Department of Economics

University of California

Los Angeles

# Adjustment Dynamics and Equilibrium Selection
# in Coordination Games

Youngse Kim †

UCLA

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# Adjustment Dynamics and Equilibrium Selection in Coordination Games

## Abstract

We consider symmetric multiperson coordination games in a perfect foresight deterministic dynamic framework with a costly adjustment. Strong doubt is cast on the equivalence between risk dominance and learning outcomes formerly alleged in two person games. Equilibrium selection is fully characterized as a function of friction, which in turn depends upon players' discount rate and the duration of action commitment. Some limiting results obtain and their links to static equilibrium concepts based on perturbation are clarified. Surprisingly enough, the limit as the friction gets smaller coincides with the selection from global perturbation and strict iterated admissibility. In any pure coordination game, a much stronger result obtains supporting the Pareto efficiency as long as friction is sufficiently small, regardless of the number of players and of the initial states. We also provide numerical results that have substantial implications for the well known experiments on coordination failures.

**Keywords**: coordination game, adjustment dynamics, global perturbation, risk dominance.

*JEL* Classification: C72, C91, D81, D90.

Youngse Kim

Department of Economics

University of California

Los Angeles    CA 90024

e-mail: Kimy@econ.sscnet.ucla.edu

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# 1 Introduction

We study the problem of equilibrium selection in coordination games with two strict Pareto-ranked Nash equilibria. This class of games represents, in a stylized fashion, the types of interactions prevalent in network externalities such as compatibility of computer softwares, video tapes, typewriter keyboards, and language, as well as many recent Keyensian macroeconomic models of coordination failures, geographical formation of core and periphery (Krugman [1991]), social convention, etc. Consideration of these games is also motivated by the simple game theoretic issue of selection in games with multiple equilibria in which the existing refinements are powerless. For instance, the most stringent solution concept proposed in the literature on refinements of Nash equilibria, such as the strategic stability of Kohlberg and Mertens [1986], are silent about selection among strict Nash equilibria. Some recent studies on learning and evolution have also addressed the question of how a particular equilibrium will emerge in a dynamic context. A very partial list of this literature includes Fudenberg and Kreps [1988], Canning [1990], Milgrom and Roberts [1990,1991], and Fudenberg and Levine [1991]. Although some convergence results are obtained, these studies do not offer an equilibrium selection criterion, since all strict Nash equilibria share the same dynamic properties in their models.

A number of researches since Blume [1990] have tried to discriminate between strict Nash equilibria and, as a consequence, relate adaptive or evolutionary dynamic learning equilibrium outcomes to Harsanyi and Selten's [1988] static notion of risk dominance. Kandori, Mailath, and Rob [1991] consider evolutionary models for a finite population in discrete time with constant flow of mutations, which generate Markov processes in the behavior patterns. Fudenberg and Harris [1992] study a version of the replicator dynamic in continuous time on a large population. The random perturbation of the system is introduced by a Brownian motion. Both works obtain the same result: for $2 \times 2$ games, as the mutation rate and noise go to zero, the distribution becomes concentrated on the risk dominant equilibrium. The Matsui and Matsuyama [1991] model, from which the present paper heavily borrows, shows an equivalence between risk dominance and dynamic stability in a two person bimatrix

game of common interests.

We apply a costly adjustment dynamics to symmetric binary action multiperson coordination games with two strict pure strategy Nash equilibria. Doubt is cast on the alleged equivalence between risk dominance and learning equilibrium outcome. In other words, two notions "happen" to coincide only in two person bimatrix game. Equilibrium selection is provided in terms of the friction parameter which in turn depends positively on the discount rate of players and negatively on the mean arrival time of the action switches.

We not only fully characterize the equilibrium outcomes prescribed by the adaptive learning dynamic, but attain some interesting limiting results. Surprisingly enough, the one limit as the friction approaches to zero turns out to coincide Carlsson and Van Damme [1990,1991] equilibrium selection from perturbing the original game in such a way that each player receives a private signal for payoffs but is unable to fully disentangle the true payoff realization and purely private noise. Lack of common knowledge among players comes into play, which make it possible the remote areas of strictly dominated strategies to exert an influence. This fact suggests solving the resulting incomplete information game using an iterative elimination of strictly dominated strategies. It is shown that either the good or the bad equilibrium will be globally attractive according only to the static payoff matrix but irrespective of the initial conditions. The other limit as friction gets infinity is closely related to concepts of evolutionary stability, especially as of Swinkels [1992a] and Matsui [1992]. Remind that any ESS-like notion comes from a "local" perturbation, whether it is made on actions or on payoffs. Both strict equilibria are simply ESS, in which the payoff structure does not matter at all and only initial conditions do. Our results might shed a light on the connection between adaptive dynamic learning outcomes and static equilibrium notions based on perturbational methods. So to speak, the dynamic outcome attained as players become more concerned about the future seems to correspond to a static equilibrium that is based on a larger perturbation, and vice-versa.

The present paper may have a substantial implication for recently developed experimental results, such as Van Huyck et al. [1990, 1991] on coordination failure

and Cooper et al. [1990] on the predictability of Nash equilibrium. In particular, we provide a theoretical and numerical evidence that can explain the following observations: weak dominance, a wide dispersion of initial effort choices, a trend to drift in small group treatments, a rapid convergence to a bad Nash equilibrium regardless of initial strategic uncertainty in large group minimum treatment, a strong history dependency in large group median treatment, and Pareto efficiency in pure coordination problem. To recapitulate, coordination failures and history dependencies are the most remarkable features, respectively, under minimum rule and median vote, when the group size is large.

The balance of the paper is organized as follows. Section 2 offers an intuitive exposition of the basic idea in a simple example. Section 3 formally defines the game of our interest. Section 4 sets up the dynamic model and then characterizes its dynamic equilibrium outcomes. Section 5 calculates important static equilibrium selections, namely global game perturbation and risk dominance. The same section proposes the equivalence between the limiting adjustment dynamic outcome and the static equilibrium selection based on global perturbation. Section 6 study two interesting subclass of games, that is, a pure coordination and a stag hunt game. Section 7 offers numerical evidence in the framework of stag hunt game on the experimental studies. The last section concludes with some suggestions on future research.

## 2 An Exposition

Consider the highly stylized game as follows. A forest is inhabited by a stag and a number of hares. Identical hunters of group size $n$ simultaneously and without communication have to choose between stag and hare. If one hunts rabbit, his payoff is \$ $x$ no matter what his opponents' choices are. If he decides to pursue stag then his payoff would be determined not only by his own choice but by the summary statistic of what others are doing. Roughly speaking, stag hunting is successful only when enough number of hunters cooperate. Minimum rule refers to the situation where even a single defection from full cooperation results in a failure. Under median vote, half cooperation is sufficient for a successful stag hunting. A successful stag hunting

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yields $100 to each of the participants, whereas a failure brings about nothing. The normal form game described by the above data is called a *stag hunt game*. which can be traced to Rousseau [1755] in his discussion of the origin of the social contract. This game may have a lot of practical applications, as enumerated in the introducing paragraph.

We first study the dynamic evolution of the social equilibrium played by a large population. Each and every hunter is randomly and repeatedly matched to form a group of $n$ players and play the stage stag hunt game. Players are perfectly rational so as to maximize their discounted average expected payoff, and the dynamic path on which they condition to calculate their expected payoffs is perfectly foreseen. However, there is a friction: Not every player is able to switch his or her own action every period. While this assumption will be incorporated in a highly stylized manner, its interpretation as transaction costs can be intuitive in many contexts. For the example of stag hunt game, the reader might want to consider it as costs to switching from arms and tools needed for stag hunting to those for rabbit catching. Given the opportunity to move, each and every hunter chooses an action that maximizes his expected utility conditioned on the perfect foresight equilibrium path. The dynamic equilibrium outcome will be fully characterized as a function of group size, $n$, and the effective discount rate, $\delta$. The effective discount rate already takes full account of both real time discount rate and the cost to switching actions. The long run steady state of the social equilibrium must end up with either everyone's hunting stag or everyone's catching hares. While not regretting their individual choice in both states, people in rabbit hunting society are neverthless unhappier than those in stag hunting community. Struggling by a single individual or a negligible number of people is simply in vain. In other words, all hunters may give a best response, but implement a Pareto inferior equilibrium.

To take a concrete example, let $n = 2$ and $\delta = 0.5$. It can be shown that the "good" stag (respectively "bad" hare) equilibrium could be obtained regardless of the initial population fraction of rabbit hunters if the sure return to rabbit hunting, $x$, is smaller than $40 (resp. greater than $ 60). In the case where $x$ is between $40 and $60, the historical accident of initial fraction of hunter types plays a crucial role

in determining exactly which long run equilibrium the society would settle down. Now as players become more patient in the sense that $\delta$ approaches to zero, the middle region of history dependency vanishes, and the limiting threshold value of $x$ is calculated as \$50. For another example, let $n = 3$ and $\delta = 0.5$ under minimum rule. Then the history dependent region is between \$23 and \$43, which will shrink to an infinitesimal area around \$33c33 if people care very much about their future. Put it another way, in the limit as people are extremely patient, the society is likely settle down on the stag (resp. hare) equilibrium if $x$ is smaller (resp. greater) than \$33c33 in the long run. Just believe me for all the numbers here!

We discuss two important static equilibrium selection concepts in turn, Harsanyi and Selten's [1988] risk dominance and Carlsson and Van Damme's [1990,1991] global perturbation. Imagine a hypothetical situation where it is common knowledge that all players think that either the stag equilibrium or the hare equilibrium must be the solution without knowing which of both equilibrium points is the solution. Risk dominance tries to capture the idea that in this state of confusion the players enter a process of expectation formation that may lead to the conclusion that in some sense one of both is less risky than the other. A plausible chain of reasoning has led us to a complete theory an outsider observer should have on the player's behavior in the hypothetical situation. The preliminary theory can be summarized as follows: (i) Each player $i$ believes that either the all the other players hunt the stag or all other players catch hares; he assigns a subjective probability $z_i$ for the former possibility and its complementary probability for the latter; (ii) Each player $i$ plays his best response to his belief.(iii) The $z_i$ are independently and uniformly distributed over $[0, 1]$. Unfortunately this simple theory will not work because this best reply strategy combination will generally not be an equilibrium point of the game, and therefore it cannot be the outcome chosen by a rational outcome selection theory. The second order best reply to the first order vector is iteratively calculated, and so forth. As the tracing procedure progresses, both the prior vector and the best response strategy combination are subjected to systematic and continuous transformations until both of them finally converges to a specific equilibrium point of the game. Thus at the end of the tracing procedure both the players' actual strategy plans and expectations

about each other's strategy plans will correspond to the same equilibrium point, representing the risk dominant outcome. Fortunately the tracing procedure will be accomplished in one round in the present stag hunt game. For the two person game, stag hunting risk dominates hare catching if $x < \$50$, and vice-versa. For the three person game under minimum rule, the critical value will be $38c20.

Global perturbation approach is based on the idea that players are uncertain about the payoffs of the game. Trembling the game is made in such a way that payoffs are almost but not perfectly common knowledge, and that there is a small but non-negligible chance that each of the actions can be a dominated strategy. To be specific, there is a real possibility that $x < 0$ where rabbit hunting is strictly dominated and $x > \$100$ where stag hunting is striclty dominated. Each hunter receives a private signal that provides an unbised estimate of the common value $x$, but the signals are noisy so the true value of $x$ will not be a common knowledge, and then chooses whether to hunt stag or rabbit. Assume that the noise can be at most \$1. For instance, if the true value of $x$ is, say, \$70 then all the private signals must be somewhere between \$69 and \$71 from the outsider's viewpoint. Imagine a situation where a particular hunter $i$ just observed his private signal $x_i$ equal to, say, \$50. Even if he knows upon having observed \$50 that the true $x$ lies between \$49 and \$51 and that all other $x_j$'s between \$48 and \$52, this is in fact not common knowledge between hunter $i$ and $j$. Now suppose that hunter $j$ observes $x_j$, say, \$48, then he knows that the true $x$ lies between \$47 and \$49, and $x_i$ between \$46 and \$50. The problem is that hunter $i$ does not know that hunter $j$ knows that his $x_i$ lies in the interval [\$46, \$50]. Lack of common knowledge expands all the way down, and therefore enables remote areas of dominated strategies that $x$ is negative or greater than \$100 to exert an influence. This argument may well be applied to all the other less extreme realizations of $x_j$ in the interval [\$48, \$52] and any smaller size of the maximum noise, say, a penny instead of a dollar. Equilibrium is charaterized using iterative elimination of strictly dominated strategies and is shown to have a cutoff property. Finally, we are interested in what happens at the payoff realization corresponding to the original game in order to select an equilibrium. For a two person stag hunt game, equilibrium selection based on global perturbation prescribes that

each hunter should hunt a stag (respectively, a hare) if his private signal is smaller (resp. larger) than $50. For another example of a three person game under majority vote, a hunter should choose a rabbit only when his private signal about the riskless return from hunting rabbit is bigger than $66c67.

Table 1 provides some calculation examples of the cutoff values for the limiting adjustment dynamic outcome, risk dominance, and global perturbation in the case of minimum and median rules when the number of players is $n = 2, 3, 15, 99$, respectively.[1] The reader may be aware that the dynamic equilibrium outcome selection in the limit as the effective discount rate $\delta$ goes to zero coincides with static equilibrium selection based on global perturbation but not risk dominace. This is no luck ! We are to verify this equivalence in coordination games in general.

## 3  The Game

We consider a symmetric $n$ person coordination game with binary actions, denoted High and Low. The normal form game denoted by $G(n, \Pi)$ has $2^n$ number of cells, but due to symmetry only $2n$ cells need to be taken into account. Considering a strategy profile in which $k$ number of persons choose H with the remaining $(n - k)$ persons choosing L, we denote $\pi_k^H$ and $\pi_{n-k}^L$ to be a player's payoff who is taking H and L, respectively, where $k = 1, 2, ..., n$. Let a vector $\Pi^\zeta = (\pi_1^\zeta, \pi_2^\zeta, ..., \pi_n^\zeta)$, for $\zeta = $ H, L, and $\Pi = (\Pi^H, \Pi^L) \in \Re^{2n}$. The games of our interest belong to:

$$\Omega \equiv \{\Pi \in \Re^{2n} | \pi_{k+1}^\zeta \geq \pi_k^\zeta, \forall \zeta, \quad \forall k \quad \text{with strict inequality for some } k;$$
$$\pi_2^H > \pi_1^L, \pi_n^L > \pi_1^H; \pi_n^H \geq \pi_n^L\}. \tag{1}$$

The first set of inequalities in (1) imply that a player taking a particular action is no worse off when the number of opponents taking the same action increases. The next two inequalities require that everyone playing a common action is a strict Nash

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1It is interesting to note that under the minimum rule, global perturbation is more conservative than risk dominance, in the sense that there is a portion of $x$ such that $x_{GP}$ prescribes subjects to choose the secure action but $x_{RD}$ the risky payoff dominant one. Under the median rule, on the contrary, risk dominance is more conservative. These observations imply that coordination failures and history dependency are more severe in global perturbation than in risk dominance.

equilibrium. The last inequality means that the equilibrium where everyone plays H. denoted **H**. is better than where everyone plays L, denoted **L**. Figure 1 depicts an example of three person coordination games with payoff specification:

$$\pi_3^H = 3, \ \pi_2^H = 1, \ \pi_1^H = -1;$$

$$\pi_3^L = 2, \ \pi_2^L = 0, \ \pi_1^L = 0.$$

Now the following preliminary result is straightforward:

**Lemma 1** *If* $\Pi \in \Omega$ *then the only pure strategy equilibrium of* $G(n, \Pi)$ *are two strict Nash, viz.* **H** *and* **L**.

*Proof.* It suffices to show that none of $k = 1, 2, ..., n - 1$ satisfies both $\pi_{n-k}^L > \pi_{k+1}^H$ and $\pi_k^H > \pi_{n-k+1}^L$, since otherwise the pure strategy profile of $k$ players choosing H's and $(n - k)$ players choosing L's would be Nash. Adding the above two inequalities yields

$$-(\pi_{n-k+1}^L - \pi_{n-k}^L) > \pi_{k+1}^H - \pi_k^H,$$

which contradicts the definition of the $\Omega$ set.　　　■

As suggested before, any of the Nash refinements including the strategic stability as of Kohlberg and Mertens is powerless in selecting between two strict Nash equilibria. Pareto efficiency is compatible with equilibrium play, so neither an incentive problem nor conflict exists. However, it is not clear whether players will be able to reach this outcome in a noncooperative situation where no direct communication is allowed. In short, a strategic uncertainty matters.

## 4　Adjustment Dynamics

### 4.1　The Model

Time is continuous from $t = 0$ to $\infty$. The game $G(n, \Pi)$ is played repeatedly in a society with a continuum[2] of identical players. At every point in time, each is matched

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2Boylan [1992] verifies that, if the population is countably infinite, there exist a probability space and a sequence of random variables which correspond to a random matching process such that the law of large numbers can nicely apply, i.e., there is no aggregate uncertainty. Green [1989] offers some big enough probability space to encompass the continuum model.

to form a group with other $n - 1$ players, randomly drawn from the population, and they play the game anonymously. All players are highly rational and choose a strategy to maximize the expected discounted payoffs. Because of the anonymity, they are engaged in this maximization without taking into account strategic considerations such as reputation, punishment, and forward induction.

The key assumption is that not every player can switch actions at every point in time. Every player needs to make a commitment to a particular action in the short run. Following Blume [1991] and Kandori and Rob [1992], we assume that the opportunity to switch actions arrives randomly; it follows the Poisson process with the parameter $\lambda$ being the mean arrival rate. Furthermore, it is assumed that the process is independent across the players and there is no aggregate uncertainty. The strategy distribution in the society as of time $t$ can be thus described as $y_t$, the fraction of the players that are committed to action H as at $t$. Due to the restriction mentioned above, the social behavior pattern $y_t$ changes continuously over time and its rate of change belongs to $[-\lambda y_t, \lambda(1-y_t)]$. Furthermore, any feasible path necessarily satisfies $y_0 e^{-\lambda t} \le y_t \le 1 - (1 - y_0)e^{-\lambda t}$, where the initial condition $y_0$ is given exogenously or "by history."

When the opportunity to switch arrives, players choose the action which results in the higher expected discounted payoffs, recognizing the future path of $y$ as well as their own inability of switching actions continuously. The value of playing action H instead of L as of time $t$ is equal to

$$
\begin{aligned}
\Phi(y_t) &= \sum_{k=1}^{n} \binom{n-1}{k-1} y_t^{k-1}(1-y_t)^{n-k}\pi_k^H - \sum_{k=1}^{n} \binom{n-1}{k-1} y_t^{n-k}(1-y_t)^{k-1}\pi_k^L \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} y_t^{k-1}(1-y_t)^{n-k}\phi_k,
\end{aligned}
\tag{2}
$$

where $\phi_k \equiv \pi_k^H - \pi_{n-k+1}^L$ is nondecreasing in $k$. Given the opportunity, players commit to take H if $V_t > 0$ and to L if $V_t < 0$ and are indifferent if $V_t = 0$, where

$$
V_t \equiv (\lambda + r) \int_0^\infty \Phi(y_{t+s})e^{-(\lambda+r)s}ds
\tag{3}
$$

with $r > 0$ being the discount rate. We define $\delta \equiv \frac{r}{\lambda}$ to be the effective discount rate or the degree of friction. Therefore, $\{y_t\}_{t=0}^\infty$ is an equilibrium path from $y_0$ if its

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righthand derivative exists and satisfies

$$\dot{y}_t^+ = \begin{cases} \lambda(1 - y_t) & \text{if } V_t \geq 0, \\ -\lambda y_t & \text{if } V_t \leq 0, \end{cases} \qquad (4)$$

for any $t$. This states that all players currently playing action H (respectively L), if given the chance, switch to L (resp. H), when $V_t < (\text{resp.} >)0$.

## 4.2 Characterization

We borrow from the Matsui and Matsuyama model the following terminology: A state $y$ is called *accessible* from $y'$, if an equilibrium path from $y'$ that reaches or converges to $y$ exists. It is called *globally attractive* if it is accessible from any $y' \in [0,1]$. A state $y$ is called *absorbing* [3] if a neighborhood $U$ of $y$ exists such that any equilibrium path from $U$ converges to $y$. It is *fragile* if it is not absorbing. The definition does not rule out the possibility that a state may be both fragile and globally attractive, or that a state may be uniquely absorbing but not globally attractive. However, we will show that these situations will not occur in this model.

We will show that the parameter $\Pi$ characterizes the game to be in one of three sets $\Omega_0, \Omega_1$ and $\Omega_{01}$, where the state $y = 0$ is globally attractive in $\Omega_0$, the state 1 is globally attractive in $\Omega_1$, and both states are absorbing in $\Omega_{01}$. For this purpose, we need the coefficients

$$\alpha_k(n, \delta) \equiv \frac{1 + \delta}{n} \prod_{j=k}^{n} \left( \frac{j}{j + \delta} \right) \text{ and } \beta_k(n, \delta) \equiv \alpha_{n-k+1}(n, \delta). \qquad (5)$$

For notational simplicity, we suppress $(n, \delta)$ whenever possible. We denote the vectors $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. The reader might be embarrassed with the complicated forms of the coefficients $\alpha_k$'s and $\beta_k$'s. According to the lemma below, however, they play a role as weights, putting higher (resp. lower) weight on larger $k$ in $\alpha(\text{resp.}\beta)$. The weight equally spreads over all $k$'s as the friction disappears, while it concentrates on an extreme $k$ as the friction grows without bound.

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[3] Although this is the same concept as *asymptotically stable* according to standard terminology in dynamical systems, we simply use *absorbing* for brevity. It should be emphasized that this is nothing to do with the Markov processes.

**Lemma 2** *For any $n$ given, (a) $\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$, $\forall \delta$;*

*(b) $\alpha_{k+1} > \alpha_k$ and $\beta_{k+1} < \beta_k$, $\forall k, \delta \in (0, \infty)$; (c) $\lim_{\delta \to 0} \alpha_k = \lim_{\delta \to 0} \beta_k = \frac{1}{n}$, $\forall k$;*

*(d) $\lim_{\delta \to \infty} \alpha = (0, ..., 0, 1)$ and $\lim_{\delta \to \infty} \beta = (1, 0, ..., 0)$.*

*Proof.* (a) Via mathematical induction. Checking the case of $n = 2$ is trivial. Supposed that it holds for $n - 1$, i.e. $\sum_{k=1}^{n-1} \prod_{j=k}^{n-1} (\frac{j}{j+\delta}) = \frac{n-1}{1+\delta}$, then for $n$

$$
\begin{aligned}
\sum_{k=1}^{n} \alpha_k &= \frac{1+\delta}{n} \sum_{k=1}^{n} \prod_{j=k}^{n} (\frac{j}{j+\delta}) \\
&= \frac{1+\delta}{n} [\frac{n}{n+\delta} + \frac{n}{n+\delta} \sum_{k=1}^{n-1} \prod_{j=k}^{n-1} (\frac{j}{j+\delta})] \\
&= \frac{1+\delta}{n} \frac{n}{n+\delta} [1 + \frac{n-1}{1+\delta}] = 1.
\end{aligned}
$$

The fact that $\sum_{k=1}^{n} \beta_k = 1$ is trivial since the elements of the vector $\beta$ are just a rearrangement of those of $\alpha$. To check (b),(c) and (d) is straightforward. ∎

The $\cdot$ denotes the inner product of two vectors. For example, $\alpha \cdot \Pi^\varsigma = \sum_{k=1}^{n} \alpha_k \pi_k^\varsigma$, etc. We attain proposition 1 together with the definition of the sets:

$$
\Omega_0 = \{\Pi \in \Omega | \alpha \cdot \Pi^H \leq \beta \cdot \Pi^L\}, \tag{6}
$$

$$
\Omega_1 = \{\Pi \in \Omega | \beta \cdot \Pi^H \leq \alpha \cdot \Pi^L\}, \tag{7}
$$

$$
\Omega_{01} = \Omega \backslash (\Omega_0 \bigcup \Omega_1). \tag{8}
$$

**Proposition 1** *The state $y$ is globally attractive iff $\Pi \in \Omega_y$ for either $y = 0$ or $y = 1$; both $y = 1$ and $y = 0$ are absorbing iff $\Pi \in \Omega_{01}$. Moreover, if an absorbing state, $y$, is globally attractive, then it is a unique absorbing state in $[0, 1]$ and any other state must be fragile.*

*Proof.* First of all, notice that $\Phi(0) = \pi_1^H - \pi_n^L < 0 < \Phi(1) = \pi_n^H - \pi_1^L$ and that $\Phi$ is strictly increasing, since

$$
\Phi'(y) = (n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} y^k (1-y)^{n-k-2} [\phi_{k+2} - \phi_{k+1}] > 0
$$

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by the definition of $\phi$ function and the nondecreasingness of $\pi_k$ sequences.

The outcome **H** can be upset when the players have an incentive to deviate for a feasible path from $y = 1$. Because of the monotonicity of $\Phi$ function, the incentive to deviate is the strongest if all players are anticipated to switch from H to L in the future, i.e. $y_t = e^{-\lambda t}$. Hence, the condition for $y = 1$ being fragile is

$$V_0 = (\lambda + r) \int_0^\infty \Phi(e^{-\lambda s}) e^{-(\lambda+r)s} ds \leq 0,$$

which would be by the change-of-variable technique

$$(1 + \delta) \int_0^1 \Phi(y) y^\delta dy \leq 0. \tag{9}$$

Using eq (2), the definition and properties of the Beta and Gamma function,[4] and some algebraic manipulations, the eq (9) becomes

$$
\begin{aligned}
0 &\geq (1 + \delta) \sum_{k=1}^n \binom{n-1}{k-1} \phi_k \int_0^1 y^{k+\delta-1} (1-y)^{n-k} dy \\
&= (1 + \delta) \sum_{k=1}^n \binom{n-1}{k-1} \phi_k \frac{\Gamma(k+\delta)\Gamma(n-k+1)}{\Gamma(n+\delta+1)} \\
&= \sum_{k=1}^n \alpha_k \phi_k,
\end{aligned}
$$

or equivalently

$$\sum_{k=1}^n \alpha_k \pi_k^H \leq \sum_{k=1}^n \alpha_k \pi_{n-k+1}^L = \sum_{k=1}^n \beta_k \pi_k^L, \tag{10}$$

which corresponds to the condition defining the $\Omega_0$ set. We claim: $y = 0$ is globally attractive if and only if $\Pi \in \Omega_0$, and that $y = 1$ is absorbing if and only if $\Pi \in \Omega \backslash \Omega_0$. To prove the if part of $y = 0$ being globally attractive and the only if part of $y = 1$ being absorbing, it suffices to show that, if eqn (10) holds, i.e. $\Pi \in \Omega_0$, a feasible path from $y = 1$ to $y = 0$, $y_t = e^{-\lambda t}$, satisfies the equilibrium condition, i.e. $V_t \leq 0 \ \forall t$ along the path. This can be checked as follows:

$$
\begin{aligned}
V_t &= (\lambda + r) \int_0^\infty \Phi(y_{t+s}) e^{-(\lambda+r)s} ds \\
&\leq (\lambda + r) \int_0^\infty \Phi(e^{-\lambda s}) e^{-(\lambda+r)s} ds \leq 0 \ \forall t.
\end{aligned}
$$

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[4] Refer to any text on mathematical statistics.

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To prove the if part of $y = 1$ being absorbing and the only if part of $y = 0$ being globally attractive, it suffices to demonstrate that, if $\Pi \in \Omega \backslash \Omega_0$, the equilibrium path is unique and converges to $y = 1$ for $y_0$ sufficiently close to 1. Reminding that any feasible path from $y_0$ satisfies $y_t \geq y_0 e^{-\lambda t}$, we get

$$V_0 \geq (\lambda + r) \int_0^\infty \Phi(y_0 e^{-\lambda s}) e^{-(\lambda+r)s} ds.$$

Since the righthand side is strictly positive at $y_0 = 1$ and continuous in $y_0$, it is still positive for $y_0$ sufficiently close to 1.

Similarly, the condition for $y = 0$ being fragile combined with the change of variable technique will be

$$
\begin{aligned}
V_0 &= (\lambda + r) \int_0^\infty \Phi(1 - e^{-\lambda s}) e^{-(\lambda+r)s} ds \\
&= (1 + \delta) \int_0^1 \Phi(y)(1 - y)^\delta dy \geq 0.
\end{aligned}
\tag{11}
$$

Again by the definition of $\Phi$ function, the properties of Gamma and Beta function, and some algebraic manipulation, we have

$$
\begin{aligned}
0 &\leq \sum_{k=1}^n \binom{n-1}{k-1} \phi_k \frac{\Gamma(k)\Gamma(n-k+\delta)}{\Gamma(n+1+\delta)} \\
&= \sum_{k=1}^n \beta_k \phi_k,
\end{aligned}
\tag{12}
$$

or equivalently

$$\sum_{k=1}^n \beta_k \pi_k^H \leq \sum_{k=1}^n \beta_k \pi_{n-k+1}^L = \sum_{k=1}^n \alpha_k \pi_k^L, \tag{13}$$

which is the condition defining $\Omega_1$. A symmetric argument as before shows that $y = 1$ is globally attractive if and only if $\Pi \in \Omega_1$, and that $y = 0$ is absorbing if and only if $\Pi \in \Omega \backslash \Omega_1$.

Combining all those facts proven thus far yields the desired result. ∎

**Proposition 2** (a) In the limit as $\delta \to 0$, the state $y = 1$ (resp. $y = 0$) is globally attractive iff $\frac{1}{n} \sum_{k=1}^n \pi_k^H > $ (resp. <) $\frac{1}{n} \sum_{k=1}^n \pi_k^L$;
(b) in the limit as $\delta \to \infty$, both states are absorbing and no state globally attractive.

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*Proof.* Part (a) is clear from Lemma 2(b) and (c). As $\delta$ goes to infinity, Lemma 2(d) together with eqn (1) implies that both $\Omega_0$ and $\Omega_1$ becomes empty, while the $\Omega_{01}$ eventually occupies the whole $\Omega$. ∎

Keep in mind that the smaller (larger) size of $\delta$ implies the more (less) patience and/or a shorter (longer) duration of an action commitment.[5] The smaller the degree of friction $\delta$ gets, the more the long run equilibrium tends to rely on the parameter specification and the less on the initial position of strategic uncertainty, and vice versa. As players are more and more patient and/or able to switch their choices, the steady state will be the good Pareto efficient equilibrium as long as the "static" unweighted average from H exceeds that from taking L.

On the other extreme case of $\delta$ approaching to infinity, sometimes called best response dynamics, both states may obtain in the long run and exactly which one would come out depends solely upon what the initial state was. In fact, Matsui [1992] verifies an equivalence between the best response dynamics and a static equilibrium concept attributed to Swinkels [1992a]. This notion, called an evolutionary stability with equilibrium entrants, imposes an additional restriction on the qualification of mutants, thus is weaker than the traditional evolutionary stability. Notice that the connection[6] of "myopic" replicator dynamics to strategic stability or rationalizability would be vacuous in coordination games, because both Nash equilibria simply survive the strict iterated admissibility.

## 5  Equivalence to Global Perturbation

The global perturbation approach of Carlsson and Van Damme [1990, 1991] is based on the idea that players are uncertain not only about the payoffs but also their modeling of the game itself. Each player $i$ will receive a private signal $\theta_i$ that provides

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5Indeed, $\theta \to 0$ implies that players are more concerned about the future. That $\lambda \to \infty$ might have two opposite effects: players are less concerned about the future whilst the current strategy distribution becomes less important. Nevertheless, a strictly positive $\theta$ guarantees the second effect always dominating the first one. Therefore, the smaller $\delta$ gets, the more players worry about the future.

6Refer to Swinkels [1992b] and Samuelson and Zhang [1992].

an unbiased estimate of $\theta$, but the signals are noisy so the true value of $\theta$ will not be common knowledge. Let $\Theta$ be a random variable and let $\{E_i\}_{i=1}^n$ be an $n$ tuple of i.i.d. random variables, each having zero mean. The $E_i$ are independent of $\Theta$, allow a continuous density and have support within $[-1, 1]$. For $\varepsilon > 0$, write $\Theta_i^\varepsilon = \Theta + \varepsilon E_i$. Notice that $\varepsilon$ measures perfectness of the common knowledge.

Given this structure, we formally define the incomplete information game $G^\varepsilon(n, \Pi)$ described by the following rules: A realization $(\theta, \theta_1, ..., \theta_n)$ of $(\Theta, \Theta_1^\varepsilon, ..., \Theta_n^\varepsilon)$ is drawn, player $i$ is informed only about $\theta_i$ and chooses between H and L, each player $i$ receives payoffs as determined by $G(n, \Pi(\theta))$ and the action taken. Even if player $i$ knows upon having observed $\theta_i$ that the true $\theta$ lies in $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and that all other $\theta_j$'s in $[\theta_i - 2\varepsilon, \theta_i + 2\varepsilon]$, this fact is not common knowledge. Now suppose that $\theta_j$ is realized as, say, $\theta_i - 2\varepsilon$, then player $j$ knows that $\theta$ lies in $[\theta_j - \varepsilon, \theta_j + \varepsilon]$, thus in $[\theta_i - 3\varepsilon, \theta_i + \varepsilon]$, and that $\theta_i$ must be in $[\theta_j - 2\varepsilon, \theta_j + 2\varepsilon]$, thus in $[\theta_i - 4\varepsilon, \theta_i]$. The problem is that player $i$ does not know that player $j$ knows that $\theta_i$ lies in $[\theta_i - 4\varepsilon, \theta_i]$. This argument applies also to all the other less extreme realizations of $\theta_j$. Lack of common knowledge expands all the way down, and thus enables remote areas of dominated strategies $(-\infty, \underline{\theta})$ and $(\bar{\theta}, \infty)$ to exert an influence, however tiny $\varepsilon$ might be as far as it is strictly positive.[7]

We confine our attention to perturbation $p_k^H$ (resp. $p_{n-k}^L$) that satisfies two conditions as follows:

**Assumption 1** *(a) They are continuous, monotonically increasing (resp. decreasing) in $\theta$, and unbounded above and below, $\forall k$; (b) the original unperturbed game obtains with $\theta = 0$, i.e. $p_\zeta^\varsigma(0) = \pi_\cdot^\varsigma$ for $\varsigma =$ H,L.*

Let $\bar{\theta}$ (resp. $\underline{\theta}$) the infimum (resp. supremum) of $\theta$'s such that H (resp. L) is a strictly dominant strategy in a game with payoff realization $\theta$. By assumption 1 above, it is obvious that $-\infty < \underline{\theta} < 0 < \bar{\theta} < +\infty$.

**Assumption 2** *The $\Theta$ is uniformly distributed over an interval $\supset [\underline{\theta}, \bar{\theta}]$.*

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7Such a remote area plays an important role in Rubinstein [1989] as well.

This would play an important role, since only order but not location of the realizations of random noise variable matters. We believe that main points would still emerge without this restriction, but we have not verified that this is the case. A guess on relaxation of this assumption will be made in the last section.

Under these assumptions, an iterative elimination of strictly dominated strategies, namely *strict iterated admissibility,* will be applied. The next lemma shows that the Bayesian Nash equilibrium is of the cutoff property, and that the game considered here is indeed dominance solvable.

**Lemma 3** *If assumption 1 and 2 hold, then the equilibrium is characterized by cutoff $\theta_{GP}$ such that player i optimally chooses H (resp. L) iff $\theta_i >$ (resp. $<$)$\theta_{GP}$. Furthermore, $\theta_{GP}$ is a unique root of the equation $\frac{1}{n}\sum_k p_k^H(\theta) = \frac{1}{n}\sum_k p_k^L(\theta)$.*

*Proof.* Notice that the existence and uniqueness of such $\theta_{GP}$ are guaranteed by assumption 1(a) and 1(c). As was suggested, we maintain the assumption that no player will choose strictly dominated strategies. Player $i$ will certainly choose H if $\theta_i > \bar{\theta}$: Since the expected value is $E(\Theta|\theta_i^\varepsilon = \theta_i) = \theta_i$, player $i$ knows that H is strictly dominant at each such observation. Consider an observation $\theta_i$ of player $i$ slightly below $\bar{\theta}$. More precisely, it must be that $|\bar{\theta} - \theta_i| < 2\varepsilon$. Player $i$ knows that his opponent will play H if $\theta_j > \bar{\theta}$, hence $i$'s payoff if he chooses H at $\theta_i$ is approximately

$$\sum_{k=1}^{n} \Pr(\theta_j > \theta_i \text{ for exactly } k-1 \text{ opponents}|\Theta_i^\varepsilon \approx \bar{\theta})p_k^H(\bar{\theta}) \tag{14}$$

$$= \sum_{k=1}^{n} \Pr(E_j > E_i \text{ for exactly } k-1 \text{ opponent})p_k^H(\bar{\theta}) \tag{15}$$

$$= \frac{1}{n}\sum_{k=1}^{n} p_k^H(\bar{\theta}). \tag{16}$$

Assumption 2 allows us to conclude that the probability in the eqn (14) is independent of $\theta_i$, at least as long as $\theta_i$ lies $\varepsilon$ inside the support of $\Theta$. This observation allows us to conclude that this probability must be equal to the a priori probability that $E_i$ is the $k+1$th smallest among the errors. Thus, the eqn (15) ensues, the probability in which is clearly the same for all players. This fact, combined with the assumption that the i.i.d. of $E_i$ has a continuous density, yields eqn (16).

A similar reasoning shows that the expected payoff to action L is at most approximately $\frac{1}{n}\sum_{k=1}^{n} p_k^L(\bar\theta)$, which is strictly lower than $\frac{1}{n}\sum_{k=1}^{n} p_k^H(\bar\theta)$ calculated above by the monotonicity assumption 1(a). Hence, if $\theta_{GP} < \bar\theta$, there exists $\bar\theta^1$ such that H is strictly dominant for any $\theta_i > \bar\theta^1$ in the reduced game where player $j$ is constrained to play H when $\theta_j > \bar\theta$. In a similar way one can construct $\bar\theta^2 < \bar\theta^1$ and continuing inductively we find sequences $\bar\theta^m$ such that H is iteratively dominant for $\theta_i > \bar\theta^m$.

On the other hand, starting from the maintained assumption that action L will be chosen when $\theta_i < \underline\theta$, we inductively find a sequence $\underline\theta^m$ such that L is iteratively dominant for $\theta_i < \underline\theta^m$. By the definition of $\theta_{GP}$, it is obvious that $\bar\theta^m \downarrow \theta_{GP}$ and $\underline\theta^m \uparrow \theta_{GP}$ as $m \to \infty$. ∎

Remind that the perturbed game will correspond to the original unperturbed game when $\theta = 0$. We are interested in what happens at $\theta = 0$ in the limit as the common knowledge about payoffs becomes arbitrarily perfect, i.e. $\varepsilon$ goes to zero. Recall that $|\theta_i| < \varepsilon$ if $\theta = 0$. So if $\theta_{GP} >$ (resp. $<$)0 for $\varepsilon$ small enough then $\theta_i <$ (resp. $>$)$\theta_{GP}$ for all $i$ when $\theta = 0$. We say that the equilibrium **H** in the unperturbed game is *robust with respect to global perturbation* if $\theta_{GP} < 0$, and that **L** is robust if $\theta_{GP} > 0$. Remind that the state $y$ be the fraction of population taking action H. Arguments thus far would yield:

**Main Theorem**    *The* $y = 1$ *(resp. 0) is the unique globally attractive state in the limit as* $\delta \to 0$ *if and only if action H (resp. L) is robust with respect to global perturbation.*

A couple of previous literature deserve mention. Harsanyi [1973] uses a similar perturbation to justify mixed strategy equilibria. His formulation requires, however, that the value of $\theta$ be common knowledge so observing $\theta_i$ implies knowing the realization of $E_i$, but not $E_{-i}$'s, and that the payoff of player $i$ depends on $\theta_i$ rather than on $\theta$. Fudenberg, Kreps, and Levine [1988] argues that an equilibrium that is unreasonable (in the sense of being eliminated by Nash refinements) in a given game may not be unreasonable in nearby games. They assert that every strict equilibrium is reasonable and they roughly show that every normal form perfect equilibrium can be

approximated by strict equilibria of nearby games, hence, that any such equilibrium is reasonable as well. Their paper differs from global perturbation in the definition of nearness of games and in the assumption that only the analyst does not know the payoffs, the payoffs are, however, common knowledge among the players themselves.

We discuss selection on the basis of Harsanyi and Selten's [1988] risk dominance,[8] and refutes its equivalence to global perturbation, thus to limiting adaptive dynamic outcome. The definition of risk dominance is based on a hypothetical process of expectation formation starting from the initial situation where it is common knowledge that either **H** or **L** will be the solution but where players do not yet know which one is the solution. Consider a process in which players first, on the basis of a preliminary theory, form priors on the strategies of their opponents. Thereafter, players gradually adapt their prior expectations to final equilibrium expectations by means of the *tracing procedure*. The players' prior beliefs $q_i$ about player $i$'s strategy should coincide with the prediction of an outside observer who reasons in the following three steps about the game: (i) Player $i$ believes that his opponents will either all choose **H** or **L**; he assigns a subjective probability $z_i$ to the first event and the complementary probability to the second; (ii) Player $i$ chooses a best response to his beliefs; (iii) The beliefs of different players are independently uniformly distributed on $[0, 1]$. From (i) and (ii), the outside observer concludes that player $i$ takes H according to $\pi_n^H z_i + \pi_1^H (1 - z_i) > \pi_1^L z_i + \pi_n^L (1 - z_i)$, or

$$z_i > \mu \equiv \frac{\pi_n^L - \pi_1^L}{\left(\pi_n^H - \pi_1^H\right) + \left(\pi_n^L - \pi_1^L\right)}.$$

Using step (iii), the outside observer forecasts player $i$'s strategy as $q_i = (1 - \mu)[H] + \mu[L]$, with different $q_i$ being independent. The tracing procedure to find a distinguished path in the graph of the correspondence from a linear combination of the naive $G(q)$ and $G(n, \Pi)$ to the set of Nash equilibria is simple in the case at hand. Player $i$'s expected payoff difference associated with H and L in $G(n, \Pi)$ when each of the opponents follows the strategy $q_{-i}$ will be

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8Payoff dominance principle a là Harsanyi and Selten [1988 p.80-1] selects **H** whereas the security a là Luce and Raiffa [1957 p.61-7] prescribes **H** (resp. **L**) according to $\pi_1^H >$ (resp. $<$)$\pi_1^L$. Note that the number of players, $n$, is totally irrelevant like in most of the Nash refinements.

$$\sum_{k=1}^{n} \binom{n-1}{k-1} (1-\mu)^{k-1} \mu^{n-k} \pi_k^H - \sum_{k=1}^{n} \binom{n-1}{k-1} \mu^{k-1} (1-\mu)^{n-k} \pi_k^L$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} (1-\mu)^{k-1} \mu^{n-k} \phi_k$$

$$\equiv \Phi(1-\mu). \tag{17}$$

Recalling that $\Phi(0) < 0 < \Phi(1)$ and $\Phi$ is monotonic increasing, write $\mu_{RD}$ the unique root of the equation $\Phi(1-\mu) = 0$. Hence, each player's best response against $q$ would be H (resp. L) iff $\mu <$ (resp. $>$)$\mu_{RD}$.

Now it is not difficult though tedious to verify the nonequivalence part, since the payoff $\Pi$ satisfying the condition $\frac{1}{n}\sum_k \pi_k^H = \frac{1}{n}\sum_k \pi_k^L$ does not generically satisfy the risk dominance solution $\Phi(1-\mu) = 0$, for $n \geq 3$. In this course, one can be aware that they just happen to be equal when $n = 2$.

## 6 Applications

### 6.1 Pure coordination

Consider a two person pure coordination game. It is often argued that, even without preplay communication, introspection alone will lead players to coordinate on the Pareto optimum. This intuition is confirmed as reasonable even in broader definition of pure coordination games. A pure coordination game specifies the payoff parameters to be

$$\pi_k^H \ (\text{resp.} \pi_k^L) = \begin{cases} a \ (\text{resp.} b) & \text{for } \kappa \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

where $\kappa$ may be any of $2, 3, ..., n$.

**Corollary 1** *There exists $\bar{\delta} > 0$ such that the only uniquely absorbing and globally attractive state is $y = 1$ for any $n$, $y_0$, and $\delta \in (0, \bar{\delta})$. Equivalently, the only equilibrium selected based on the global perturbation must be the Pareto efficient* **H** *for any $n$.*

*Proof.* Since $\frac{1}{n}\sum_k \pi_k^H = \frac{n-\kappa+1}{n}a > \frac{n-\kappa+1}{n}b = \frac{1}{n}\sum_k \pi_k^L$ always holds. it is straightforward that, as $\delta \to 0$, the $\Omega_1$ set will ultimately occupy the whole $\Omega$ and the remaining region $\Omega_0$ and $\Omega_{01}$ be empty sets. The second part is direct from our main theorem. ∎

## 6.2 Stag hunt

The most general payoff specification that includes the game discussed in the expository section will be as follows:

$$\pi_k^L = x \in (0,1) \text{ all } k$$

$$0 = \pi_1^H \leq \pi_2^H \leq ... \leq \pi_n^H = 1.$$

Besides its practical applicability, this game has a couple of merits to analyze. First, the Pareto optimality is at odds with the security, so which outcome would actually appear may be controversial. Second, it reduces the $\Omega$ sets to a one dimensional space, which makes the results extremely intuitive and facilitates numerical studies. Recalling that $\cdot$ denotes a dot product of two vectors, we define

$$u(n,\delta) \equiv \alpha \cdot \Pi^H \text{ and } \ell(n,\delta) \equiv \beta \cdot \Pi^H \tag{18}$$

where $\alpha_k$'s and $\beta_k$'s are as in eqn (5). Directly applying proposition 1 and 2 yields:

**Lemma 4** *(a) The state $y = 1$ is globally attractive iff $x \geq u(n,\delta)$; $y = 0$ is globally attractive iff $x \leq \ell(n,\delta)$; both $y = 1$ and $y = 0$ are absorbing iff $\ell(n,\delta) \leq x \leq u(n,\delta)$; (b) in the limit as $\delta \to 0$, the state $y = 1$ (resp. $y = 0$) is globally attractive iff $x < (resp. >)\frac{1}{n}\sum_k \pi_k^H \equiv x_{AD}$; (c) in the limit as $\delta \to \infty$, both states are absorbing.*

The AD in the expression of $x_{AD}$ stands for 'adjustment dynamic'. We now discuss selection on the basis of risk dominance and global perturbation in turn. Carefully following the linear tracing procedure described in section 5.1, one can reproduce Carlsson and Van Damme [1991] result that: each player's best response against $q_{-i}$, where $q_i = (1 - x)[\text{Stag}] + x[\text{Rabbit}]$ with different $q_i$ being independent, would be

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to hunt stag (resp. rabbit) if and only if $x <$ (resp. $>)x_{RD}$, where $x_{RD}$ is the fixed point of the mapping

$$B(x) = \sum_{k=1}^{n} \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) (1-x)^{k-1} x^{n-k} \pi_k^H .$$  (19)

For the study of global payoff uncertainty, consider a specific perturbation. That is, we assume that all data of $G(n, x, \Pi^H)$ are common knowledge, except for the payoff $x$ associated with the safe action L. Remind that the $X$ is uniformly distributed over an interval containing $[0, 1]$. Denote $x_{GP}$ to be the cutoff calculated through the strict iterated dominance, we are ready to state:

**Corollary 2**

$$x_{AD} = \frac{1}{n} \sum_{k=1}^{n} \pi_k^H = x_{GP} \neq x_{RD}$$  (20)

*Proof.* It suffices to demonstrate that $\frac{1}{n} \sum_{k=1}^{n} \pi_k^H$ is not a fixed point of $B(\cdot)$ mapping defined in eqn (19), i.e. $B(\frac{1}{n} \sum_{k=1}^{n} \pi_k^H) \neq \frac{1}{n} \sum_{k=1}^{n} \pi_k^H$ for $n \geq 3$. All the remaining proofs are nothing but a duplicate of the proof of proposition 1 and 2 and the main theorem, which is available from the author upon request.  ∎

## 7 Experimental Implications

A brief survey of Van Huyck et al. [1990; 1991] experimental results is offered. Each treatment typically lasts ten stages but the number of stages was not announced in advance in some experiments. A summary statistic of subjects' strategy choices was publicly announced after each stage. At the end of each experiment, subjects were paid the sum of their payoffs in the games they played. In each of the games, each player $i$ chooses a pure strategy, denoted $e_i$ and called effort, from the set $\{1, ..., 7\}$. In each stage, each player's payoff was determined by his own effort and a simple summary statistics of those of the players in his group. This statistic was either minimum or median of group effort choices. The parameter values were given for these normal forms[9] to be of coordination games with seven strict Pareto ranked

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[9]Van Huyck et al.1990 article for minimum treatments and 1991 research for median ones contain parameter values actually used in the experiments and the resulting normal forms.

symmetric pure strategy Nash equilibria. In every game, the payoff dominance selects all players' choosing the highest effort, i.e. **7**, irrespective of the number of subjects in a group. Finally, a large group consists of 14 to 16 players whilst a small of only two persons.

Despite payoff dominance, in large group minimum treatments subjects initially choose widely dispersed efforts and then rapidly approached the lowest effort, $e = 1$ : up to 84% of the subjects reached that effort within a few stages. In one treatment in which the parameters were adjusted so as for the highest effort $e = 7$ to be weakly dominant, approximately 96 percentage reached that effort by the fifth stage. This result may justify our maintained assumption that no strictly dominated strategy will be played at all.[10] In small group experiments, subjects' initial choices varied substantially and then drifted over time with no clearly discernable trend. By contrast, subjects in every median treatment converged completely and promptly to the Nash equilibrium determined by the "historical accident" of their initial stage median, despite considerable variation in the initial median across treatments. In brevity, it exhibits a strong history dependency. Finally, in large group median experiments with pure coordination game, players move swiftly to the Pareto best equilibrium action. This last observation can be at least partially explained by our corollary 1 and the fact that subjets were allowed to switch their choices every period.

Our simple model captures many salient features that were reported above. To see this, we consider a stag hunt game as follows:

$$\pi_k^H = \begin{cases} 1 & \text{if } \kappa \le k \le n \\ 0 & \text{otherwise,} \end{cases}$$

where $\kappa$ denotes the minimum number of players necessary for a successful stag hunt. Note that the minimum rule specifies $\kappa = n$ and that the median vote does $\kappa = \frac{n+1}{2}$. Plugging into eqn (19) gives rise to

$$u = \sum_{k=\kappa}^{n} \alpha_k \quad \text{and} \quad \ell = \sum_{k=\kappa}^{n} \beta_k,$$

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10No conflict arises with Cooper et al. [1990] experimental evidence, which just asserts that any addition or deletion of dominated strategies may affect the equilibrium actually selected.

with $\alpha_k$ and $\beta_k$ as defined in eqn (5). Remember from lemma 4 that the steady state could be **H** and **L** regardless of the initial state, respectively, according as $x < \ell$ and $x > u$. Certainly there might exist an equilibrium path converging to, say, **H** when $x > u$, if the initial population fraction of stag hunters is very high. However, we execute the numerical analysis as if the globally attractive state was globally stable. This is silly but can be tolerated reflecting the fact that the two regions of global attraction roughly offset each other. If $x \in [\ell, u]$, exactly which equilibrium will be obtained in the long run hinges on $y_0$, the historical accident of initial states. For the sake of calculation, we impose the monotonicity requirement, that is, only the paths monotonically converging either to **H** and **L** will be taken into account. We rule out any cyclical path. Deterministic nature of the present dynamic together with monotonicity implies the existence of a unique critical value of $x$, below which the path converges to **H**, and vice-versa.

We assume throughout that $x$ is uniformly distributed over $[0, 1]$. Two remarks are in order. First, the strategic uncertainty as has been understood should imply the distribution over the initial $y_0$ with $x$ being fixed. It causes analytically little problem to consider $y_0$ as deterministic and instead $x$ as uncertain. Another justification might be the uncertainty on the part of the experimenter about subjects' subjective evaluations of the fixed monetary compensation $x$. Second, relaxing reasonably the uniformity of the $x$-distribution here only seems to make our result stronger. As in Van Huyck et al.'s experiments, we let $n = 2$ for a small group and $n = 15$ for a large group.

Let $y(x)$ denote the inverse function of

$$\sum_{k=\kappa}^{n} \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} = x,$$

where the left (resp. right) hand side is the expected payoff from H (resp. L). From deterministic nature of the present model, it is clear that a player should choose action H (resp. L) if $y_0 >$ (resp. $<$)$y(x)$ in the intermediate history dependency region, given an opportunity to switch. The probability that the steady state be the

Pareto inferior Nash **L** will be at least approximately:

$$\Pr(\mathbf{L} \text{ is a steady state}) = \Pr(x \geq u) + \Pr(\ell < x < u, y_0 < y(x))$$
$$= (1 - u) + \int_\ell^u y(x)dx. \tag{21}$$

In small group treatment, the threshold value will be

$$u = \alpha_2 = \frac{1 + \delta}{2 + \delta} \quad \text{and} \quad \ell = \beta_2 = \frac{1}{2 + \delta}.$$

The probability that the steady state is **L** would be $(1 - \frac{1+\delta}{2+\delta}) + \int_\ell^u xdx = 0.5$ regardless of $\delta$. Under a large group minimum rule, we have

$$u = \alpha_{15} = \frac{1 + \delta}{15 + \delta} \quad \text{and} \quad \ell = \beta_{15} = \frac{14!}{\prod_{j=2}^{15}(j + \delta)},$$

thereby the probability that **L** obtains in the long run would be expressed as:

$$(1 - u) + \int_\ell^u x^{\frac{1}{14}}dx = \frac{14}{15 + \delta} + \frac{14}{15}(u^{\frac{15}{14}} - \ell^{\frac{15}{14}}).$$

Table 2 provides several simulations according to varying parameters.[11] The range of $x$ in which the Pareto inferior equilibrium **L** could be selected irrespective of the initial states is very broad, unless subjects are extremely impatient. On the other hand, with a big $\delta$ value, the portion of which both strict equilibria are absorbing is large. But even in such a situation, the basin of attraction with respect to initial strategic uncertainty is much larger for **L** than that for **H** under maintained assumption of the path monotonicty. These are reflected on the fact that the steady state is likely to settle down on the inferior equilibrium **L** with probability of at least 93.3 percent and up to 97 percent.

Table 3 analogously analyzes the large group median treatments. The probability that the superior Nash equilibrium **H** will be selected as the long run state is shown to be stable around 54 percent. For each $\delta$ given, a relatively wide range between $\ell$ and $u$ indicates a strong dependence on the initial state, or put differently, "historical accident." For instance, with $\delta = 1$ and large group, the history dependence region [.008, .125] of a minimum rule is in sharp contrast to [.300, .767] of median vote. We should mention that our results are fairly robust to the somewhat arbitrary parameter $\delta$ size.

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11All the simulations were carried out using *Mathematica.*

# 8 Concluding Remarks

Consider a "weakest link" model where people have to decide whether to contribute to a public good, non-contributors are excluded from consuming the public good, contributions are not refunded, and if the public good is provided only when enough people contribute. In contrast, the best shot refers to the case where only one's provision or success is enough for all, such as rats trying to bell the cat. Harrison and Hirshleifer [1989] convincingly argues that the 'free-rider' problem would be less (more) serious, thus cooperation would be more (less) likely to obtain, in the Weakest Link (Best Shot, respectively) model. Reflecting the fact that the Weakest Link is strategically equivalent to the stag hunt game under the minimum rule, their insight and the basic theme here seemingly contradict each other. This is not the case. Take the example of military units defending segments of the front against an enemy offensive. If all other units are successfully defending their own segments and if this fact is common knowledge then it certainly would be in my interest to defend my own. However, once even a single segment is broken through, running away will be everyone else's best response. How does one know the others are doing well ? As an obvious guess, it seems likely that some means of signalling, such as cheap talk and sequential move structure, could enhance the possibility of cooperation. On the contrary, the actual failure of or little doubt about the perfect defense will make the good equilibrium collapse.[12] We view this as an underlying reason for HH's experimental outcomes, in which subjects show a substantial cooperation with the sequential protocol while little clearcut evidence on cooperation or behavioral pattern is perceived with the sealed bid protocol. Related to experimental results as of Harrison and Hirshleifer and Cooper et al. [1992], the other direction of research will be to introduce a cheap talk argument, thus to see whether the possibility of cooperation could be enhanced through a costless preplay communication with more than two players.

The present paper of course has shortcomings, especially its critical dependence on a somewhat arbitrary parameter $\delta$, the effective discount rate or friction. Uniformity

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[12]It is a *contagious* equilibrium in Kandori [1992] sense.

of the distribution of random noise variables looks also restrictive, although a scrutiny of the proofs in Carlsson and Van Damme [1990] might suggest a relaxation of this assumption. Our conjecture is as follows: under a general distribution with compact support and given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ satisfying $\lim_{\varepsilon \to 0} \eta = 0$ such that player $i$ optimally chooses action H (resp. L) if his private signal $\theta_i > \theta_{GP} + \eta$ (resp. $< \theta_{GP} - \eta$). Moreover, this almost dominance solvability in the limit is reduced to exact dominance solvability under uniform distribution.

It needs to be generalized to encompass multi actions and/or asymmetric payoffs. Technical difficuties arise from a huge amount of case distinctions and calculations. With $m$ actions, we have to consider $2^m - 1$ number of $\Omega_{(.)}$ sets, where only $a$ is globally attractive if $\Pi \in \Omega_a$ and only $a_1$, $a_2$, ..., $a_i$ are absorbing if $\Pi \in \Omega_{a_1 a_2 \ldots a_i}$ for $2 \leq i \leq m$. Payoff asymmety in $n$ person $m$ action game requires considering $n^m$ dimensional space. While there is, at least in principle, no reason why adjustment dynamics or global perturbation fails to be well-defined even in the general setting, it is known that risk dominance may well be troublesome because of intransitivities. In view of our corollary 1, this line of research seems to include as a special example the former part of Kandori and Rob [1992], which abandons risk dominance even in a two person $m$ action coordination game.

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| $n$ | 2 | 3 | | 15 | | 99 | |
|---|---|---|---|---|---|---|---|
| rule | | mim | med | min | med | mim | med |
| Limiting Adjustment Dynamics | 50 | 33.3 | 66.7 | 6.7 | 53.3 | 1.0 | 50.5 |
| Global Perturbation | 50 | 33.3 | 66.7 | 6.7 | 53.3 | 1.0 | 50.5 |
| Risk Dominance | 50 | 38.2 | 61.8 | 13.4 | 52.6 | 3.4 | 50.1 |

Table 1: Cutoff $x$ in Stag Hunt Game.

|   | H | L | H | L |
|---|---|---|---|---|
| H | 3,3,3 | 1,0,1 | 1,1,0 | -1,0,0 |
| L | 0,1,1 | 0,0,-1 | 0,-1,0 | 2,2,2 |

|  H  |  | L |  |
|---|---|---|---|

Figure 1. ▬

| $\delta$ | 0 | 0.1 | 0.5 | 1 | 10 | $10^2$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $u$ | .067 | .073 | .097 | .125 | .440 | .878 | 1.000 |
| $\ell$ | .067 | .053 | .020 | .008 | .001 | .000 | .000 |
| **H** | .067 | .056 | .035 | .030 | .053 | .066 | .067 |
| **L** | .933 | .944 | .965 | .970 | .947 | .934 | .933 |

Table 2: Large Group Minimum Rule

| $\delta$ | 0 | 0.1 | 0.5 | 1 | 10 | $10^2$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $u$ | .533 | .566 | .673 | .767 | .997 | 1.000 | 1.000 |
| $\ell$ | .533 | .502 | .398 | .300 | .183 | .000 | .000 |
| **H** | .533 | .536 | .541 | .544 | .534 | .533 | .533 |
| **L** | .467 | .464 | .459 | .456 | .466 | .467 | .467 |

Table 3: Large Group Median Vote