - c[z,x(n)]\}dF(z) + w[1 - x(n)]$$

$$= \int_0^n \{a(z) - c[z,x(n)] - w\}dF(z) + w,$$

where s and t no longer enter the social welfare function explicitly. Solving the budget constraint for t and substituting the result into the self-selection constraint yields

$$(16) \qquad a(n) - [1 + \frac{1 - x(n)}{x(n)}]s - c(n,x) - w = 0.$$

The Lagrangian for maximizing (15) subject to the budget and self-selection constraint (16) is

$$(17)\ L(n,s,\phi)= \int_0^n \{a(z) - c[z,x(n)] - w\}dF(z) + w + \phi\{a(\hat{n}) - [1 + \frac{1-x(n)}{x(n)}]s - c(n,x) - w\},$$

where ϕ is the Lagrange multiplier associated with the constraint. If $s > 0$ at the solution the Kuhn-Tucker condition with respect to s requires that

$$(18)\qquad \phi[1 + \frac{1-x(n)}{x(n)}] = 0.$$

The fact that $[1 + \frac{1-x(n)}{x(n)}] > 0$ implies that, at the optimum, $\phi = 0$. The first-order condition with respect to n is therefore

$$(19)\qquad \frac{dW(n,s,t)}{dn} = a(n) - c[n,x(n)] - w - \int_0^n c_2[z,x(n)]x'(n)dF(z) = 0$$

which is identical to equation (11), so the optimal value of n is n^{**}. Thus, when all individuals are risk neutral the socially optimum value for the cutoff level is also n^{**}.•

IV. Private Ownership versus Free-Access Rights

A well known result due to Weitzman states that if the variable factor is homogeneous then the variable factor will "always be better off with (inefficient) free-access rights than under (efficient) private ownership."[16] An interesting question is whether this result remains valid if the assumption that the variable factor is homogeneous is dropped. It turns out that if the variable factor is heterogeneous, examples can be constructed under which private ownership leads to a *higher* income to the variable factor. Intuition for this result can be provided by the example of

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[16] See Weitzman p. 225. Our model also differs from his not only in allowing for heterogeneity, but also in assuming constant marginal cost for individual firms while Weitzman assumes increasing marginal costs. We relax this assumption in Section VI.

large and small aircraft queuing to land. A relatively small landing fee that is insignificant for the large aircraft may deter the smaller aircraft from using the airport. The savings to the large aircraft may, in fact, be sufficient to increase the aggregate income of *all* aircraft. It is necessary, however, to check that such a fee is profit maximizing for the operators of the airport.

This result can be combined with the result of the last section to show that it is possible to construct examples where taxation of the use of the commons can lead to a Pareto superior allocation. Assume revenue is collected from the use of the commons (To avoid awkward phrasing, the sector that was the commons will be referred to as "the commons" even under private ownership.). If this revenue is then distributed to the nonusers, they can also be made better off by imposing a tax on the use of the commons. In constructing the example care has to be taken, however, to ensure that those excluded from the use of the commons by the tax are not made worse off. The example presented in this section satisfies this condition.

The profit function of the owner of the fixed resource charging an access fee r would be

$$\Pi(r) = r \int_0^n dF(z), \tag{20}$$

which the owner would maximize subject to the self-selection constraint, which in this case requires that

$$a(n) - r - c(n,x) = w. \tag{21}$$

The Lagrangian for this maximization is

$$L(n,r,x,\lambda,\delta) = rx + \lambda\{a(n) - r - c(n,x) - w\} + \delta\{x - \int_0^n dF(z)\} \tag{22}$$

where λ and δ are Lagrange multipliers.

Denoting the private ownership equilibrium by the subscript p, the first-order conditions for an interior maximum are

(23) $$\frac{\partial L}{\partial r} = x_p - \lambda = 0,$$

(24) $$\frac{\partial L}{\partial x} = r_p - \lambda\, c_2(n_p,x_p) + \delta = 0,$$

(25) $$\frac{\partial L}{\partial n} = \lambda[a'(n_p) - c_1(n_p,x_p)] - \delta f(n_p) = 0.$$

Combining these first-order conditions implies that

(26) $$r_p = x_p\{c_2(n_p,x_p) - \frac{[a'(n_p) - c_1(n_p,x_p)]}{f(n_p)}\} > 0,$$

where from (21)

(27) $$r_p = a(n_p) - c(n_p,x_p) - w > a(n^*) - c(n^*,x^*) - w = 0.$$

As expected, the private ownership equilibrium level, n_p and x_p, are less than the free-access ownership equilibrium level, n^* and x^*.

Define ρ as the rents earned by the marginal individual under private ownership in the free-access equilibrium

(28) $$\rho = a(n_p) - c(n_p,x^*) - w.$$

Since $r_p = a(n_p) - c(n_p,x_p) - w$, adding and subtracting $c(n_p,x^*)$ we get,

(29) $$r_p = \rho + [c(n_p,x^*) - c(n_p,x_p)],$$

so the monopolist is able to capture the rents and the savings due to decreased use of the commons.

Proposition 6. If the expected costs are independent of n or x, so $c_1(n,x)$ or $c_2(n,x) = 0$, then the variable factor will be better off with free-access rights than under private ownership.

Proof

The difference in the income of the variable factor under the free-access and the private ownership equilibria, Δ, is

$$\Delta = \int_0^{n^*} [a(z) - c(z,x^*)]dF(z) - \int_0^{n_p} [a(z) - r - c(z,x_p)]dF(z) - \int_{n_p}^{n^*} w\,dF(z). \tag{30}$$

If $\Delta > 0$, then income is higher in the free-access equilibrium. Equation (30) can be written as

$$\Delta = \int_{n_p}^{n^*} [a(z) - c(z,x^*) - w]dF(z) + \int_0^{n_p} \{r - [c(z,x^*) - c(z,x_p)]\}dF(z). \tag{31}$$

The difference in income of those individuals that are excluded by the access fee is $a(n) - c(n,x^*) - w > 0$ for all $n < n^*$, so the first term is nonnegative. The second term is the difference in income of those individuals that remain. If the expected costs are independent of n, then $c(z,x_p) = c(n_p,x_p) = a(n_p) - w - r$ and $c(z,x^*) = c(n^*,x^*) = a(n^*) - w$ for all z, so equation (31) becomes

$$\Delta = \int_{n_p}^{n^*} [a(z) - c(z,x^*) - w]dF(z) + \int_0^{n_p} [a(n_p) - a(n^*)]dF(z) > 0, \tag{32}$$

where the last inequality follows from the fact that $a(n_p) - a(n^*) > 0$.

If expected costs are independent of x, then $[c(z,x^*) - c(z,x_p)] = 0$ and

$$\Delta = \int_{n_p}^{n^*} [a(z) - c(z,x^*) - w]dF(z) + \int_0^{n_p} r\,dF(z) > 0. \bullet \tag{33}$$

It is possible to construct examples where this result will not hold. Consider an expected loss function of the form

(34) $$c(n,x) = a(n)p(x) - w,$$

where an individual's expected loss is proportional to his or her product less a constant which can be chosen to be equal to the wage, w. Note that the cost function is not independent of n or x so the previous proposition does not apply. An individual employed in the commons in the free-access equilibrium will have an income of

(35) $$a(n) - c(n,x) = a(n)[1 - p(x)] + w.$$

There will be entry into the commons in this example as long as $p(x) < 1$. Since free access will eliminate all rents, $p(x^*) = 1$ and

(36) $$\int_0^{n^*} \{a(z)[1 - p(x^*)] + w\}dF(z) = \int_0^{n^*} wdF(z).$$

Since $x^* > x_p$, $p(x^*) > p(x_p)$. The boundary condition $a(n_p)[1 - p(x_p)] + w - r = w$ and $a'(n)[1 - p(x_p)] < 0$ imply that $a(n)[1 - p(x_p)] + w - r > w$ for all $n < n_p$. Thus,

(37) $$\int_0^{n_p} [a(z)[1 - p(x_p)] + w - r\,]dF(z) + \int_{n_p}^{n^*} wdF(z) > \int_0^{n^*} wdF(z).$$

The variable factor will be better off with private ownership than with free-access rights. Since those individuals who were forced from the use of the commons by the access fee are no worse off, private ownership of the commons is Pareto superior .

In this special case individuals were earning no rents under free-access. In the general case, the change in income for those individuals that remain employed in the commons is, from second term in equation (31),

(38) $$\Delta_p = \int_0^{n_p} r_p\, dF(z) - \int_0^{n_p} [c(z,x^*) - c(z,x_p)]dF(z).$$

Using equation (29) to substitute for r_p yields

$$\Delta_p = \int_0^{n_p} \rho + [c(n_p,x^*) - c(n_p,x_p)]dF(z) - \int_0^{n_p} [c(z,x^*) - c(z,x_p)]dF(z). \tag{39}$$

Define $\delta(n_p,x)$ as the average expected cost of individuals in the commons, where

$$\int_0^{n_p} c(z,x)\, dF(z) = \delta(n_p,x) \int_0^{n_p} dF(z). \tag{40}$$

Then if $\delta(n_p,x^*) - \delta(n_p,x_p) > \rho + [c(n_p,x^*) - c(n_p,x_p)]$, the private ownership equilibrium will yield higher income to the free factor than the free-access equilibrium. This result highlights the importance of heterogeneity. If individuals were identical, the average expected cost of the users of the commons and the expected cost of the marginal user would be identical, so the Weitzman result would hold.

V. Non-Discriminatory Exploitation

It is frequently the case that property rights are defined in such a way that any reduction of the exploitation of the commons must be done in a fashion that is non-discriminatory. For example the governing body of the commons may be obligated to treat all members equally or there may be legal constraints such as Title VII of the Civil Rights Act of 1964 or equal protection, which may not permit discriminatory allocation of rights to the commons. Attempts to discriminate on the basis of economic efficiency may be viewed as illegitimate. For example, it may have a disparate impact on a disadvantaged group.[17] A recent example of non-discriminatory allocation of rights is the Middle Atlantic council's 1991 action which privatized fishing rights and restricted boats engaged in clam fishing to a total of 144 hours of fishing regardless of the size or efficiency of the boats. In this case, the rights were transferable, and a market in the rights was established.[18] If the rights are transferable and the aggregate allocation is correct, then the only

[17] See Kelman (1991).

[18] See Passell (1991).

issues involved are redistributional. It is interesting, however, to investigate the implications of a non-discriminatory proportional reduction in the exploitation when the rights are not transferable.[19]

A non-discriminatory proportional reduction in the exploitation by the commons of an amount λ from the level of the free-access equilibrium is equivalent to

$$\tilde{x}(n^*,\lambda) = \int_0^{n^*} (1-\lambda)dF(z)) = (1-\lambda)x(n^*) \tag{41}$$

where $\tilde{x}(n^*,\lambda)$ is the amount of effort employed in the commons. Residual effort is assumed to be employed in the safe sector. Such a non-discriminatory proportional reduction yields for social welfare

$$\tilde{W}(n^*,\lambda) = w[1 - \tilde{x}(n^*,\lambda)] + \int_0^{n^*} (1-\lambda)\{a(z) - c[z,\tilde{x}(n^*,\lambda)]\}dF(z). \tag{42}$$

Define the *average net product* in the commons as

$$\mu(n) = \frac{\int_0^n [a(z)- c(z,x(n))]dF(z)}{\int_0^n dF(z)}. \tag{43}$$

Proposition 5. If the difference between the average and marginal productivity in the commons, as given by

$$\mu(n^*) - [a(n^*) - c[n^*,x(n^*)] = \mu(n^*) - w, \tag{44}$$

is larger than the average of the marginal reduction in costs,

[19] We are assuming ultracomplete property rights in that there exists a right to control entry. See Hamilton, Sheshinski and Slutsky (1989) p. 458.

(45)
$$\frac{\int_0^{n^*} c_2(z,x(n^*))x(n^*)dF(z)}{\int_0^{n^*} dF(z)},$$

then marginal non-discriminatory proportional reduction of the use of the commons is inferior to the free-access equilibrium.

Proof

It is sufficient to show

(46)
$$\frac{\partial\tilde{W}(n^*,\lambda)}{\partial\lambda} = wx(n^*) - \int_0^{n^*} \{a(z) - c[z,\tilde{x}(n^*,\lambda)]\} + c_2[z,\tilde{x}(n^*,\lambda)]\tilde{x}(n^*,\lambda)dF(z) < 0.$$

where (41) has been used to replace $\frac{\partial\tilde{x}(n^*,\lambda)}{\partial\lambda}$ with $-x(n^*)$.

At $\lambda = 0$, equation (46) can be rewritten as

(47)
$$\frac{\partial\tilde{W}(n^*,\lambda)}{\partial\lambda} = -x(n^*)\left\{[\mu(n^*) - w] - \frac{\int_0^{n^*} c_2(z,x(n^*))x(n^*)dF(z)}{\int_0^{n^*} dF(z)}\right\}.$$

Thus, if the difference between the productivity of the average and marginal user of the commons is larger that the average of the marginal reduction in costs, then a marginal non-discriminatory proportional reduction of the use of the commons is not optimal.•

From Proposition 2 it follows that $\tilde{W}(n^*,\lambda) < W(n^*)$ for $\lambda = 1$. Thus, if $\tilde{W}(n^*,\lambda)$ is initially deceasing, satisfying Proposition 5, a non-discriminatory proportional reduction of the use of the commons can dominate the free-access equilibrium only if equation (46) has a minimum and a maximum for λ in [0,1]. This requires at least two distinct real roots for λ. Since

$$wx(n^*) - \int_0^{n^*} a(z)dF(z) \equiv - K,$$

is not a function of λ, equation (46) will have multiple distinct real roots only if

(48) $$\int_0^{n^*} c[z,(1-\lambda)x(n^*)] - c_2[z,(1-\lambda)x(n^*)](1-\lambda)x(n^*)dF(z) = K$$

has multiple distinct real roots for λ in [0,1]. The condition that equation (48) not have multiple distinct real roots is easy to satisfy. For example, homogeneity of any degree of c(n,x) in x will satisfy the condition. More generally, either convexity or concavity of c(n,x) in x will satisfy the condition. The key is that $c_{22}(n,x)$ not change signs, leading to the following proposition:

Proposition 6. If the conditions of Proposition 5 are satisfied and $c_{22}(n,x)$ does not change signs, a non-discriminatory proportional reduction of the use of the commons is inferior to the free-access equilibrium.

Proof

From (48)

(49) $$\frac{\partial}{\partial \lambda} \int_0^{n^*} c[z,(1-\lambda)x(n^*)] - c_2[z,(1-\lambda)x(n^*)](1-\lambda)x(n^*)dF(z)$$

$$= \int_0^{n^*} c_{22}[z,(1-\lambda)x(n^*)](1-\lambda)x(n^*)dF(z),$$

so monotonicity of c(n,x) is sufficient to preclude multiple distinct real roots.•

VI. Extensions

The analysis in this paper was carried out under the assumption that the externality is a function of the number of individuals active in the commons. Another possible assumptions is that the externality depends on the level of output. Alternatively, it could be a negative stock externality such pollution, or a positive stock externality such as fisheries and pastures where increasing the amount of the stock increases the productivity of users of the commons. Most of the results in this paper continue to hold for this broader class of externalities

If the externality is generated by the level of output, equation (3), defining x(n) can be written as

$$x(n) = \int_0^n a(z)f(z)dz \equiv \int_0^n dF(z) \tag{50}$$

and redefining dF(z) as a(z)f(z)dz. No special assumptions were made about the density function, f(n). Since a(n) was assumed to be non-negative, most results that hold for dF(z) ≡ f(n)dz will also hold for dF(z) ≡ a(z)f(n)dz. One exception is in the proof of Proposition 3 because of the definition of the budget constraint in the proof. This is the proposition that states that the government can obtain the socially optimum level of employment in the commons by subsidies and taxes. If the externality is a function of output, the proposition is no longer valid as stated, but the government can obtain the socially optimum level of employment if a tax is levied on output. In that case, equation (14) can be written as

$$W(n,s,t) = \int_0^n \{a(z) - c[z,x(z)] - a(z)t\}dF(z) + \int_n^{\bar{n}} (w + s)dF(z). \tag{51}$$

As before, the budget constraint can be used to eliminate the tax and subsidy, and the rest of the argument in the proof of Proposition 3 remains unchanged.

If the externality is a negative stock externality such as pollution, most of the results will hold in the steady state. Assume that the stock is given by the differential equation

$$\dot{x} = \int_0^n a(z)dF(z) - \delta x. \tag{52}$$

In a steady state

$$(53) \qquad x = \frac{1}{\delta}\int_0^n a(z)dF(z),$$

if $dF(z) \equiv \frac{a(z)}{\delta}f(n)dz$, most results will continue to hold.

The problem is slightly more complicated if the externality is a positive stock externality such as fish since the level of the stock then depends on the technology for harvesting the resource as well as the process that generates the resource. This problem is addressed in the Appendix, but the results are essentially the same.

Another simplifying assumption that was made was that individuals had constant marginal products. This assumption can be dropped. Assume that individuals can be indexed by type t, where t is in the interval $[0,\bar{T}]$ with a density function $\phi(t)$. The output of a individual of type t is given by

$$(54) \qquad y(t,\theta) = h(t,\theta),$$

where θ is the level of effort . Let $m = h_2(t,\theta) > 0$ and $\theta = \eta(t,m)$ be the inverse function. Assume that there is no output without effort, the highest marginal product of an all individuals is and marginal product is a declining function of effort. That is $h(t,0) = 0$, $h_2(t,0) = \bar{m}$ and $h_{22}(t,\theta) < 0$ for all t . Then

$$(55) \qquad y(t,\theta) = \int_0^\theta h_2(t,s)ds = y(t,m) = \int_m^{\bar{m}} \xi(t,z)dz$$

where $\xi(t,z) = \frac{-z}{h_{22}(t,\eta(t,z))}$ and where m is the marginal product of labor.

Total output is

(56) $$y(m) = \int_0^{\bar{T}} \phi(t) \int_m^{\bar{m}} \xi(t,z)dzdt = \int_m^{\bar{m}} b(z)dF(z),$$

where $b(z)dF(z) \equiv \int_0^{\bar{T}} \phi(t)\xi(t,z)dzdt$. Except for minor sign differences this is essentially equivalent to the definition of output in Section II. A similar definition can be developed for c(n,x).

Distributional results will depend on the assumptions made about the distribution of endowments and of individuals. This structure and the example in Section V can be used to gain some insights into the Weitzman's result that the income of the free factor falls under private ownership This example requires special assumptions about the institutional arrangement even if individuals are identical.

Assume that all individuals are identical so that $\xi(t,s) = b(s)$ for all t. Assume a technology similar to the example and let

(57) $$x(m) = \int_m^{\bar{m}} dF(z),$$

where the cost function is similar to that of (34),

(58) $$c(m,x) = b(m)p(x) - w.$$

Then, as before, $b(m) - c(m,x) = b(m)[1 - p(x)] + w$, so there will be entry into the commons as long as $p(x) < 1$. Since free access will eliminate all rents, $p(x^*) = 1$ and

(59) $$\int_m^{\bar{m}} \{b(z)[1 - p(x^*)] + w\}dF(z) = \int_m^{\bar{m}} wdF(z).$$

This is an economy with identical individuals with decreasing marginal product in the commons. The equilibrium condition under free entry is consistent with the model used by Weitzman. Now impose an entry fee r which reduces employment in the commons, as before, and denote this by x_p, where the marginal product is m_p. Since $x^* > x_p$, $p(x^*) > p(x_p)$ and

(60) $$\int_{m_p}^{\bar{m}} \{b(z)[1 - p(x_p)] + w\} dF(z) > \int_{m}^{\bar{m}} w dF(z).$$

Suppose that the fixed entry fee is independent of the level of effort and equal to the rents

(61) $$r_f = \int_{m_p}^{\bar{m}} \{b(z)[1 - p(x_p)] - w\} dF(z).$$

Then the self-selection constraint faced by each individual will be

(62) $$\int_{m}^{\bar{m}} \{b(z)[1 - p(x)] + w\} dF(z) - r_f = \int_{m}^{\bar{m}} w dF(z),$$

and each individual will want to chose a level of effort in the commons m*. This is not a Nash equilibrium since if everyone adopted a level of effort m* the joint outcome would lead to a loss for all. Thus, the owner of the commons cannot appropriate the rents with a lump-sum fee.

Now suppose that the owner imposes a fee on the level of effort. The income of the individual will be

(63) $$\int_{m}^{\bar{m}} \{b(z)[1 - p(x)] + w\} dF(z) + \int_{0}^{m} w dF(z) - \int_{m}^{\bar{m}} r dF(z),$$

and the first-order condition for determining the level of effort in the commons given an entry fee is

(64) $$\{b(m_p)[1 - p(x_p)] + w\} - r = w.$$

Since b'(m) > 0,

(65) $$\int_{m_p}^{\bar{m}} \{b(z)[1 - p(x)] + w\}dF(z) + \int_0^{m_p} wdF(z) - \int_{m_p}^{\bar{m}} rdF(z) > \int_0^{\bar{m}} wdF(z),$$

and the free factor has a higher income under private ownership even if all individuals are identical and have declining marginal products. A two-part tariff, however, will permit the owner to capture all the gains from restricted access if all workers are identical. For example, using a nonlinear entry fee of the form

(66) $$R(m) = r_f + w(m - m_p)$$

Since the free factor's income in the commons is R(m) +wm, this will result in the owner capturing the gains from restricted access, with the free factor supplying m_p amount of labor. However, if all workers are not identical, r_f must be the rents associated with the marginal agent. Thus, inframarginal individuals will be able to appropriate some of the gains from restricted access.

Weitzman assumed that the owner of the commons behaved as a profit maximizing firm and purchased all labor at a fixed wage rate which was equal to the marginal product of the last unit supplied. It is this institutional arrangement that permits the owner of the commons to capture all of the gains from restricted access. While this is a plausible arrangement, it is equally plausible that the owner of the commons will act as a licensor. In that case the gains from restricted access will be shared between the owner and the free factor.

VII. Conclusion

This paper is motivated by the question as to whether individuals should be restricted from voluntary behavior to protect them, and perhaps also other individuals, from consequences which are viewed as detrimental to their general welfare. This question is posed in the more general context of regulating an activity that produces an externality, but where the externality is restricted to those individuals that engage in the activity. Individuals are assumed to be heterogeneous in their ability to exploit the risky activity, and the externality involved is a negative externality that is a function of inputs, rather than output, i.e. a production externality. This is the problem of the *commons*, defined as a resource in which a group of individuals have a common right of access, but for which each individual's use has a negative impact on the others using the resource.

The explicit treatment of heterogeneity in the users of the commons leads to five results that differ from what are now generally accepted as standard results in the externalities/commons literature. First, the free-access equilibrium Pareto dominates the equilibrium in which the commons is prohibited or shut down, even in the case in which the commons involves the use of a risky technology. Second, the optimum level of employment in the commons can be supported by a tax/subsidy scheme and, given risk neutrality, such a scheme is optimal. Third, examples can be constructed where the private ownership equilibrium leads to a *higher* level of income to the free factor than that of the free-access equilibrium. Fourth, under reasonable conditions, the income of the variable factor that remains employed in the commons will be higher in the private ownership equilibrium. Fifth, a non-discriminatory proportional reduction of the use of the commons is inferior to the free-access equilibrium if there is sufficient heterogeneity among the users.

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Appendix

This appendix will consider the case where externality is the result of the overexploitation of a stock, such as the case of a fishery or a pasture. It will be shown that, with minor modifications, most of the results derived in the body of the paper will also hold in this case. Assume that vessels that operate in a fishery vary in capacity which is denoted by v. The distribution function of capacity, F(v) is non-degenerate over $[0, \bar{v}]$, and it has a corresponding density function, f(v). The rate of exploitation by a vessel of capacity v, given a fish stock level of y is given by C(v,y), where C(v,y) is concave in both v and y with positive first derivatives. The growth rate of the fish stock is given by the differential equation

$$\text{(A-1)} \qquad \dot{y} = g(y) - \int_{v}^{\bar{v}} C(z,y)dF(z);$$

where g(y) gives the biological net growth of the bio-mass. It will be assumed that for a fixed number of vessels, the fish stock is stable so that

$$\text{(A-2)} \qquad \frac{d\dot{y}}{dy} = g'(y) - \int_{v}^{\bar{v}} C_2(z,y)dF(z)) < 0.$$

In a steady state the equation

$$\text{(A-3)} \qquad 0 = g(y) - \int_{v}^{\bar{v}} C(z,y)dF(z)$$

can be solved for y(v). For simplicity, it will be assumed that (A-3) has a unique stable equilibrium. The stability condition implies that

(A-4) $$\frac{dy(v)}{dv} = \frac{C(v,y)}{g'[y(v)] - \int_0^v C_2(z,y(v))dF(z)} < 0.$$

Let the price of fish be the numeraire and w(v) be the private cost of operating a vessel of capacity v. The equilibrium condition for the marginal vessel in the fishery will be

(A-5) $$C[v^*,y(v^*)] - w(v^*) = 0.$$

The social return to the fishery if v is the capacity of the marginal vessel employed in the fishery is given by

(A-6) $$W(v) = \int_v^{\bar{v}} \{C[z,y(v)] - w(z)\}dF(z),$$

and the social optimum is given by

(A-7) $$\frac{dy(v)}{dv} = -\{C[v,y(v)] - w(v)\} + \int_v^{\bar{v}} C_2[z,y(v)]y'(v)dF(z) = 0.$$

Since $\int_v^{\bar{v}} C_2[z,y(v)]y'(v)dF(z) < 0$, it follows that $C[v,y(v)] - w(v) > 0$. As expected, the social optimum involves less fishing than the free-access equilibrium.

A non-discriminatory proportional reduction in the exploitation of the commons of an amount λ from the level of the free-access equilibrium will be assumed to restrict the amount of time vessels are allowed to fish by a common proportion. This is equivalent to

(A-8) $$\tilde{C}(\lambda,v,y) = (1-\lambda)C(v,y),$$

where restricted variables are denoted by ~. In equilibrium, as in (A-3),

$$\text{(A-9)} \qquad 0 = g[y] - \int_{v^*}^{\bar{v}} (1-\lambda)C[z,y]dF(z).$$

This equation can be solved for $\tilde{y}(v^*,\lambda)$, where the stability condition implies that

$$\text{(A-10)} \qquad \frac{\partial \tilde{y}(v^*,\lambda)}{\partial \lambda} = \frac{\int_{v^*}^{\bar{v}} C(z,\tilde{y}(v^*,\lambda))dF(z)}{g'[y(v)] - \int_{v^*}^{\bar{v}} (1-\lambda)C_2(z,\tilde{y}(v^*,\lambda))dF(z)} > 0.$$

A non-discriminatory proportional reduction yields for social welfare

$$\text{(A-11)} \qquad \tilde{W}(v^*,\lambda) = (1-\lambda) \int_{v^*}^{\bar{v}} \{C[z,\tilde{y}(v^*,\lambda)] - w(z)\}dF(z),$$

and the change in social welfare as a result of such a proportional reduction in the exploitation in the commons is given by,

$$\text{(A-12)} \quad \frac{\partial \tilde{W}(n^*,\lambda)}{\partial \lambda} = - \int_{v^*}^{\bar{v}} \{C[z,\tilde{y}(v^*,\lambda)] - w(z)\}dF(z) + (1-\lambda) \int_{v^*}^{\bar{v}} C_2[z,\tilde{y}(v^*,\lambda)]\frac{\partial \tilde{y}(v^*,\lambda)}{\partial \lambda}dF(z).$$

At $\lambda = 0$, equation (A-12) can be rewritten as

$$\text{(A-13)} \quad \frac{\partial \tilde{W}(n^*,0)}{\partial \lambda} = - \int_{v^*}^{\bar{v}} [C(z,y^*) - w(z)]dF(z) + \int_{v^*}^{\bar{v}} C_2(z,y^*)\frac{\partial \tilde{y}(v^*,\lambda)}{\partial \lambda}dF(z),$$

where $\tilde{y}(v^*,\lambda)$ is replaced by y^*. The first term is negative and the second term is positive. Thus, if the net product, given by

$$\int_{v^*}^{\bar{v}} [C(z,y^*) - w(z)]dF(z)$$

is larger than the sum of the marginal reduction in costs, given by

$$\int_{v^*}^{\bar{v}} C_2(z,y^*)\frac{\partial\tilde{y}(v^*,\lambda)}{\partial\lambda}dF(z),$$

then a marginal non-discriminatory proportional reduction of the use of the commons is not optimal. This is similar to Proposition 3 in the paper. Closing the fishery is clearly Pareto dominated, so $\tilde{W}(v^*,0) > \tilde{W}(v^*,1)$. If $\tilde{W}(v^*,\lambda)$ is initially deceasing in λ, a non-discriminatory proportional reduction of the use of the commons can dominate the free-access equilibrium only if

$$\text{(A-14)} \quad \int_{v^*}^{\bar{v}} \{C[z,\tilde{y}(v^*,\lambda)] - (1-\lambda)C_2[z,\tilde{y}(v^*,\lambda)]\frac{\partial\tilde{y}(v^*,\lambda)}{\partial\lambda}\}dF(z) = \int_{v^*}^{\bar{v}} w(z)dF(z)$$

has at least two distinct real roots for λ in [0,1]. Conditions that imply that equation (A-14) does not have multiple distinct real roots depend on the properties of $\tilde{y}(v^*,\lambda)$. These properties involve assumptions about the biological process as well as the fishing technology. Define

$$\text{(A-15)} \qquad Z = \int_{v^*}^{\bar{v}} \{C[z,\tilde{y}(v^*,\lambda)] - (1-\lambda)C_2[z,\tilde{y}(v^*,\lambda)]\frac{\partial\tilde{y}(v^*,\lambda)}{\partial\lambda}\}dF(z)$$

A sufficient condition for equation (A-14) not to have multiple roots is that $\frac{\partial^2\tilde{y}(v^*,\lambda)}{\partial\lambda^2} \leq 0$, since this will imply that

$$\text{(A-16)} \qquad \frac{\partial Z}{\partial\lambda} = \int_{v^*}^{\bar{v}} \{C_2[z,\tilde{y}(v^*,\lambda)][2\frac{\partial\tilde{y}(v^*,\lambda)}{\partial\lambda} - (1-\lambda)\frac{\partial^2\tilde{y}(v^*,\lambda)}{\partial\lambda^2}] - (1-\lambda)C_{22}[z,\tilde{y}(v^*,\lambda)]\frac{\partial\tilde{y}(v^*,\lambda)}{\partial\lambda}\}dF(z)$$

is monotonic. It is easy to construct examples that work. Assume that the growth of the bio-mass is given by,

(A-16) $$\dot{y} = ay - by^2 - (1 - \lambda) \int_{v^*}^{\bar{v}} C(z,y)dF(z).$$

Further assume that the fishing technology is quadratic in fish and linear in the capacity of the vessels,

(A-17) $$C(v,y) = v[\alpha y - \beta y^2]$$

where the parameters satisfy the conditions $\frac{\alpha}{\beta} > \frac{a}{b}$ and $b - F(\bar{v})\beta > 0$. The first assumption implies that $C(v,y) > 0$ at the point of maximum bio-mass, while the second condition guarantees that a maximum exits when the fishery is operating without appealing to Kuhn-Tucker conditions. At the equilibrium

(A-18) $$\int_{v^*}^{\bar{v}} C(z,y)dF(z) = \delta(v^*)[\alpha y - \beta y^2],$$

where

(A-19) $$\delta(v^*) = \int_{v^*}^{\bar{v}} F(z).$$

so the equilibrium is given by

(A-20) $$0 = ay - by^2 - (1 - \lambda)\delta(v^*)[\alpha y - \beta y^2].$$

Solving for $\tilde{y}$,

(A-21) $$\tilde{y}(v^*,\lambda) = \frac{a - (1 - \lambda)\delta(v^*)\alpha}{b - (1 - \lambda)\delta\beta}.$$

The derivatives of $\tilde{y}$ with respect to λ are,

(A-22) $$\frac{\partial \tilde{y}(v^*,\lambda)}{\partial \lambda} = \frac{\delta(v^*)[\alpha b - a\beta]}{[b - (1 - \lambda)\delta(v^*)\beta]^2} > 0,$$

and

(A-23) $$\frac{\partial^2 \tilde{y}(v^*,\lambda)}{\partial \lambda^2} = \frac{-\delta^2\beta[\alpha b - a\beta]}{[b - (1 - \lambda)\delta(v^*)\beta]^3} < 0,$$

which guarantee that equation (A-14) does not have multiple roots.

Finally consider the effect of private ownership on the income of vessels that remain in the commons. As in the body of the paper, assume, that the owner of the commons (to avoid awkward phrasing, the sector that was the commons will be referred to as "the commons," even under private ownership) charges an an access fee r. Denoting the private ownership equilibrium by the subscript p, the equilibrium condition for the marginal vessel in the fishery under private ownership will be

(A-24) $$C[v_p,y(v_p)] - w(v_p) - r_p = 0.$$

The difference between the income of the variable factor under free access and its value in the private ownership equilibria is

(A-25) $$\Delta_p = \int_{v_p}^{\bar{v}} [C(z,y^*) - w(z)]dG(z) - \int_{v_p}^{\bar{v}} [C(z,y_p) - w(z)- r_p]dG(z).$$

This is analogous to equation (38) in the body of the paper. This equation can be written as

(A-26) $$\Delta_p = \frac{\int_{v_p}^{\bar{v}} [C(z,y^*) - C(z,y_p)]dG(z)}{\int_{v_p}^{\bar{v}} dG(z)} - r_p$$

The results are similar to those in the paper. As in the paper, the sign of Δ_p will depend on the change in the average product and the product of the marginal user which is reflected by the access fee, r_p.