OPTIMAL AUCTIONS

by

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In the two decades since the seminal paper by William Vickrey (1961) literature on the theory of auctions has developed at a rapid though uneven pace.[1] Much of this literature is fragmentary, varies widely in scope and is not easily accessible to economists. As a result, the implications of different auction rules in various settings remain relatively unknown. This paper provides a systematic examination of alternative forms of auctions. In so doing it presents a general characterization of the implications for resource allocation of different auction designs within the model originally proposed by Vickrey.[2]

The auction model is a useful description of "thin markets" characterized by a fundamental asymmetry of market position. While the standard model of perfect competition posits buyers and sellers sufficiently numerous that no economic agent has any degree of market power, the bare bones of the auction model involves competition on only one side of the market. In this setting a single seller of an indivisible good faces a number (n) of potential buyers. Competition among the (possibly small number of) buyers takes place according to a well-defined set of auction rules calling for the submission of price offers from the buyers. Most commonly the choice of auction method employed rests with the monopolistic seller.

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[1] A current bibliography by Robert Stark and Michael Rothkopf (1979) lists nearly five hundred papers written over this period. For a recent survey of this literature see Richard Engelbrecht-Wiggans (1978).

[2] A number of unpublished dissertations have discussed auctions. Versions of Propositions 1 and 2 are contained in Armando Ortega-Reichert (1968), Gerard R. Butters (1975) and William F. Samuelson (1978). Butters also provides examples illustrating Propositions 3 and 7, both of which are derived by Samuelson.

These brief observations suggest two natural questions for analysis: First, what form does the competition among the few buyers take under the most common auction procedures? In turn, how is a sale price determined? Second, by what means can the seller best exploit his monopoly position? For example, would it be more profitable for the seller to assign payment not just to the high bidder but also to those with lower ranked bids?

As one might expect, any change in the rules of the auction results in different bidding strategies on the part of the buyers. In particular, if the auction rules posit a minimum payment for one or more of the bidders (determined by rank), those with sufficiently low valuations will be discouraged from entering a bid. Our analysis will demonstrate that in a risk neutral setting it is the minimum entry value (below which a buyer opts to remain out of the auction) which is crucial. To be precise all auctions which have the same entry value yield the same expected profit to the seller. Moreover, the seller maximizes expected profits by setting the entry value strictly above his own reservation value. A further rather surprising result emerges: The optimum entry value is independent of the number of competing buyers.

Throughout the paper we shall retain the following basic assumption.

a) A single seller with reservation prive v_0 faces n potential buyers, where buyer i holds reservation price v_i, $i = 1, \ldots n$.

b) The reservation prices of the parties are independent and identically distributed, drawn from the common distribution $F(v)$ with $F(\underline{v}) = 0$, $F(\bar{v}) = 1$ and $F(v)$ strictly increasing and differentiable over the interval $[\underline{v}, \bar{v}]$. We will refer to this as the IID assumption.

The IID assumption was first presented by William Vickrey (1961) and has been frequently employed in the bidding literature. In practical terms, each party is uncertain about the others' reservation prices believing that each individual formulates his price (or measure of value) independently of the others. In addition, the parties share common priors with respect to the possible reservation prices of each individual.[3] With the IID assumption, the bidding procedures we outline below belong to the class of games of incomplete information first formulated by John Harsanyi (1968).

Given the practical importance of the "English" or "ascending bid" auction and the sealed "high bid" auction in which the highest bid is accepted by the seller we consider these separately in Section 1. It is shown that in each case expected seller profit is maximized by the introduction of a reserve price. Then in Section 2 we present our central result on the partitioning of auction designs into seller equivalence classes. It is shown that

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[3] In recent years Robert Wilson (1975) and Matthew Oren and Albert Williams (1975) have studied a different model of competitive bidding -- one relevant to the auctioning of off-shore oil field leases. In this model buyers begin with common prior beliefs about the value of a resource but have different posterior beliefs as a result of independent sampling.

For discussions of auctions in which buyers have different prior beliefs see Wilson (1967).

the "English" and sealed "high bid" auctions, cum reserve price, are members of the equivalence class optimal for the seller. It is also shown that the seller cannot profit by concealing his reserve price. In Section 3 several alternative designs are examined in detail and their implications for the seller are compared. Finally in Section 4 the two commonly used auctions are once again compared under the assumption that the buyers are risk averse rather than risk neutral. It is shown in this setting that the English auction is dominated by the sealed "high bid" auction and that the optimal reserve price is a declining function of the degree of buyer risk aversion.

1. "HIGH BID" AND "SECOND BID" AUCTIONS

Because of their overwhelming practical importance, we begin by contrasting the sealed "high bid" auction with the "English" or "ascending bid" auction. The rules of the latter bear some explanation. Commonly, when antiques, estate objects and works of art are auctioned, the good is awarded to the buyer who makes the final and highest bid. The buyer placing the highest valuation on the good therefore pays approximately the maximum of the reservation prices of the other n-1 buyers. As Vickrey noted, this is equivalent to a sealed bid auction in which each buyer submits a bid and the high bidder pays the <u>second</u> rather than the high bid.[4] To see this, suppose the ith buyer considers shading his bid b_i below his reservation value v_i. If $b_* = \max_{j \neq i} b_j$ exceeds v_i another buyer is

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[4]This type of auction is sometimes referred to as a Vickrey auction.

the high bidder so such shading has no effect on buyer i's profit. If $b_* < b_i$, buyer i remains the high bidder and continues to gain a profit of $v_i - b_*$. However, if $b_i < b_* < v_i$, the shading yields a zero profit whereas without shading the profit is $v_i - b_*$. The optimal strategy of each buyer is therefore to submit his reservation value. It follows that just as in the English auction the high bidder ends up paying the second highest reservation value.

This equivalence greatly simplifies the comparison between the English and sealed high bid auctions[5] since it implies that we need only compare the two sealed bid auctions. For each auction we allow the seller to announce a reserve price b_0. Unless there is a bid higher than b_0 the good is withdrawn by the seller. There are two reasons for introducing such a reserve price. First, the results are presented in anticipation of the general theorem in Section 2. Second, at the practical level, the establishment of a seller reserve price is a frequent occurence in auction sales involving art as well as basic commodities. Announcing the reserve price is the more common procedure. In an English auction, however, the seller can establish a silent reserve price by instructing an ally in the audience to buy back the good if it would otherwise sell for too low a price.

For the high bid auction buyers have an incentive to behave strategically shading their bids below their reservation values in order to make an expected profit. Because of the symmetry of the problem we seek a strategy with the property that, when adopted by

[5]The sealed high bid auction also has its open auction equivalent. In this "Dutch" auction the sale price is initially set at a high level and is then lowered until a bid is made.

n-1 buyers it is optimal for the nth buyer to adopt it also. We begin by characterizing this strategy.

Proposition 1: Suppose the IID assumption holds, all buyers are risk neutral and the seller announces a reserve price b_0. Under the high bid auction the equilibrium bidding strategy of a typical buyer is

$$b = O(v) = v - \int_{b_0}^{v} \frac{F(x)^{n-1}dx}{F(v)^{n-1}} = \int_{0}^{v} \frac{\max\{b_0, x\}d(F(x)^{n-1})}{F(v)^{n-1}}$$

While the literature contains several proofs of this proposition the following derivation is especially direct. It also provides an introduction to the more general analysis of Section 2.

Suppose all but buyer i adopt the strategy of bidding according to the increasing function $b = O(v)$. Since buyer i can win with probability 1 by bidding $b_i = O(\bar{v})$ he has no incentive to bid outside the range of possible bids by the other buyers. Then there is some $v \varepsilon [\underline{v}, \bar{v}]$ such that the profit maximizing bid by buyer i, b_i, satisfies

(1) $$b_i = O(v)$$

By assumption $O(v)$ is increasing in v. Therefore buyer i wins if and only if v exceeds the reservation values of all the other buyers, that is, with probability $F(v)^{n-1}$. His expected profit can therefore be written as

$$(2) \qquad \Pi^i(v, v_i) = (v_i - O(v))F(v)^{n-1}$$

Differentiating with respect to v we have

$$(3) \qquad \Pi_1^i(v, v_i) = v_i \frac{d}{dv}(F(v)^{n-1}) - \frac{d}{dv}(O(v)F(v)^{n-1})$$

For $b = O(v)$ to be the equilibrium strategy the profit maximizing choice of buyer i must be to adopt it also. That is, $\Pi^i(v, v_i)$ must take on its maximum at $v = v_i$. Then from (3), $O(v)$ is the solution to the first order ordinary differential equation

$$(4) \qquad v \frac{d}{dv}(F(v)^{n-1}) - \frac{d}{dv}(O(v)F(v)^{n-1}) = 0 \quad .$$

Also combining (3) and (4) yields

$$\Pi_1^i(v, v_i) = (v_i - v)\frac{d}{dv}(F(v)^{n-1})$$

$$\gtrless 0 \qquad \text{as} \qquad v \lessgtr v_i \quad .$$

Thus the first order condition (3) indeed defines the global maximum for buyer i.

Integrating (4) by parts yields

$$(5) \qquad O(v)F(v)^{n-1} - O(b_0)F(b_0)^{n-1} = vF(v)^{n-1} - b_0F(b_0)^{n-1} - \int_{b_0}^{v} F(x)^{n-1}dx$$

But all bids $O(v)$ yielding a positive expected profit must satisfy the inequalities

$$b_0 < O(v) < v$$

Therefore as $v \to b_0$, $O(v) \to b_0$ and the second terms on each side of the equality in (5) are equal. Dividing through by $F(v)^{n-1}$ then yields the first equality of Proposition 1. Integrating by parts yields the second equality.

The first equality tells us directly by how much a buyer should shade his bid.[6] The second equality also has a simple interpretation. $F(x)^{n-1}$ is the probability distribution function of the (n-1)th order statistic of n-1 independent drawings from F. Dividing by $F(v)^{n-1}$ the domain of positive density is transformed from $[\underline{v}, \overline{v}]$ to $[\underline{v}, v]$. Thus

$$\int_0^v \frac{\max\{b_0, x\} d(F(x)^{n-1})}{F(v)^{n-1}}$$

represents the expected value of the maximum of the seller's reserve price and the highest reservation value of the n-1 buyers given that the buyers values are below v. A risk neutral buyer therefore follows the following bidding rule:

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[6]Also by differentiating it is readily confirmed that the assumption $O'(v) > 0$ is satisfied everywhere on the interval $(b_0, \overline{v})$.

At reservation value v_i place a bid b_i equal to the expected value of the maximum of the seller's reserve price and the other buyers' reservation values on the assumption that each of the latter are less than v_i.

The last stipulation makes intuitive sense. If there exists a $v_j > v_i$, then, because the common bidding strategy, $O(v)$, is increasing buyer i will be outbid. The only relevant event then is $v_i > v_j$ for all $j \neq i$ and in this instance the rule of thumb above maximizes expected profit.[7]

This rule is also helpful in understanding our first equivalence result.

Proposition 2: Suppose assumption IID holds, all buyers are risk neutral and the seller announces a reserve price b_0. Then the expected profit to the seller is the same under the high bid and second bid auctions for arbitrary distributions $F(v)$.

This is an immediate implication of the general proposition in Section 2. However, the following informal derivation is revealing. In the second bid auction the winner pays the higher of the second bid and the reserve price b_0. Therefore, if buyer i is the winner his expected profit is

$$E(\Pi^i \mid i \text{ submits winning bid}) = v_i - E(\max_{j \neq i}\{b_0, v_j\}) \tag{6}$$

[7] Robert Wilson (1977) analysing a slightly different auction model notes a similar interpretation of the optimal buyer strategy.

Moreover, from the optimal bidding rule described above it follows immediately that (6) also describes the expected profit to buyer i in the high-bid auction. That is, the expected profit conditional upon b_i being the highest bid is the same in the two auctions. Since both auctions are efficient in the sense that the successful bidder is always the buyer with the highest reservation value, the unconditional expected profit to the buyers taken as a group is the same in the two auctions. Also the expected reservation value of the successful bidder in each case is

$$E(\max_j\{v_j\})$$

Since the expected payment to the seller is just the difference between the expected reservation value of the successful buyer and the buyers' expected profits, it follows immediately that the seller is indifferent between the two auctions.

The natural next question is what reserve price maximizes seller expected profit.

Proposition 3: If assumption IID holds and all buyers are risk neutral, then under either auction rule the seller maximizes expected profit by announcing a reserve price b_0 satisfying

$$b_0 = v_0 + \frac{1 - F(b_0)}{F'(b_0)}$$

independent of the number of buyers.

Again, formal derivation is left until Section 2. However the extreme case of only one potential buyer is easily analyzed. Far from being an example of "competitive" bidding, this case belongs to the category of bilateral monopoly -- for which a bargaining solution is customarily sought. The bargaining procedure (if one wishes to think of it as such) is particularly simple. By announcing a reserve price the seller is, in effect, making a first and final offer. The buyer accepts this offer by making a matching bid and rejects it by making any smaller bid. The expected profit to the seller is just the difference between the announced reserve price b_0 and his own reservation value v_0 multiplied by the probability of a sale $(1 - F(b_0))$. Therefore the seller chooses b_0 to maximize

$$(1 - F(b_0))(b_0 - v_0)$$

Differentiating this expression with respect to the reserve price, b_0, yields the condition in Proposition 3.

Note that it is always optimal for the seller to set a reserve price, b_0, in excess of his own reservation value, v_0. More important, this optimal markup is independent of the number of potential buyers.

2. GENERAL BIDDING RULES

In the previous section we compared the two most common types of auctions. Here we shall consider a broad class of auction rules in which payments are made by both the successful and unsuccessful bidders. Perhaps the simplest example of such a rule, and one quite commonly used, is the English auction plus the additional feature that all potential buyers wishing to bid must first pay an entry fee.[8] As this example makes clear, one decision buyers must make under general auction rules is whether or not to enter a bid. We therefore allow the payment by each buyer to depend not only upon the rank, r, of his bid and the vector of bids entered but also the number of such bids. Expressing this formally, if $b_{(r)}$ is the rth ranked bid, the buyer bidding $b_{(r)}$ pays an amount

$$(7) \qquad p^{mr} = p^{mr}(b_{(1)}, b_{(2)}, \ldots b_{(m)}) \qquad r = 1, 2, \ldots m. \quad m = 1, \ldots n.$$

To illustrate, in the high-bid auction with a fixed entry fee, c, we have:

$$(8) \qquad p^{mr} = \begin{cases} c + b_{(1)}, & r = 1 \\ c & , r > 1 \end{cases}$$

[8] The interpretation here is that all such fees accrue to the seller. A second possible interpretation is that each buyer irrevocably commits resources in an attempt to win a good or (more likely) a contract. However, the latter situation is more aptly modelled by making such precommitments endogenous. For an introductory discussion of these issues see Hirshleifer and Riley (1978).

Similarly, in the second-bid auction with fixed entry fee we have:

$$(9) \qquad P^{mr} = \begin{cases} c + b_{(2)}, & r = 1 \\ c & , r > 1 \end{cases}$$

In what follows we shall refer to these rules as the family of symmetric auction rules.

We now seek to characterize the equilibrium bidding strategy for a general member of this family. Suppose all but buyer i enter the auction for those reservation values v in some interval $V \subset [\underline{v}, \bar{v}]$, and make bids $b = O(v)$ where $O(\cdot)$ is strictly increasing over V. The profit maximizing response of buyer i will be either to remain out of the auction or to make a bid b_i in the range of the function $O(v)$. Thus if buyer i enters the auction there is some $v \in V$ such that

$$b_i = O(v) \quad .$$

In seeking the optimal bid b_i we may therefore express the expected payment P_i as a function of v. Buyer i is the successful bidder if and only if $v_j < v$ for all $j \neq i$, that is, with probability $F(v)^{n-1}$. His expected profit is therefore

$$(10) \qquad \Pi^i(v, v_i) = v_i F(v)^{n-1} - P_i(v) \quad .$$

Since expected gross benefit is increasing in v_i and expected payment is independent of v_i, a simple revealed preference argument establishes that the higher the reservation value v_i, the higher the profit maximizing level of v (and hence the higher the optimal bid b_i). Therefore the equilibrium bid function must have the simple form

$$b_i \equiv O(v_i), \quad v_i \geq v_* \quad .$$

We shall refer to v_*, the lowest reservation value for which buyers will enter the auction, as the "entry value".

For a symmetric equilibrium the optimal bid by buyer i $b_i \equiv O(v)$ must equal $O(v_i)$. Thus $\Pi^i(v, v_i)$ must take on its maximum at $v = v_i$. Assuming that the payment functions $P^{mr}(b_{(1)}, \ldots b_{(m)})$ are continuous, $P'(v)$ is differentiable and we may differentiate (10) to obtain

$$\text{(11)} \qquad \Pi_1^i(v, v_i) = v_i \frac{d}{dv} (F(v))^{n-1} - P_i'(v) \quad .$$

Then the equilibrium bid function $O(v)$ must yield a payment function $P(v)$ satisfying the first order ordinary differential equation.

$$\text{(12)} \qquad \Pi_1^i(v, v) = v \frac{d}{dv} (F(v))^{n-1} - P'(v) = 0 \qquad v \geq v_*$$

Finally we note that, since expected profit is zero for any buyer remaining out of the auction, we also have the boundary condition:[9]

(13) $\Pi^i(v_*, v_*) = 0$

For any set of apyment functions it is, in principle, possible to solve for the equilibrium bid function $O(v)$ by integrating (12) and using condition (13) to determine the constant of integration. We shall consider some examples below. However first we demonstrate the following general equivalence result.

Proposition 4: Suppose the IID assumption holds and all buyers are risk neutral. The equilibrium bid function for the family of symmetric auction rules yields an expected profit to the seller, $E(\Pi_0)$, a function only of the entry value v_*, satisfying

$$E(\Pi_0) = n \int_{v_*}^{\bar{v}} [(v - v_0)F'(v) + F(v) - 1]F(v)^{n-1}dv.$$

The proof follows rather easily from the characterization of the equilibrium bidding rule. First we note that the rate of change of expected profit of the ith buyer with respect to the reservation value v can be written as

$$\frac{d}{dv} \Pi^i(v, v) = \Pi_1^i(v, v) + \Pi_2^i(v, v)$$

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[9] In appealing to this boundary condition we are ruling out auctions in which all participants earn a positive profit. As we shall later see, such auctions are never optimal for the seller.

From (12) $\Pi_1^i(v, v) = 0$ and from (10) $\Pi_2^i(v, v) = F(v)^{n-1}$. We therefore have:

(14) $$\frac{d}{dv}\Pi^i(v, v) = F(v)^{n-1}$$

Next taking the expectation over the prior distribution $F(v)$, the unconditional expected profit for buyer i is given by

$$E(\Pi^i) = \int_{v_*}^{\bar{v}} \Pi^i(v, v)\,dF(v)$$

Integrating by parts we have

$$E(\Pi^i) = -[(1-F(v))\Pi^i(v,v)]_{v_*}^{\bar{v}} + \int_{v_*}^{\bar{v}} (1-F(v))\frac{d}{dv}\Pi^i(v,v)\,dv$$

Substituting from the boundary condition (13) and utilizing condition (14) this reduces to

(15) $$E(\Pi^i) = \int_{v_*}^{\bar{v}} (1-F(v))F(v)^{n-1}dv$$

Furthermore from the definition of expected profit, (10), the unconditional expected payment by buyer i satisfies

$$E(P_i) = \int_{v_*}^{\bar{v}} P_i(v)\,dF(v) = \int_{v_*}^{\bar{v}} vF'(v)F(v)^{n-1}dv - E(\Pi^i)$$

Then substituting from (15)

(16) $$E(P_i) = \int_{v_*}^{\bar{v}} (vF'(v) + F(v) - 1)F(v)^{n-1}dv.$$

In turn the expected payment to the seller is just n times the expected payment by a typical buyer -- that is,

$$E(P_0) = nE(P_i).$$

The probability that the object will remain unsold is $F(v_*)^n$. Therefore with probability $1 - F(v_*)^n$ the seller relinquishes an object which he values at v_0. Expected profit to the seller is therefore the sum of the expected payments, less the reservation value multiplied by the probability of a sale, that is,

$$\begin{aligned} E(\Pi_0) &= nE(P_i) - v_0(1 - F(v_*)^n) \\ &= n \int_{v_*}^{\bar{v}} [(v-v_0)F'(v) - F(v) - 1]F(v)^{n-1}dv, \end{aligned}$$

which completes the proof. Differentiating to solve for the profit maximizing entry value we then have the following further result.

Proposition 5: If assumption IID holds and buyers are risk neutral, the members of the family of symmetric auction rules which maximize expected profit are those for which the entry value, v_*, satisfies

$$v_* = v_0 + \frac{1 - F(v_*)}{F'(v_*)}.$$

The optimal entry value may therefore be written as $v_* = v_*(v_0)$, a function only of the sellers reservation value.

An immediate implication of Proposition 5 is that the sealed high bid and second bid auctions, cum reserve price, are both optimal. By announcing a reserve price $b_0 = v_*$ the seller attracts all those buyers with a reservation value v_i in excess of v_* --- hence Proposition 3.

However, the reserve price is only one of many ways in which an optimal entry value can be generated. Suppose in the absence of a reserve price, the seller announces a fixed entry fee c. For all buyers with valuations less than some number v_c it will be optimal to remain out of the auction. Consider a buyer with the borderline reservation value v_c. In the second bid auction he enters and, since the entry fee is now sunk, bids his true value v_c. He wins if and only if there are no other bidders, in which case there is no additional payment. Since this occurs with probability $F(v_c)^{n-1}$ his expected profit is

$$(17) \qquad v_c F(v_c)^{n-1} - c$$

But for v_c to be the borderline reservation value, the expected profit must be zero. The seller then chooses an entry fee c_* satisfying,

$$(18) \qquad c_* = v_* F(v_*)^{n-1}.$$

A similar argument holds for the high bid auction. If a buyer has the borderline reservation value v_c he wins if and only if there are no bidders. The optimal bid in such circumstances is zero, therefore, the expected profit is again given by (17) and the optimal entry fee by (18).

Our general results are also helpful in analyzing the expected payoff to multiple rounds of bidding. Suppose, for example, that a seller with a minimum reservation value ($v_0 = \underline{v}$) adopts the second bid auction rules and charges the profit maximizing entry fee c_*. Since for this fee there is a corresponding entry value v_*, it is possible that all reservation values, v_i, are so low that no one submits a bid. But for all i

$$v_0 = \underline{v} \leq v_i < v_c.$$

Thus there remain potential gains from trade, that is, the auction is inefficient <u>ex post</u>. Thus one might argue that the seller could increase expected profit by announcing a second round of bidding with no entry fee.

However, suppose buyers are sophisticated enough to anticipate the second round. In this case only those buyers with reservation values in excess of some $v_{**} > v_c$ will enter the first round. Each of the other buyers can be thought of as bidding his true value and marking his bid "second round only". But this, in effect, is a single round auction with the following payment scheme.

$$p^{m1}(b_{(1)},\ldots,b_{(m)}) = \begin{cases} b_{(2)} & b_{(1)} < v_{**} \\ c_* & b_{(2)} < v_{**} < b_{(1)} \\ c_* + b_{(2)} & v_{**} < b_{(2)} \end{cases}$$

$$p^{mr}(b_{(1)},\ldots,b_{(m)}) = \begin{cases} 0 & b_{(r)} < v_{**} \\ c_* & b_{(r)} > v_{**} \end{cases}$$

Since all buyers will enter such an auction it follows from Proposition 4 that the expected seller profit is exactly the same as in the ordinary second bid auction with no entry fee or reserve price. Then, from Proposition 5 the second round, if anticipated, lowers expected seller profit.

A final point concerns the decision of the seller whether or not to announce a reserve price. In the second bid auction the strategy of bidding one's reservation value is a dominant strategy. Therefore the seller cannot influence bids by concealing his reserve price. It follows that the optimal silent reserve price is the same as the optimal announced reserve price and that expected seller revenue is identical.

Comparison of the high bid auction with and without an announced reserve price appears in the appendix, where the following result is established.

Proposition 6: If assumption IID holds and all buyers are risk neutral, then for either the high bid or second bid auction rule an announced reserve price is at least as good in terms of expected seller profit as a silent price.

3. ALTERNATIVE AUCTIONS

To illustrate the general equivalence result of the previous section we now present an unusual pair of auction designs which happen to belong to the class of optimal auctions. In contrast, a seemingly natural (and commonly employed) auction procedure is shown to be suboptimal.

Example 1:

Suppose there are just two buyers and the seller announces the following auction rules.

i) Each buyer deciding to submit a sealed bid must pay an entry fee c.

ii) The high bidder receives the good but retains his bid.

iii) The low bidder (if there is one) loses his bid.

It is tempting to conjecture that there is no equilibrium bidding strategy for this set of rules. However not only is such a conjecture false, but the derivation of the equilibrium bid function is relatively straightforward. Suppose buyer 2 adopts the strategy of bidding according to $b_2 = O(v_2)$, for all $v_2 \geq v_*$, where $O(\cdot)$ is a strictly increasing function. Then if buyer 1 bids $b_1 = O(v)$ he wins if and only if $v_2 < v$, that is, with probability $F(v)$. His expected profit is therefore

$$\Pi^1(v, v_1) = v_1 F(v) - O(v)(1 - F(v)) - c .$$

Differentiating with respect to v, expected profit is maximized by choosing v so that

$$\Pi^1_1(v, v_1) = v_1 F'(v) - \frac{d}{dv}\,(O(v)(1 - F(v))) = 0 \tag{19}$$

But for $O(v)$ to be the equilibrium strategy we require $b_1 = O(v_1)$, that is, $v = v_1$. Substituting for v_1 in (19) we therefore have the first order ordinary differential equation

$$vF'(v) = \frac{d}{dv}\,(O(v)(1 - F(v)))$$

Integrating then yields

$$(20) \qquad O(v) = \frac{\int_{v_*}^{v} xF'(x) + A}{1 - F(v)}$$

But if $v_1 = v_*$ buyer 1 is successful if and only if he is the only bidder. In this case he has no incentive to submit a positive bid. Then $O(v_*) = 0$ implying that the constant of integration is zero. Finally, c is determined by the requirement that the marginal buyer should make zero profit, that is,

$$\Pi^1(v_*, v_*) = v_*F(v_*) - c = 0$$

Note that the numerator of expression (20) is positive and increasing in v. Moreover, as v approaches its upper bound, $\bar{v}$, the denominator approaches zero. Therefore the equilibrium bid $b = O(v)$ increases without bound as v approaches $\bar{v}$! Nevertheless, it is easy to confirm that expected seller profit under this scheme matches that of the familiar high bid and second bid auctions, cum optimal reserve price.

Example 2:

Under the high bid and second bid auctions only the recipient of the good profits. In contrast, the following auction distributes positive profits to all participants and is equivalent in terms of expected seller price to the high bid and second bid auctions.

i) Any buyer who submits a bid, $b \geq v_*$ receives from the seller an amount

$$R(b) = \int_{v_*}^{b} F(v)^{n-1} dv$$

ii) The high bidder obtains the good for his bid price so that his net payment is $b_{(1)} - R(b_{(1)})$.

To confirm that the equilibrium strategy of each buyer is to bid truthfully, suppose that all buyers but the first are so bidding, that is, $b_i = v_i$, $i=2 \ldots n$. Then if buyer 1 bids b_1 his expected profit is given by

$$\Pr\{b_1 \text{ is high bid}\}\,(v_1-b_1) + R(b_1) = F(b_1)^{n-1}(v_1-b_1) + R(b_1).$$

It is straightforward to check that this expression is maximized at $b_1 = v_1$.

The expected payment to the seller (net of his payments made to losing buyers) is

$$E(P_0) = E(b_{(1)}) - nE(R(v))$$

$$= n\int_{v_*}^{\bar{v}} vF(v)^{n-1}F'(v)dv - n\int_{v_*}^{\bar{v}}\int_{v_*}^{v} F(x)^{n-1}dxF'(v)dv$$

Noting that $F'(v) = \frac{d}{dv}(F(v) - 1)$, the second term can be integrated by parts allowing it to be rewritten as

$$n\int_{v_*}^{\bar{v}} (1 - F(v))F(v)^{n-1}dv$$

Thus

$$E(P_0) = n\int_{v_*}^{\bar{v}} [vF'(v) + F(v) - 1]F(v)^{n-1}dv$$

which is precisely equation (16) in Section 2. Choosing v_* to satisfy the condition in Proposition 5 this auction becomes a member of the class of optimal auctions.

It is interesting to note that this auction procedure can be generalized and applied in situations where buyer reservation prices are drawn from different probability distributions. By using n different payment schedules the seller can induce each potential buyer to bid truthfully and thus ensure an efficient allocation of the good <u>ex post</u>. Of course the formidable information requirements of such a procedure limits its practical usefulness.

Since the implication of Proposition 4 is that many seemingly different auction techniques lead to the same ultimate results, it is important to illustrate the range of exceptions.

Example 3:

Suppose there are just two buyers and the seller employs the following auction rules.

i) There will be m (a finite number) rounds of bidding, each round beginning with buyer 1 quoting a price.

ii) In each round buyer 2 can obtain the current right to the good simply by matching the first buyer's bid.

Though this auction procedure is quite common (e.g., in house sales, a renter occupant is frequently given the right to match the offer of any potential buyer), it is inefficient, not only from the point of view of the seller, but also <u>ex post</u>. To illustrate this point, suppose that the reservation prices of the individuals are drawn from a uniform distribution on [0, 1] -- that is, $F(v) = v$.

The strategy of buyer 2 is straightforward. With k rounds still to go he matches his opponents bid, b_k if and only if

$b_k \leq v_2$. Buyer 1, of course, anticipates this response and determines his optimal strategy by solving a standard dynamic programming problem over m stages. We let $E(b_k, v_1)$ denote the expected profit of buyer 1 holding reservation price v_1 when his current bid, b_k, has just been matched by his opponent, on the assumption that his remaining k-1 bids are optimal. By definition $E(b_k, v_1)$ satisfies the functional equation

$$E(b_k, v_1) = \max_{b_k \leq b_{k-1} \leq 1} \left[(v_1 - b_{k-1}) \frac{(b_{k-1} - b_k)}{1 - b_k} + \frac{(1 - b_{k-1})}{1 - b_k} E(b_{k-1}, v_1) \right]$$

and $E(b_0, v_1) = 0$. The optimal bidding strategy of buyer 1 can be determined from this functional equation starting from the last stage and applying the technique of backward induction. For the uniform case the optimal bidding strategy can be simply stated. If the buyer's last bid, b_k, has been matched, then he should place his k remaining bids to partition the interval $[b_k, v_1]$ into k+1 equal parts. In particular, when m rounds of bidding are allowed, the buyer's optimal bidding strategy is

$$b_{m-1} = \frac{1}{m+1} v_1, \quad b_{m-2} = \frac{2}{m+1} v_1 \ \dots \ b_0 = \frac{m}{m+1} v_1$$

For instance, if m = 1 then the expected profit of buyer one is $\int_0^{b_0} (v_1 - b_0) dv$ which achieves a maximum at $b_0 = 1/2\, v_1$.

Similarly, it is straightforward to check the solution in the case of m rounds.

It is evident that this auction procedure is inefficient <u>ex post</u>. When $\frac{m}{m+1} v_1 \leq v_2 \leq v_1$, the good will be awarded to the second buyer who values it less highly than the first. The expected group profit that is lost amounts to

$$L = \int_0^1 \int_{\frac{m}{m+1} v_1}^{v_1} (v_1 - v_2) dv_2 \, dv_1 = \frac{1}{6} \left(\frac{1}{m+1}\right)^2.$$

A second question concerns the distributional impact of the matching bid auction. It is easy to confirm that the first and second bid auctions imply the following expected price and profits to the parties.

$$E(P_0) = \frac{1}{3} \qquad E(\Pi_1) = E(\Pi_2) = \frac{1}{6}$$

For simplicity suppose the seller has a zero reservation value so that $E(P_0) = E(\Pi_0)$. Then

$$E(S) \equiv E(\Pi_0 + \Pi_1 + \Pi_2) = \frac{2}{3}$$

The matching bid auction on the other hand implies

$$E(P_0) = \frac{1}{3} \frac{m(m+2)}{(m+1)^2}, \quad E(\Pi_1) = \frac{1}{6} \left(1 - \frac{1}{m+1} + \frac{1}{(m+1)^2}\right)$$

$$E(\Pi_2) = \frac{1}{6} \left(\frac{m+2}{m+1}\right), \quad E(S) = \frac{2}{3} - \frac{1}{6} \left(\frac{1}{m+1}\right)^2$$

The obvious advantage that goes with the opportunity to match an opponent's bid is reflected in these profit expressions. Not only does the profit of the second buyer exceed that of the first, it also surpasses what he could expect under a symmetric auction. The first buyer, on the other hand, is at a disadvantage in the matching auction relative to a symmetric auction. More important, the seller also sacrifices profit by extending the matching privilege to a buyer. For instance, employing a one bid matching auction causes a 25% loss in profit to the seller on average (relative to a symmetric auction). Buyer one also suffers a 25% average loss, while buyer two enjoys a 50% gain in profit. As common sense suggests, the outcome of the matching auction approaches that of the sealed bid auctions as the number of rounds increases. The profit expressions above confirm this convergence and also indicate that, as in the sealed bid auctions, the matching auction attains *ex post* efficiency.

4. BUYER RISK AVERSION

When potential buyers are risk averse, the fundamental equivalence result outlined in Section 2 is no longer valid. Retaining the assumption of buyer symmetry we now show that the high bid auction dominates the second bid auction under buyer risk aversion.

Proposition 7: Suppose assumption IID holds and all buyers share a common utility function displaying risk aversion. Then

i) Under the second bid auction, buyers continue to bid truthfully, that is $b_i = v_i$.

ii) Under the high bid auction, the risk averse

ii) ... buyer makes uniformly higher offers than his risk neutral counterpart.

iii) Consequently, the seller enjoys a greater expected profit under the high bid auction than under the second bid auction.

It is evident that the introduction of risk aversion does not affect the strategy dominance of truthful bidding under the second bid auction.

Let $b(v)$ be the common equilibrium strategy of n risk averse buyers, each of whom has the same von Neumann-Morgenstern utility function $u(x)$. We assume that $u(x)$ is a strictly increasing, concave function of x and normalize so that $u(0)=0$. With all other buyers using the equilibrium bidding strategy and buyer j bidding $b(x)$, j's expected utility is

(A1) $F^{n-1}(x)u(v_j-b(x))$

For $b(x)$ to be the equilibrium strategy (A1) must have its maximum at $x=v_j$. Differentiating with respect to x and setting the derivative equal to zero at $x=v_j$, we have the necessary condition

$$(n-1)F^{n-2}(v_j)F'(v_j)u(v_j-b(v_j))-F^{n-1}(v_j)u'(v_j-b(v_j)\ \frac{db}{dv_j}=0$$

Rearranging yields the following differential equation for $b(v)$

(A2) $b'(v) = (n-1)\dfrac{F'(v)}{F(v)}\ \dfrac{u(v-b)}{u'(v-b)}$

With reserve price $b_0=v_*$ we also have the boundary condition

(A3) $b(v_*)=v_*$

We wish to compare the solution for two different utility functions, $u_1(\cdot)$ and $u_2(\cdot)$ where the latter exhibits a higher degree of risk aversion, that is,

(A4) $-u_2''(x)/u_2'(x) > -u_1''(x)/u_1'(x) \geq 0$

By inspection of (A2), if we can establish that

(A5) $\phi(x) = u_2(x)/u_2'(x) - u_1(x)/u(x) > 0$, for $x>0$,

then $b_2'(v) > b_1'(v)$ and hence $b_2(v) > b_1(v)$ for all $v > v_*$. To demonstrate (A5) we note first that, since $u(0)=0$ and $u(x)$ is strictly increasing,

(A6) $\frac{u(x)}{u'(x)} > \frac{u(0)}{u'(0)} = 0$ for all $x>0$

Inequality (A5) holds if we can establish that for all x such that $\phi(x)=0$, $\phi(x)$ is strictly increasing. Differentiating (A5) we have

(A7) $\phi'(x) = \left(\frac{-u_2''}{u_2'}\right) \left(\frac{u_2}{u_2'}\right) - \left(\frac{-u_1''}{u_1'}\right) \left(\frac{u_1}{u_1'}\right)$

From (A4) - (A6), $x>0$ and $\phi(x)=0$ implies that $\phi'(x)>0$. Moreover, differentiating (A7) and setting $x=0$ we also have

$\phi''(0) > \phi'(0) = 0$

Thus $\phi(x)$ is strictly increasing at $x=0$.

Q.E.D.

The intuition behind these results is that with risk aversion the marginal increment in wealth associated with a successful slightly lower bid is weighted less heavily than the possible loss of profit $(v_i - b_i)$ if, as a result of lowering the bid, the buyer is no longer the high bidder. This leads risk averse bidders always to shade their bids less than risk neutral bidders.

Under risk aversion, the general equivalence result obtained in Proposition 5 no longer holds. For instance, an auction employing a seller reserve price will not, in general, be equivalent to one that specifies a buyer entry fee -- even when the same buyer minimum entry value is implied. Still it is natural to explore the effect that buyer risk aversion has on the optimal seller reserve price.

Proposition 8: Suppose assumption IID holds and all buyers share a common cardinal utility function. Then the optimal seller reserve price is a declining function of the degree of risk aversion.

The proposition is intuitively plausible in view of the fact that as buyers become risk averse in the extreme, their bids approach the truth, $b_i = v_i$. Naturally, the seller can do no better than to announce his true reservation value as his reserve price, $b_0 = v_0$. To quote a higher price cannot "push up" buyer offers and risks the loss of beneficial sales. Of course when $b_0 = v_0$ and $b_i = v_i$, the high bid auction is also efficient ex post.

The method of proof is to compare the effect of a change in the reserve price v_* on the equilibrium bid function $b=b(v,v_*)$ for different degrees of risk aversion. Expected seller revenue, $R(v_*)$, is the expected value of the highest ranked bid, that is,

$$R(v_*) = \int_{v_*}^{\bar{v}} b(v, v_*)\, dF^{n-1}(v)$$

Then the net advantage to the seller if utility is $u_2(\cdot)$ rather than $u_1(\cdot)$ can be expressed as

$$R_2(v_*) - R_1(v_*) = \int_{v_*}^{\bar{v}} [b_2(v, v_*) - b_1(v, v_*)]\, dF^{n-1}(v)$$

Differentiating with respect to v_* we have

$$\text{(A7)} \qquad R_2'(v_*) - R_1'(v_*) = \int_{v_*}^{\bar{v}} [\frac{\partial b_2}{\partial v_*} - \frac{\partial b_1}{\partial v_*}]\, dF^{n-1}(v)$$

It suffices to show that the bracketed expression in (A7) is negative, for then $R_2'(v_*)$ is negative when $R_1'(v_*)$ is zero.

The equilibrium bid function $b(v, v_*)$ is the solution to,

$$\text{(A8)} \qquad \frac{\partial}{\partial v} b(v, v_*) = (n-1) \frac{F'(v)}{F(v)} \frac{u(v-b)}{u'(v-b)},$$

with the boundary condition,

$$\text{(A9)} \quad b(v_*, v_*) = v_*.$$

Assuming $u(\cdot)$ is twice differentiable we can differentiate (A8) with respect to the reserve price v_* and so obtain the following differential equation for $\partial b/\partial v_*$

$$\text{(A10)} \quad \frac{\partial}{\partial v}(\frac{\partial b}{\partial v_*}) = -(n-1) \frac{F'(v)}{F(v)} [1 + (\frac{-u''}{u'}) (\frac{u'}{u})] (\frac{\partial b}{\partial v_*})$$

From (A4) and (A5) the bracket in (A10) is larger for the utility function $u_2(x)$ exhibiting greater risk aversion. Then if we can establish that $\partial b_2/\partial v_* = \partial b_1/\partial v_* > 0$ at $v=v_*$, it will follow from (A10) that

$$\frac{\partial}{\partial v}\left(\frac{\partial b_2}{\partial v_*}\right) > \frac{\partial}{\partial v}\left(\frac{\partial b_1}{\partial v_*}\right), \text{ for } v>v_* ,$$

and hence that $\partial b_2/\partial v_* > db_1/\partial v_*$ for $v > v_*$.

From (A9) we have,

$$\text{(A11)} \qquad \left.\frac{\partial b(v,v_*)}{\partial v}\right|_{v=v_*} + \left.\frac{\partial b(v,v_*)}{\partial v_*}\right|_{v=v_*} = 1.$$

Since $b(v_*,v_*)=v_*$ and $u(0)=0$, it follows from (A2) that for any concave utility function and any $v_*>0$, the first term in (A11) is zero. Then the second term in (A11) is equal to unity for both $u_1(x)$ and $u_2(x)$.

5. CONCLUDING REMARKS

While a general result concerning the design of optimal auctions under uncertainty has been presented, it is important to point out the limitations and special assumptions of the present model. We have assumed that:

a) A single indivisible good is the object of sale.

b) Buyers are risk neutral.

c) Buyer roles are symmetrical (i.e., buyer values are drawn from a common distribution).

d) These values are independent.

Additional difficulties are raised when multiple goods are auctioned or when a divisible good must be allocated. Unless buyer valuations are additive and income independent, auctioning the goods in sequence will be inefficient (*ex post* and *ex ante*). When multiple goods are auctioned, each buyer should logically submit a bid for each subset of goods. Roughly speaking, the seller will allocate goods to maximize revenue under one of a number of auction schemes. In the case of a divisible good, each buyer will submit a "demand schedule" indicating the price he is willing to pay for any given quantity of the good. The seller must formulate an auction rule which specifies the allocation of the good and appropriate payments of buyers. In either instance the determination of optimal auctions for these more general environments lies beyond the bounds of the present analysis.

Reviewing the proofs of Section 2, it is evident that the central result of this paper -- the equivalence in terms of seller profit of a number of seemingly different auctions -- depends critically on the assumption of buyer risk neutrality. Given the latter, the only possible effect of a change in the distribution of payments is a change in the valuation below which entry into the auction is unprofitable. As a result the seller can do no better than adopt the second bid auction in which bidding truthfully is a dominant strategy and hence buyers need not know the form of the distribution function $F(v)$. As pointed out in Section 4, the situation changes drastically when this assumption is dropped. Against risk averse buyers, the high bid auction dominates the second bid auction. Under the high bid auction, the seller should lower his announced reserve price as buyers become more risk averse. These propositions are indicative of the partial results available in this more general environment.

Dropping the assumption of buyer symmetry causes similar complications in the analysis. The derivation of the class of optimal auctions relied explicitly on the existence of a <u>common</u> equilibrium bidding strategy. Without this, these propositions no longer hold. The asymmetric model, though far more complex, is, nevertheless, amenable to the basic approach developed herein. Suppose the reservation prices of the buyers are drawn from the independent distributions, F_1, F_2, ... F_n. Some partial results from this setting suggest a basic conclusion. An optimal auction extends the asymmetry of the buyer roles to the allocation rule itself. The assignment of the good and the appropriate buyer payment will depend not only on the list of offers, but also on the identities of the buyers who submit the bids. In short, an optimal auction under asymmetric conditions violates the principle of buyer anonymity.[13]

Finally, one must consider the appropriateness of the model's most basic assumption, value independence. The analysis has assumed that each buyer is informed of his own reservation price and, more important, that this price conveys no information about any other buyer's value. By way of contrast, consider the auction model usually applied to off-shore oil leases. Here, a tract being auctioned is assumed to have a common value for all parties. Moreover, the tract value is unknown, though buyers may possess (differing) sample information allowing inferences about this value. In this setting, each buyer must determine a strategy for acquiring information concerning the value of the tract and for submitting a bid based on a correct estimate of this value.[14] These features

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[13]As an extreme illustration of this proposition, consider a seller with $v_0 = 0$, and a buyer whose reservation price is distributed uniformly on the unit interval. The optimal seller reserve price is $b_0 = 1/2$ in this case. Now suppose a second buyer enters the competition with a reservation price distributed uniformly on $[0, 1/2]$.

Suppose that the seller adopts the following rule. He awards the good to the first buyer if and only if $b_1 \geq b_{01}$ and to the second buyer if and only if $b_1 < b_{01}$ and $b_2 \geq b_{02}$. It is easy to check that the profit maximizing choice of these reserve prices is $b_{01} = 9/16$ and $b_{02} = 1/4$, showing that symmetric treatment $(b_{01} = b_{02})$ is suboptimal.

[14]Concerning the issue of value estimation and bidding strategy, a number of authors have noted the phenomena of the "winner's curse." A buyer errs in using a naive estimate of the tract's value based on his sample information alone. A sophisticated assessment determines the expected value, conditional on winning the tract. Since the buyer wins only if his opponent's sample information is less favorable than his own, a sophisticated buyer discounts his own sample information in making a bid. For discussions of the informational issues in this model, see Mathews (1979) and Reece (1978).

have a direct influence on the determination of an optimal auction and raise additional policy issues. (Should the seller maintain a stake in an awarded tract for the purpose of risk sharing? Should the seller undertake measures to facilitate information acquisition or to allow information pooling?)

The choice of an appropriate auction model can be illustrated by a practical example. The largest auction houses (e.g., Southeby Park Bernet, Inc. and Christie's) employ the English auction to sell rare and valuable items (art, antiques, and jewelry). A buyer can bid personally for an item on the day of the auction or can submit a prior written offer, designating a representative from the auction house to bid in his behalf. This same procedure establishes a silent seller reserve price, since a house representative is instructed to buy back the good if the sale price is insufficient. In addition, an estimated value (prepared by a house expert) is listed for each sale item.

Under value independence, our results concerning optimal auctions suggest two immediate conclusions. First, the estimated values should be irrelevant. A buyer who holds purely personal values for items should not be influenced by the estimated prices. He should use the pre-auction viewing period to establish his reservation prices, resisting the temptation to examine first the listed prices. The same self-restraint should be exercised when he participates in the open auction. Holding firm to his reservation value, he should not be influenced by competitors' bids.[15]

Second, the confidentiality of the seller reserve price is an illusory benefit for the owner of the good. As discussed in the conclusion of Section 2, the seller should be indifferent between announcing an optimal reserve price or keeping it confidential. Since the desire for confidentiality seems to be a practical fact, it is perhaps best to recognize this as another psychological element present in a "live" auction.

What if some degree of value dependence is present, as is the case when the worth of the good is determined in part by its potential resale value? In this instance, an estimated price has an obvious bearing. Such an estimate will convey partial information about the value of the good and ideally serve to elevate buyer bids. An optimal seller strategy might authorize appraisal of an item which the seller believes to be worth more than its face value and prohibit appraisal in the opposite circumstances. In the interest of "fair" auctions, however, the house does not permit this kind of buyer discretion. Concerning the appropriate reserve price policy under value dependence, it appears that an announcement dominates a silent price -- for the same reasons that apply under value independence. The optimal announcement (a mark

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[15] It is a common observation that the competitive features of the open ascending auction serve to elevate buyer offers (above their prior values). If value independence is taken as the appropriate model, buyers are behaving irrationally. This implies that the open ascending auction enjoys a practical advantage over the sealed bid auction. The "mixed" auction currently employed allows written bids to promote the greatest possible participation while maintaining the "uplifting" features of the open ascending auction.

up above v_0) directly elevates buyer offers and, consequently, permits increased seller profit.

It is easy to imagine (though not to solve) a hybrid model specifying both dependent and independent components of buyer reservation prices. A formal analysis of optimal auction design in this more general environment remains to be undertaken.

APPENDIX

Proposition 6: If assumption IID holds and all buyers are risk neutral, then for either the high or second bid auction rule an announced reserve price is at least as good in terms of expected seller revenue as a silent price.

Proof:

In section 2 we argued that, for the second bid auction expected seller revenue is independent of whether or not the reserve price is announced in advance. To analyze the high bid auction we must take account of the fact that with a silent reserve price the probability of winning,conditional upon having the highest valuation, will be of the general form $G(v)$. We wish to show that the seller can do no better than announce a reserve price of v_* so that $G(v)$ becomes

$$\text{(A1)} \quad G^*(v) = \begin{cases} 0, & v < v_* \\ 1, & v \geq v_* \end{cases}$$

To obtain an expression for expected seller revenue we follow exactly the derivation of Proposition 1. Expression (10) for the expected buyer gain becomes

$$\text{(A2)} \quad \Pi^i(x,v_i) = v_i G(x) F^{n-1}(x) - P_i(x)$$

Then, since $\Pi_1^i(x,v_i) = 0$ at $x = v_i$ we have

$$\text{(A3)} \quad \frac{d\Pi^i}{dv}(v,v) = \Pi_2^i(v,v) = G(v)F^{n-1}(v)$$

From the seller's viewpoing buyer i's expected gain is

$$(A4) \quad \int_{\underline{v}}^{\overline{v}} \Pi^i(v,v)\,dF(v)$$

Since the seller has a reservation value in $[\underline{v},\overline{v}]$ he will never sell to a buyer with valuation $\underline{v}$. Then $G(\underline{v}) = 0$. Integrating (A4) by parts and making use of (A3) we have

$$(A5) \quad \int_{\underline{v}}^{\overline{v}} \Pi^i(v,v)\,dF(v) = \int_{\underline{v}}^{\overline{v}} G(v)F^{n-1}(v)(1 - F(v))\,dv$$

Finally, setting $x = v_i$ in (A2) and taking the expectation over v_i we have

$$(A6) \quad \overline{P}^i = \int_{\underline{v}}^{\overline{v}} P_i(v_i)\,dF(v_i) = \int_{\underline{v}}^{\overline{v}} G(v)(vF'(v) + F(v) - 1)F^{n-1}(v)\,dv$$

In the high bid auction, for any silent reserve price strategy of bidding $b_0 = \phi(v_0)$, there is an implied conditional probability function $G(v)$. From (A6) the expected payment by i, for any given $G(v)$, is the same in the high and second bid auctions. But we have already argued that, in the latter, $\overline{P}_i$ is maximized if $G(v)$ satisfies (A1). Then the same is true for the high bid auction. Actually, in practice there is an additional problem of verification in the high bid auction. For whatever the strategies of the buyers, the seller can do no better <u>ex post</u> than to "quote" his true reservation value v_0. Raising b_0 above v_0 does not affect the sale price but can cause the loss of a profitable sale if b_0 is raised above the high bid. Thus the seller will have difficulty convincing buyers that his strategy will be to submit a silent bid other than $b_0 = v_0$. Indeed, if buyers believe that the true strategy of the seller is $b_0 = v_0$ the best the seller can do is to adopt this strategy. But then, as we have already argued, the silent bid strategy yields the seller a strictly lower expected revenue.

Q.E.D.

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