A Characterization of Equilibria in Common Value Second–Price and Open–Exit Auctions

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Abstract

A sufficient condition for uniqueness of Nash equilibrium in second–price, common value auctions is identified. We establish that there exists a continuum of equilibria in open–exit, common value auctions. For the case of two bidders, we provide plausible sufficient conditions under which the expected seller revenue is maximized at the symmetric equilibrium. An example in which the auctioneer's revenue at the symmetric equilibrium in a second–price auction is strictly greater than the revenue at many asymmetric equilibria in the corresponding open–exit auction is provided.

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1 Introduction

In the past decade there has been a vast increase in the literature on the theory of auctions. The interested reader is referred to two comprehensive surveys — McAfee and McMillan (1987), and Wilson (1987). A question that is often addressed in this literature is which of several common auction mechanisms yield the greatest revenue to the auctioneer under different informational assumptions.[1] In an important paper, Milgrom and Weber (1982) ranked the auctioneer's revenue at the symmetric equilibrium of three common auctions under fairly general conditions. They established that expected revenue in an open ascending bid auction tends to be higher if a seller is able to impose the constraint that exit from bidding is irreversible. That is, once a bidder becomes inactive, he is not allowed to reenter the auction.

Since, in the common "English" auction, buyers take pains to conceal the fact that they are still interested parties, the implication is that a seller would be better off if the bidding process were to proceed as follows. All potential bidders are asked to stand. Bids are then taken in the usual manner. However, when a bidder wishes to exit, he does so by sitting down again. Further bids are accepted only from those still standing.

Despite the apparent similarity of the two auctions, we shall argue below that there is an important difference which casts doubt upon the superiority of what we call here the "open–exit" auction over the common English auction.

Milgrom and Weber established the existence of symmetric equilibria for the two auctions and then contrasted expected revenue for these auctions. They did not, however, ask whether there might be other equilibria as well. In this paper we show that, in general, there is a very large family of asymmetric equilibria for the open–exit auction. On the other hand, by imposing a moderate restriction upon beliefs, we are able to derive a strong uniqueness theorem for the common English auction.

Such an inquiry is interesting for several, not unrelated, reasons. First, it helps delineate the set of equilibria and identify conditions under which there exists a unique equilibrium.

Second, in comparing the auctioneer's revenues from common, symmetric, auctions the auctioneer's revenues are computed at the symmetric equilibrium. Although a focus on the symmetric equilibrium in symmetric auctions is an appropriate starting point, we think that it is also important to investigate other equilibria. Third, as argued in Bikhchandani (1988), in the class of equilibria in increasing and continuous strategies, the symmetric equilibrium in an open–exit or English, common value, auction with two bidders is unstable under small departures from symmetry of the players. Therefore, a first step towards investigating the stability of equilibria in such auctions, when there are more than two bidders, is to characterize the equilibrium set.

If, as we have argued, the only information available to the other bidders in the English auction is that there is at least one other active bidder, this auction is equivalent to a sealed bid auction in which the winning bidder pays the second highest of the bids received. It is the equilibria of the sealed "second–price" auction which we formally analyze and contrast with those of the open–exit auction.

The other common auction is, of course, the sealed "first–price" auction. In a related paper, Maskin and Riley (1986) prove that the symmetric equilibrium in first–price and second–price auctions with private, independent, values, is unique.[2] They also establish that in first–price auctions with affiliated values, the symmetric equilibrium is unique when there are two bidders. As shown in Milgrom (1981), there exists a continuum of asymmetric equilibria in common value, second–price or open–exit auctions with two bidders.[3] We find that this result generalizes to open–exit auctions with more than two bidders. In the second–price auction there can also be a continuum of equilibria when there are more than two equilibria. We present an example illustrating this possibility and then establish conditions under which there are no asymmetric equilibria in the class of equilibria with increasing and continuous strategies. Under these same conditions we can also rule out equilibria with step–function strategies. However, it remains to determine whether there exist asymmetric equilibria in discontinuous, non–decreasing strategies in second–price auctions with more than two bidders.

Given the multiplicity of equilibria in the open–exit auction, the question arises as to whether the seller's expected revenue is higher or lower if the bidding is asymmetric. Our initial intuition about this went as follows. If one bidder is aggressive and stays in the auction until the asking price is high relative to his signal, opposing bidders will be more passive in response. The aggressive bidder will then tend to win big, as his opponents drop out relatively quickly. Given this behavior, the expected selling price will tend to be lower than under symmetric bidding. Surprisingly, however, we are able to show that this intuition is incomplete. For some examples it is true that expected seller revenue is highest under symmetric bidding. On the other hand, we also present an example in which expected seller revenue is minimized under symmetric bidding. We do, however, provide plausible sufficient conditions which guarantee that the seller's expected revenue is maximized at the symmetric equilibrium. An example in which the auctioneer's revenue at the symmetric equilibrium in a second–price auction is strictly greater than the revenue at many asymmetric equilibria in the corresponding open–exit auction is provided. Thus Milgrom and Weber's revenue ranking between second–price and open–exit auctions at the symmetric equilibrium cannot be extended to other equilibria.

In the next section we present the model. In section 3, we investigate equilibria of the second–price auction with more than two bidders. We investigate equilibria in open–exit auctions in section 4. In section 5, the auctioneer's revenues at the symmetric and asymmetric equilibria are compared. Section 6 contains concluding remarks.

2 The Model

Consider a common value auction with n symmetric, risk-neutral bidders, indexed $i \in \mathcal{N}$, where $\mathcal{N} \equiv \{1, 2, ...n\}$. The true value of the object being auctioned is the same for all bidders and is an unknown random variable, $\tilde{V}$. Each bidder i has a common prior on $\tilde{V}$, and observes a private signal, $\tilde{X}_i$, about the true value. The signals are identically distributed.

Let $f(v, \mathbf{x})$ denote the joint density function of $\tilde{V}$ and the vector of signals $\tilde{X} \equiv$

$(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)$. It is assumed that f is symmetric in its last n arguments. Let $[\underline{V}, \overline{V}] \times [\underline{X}, \overline{X}]^n$ be the support of f. Further, it is assumed that all the random variables in this model are affiliated. That is, for all $\mathbf{x}$, $\mathbf{x}' \in [\underline{X}, \overline{X}]^n$, and for all v, $v' \in [\underline{V}, \overline{V}]$,

$$f((v,\mathbf{x}) \vee (v',\mathbf{x}'))\, f((v,\mathbf{x}) \wedge (v',\mathbf{x}')) \geq f(v,\mathbf{x}) f(v',\mathbf{x}') \tag{2.1}$$

where $\vee$ denotes the componentwise maximum, and $\wedge$ denotes the componentwise minimum. Affiliation implies that $\mathbf{E}[\tilde{V} | c_i \leq \tilde{X}_i \leq d_i,\ i \leq n]$ is a non–decreasing function of c_i, d_i. See Milgrom and Weber (1982) for some implications of affiliation. Where it simplifies proofs we shall assume that affiliation is strict. That is (2.1) holds everywhere with strict inequality. Then $\mathbf{E}[\tilde{V} | c_i \leq \tilde{X}_i \leq d_i,\ i \leq n]$ is strictly increasing in c_i, d_i over $[\underline{X}, \overline{X})$. It is also assumed that $\mathbf{E}[\tilde{V} | c_i \leq \tilde{X}_i \leq d_i,\ i \leq n]$ is continuous in c_i, d_i. A sufficient condition for this is that the joint density of the random variables, $f(v, \mathbf{x})$, is continuous in the last n arguments.

One much discussed example of a common value auction is bidding by oil companies for off–shore oil–field tracts. In this example, each bidder's signal is interpreted as the outcome of seismic testing, sample boring, etc. Given relatively small and homogenous tracts, it is natural to assume that signals are strictly affiliated.

However, more generally, in the bidding for extraction rights, suppose a tract is large and that the location and scale of the mineral deposits being sought are very heterogenous. In this case it becomes a reasonable first approximation to assume that each potential bidder samples from different parts of the tract and that estimates are independent. Under these alternative assumptions, each $\tilde{X}_i$ is an independent draw from some c.d.f. $F(x)$ and the expected value of the entire tract, conditional on a vector $(x_1, x_2, ..., x_n)$ of signal realizations is

$$\mathbf{E}[\tilde{V} | \tilde{X}_1 = x_1, \tilde{X}_2 = x_2, ..., \tilde{X}_n = x_n] = \alpha + \beta \sum_{j=1}^{n} \tilde{X}_j.$$

This may be seen as a limiting case of the common value model. Given its analytical simplicity, it is especially useful for illustrative purposes.

3 Second–Price Auctions

In a second–price, sealed bid auction, the bidders submit sealed bids, after observing their private signals. The highest bidder wins the auction and pays a price equal to the second highest bid. The other bidders pay nothing. All this is common knowledge.

As shown in Milgrom (1981), when there are two bidders there exist many asymmetric Nash equilibria in common value, second–price auctions. The easiest way to see this is to define $b_i(x)$, $i = 1, 2$ to be the equilibrium bid functions and $\phi_i(b) \equiv b_i^{-1}(b)$ to be the equilibrium inverse bid functions.[4] Also define $V(\tilde{X}_1, \tilde{X}_2)$ to be the expected value of the object given signals $\tilde{X}_1$ and $\tilde{X}_2$, that is

$$V(x_1, x_2) \equiv \mathbf{E}[\tilde{V} | \tilde{X}_1 = x_1, \tilde{X}_2 = x_2].$$

If bidder 1 bids b he wins as long as bidder $2's$ bid b_2 is less than b. His expected payoff, given $\tilde{X}_1 = x_1$, is therefore

$$U_1 = \int_0^b (V(x_1, \phi_2(b_2)) - b_2) G_2'(b_2 | x_1) db_2$$

where $G_2(b_2|x_1)$ is bidder $2's$ equilibrium bid distribution, given $\tilde{X}_1 = x_1$. Bidder $1's$ best reply is therefore to choose a bid $b_1(x_1)$ to satisfy

$$\frac{\partial U_1}{\partial b}\Big|_{b=b_1(x_1)} = (V(x_1, \phi_2(b)) - b) G_2'(b_2|x_1)\Big|_{b=b_1(x_1)} = 0.$$

Since $\phi_i(b_1(x_1)) \equiv b_1^{-1}(b(x_1)) = x_1$, this equilibrium condition can be written more simply as

$$V(\phi_1(b), \phi_2(b)) - b = 0, \qquad \forall b. \tag{3.0.1}$$

By symmetry, the analysis of buyer $2's$ best reply yields the same condition. Clearly there is a continuum of equilibrium inverse bid functions satisfying (3.0.1). To be precise, for any increasing functions $\phi_1(b)$, $\phi_2(b)$ we can implicitly define an increasing function $h(\cdot)$ such that

$$\phi_2(b) \equiv h(\phi_1(b)).$$

Then from (3.0.1) we can invert $\phi_1(b)$ and $\phi_2(b)$ to obtain

$$\begin{aligned} b_1(x) &= V(x, h(x)) \\ b_2(x) &= V(h^{-1}(x), x) \end{aligned}$$

As long as $h(\cdot)$ is increasing and surjective, these are equilibrium bid functions.

We now turn to the case of more than two bidders. First we illustrate, by example, the possibility that there can again be a continuum of equilibria. We then establish that the example is somewhat special and present conditions which rule out asymmetric, increasing, and continuous bidding strategies in equilibrium. This condition also rules out equilibria in which the players' strategies are step–functions. However, whether there exists an asymmetric equilibrium in discontinuous, non–decreasing strategies remains an open question.

3.1 Asymmetric Equilibria with n Bidders

Suppose each of n bidders has a common prior on the true value of the object, given by the marginal density function

$$f_{\tilde{V}}(v) = \frac{1}{x_0} \exp^{\frac{-v}{x_0}}, \quad v \in [0, \infty].$$

Bidder i observes a signal $\tilde{X}_i$ which is a random draw from the conditional density function

$$f_{\tilde{X}_i|\tilde{V}}(x_i|v) = \frac{v}{x_i^2} \exp^{\frac{-v}{x_i}}, \quad x_i \in [0, \infty].$$

The corresponding conditional distribution function is

$$F_{\tilde{X}_i|\tilde{V}}(x_i|v) = \exp^{\frac{-v}{x_i}}, \quad x_i \in [0, \infty].$$

Further, it is assumed that the signals are conditionally independent, given the true value. Hence the joint density function is

$$\begin{aligned} f(v, \mathbf{x}) &= f_{\tilde{V}}(v) \textstyle\prod_{i=1}^{n} f_{\tilde{X}_i|\tilde{V}}(x_i|v) \\ &= x_0 v^n \exp^{-vA_n(\mathbf{x})} \textstyle\prod_{i=0}^{n} \frac{1}{x_i^2}, \end{aligned}$$

where $A_n(\mathbf{x}) \equiv \sum_{i=0}^{n} \frac{1}{x_i}$. It is readily confirmed that the $n+1$ random variables $\tilde{V}$, $\tilde{X}_1$, $\tilde{X}_2, ..., \tilde{X}_n$ are affiliated. The posterior density of $\tilde{V}$, given signals $\tilde{X}_1$, $\tilde{X}_2, ..., \tilde{X}_m$ and the knowledge that the remaining signals are no greater than $x_{m+1}, ..., x_n$ respectively is

$$f_{\tilde{V}|\tilde{X}}(v|\tilde{X}_i = x_i, \quad i \leq m, \quad \tilde{X}_j \leq x_j, \; j > m)$$

$$= \frac{\int_0^{x_n} \cdots \int_0^{x_{m+1}} f(v,\mathbf{x}) \; dx_{m+1} ... dx_n}{\int_{v=0}^{\infty} \int_0^{x_n} \cdots \int_0^{x_{m+1}} f(v,\mathbf{x}) \; dx_{m+1} ... dx_n dv}$$

$$= \frac{\int_0^{x_n} \cdots \int_0^{x_{m+1}} x_0 v^n \exp^{-vA_n(\mathbf{x})} \prod_{i=0}^{n} \frac{1}{x_i^2} \; dx_{m+1} ... dx_n}{\int_{v=0}^{\infty} \int_0^{x_n} \cdots \int_0^{x_{m+1}} x_0 v^n \exp^{-vA_n(\mathbf{x})} \prod_{i=0}^{n} \frac{1}{x_i^2} \; dx_{m+1} ... dx_n dv}$$

$$= \frac{v^m \exp^{-vA_n(\mathbf{x})}}{\int_{v=0}^{\infty} v^m \exp^{-vA_n(\mathbf{x})} \, dv}.$$

Next define $V_m(x_1, x_2, ..., x_m; x_{m+1}, ..., x_n)$ to be the expected value of $\tilde{V}$ given the knowledge that the first m signals are $x_1, x_2, ..., x_m$ and that the remaining signals are no greater than $x_{m+1}, ..., x_n$ respectively. We have

$$V_m(x_1, x_2, ..., x_m; x_{m+1}, ..., x_n) = \int_{v=0}^{\infty} v f_{\tilde{V}|\tilde{X}}(v|\tilde{X}_i = x_i, i \leq m, \; \tilde{X}_j \leq x_j, \; j > m) dv$$

$$= \frac{\int_{v=0}^{\infty} v^{m+1} \exp^{-vA_n(\mathbf{x})} \, dv}{\int_{v=0}^{\infty} v^m \exp^{-vA_n(\mathbf{x})} \, dv}$$

$$= \frac{m+1}{n+1} \left[\left(\frac{1}{n+1} \sum_{i=0}^{n} \frac{1}{x_i} \right)^{-1} \right] \tag{3.1.1}$$

Note the bracketed expression is the harmonic mean of the signal levels $x_1, x_2, ..., x_n$ and the prior expected value of $\tilde{V}$. Therefore the conditional expectation V_m is just a fraction of the harmonic mean.

For the two buyer case, the inverse equilibrium bid functions must satisfy

$$V_2(\phi_1(b), \phi_2(b)) = b$$

that is

$$3\left(\frac{1}{x_0}+\frac{1}{\phi_1(b)}+\frac{1}{\phi_2(b)}\right)^{-1}=b.$$

We now show that, with n buyers, the equilibrium inverse bid functions must satisfy

$$V_2(\phi_1(b),\phi_2(b);\phi_3(b)...,\phi_n(b))=b$$

that is

$$3(\frac{1}{x_0}+\frac{1}{\phi_1(b)}+...+\frac{1}{\phi_n(b)})^{-1}=b. \tag{3.1.2}$$

To see this, suppose that all buyers adopt strategies $(\phi_1,\phi_2,...,\phi_n)$ satisfying (3.1.2). Suppose buyer 1 were to find out that the maximum of his opponents' bids was b and that this is a bid made by buyer i. Buyer $1's$ expected return is then

$$V_2(x_1,\phi_i(b);\phi_2(b),...,\phi_{i-1}(b),\phi_{i+1}(b),...,\phi_n(b))-b=3\left(\frac{1}{x_0}+\frac{1}{x_1}-\frac{1}{\phi_1(b)}+\sum_{i=1}^{n}\frac{1}{\phi_i(b)}\right)^{-1}-b$$

From (3.1.2) this is positive if and only if $\phi_1(b)>x_1$. Therefore, even with this information about his opponents, buyer $1's$ best reply is to bid b satisfying

$$\phi_1(b)=x_1.$$

It follows that $\phi_1(b)$ is his equilibrium bid function as claimed.

As before, if we implicitly define increasing functions $h_{ji}(\cdot)$ to satisfy[5]

$$\phi_j(b)\equiv h_{ji}(\phi_i(b)),\quad i,j=1,2,...,n$$

and, for all i,j, $h_{ji}(\cdot)$ as a mapping from $[\underline{X},\overline{X}]$ onto itself, then buyer $i's$ equilibrium bid function can be written as

$$b_i(x_i)=3\left(\frac{1}{x_0}+\sum_{j=1}^{n}\frac{1}{h_{ji}(x_i)}\right)^{-1}$$

From (3.1.1) we see that this example satisfies

$$V_2(x_1,x_2,;x_3,...,x_n)=V_2(x_1,x_i,;x_3,..x_{i-1},x_2,x_{i+1},...,x_n),\quad \forall x_1,x_2...,x_n \tag{3.1.3}$$

Consequently, in any of the asymmetric equilibria in this example, if, after learning the price he has to pay, the winner also learns the identity of the second highest bidder, his valuation of the object does not change. This seems to be a necessary condition for an equilibrium in a second–price auction, and is clearly satisfied at the symmetric equilibrium. When there are only two bidders, this condition is trivially satisfied, and hence we have multiple equilibria.

We prove that the (3.1.3) is sufficient to imply the existence of multiple equilibria. First, in the next lemma (proved in Appendix I) we obtain a sufficent condition for (3.1.3) when the signals are conditionally independent, as in the example. It is shown that (3.1.3) is satisfied if $\frac{f_{\tilde{X}_1|\tilde{V}}(x_1|v)}{F_{\tilde{X}_1|\tilde{V}}(x_1|v)}$ is multiplicatively separable in x_1 and v. Clearly, this will not usually be the case.

Lemma 1: *Suppose that the signals $\tilde{X}_i$ are conditionally independent given $\tilde{V}$. That is the joint density function of the random variables is*

$$f(v,\mathbf{x}) = f_{\tilde{V}}(v)\prod_{i=1}^{n} f_{\tilde{X}_i|\tilde{V}}(x_i|v).$$

Let $F_{\tilde{X}_i|\tilde{V}}(x|v)$ be the conditional distribution function of $\tilde{X}_i$ given $\tilde{V}$. If there exist functions $K_1 : [\underline{X},\overline{X}] \rightarrow R$ and $K_2 : [\underline{V},\overline{V}] \rightarrow R$ such that

$$\begin{aligned}\frac{\partial \ln F_{\tilde{X}_i|\tilde{V}}(x|v)}{\partial x} &\equiv \frac{f_{\tilde{X}_i|\tilde{V}}(x|v)}{F_{\tilde{X}_i|\tilde{V}}(x|v)} \\ &= K_1(x)K_2(v), \quad \forall x, v\end{aligned}$$

then (3.1.3) is satisfied.

The example satisfies the conditions of Lemma 1 with $K_1(x) = \frac{1}{x^2}$ and $K_2(v) = v$. More generally, we have the following lemma.

Lemma 2: *If (3.1.3) is satisfied then there exists a continuum of equilibria in second–price auctions.*

Proof: Let $\phi_1(b), \phi_2(b), ..., \phi_n(b)$ be inverse bidding functions satisfying

$$V_2(\phi_1(b), \phi_2(b); \phi_3(b)..., \phi_n(b)) = b, \tag{3.1.4}$$

We show that $\phi_1(b), \phi_2(b), ..., \phi_n(b)$ are equilibrium inverse bidding functions.

Suppose that all buyers adopt strategies $(\phi_1, \phi_2, ..., \phi_n)$ satisfying (3.1.4). Suppose buyer 1 were to find out that the maximum of his opponents' bids was b and that this is a bid made by buyer i. Buyer $1's$ expected return is then

$$V_2(x_1, \phi_i(b); \phi_2(b), ..., \phi_{i-1}(b), \phi_{i+1}(b), ..., \phi_n(b)) - b$$

which, from (3.1.3), is equal to

$$V_2(x_1, \phi_2(b); \phi_3(b), ..., \phi_{i-1}(b), \phi_i(b), \phi_{i+1}(b), ..., \phi_n(b)) - b$$

Since by affiliation V_2 is increasing in all its arguments, it follows from (3.1.4) that the buyer's expected return is positive if and only if $\phi_1(b) > x_1$. Therefore, even with this information about his opponents, buyer $1's$ best reply is to bid b satisfying

$$\phi_1(b) = x_1.$$

Thus $\phi_1(b)$ is his equilibrium bid function. ∎

3.2 A Sufficient Condition for Uniqueness

While Lemma 2 establishes the possibility of multiple equilibria, (3.1.3) has a feature which seems unlikely to be satisfied in many applications. We believe that in most cases the two conditional expectations in (3.1.3) will not be equal. In particular, (3.1.3) implies that for $x >> \epsilon > 0$

$$\mathbf{E}[\tilde{V} | \tilde{X}_1 = x, \ \tilde{X}_2 \leq \underline{X} + \epsilon] = \mathbf{E}[\tilde{V} | \tilde{X}_1 \leq x, \ \tilde{X}_2 = \underline{X} + \epsilon].$$

Plausibly, especially for small ϵ, one would expect the left hand expression to be strictly greater. We now consider distributions which satisfy

Assumption 1: *If $\alpha > \beta$ then*

$$\mathbf{E}[\tilde{V}|\tilde{X}_1 = \alpha,\ \tilde{X}_2 \leq \beta,\ c_i \leq \tilde{X}_i \leq d_i,\ i \geq 3] \ > \ \mathbf{E}[\tilde{V}|\tilde{X}_1 \leq \alpha,\ \ \tilde{X}_2 = \beta,\ c_i \leq \tilde{X}_i \leq d_i,\ i \geq 3],\ \ \forall c_i, d_i.$$

While it would be useful to provide conditions on the underlying distributions which are sufficient for Assumption 1, this is not a straightforward matter. For the borderline case discussed in section 2 where the signals are independent, we provide such conditions.

When $\tilde{V} = \tilde{X}_1 + \tilde{X}_2 + ... + \tilde{X}_n$, and each $\tilde{X}_i$ is independently distributed with c.d.f $F(x)$, Assumption 1 is equivalent to

$$\alpha + \mathbf{E}[\tilde{X}_2|\tilde{X}_2 \leq \beta] > \beta + \mathbf{E}[\tilde{X}_1|\tilde{X}_1 \leq \alpha], \quad \forall \alpha > \beta$$

Define

$$D(\alpha) = \alpha - \mathbf{E}[\tilde{X}_1|\tilde{X}_1 \leq \alpha]$$

For this example, Assumption 1 is equivalent to requiring that $D(\alpha)$ be everywhere increasing. The following lemma (proved in Appendix I) provides sufficient conditions for the monotonicity of $D(\alpha)$. These conditions can be used to confirm that $D'(\cdot)$ is positive under a wide range of alternative assumptions. For example, the first condition holds if $F(x) = (x - \underline{X})^c$ and the second condition holds if $\tilde{X}$ is normally distributed.

Lemma 3: *Suppose $\tilde{X}$ has support $[\underline{X}, \overline{X}]$ and c.d.f. $F(\cdot)$, and density function $f(\cdot) \in C^1$. If (i) $\ln F(x)$ is strictly concave*

or

(ii) $\ln f(x)$ is strictly concave

then $D(x) \equiv x - \mathbf{E}[\tilde{X}|\tilde{X} \leq x]$ is strictly increasing on $[\underline{X}, \overline{X}]$.

Milgrom (1981), and Milgrom and Weber (1982) have established that a symmetric equilibrium in the second–price auction is $b_i(x) = \mathbf{E}[\tilde{V}|\tilde{X}_i = x,\ \max_{j \neq i}\{\tilde{X}_j\} = x]$. In Proposition 1 we show that for $n \geq 3$, when Assumption 1 is satisfied there do not exist any other equilibria in continuous and increasing strategies.

Proposition 1: *If there are three or more bidders in a second–price auction, and Assumption 1 is satisfied, then the symmetric equilibrium is the unique equilibrium in increasing and continuous strategies.*

Proof: Define

$$N_{ij}(b,x) \equiv \mathbf{E}[\tilde{V}|\tilde{X}_i = x, \tilde{X}_j = \phi_j(b), \tilde{X}_k \leq \phi_k(b), k \neq i,j] - b.$$

$N_{ij}(b,x)$ is player i's expected profit if he wins the auction, $\tilde{X}_i = x$, and player j, the second highest bidder, submits a bid b. Since the player's strategies and the conditional expectation of $\tilde{V}$ are increasing and continuous, $N_{ij}(b,x)$ is continuous. Without further loss of generality we may assume that for some bid b

$$\phi_1(b) \geq \phi_2(b) \geq ... \geq \phi_n(b). \tag{3.2.1}$$

By Assumption 1, and given that signals are identically distributed, $N_{ij}(b,x) \geq N_{ik}(b,x)$ for all i, and all $j \leq k$. In particular

$$\begin{aligned} N_{12}(b,\phi_1(b)) \geq \quad & N_{13}(b,\phi_1(b)) \geq ... \geq \; N_{1n}(b,\phi_1(b)) = \\ & N_{n1}(b,\phi_n(b)) \geq N_{n2}(b,\phi_n(b)) \geq ... \geq N_{n,n-1}(b,\phi_n(b)). \end{aligned} \tag{3.2.2}$$

Let $p_{ij}(b,x)$ be the probability that j is the highest bidder among $\mathcal{N}\backslash\{i\}$, given that $\tilde{X}_i = x$ and that the highest bid among $\mathcal{N}\backslash\{i\}$ is b. If player 1 wins the auction at a price b and his signal is $\tilde{X}_1 = x$, his expected profit is

$$\Pi_1(b,x) \equiv \sum_{j=2}^{n} p_{1j}(b,x)N_{1j}(b,x).$$

This must be zero for $b = b_1(x)$, for if $\Pi_1(b,\phi_1(b)) > 0$, then player 1 is better off bidding a little higher than $b_1(x)$ when $\tilde{X}_1 = x$, and if $\Pi_1(b,\phi_1(b)) < 0$, he is better off bidding a little less. Therefore we have,

$$\sum_{j=2}^{n} p_{1j}(b,\phi_1(b))N_{1j}(b,\phi_1(b)) = 0. \tag{3.2.3}$$

Similarly, the optimality of $b_n(\cdot)$ implies

$$\sum_{j=1}^{n-1} p_{nj}(b,\phi_n(b))N_{nj}(b,\phi_n(b)) = 0. \tag{3.2.4}$$

From (3.2.2) $N_{1n}(b,\phi_1(b)) > 0$ implies $N_{1j}(\phi_1(b)) > 0$ for all j. But this contradicts (3.2.3), and so $N_{1n}(b,\phi_1(b)) \leq 0$. Again from (3.2.2), $N_{1n}(b,\phi_1(b)) = N_{n1}(b,\phi_n(b)) < 0$ implies $N_{nj}(b,\phi_n(b)) < 0$, $\forall j$. But this contradicts (3.2.4). It follows that $N_{1n}(b,\phi_1(b)) = 0$. Using the same argument iteratively, one can conclude that $N_{1,n-1}(b,\phi_1(b)) = N_{n-1,1}(b,\phi_{n-1}(b)) = 0$, and so $N_{ij}(b,\phi_i(b)) = 0$ for all $i,j,\ j \neq i$. In particular, $N_{12}(b,\phi_1(b)) = N_{n2}(b,\phi_n(b)) = 0$ implies

$$\mathbf{E}[\tilde{V}|\tilde{X}_1 = \phi_1(b),\ \tilde{X}_2 = \phi_2(b),\ \ \tilde{X}_j \leq \phi_j(b),\ j \neq 1,2] = \\ \mathbf{E}[\tilde{V}|\tilde{X}_n = \phi_n(b),\ \tilde{X}_2 = \phi_2(b),\ \tilde{X}_j \leq \phi_j(b),\ j \neq 2,n].$$

Given Assumption 1, and (3.2.1), it follows that

$$\phi_1(b) = \phi_2(b) = \ldots = \phi_n(b). \tag{3.2.5}$$

The only strategies which satisfy $N_{ij}(b,\phi_i(b)) = 0$ and (3.2.5) are $b_i(x) = \mathbf{E}[\tilde{V}|\tilde{X}_i = x,\ \max_{j\neq i}\{\tilde{X}_j\} = x]$. Thus the symmetric equilibrium is the unique equilibrium in increasing and continuous strategies. ∎

In Appendix II we generalize this result and show that when Assumption 1 is satisfied there do not exist any partially poolling equilibria, i.e., equilibria in step–function strategies. On the other hand, if (3.1.3) is satisfied there exists a continuum of partially pooling equilibria. Whether an equilibrium bid function must be strictly increasing and continuous when there are three or more bidders in a second–price auction and Assumption 1 is satisfied, that is, whether the symmetric equilibrium is the unique equilibrium, remains to be established.

4 Open–Exit Auctions

There are several variations of an open–exit auction. The variant examined here is the same as the one in Milgrom and Weber (1982). Starting with a very low price, the auctioneer raises the price continuously. The bidders indicate whether they are active at the current price by depressing a button. The current price and the identities of the active bidders are posted on an electronic display. Once a bidder withdraws, he cannot reenter the auction. The auction can end in one of two possible ways. If at any time there is only one active bidder, then this bidder is declared the winner and the auction ends. Else, if at any instant the remaining active bidders withdraw simultaneously, then the auction ends and one of the last active bidders is randomly chosen as the winner. The winner gets the object and pays the current price. The other bidders pay nothing. At each point, it is common knowledge whether a bidder is active or not, and if inactive, the price at which he dropped out is also common knowledge.

Each player's pure strategy specifies the price level up to which he will remain active, given his private signal, and given the number and identities of the bidders who have dropped out, and the price levels at which they quit. Thus bidder $i's$ strategy consists of a vector of $n-1$ functions, $\mathbf{b_i} = (b_{i,0}, b_{i,1}, b_{i,2}, ...b_{i,k}, ..., b_{i,n-2})$, where $b_{i,k} : [\underline{X}, \overline{X}] \times R^k \to R$, $k = 0, 1, 2, ...n-2$. When bidders $i_1, i_2, ..., i_k$ quit at price levels $p_1 \leq p_2 \leq ... \leq p_k$, and $\tilde{X}_i = x$, then bidder i will remain active till the price reaches $b_{i,k}(x; p_1, p_2, ..., p_k)$. If bidder i_{k+1} quits before the price reaches this level, then $b_{i,k+1}(x; p_1, p_2, ..., p_k, p_{k+1})$ specifies the new price level until which bidder i remains active.

A symmetric equilibrium for an open–exit auction has been obtained by Milgrom and Weber [1982]. In the next section this equilibrium is shown to be the unique symmetric equilibrium. In section 4.2, we show that there exists a continuum of asymmetric equilibria in open–exit auctions with $n \geq 3$ bidders. Thus, the existence of asymmetric equilibria in open–exit auctions with two bidders generalizes. Note that after $n-2$ bidders drop out in an open–exit auction, it is identical to an open–exit auction with two bidders.

4.1 The Symmetric Equilibrium

A symmetric Nash equilibrium in the open–exit auction has been obtained by Milgrom and Weber (1982). In this equilibrium, each player's strategy is $\mathbf{b}^* = (b_0^*, b_1^*, b_2^*, ...b_k^*, ..., b_{n-2}^*)$, where,

$$\begin{aligned} b_0^*(x) &= \mathbf{E}[\tilde{V}|\tilde{X}_j = x,\ \forall j], \\ b_k^*(x; p_1, p_2, ..., p_k) &= \mathbf{E}[\tilde{V}|\tilde{X}_j = x,\ \forall j \in \mathcal{N}\backslash\{i_1, i_2, ..., i_k\}, \\ &\qquad b_{k-1}^*(\tilde{X}_{i_k}; p_1, p_2, ..., p_{k-1}) = p_k, ..., b_0^*(\tilde{X}_{i_1}) = p_1], \\ &\qquad k = 1, 2, ..., n-2. \end{aligned} \tag{4.1.1}$$

As proved in the next lemma, this is the unique symmetric equilibrium.

Lemma 4: *The strategies specified in (4.1.1) constitute the unique symmetric perfect equilibrium in non–decreasing strategies.*

Proof: The proof is by induction on the number of players, n.

Suppose $n = 2$, and player 1 uses a strategy $b(x)$. It is shown that player $2's$ best response is $b(x)$ only if $b(x) = \mathbf{E}[\tilde{V}|\tilde{X}_1 = x, \tilde{X}_2 = x]$, $\forall x$. Suppose $b(x) < \mathbf{E}[\tilde{V}|\tilde{X}_1 = x,\ \tilde{X}_2 = x]$ for some x. Then if $\tilde{X}_2 = x$ and player 2 bids $b(x) + \epsilon$, ϵ small, positive, instead of $b(x)$, then since $b(x)$ is non–decreasing his expected payoff increases by at least $\mathbf{E}[\tilde{V}1_{\{b(x)<b(\tilde{X}_1)<b(x)+\epsilon\}}|\tilde{X}_2 = x] - b(x) - \epsilon$. Thus b(x) is not a best response. Similarly, if $b(x) > \mathbf{E}[\tilde{V}|\tilde{X}_1 = x, \tilde{X}_2 = x]$, player 2 is better off bidding $b(x) - \epsilon$ rather than $b(x)$, when $\tilde{X}_2 = x$. Thus the unique symmetric equilibrium for $n = 2$ is, as specified in (4.1.1), $b_0^*(x) = \mathbf{E}[\tilde{V}|\tilde{X}_1 = x, \tilde{X}_2 = x]$.

Suppose the equilibrium in (4.1.1) is the unique symmetric perfect equilibrium in non–decreasing strategies. when there are $n-1$ bidders. Consider an open–exit auction with n bidders. After one bidders drops out at a price level p_1, we have an open–exit auction with $n-1$ bidders. By the induction hypothesis, the unique symmetric equilibrium is as specified in (4.1.1), with the prior on $\tilde{V}$ updated to take into account that a bidder dropped out at a price level p_1. Thus in any symmetric equilibrium, $(\hat{b}_0, \hat{b}_1, \hat{b}_2, ..., \hat{b}_{n-2})$, in an open–exit auction with n bidders, $\hat{b}_k$, $k = 1, 2, ..., n-2$, must be given by (4.1.1).

Consider $\hat{b}_0$. Suppose $\hat{b}_0(x) < \mathbf{E}[\tilde{V}|\tilde{X}_j = x,\ \forall j]$, for some x. Let $y > x$ be such that $\hat{b}_0(y) = \mathbf{E}[\tilde{V}|\tilde{X}_j = x,\ \forall j]$. Suppose $\tilde{X}_1 = x$, the price has been bid upto $\hat{b}_0(x)$, and no bidder has dropped out. Player 1 can profit from the following deviation. He stays in the auction until the price reaches $\hat{b}_0(y)$. If someone else drops out before the price reaches this level, he plays according to $\hat{b}_k$, $k = 1, 2..., n-2$, as before. Suppose that $\tilde{X}_j = z$, $\forall j = 2, 3, ...n$, for any $z \in (x, y)$. This is a zero probability event when $n \geq 3$, but must be considered in a perfect equilibrium. If all other bidders quit at a price $\hat{b}_0(z)$, $x < z < y$, he wins the auction and obtains a positive expected profit equal to $\mathbf{E}[\tilde{V}|\tilde{X}_1 = x, \tilde{X}_j = z, j \geq 2] - \hat{b}_0(z)$. Thus $\hat{b}_0(x)$ cannot be a perfect equilibrium strategy unless $\hat{b}_0(x) \geq \mathbf{E}[\tilde{V}|\tilde{X}_j = x, \forall j]$, $\forall x$. By a symmetric argument we must have $\hat{b}_0(x) \leq \mathbf{E}[\tilde{V}|\tilde{X}_j = x, \forall j]$, $\forall x$, and thus $\hat{b}_0(x) = \mathbf{E}[\tilde{V}|\tilde{X}_j = x, \forall j]$, $\forall x$. ∎

4.2 Asymmetric Equilibria

As we have seen in section 3, when there are two bidders, there exist many asymmetric Nash equilibria in common value, second–price auctions. In a common value auction with two bidders, second–price and open–exit auctions are equivalent. We show below that there exists a continuum of asymmetric equilibria in open–exit auctions with more than two bidders.

As in section 3.1, we define $n-1$ independent functions, $h_{i1}(\cdot)$ which map player $1's$ inverse bidding strategy into player $i's$ inverse bidding strategy. Let $h_{i1} : [\underline{X}, \overline{X}] \rightarrow [\underline{X}, \overline{X}]$, $i = 2, ..., n$, be increasing, continuous functions with $h_{i1}(\underline{X}) = \underline{X}$ and $h_{i1}(\overline{X}) = \overline{X}$. Let $h_{11}(x) \equiv x$. Define

$$h_{1i}(x) \equiv h_{i1}^{-1}(x), \quad \forall i = 1, 2, ..., n,$$

$$h_{ij}(x) \equiv h_{i1}(h_{1j}(x)), \quad \forall i, j = 1, 2, ..., n.$$

Note that $h_{ii}(x) = x$, $h_{ji}(x) = h_{ij}^{-1}(x)$, and $h_{il}(x) = h_{ij}(h_{jl}(x))$ for all i, j, l. As we shall see, h_{ij} maps player $j's$ inverse bidding strategy into player $i's$ inverse bidding strategy.

For $i \in \mathcal{N}$ define $\hat{\mathbf{b}}_i = (\hat{b}_{i,0}, \hat{b}_{i,1}, ..., \hat{b}_{i,k}, ..., \hat{b}_{i,n-2})$ as

$$\begin{aligned} \hat{b}_{i,0}(x) &\equiv \mathbf{E}[\tilde{V} | \tilde{X}_j = h_{ji}(x),\ \forall j], \\ \hat{b}_{i,k}(x; p_1, p_2, ..., p_k) &\equiv \mathbf{E}[\tilde{V} | \tilde{X}_j = h_{ji}(x),\ \forall j \in \mathcal{N} \backslash \{i_1, i_2, ..., i_k\}, \\ &\qquad \hat{b}_{i_k,k-1}(\tilde{X}_{i_k}; p_1, p_2, ..., p_{k-1}) = p_k, \\ &\qquad \hat{b}_{i_{k-1},k-2}(\tilde{X}_{i_{k-1}}; p_1, p_2, ..., p_{k-2}) = p_{k-1}, \\ &\qquad ...., \hat{b}_{i_1,0}(\tilde{X}_{i_1}) = p_1], \\ &\qquad k = 1, 2, ..., n-2. \end{aligned}$$

By affiliation $\hat{b}_{i,k}$ is increasing in all its arguments. Also,

$$\hat{b}_{i,k}(x; p_1, p_2, ..., p_k) = \hat{b}_{l,k}(h_{li}(x); p_1, p_2, ..., p_k),$$

$$\forall k = 0, 1, 2, ..., n-2,\ \forall p_1 \le p_2 \le ... \le p_k,\ \forall i, l \in \mathcal{N} \backslash \{i_1, i_2, ..., i_k\}. \tag{4.2.1}$$

Thus when the bidders play the strategy $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, ..., \hat{\mathbf{b}}_n)$ then bidder l will drop out earlier than bidder i if and only if $\{h_{li}(\tilde{X}_i) > \tilde{X}_l\}$.

Proposition 2: $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, ..., \hat{\mathbf{b}}_n)$ *is an equilibrium in the open–exit auction.*

Proof: It is shown below that no bidder could do better by deviating from his strategy, even if he knew the other bidders' signal realizations.

Suppose that for some realization of the bidders' signals, bidders $i_1, i_2, ..., i_k, ..., i_{n-1}$ drop out in that order, at price levels $p_1 \le p_2 \le ... \le p_k \le ... \le p_{n-1}$, when the strategies $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, ..., \hat{\mathbf{b}}_n)$ are played. Let the signal realizations of the bidders be $\tilde{X}_{i_k} = x_{i_k}$, $k = 1, 2, ..., n$. Since $p_m \le p_k$, $\forall m \le k$, (4.2.1) implies that

$$h_{i_k i_m}(x_{i_m}) \le x_{i_k}, \qquad \forall m \le k. \tag{4.2.2}$$

Bidder i_n wins the auction and pays a price $p_{n-1} = \mathbf{E}[\tilde{V} | \tilde{X}_{i_k} = x_{i_k},\ k = 1, 2, ..., n-1,\ \tilde{X}_{i_n} = h_{i_n i_{n-1}}(x_{i_{n-1}})]$. He values the object at $\mathbf{E}[\tilde{V} | \tilde{X}_{i_k} = x_{i_k},\ k = 1, 2, ..., n-1,\ \tilde{X}_{i_n} = x_{i_n}]$, which, by (4.2.2) and affiliation is greater than p_{n-1}. Hence, $\hat{\mathbf{b}}_{i_n}$ is an optimal strategy for bidder i_n.

Suppose some bidder i_l, $l = 1, 2, ..., n-1$ deviates from his strategy $\hat{\mathbf{b}}_{\mathbf{i}_l}$ and bids as if he observed a signal $\tilde{X}_{i_l} = y$. He will win the auction only if $y \geq h_{i_l i_n}(x_{i_n})$, so we may as well assume that. The object is worth $\mathbf{E}[\tilde{V}|\tilde{X}_{i_k} = x_{i_k},\ k = 1, 2, ..., n]$ to him. The price he will pay if he deviates and wins the auction is $\mathbf{E}[\tilde{V}|\tilde{X}_{i_k} = x_{i_k},\ \forall k \neq l, \tilde{X}_{i_l} = h_{i_l i_n}(x_{i_n})]$, which by (4.2.2) and affiliation, is greater than his valuation of the object. Hence $\hat{\mathbf{b}}_{\mathbf{i}_l}$ is an optimal strategy for bidder i_l. ∎

Remark: *A larger class of equilibria is obtained if, at each stage k, we allow the functions h_{ij} to depend on $(p_1, p_2, ..., p_k)$. For each i, j, and each k, the functions, h^k_{ij}, must satisfy $h^k_{ij}(h^k_{ji}(x; p_1, p_2, ..., p_k); p_1, p_2, ..., p_k) = x$. $\hat{b}_{i,k}$ is defined as before with h^k_{ji} replacing h_{ji}. In addition, h^k_{ji} must be chosen so that if $\hat{b}_{i,k-1}(x; p_1, p_2, ..., p_{k-1}) \geq p_k$, then $\hat{b}_{i,k}(x; p_1, p_2, ..., p_{k-1}, p_k) \geq p_k$. The proof of Proposition 2 easily generalizes.*

5 The Auctioneer's Revenues

Consider a second–price/open–exit auction with two bidders. In any Nash equilibrium in weakly undominated strategies, the players' strategies satisfy the following constraints

$$\mathbf{E}[\tilde{V}|\tilde{X}_1 = x,\ \tilde{X}_2 = \underline{X}] \leq b^*_1(x),\ b^*_2(x) \leq \mathbf{E}[\tilde{V}|\tilde{X}_1 = x,\ \tilde{X}_2 = \overline{X}].$$

Also, $b^*_1(x) = \mathbf{E}[\tilde{V}|\tilde{X}_1 = x,\ \tilde{X}_2 = \underline{X}]$, $b^*_2(x) = \mathbf{E}[\tilde{V}|\tilde{X}_1 = x,\ \tilde{X}_2 = \overline{X}]$, is an equilibrium. Since bidder 1 always loses in this equilibrium, the auctioneer's revenues are $b^*_1(\tilde{X}_1) = \mathbf{E}[\tilde{V}|\tilde{X}_1,\ \tilde{X}_2 = \underline{X}]$. As noted earlier, there exist a continuum of asymmetric equilibria given by $b^*_1(x) = \mathbf{E}[\tilde{V}|\tilde{X}_1 = x,\ \tilde{X}_2 = h(x)]$, $b^*_2(x) = \mathbf{E}[\tilde{V}|\tilde{X}_1 = h^{-1}(x),\ \tilde{X}_2 = x]$, where $h(x)$ is any increasing, surjective function. $h(x) = x$ gives us the symmetric equilibrium.

When comparing the auctioneer's expected revenues from an open–exit, a second–price, and a first–price auction, Milgrom and Weber (1982) compare revenues at the symmetric equilibrium in these auctions. Our results in the preceding sections point in the direction of uniqueness of equilibrium for the second–price auction whereas we construct a continuum of equilibria for the open–exit auction. Therefore, comparisons of the auctioneer's revenues at the asymmetric equilibria in the open–exit auction and the symmetric equilibrium in the second–price auction are also of interest. First, we provide sufficient conditions under which the auctioneer's revenue in an open–exit auction with two bidders is maximized at the symmetric equilibrium.

Proposition 3: *Suppose that in an open–exit auction with two bidders, the players' signals are independently distributed conditional on the true value with conditional density function $f(x_i|v)$ and conditional distribution function $F(x_i|v)$. If the decay rate*

$$\frac{f(x_i|v)}{1-F(x_i|v)}$$

is non–decreasing in x_i and if $V(x_1,x_2) \equiv \mathbf{E}[\tilde{V}|\tilde{X}_1 = x_1, \tilde{X}_2 = x_2]$ is quasiconcave, then the distribution of the selling price at the symmetric equilibrium first order stochastically dominates the distribution of the selling price at any asymmetric equilibrium. Thus the auctioneer's revenue is maximized at the symmetric equilibrium.

Proof: Let $\hat{\phi}(\cdot)$ be the inverse bidding strategy at the symmetric equilibrium. That is,

$$V(\hat{\phi}(b),\hat{\phi}(b)) = b, \quad \forall b$$

Suppose that $\phi_1(\cdot)$, $\phi_2(\cdot)$ is an asymmetric equilibrium. That is

$$V(\phi_1(b),\phi_2(b)) = b, \quad \forall b$$

Without loss of generality suppose that for some bid b, $\phi_1(b) = \hat{\phi}(b) - \lambda$, $\lambda \geq 0$. Since $V(\cdot,\cdot)$ is quasiconcave and symmetric

$$\begin{aligned} V(\hat{\phi}(b)-\lambda,\hat{\phi}(b)+\lambda) &\leq V(\hat{\phi}(b),\hat{\phi}(b)) \\ &= b \end{aligned}$$

Therefore

$$\hat{\phi}(b)+\lambda \leq \phi_2(b) \tag{5.1.1}$$

Let $\tilde{P}^A$ be the selling price at this asymmetric equilibrium and let $\tilde{P}^S$ be the selling price at the symmetric equilibrium. $G^A(b)$ and $G^S(b)$ denote the respective distribution functions.

The proof is complete if we can show that $G^A(b) \geq G^S(b)$. We have

$$
\begin{aligned}
G^A(b) &= \Pr\{\tilde{P}^A \leq b\} \\
&= 1 - \int_{\underline{V}}^{\overline{V}} (1 - F(\phi_1(b) \mid v))(1 - F(\phi_2(b) \mid v)) f_{\tilde{V}}(v) dv
\end{aligned}
$$

where $f_{\tilde{V}}(v)$ is the marginal density of $\tilde{V}$. Note that the right hand side is increasing in $\phi_2(b)$. Thus

$$
\begin{aligned}
G^A(b) &\geq 1 - \int_{\underline{V}}^{\overline{V}} (1 - F(\hat{\phi}(b) - \lambda \mid v))(1 - F(\hat{\phi}(b) + \lambda \mid v)) f_{\tilde{V}}(v) dv \\
&= 1 - \int_{\underline{V}}^{\overline{V}} H(\lambda, v, b) f_{\tilde{V}}(v) dv
\end{aligned}
$$

where

$$
\begin{aligned}
H(\lambda, v, b) &\equiv (1 - F(\hat{\phi}(b) - \lambda \mid v))(1 - F(\hat{\phi}(b) + \lambda \mid v)) \\
&\geq 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\ln H(\lambda, v, b) &= \ln(1 - F(\hat{\phi}(b) - \lambda \mid v)) + \ln(1 - F(\hat{\phi}(b) + \lambda \mid v)) \\
\frac{1}{H}\frac{\partial H}{\partial \lambda} &= \frac{f(\hat{\phi}(b) - \lambda \mid v)}{1 - F(\hat{\phi}(b) - \lambda \mid v)} - \frac{f(\hat{\phi}(b) + \lambda \mid v)}{1 - F(\hat{\phi}(b) + \lambda \mid v)} \\
&\leq 0
\end{aligned}
$$

where the inequality follows from the fact that the decay rate is non–decreasing. Hence

$$
H(0, v, b) \geq H(\lambda, v, b), \quad \forall \lambda \geq 0, \forall v
$$

Thus

$$
\begin{aligned}
G^A(b) &\geq 1 - \int_{\underline{V}}^{\overline{V}} H(\lambda, v, b) f_{\tilde{V}}(v) dv \\
&\geq 1 - \int_{\underline{V}}^{\overline{V}} H(0, v, b) f_{\tilde{V}}(v) dv \\
&= G^S(b)
\end{aligned}
$$

∎

The following example, in which the auctioneer's revenue is minimized at the symmetric equilibrium, establishes that the assumption on the quasiconcavity of $V(x_1, x_2)$ is crucial. Although the signals are conditionally independent and $\frac{f(x_i|v)}{1-F(x_i|v)}$ is decreasing in x_i in this example, Proposition 3 does not apply because $V(x_1, x_2)$ is quasiconvex. Each of the two bidders has a common prior on the true value of the object given by the uniform density function

$$f_{\tilde{V}}(v) = 1, \quad v \in [0, 1].$$

Bidder i observes a signal $\tilde{X}_i$ which is a random draw from the conditional density function

$$f_{\tilde{X}_i|\tilde{V}}(x_i|v) = \frac{1}{v}, \quad x_i \in [0, v].$$

Clearly $V(x_1, x_2) = p(\max\{x_1, x_2\})$, for some increasing function $p(\cdot)$. Let $R(h)$ denote the auctioneer's revenue at the equilibrium $b_1(x) = V(x, h(x))$, $b_2(x) = V(h^{-1}(x), x)$. Note that $R(h) = R(h^{-1})$. Then if $h(x) \geq x$, $\forall x$, $h(x) \not\equiv x$,

$$
\begin{aligned}
R(h) &= \mathbf{E}[V(\tilde{X}_1, h(\tilde{X}_1)) 1_{\{\tilde{X}_2 \geq h(\tilde{X}_1)\}} + V(h^{-1}(\tilde{X}_2), \tilde{X}_2) 1_{\{\tilde{X}_2 < h(\tilde{X}_1)\}}] \\
&= \mathbf{E}[p(h(\tilde{X}_1)) 1_{\{\tilde{X}_2 \geq h(\tilde{X}_1)\}} + p(\tilde{X}_2) 1_{\{\tilde{X}_2 < h(\tilde{X}_1)\}}] \\
&> \mathbf{E}[p(h(\tilde{X}_1)) 1_{\{\tilde{X}_2 \geq h(\tilde{X}_1)\}} + p(\tilde{X}_1) 1_{\{\tilde{X}_1 \leq \tilde{X}_2 < h(\tilde{X}_1)\}} + p(\tilde{X}_2) 1_{\{\tilde{X}_2 < \tilde{X}_1\}}] \\
&= \mathbf{E}[p(\tilde{X}_1) 1_{\{\tilde{X}_2 \geq \tilde{X}_1\}} + p(\tilde{X}_2) 1_{\{\tilde{X}_2 < \tilde{X}_1\}}] \\
&= \mathbf{E}[V(\tilde{X}_1, \tilde{X}_1) 1_{\{\tilde{X}_2 \geq \tilde{X}_1\}} + V(\tilde{X}_2, \tilde{X}_2) 1_{\{\tilde{X}_2 < \tilde{X}_1\}}] \\
&= R(\mathbf{x}),
\end{aligned}
$$

where $R(\mathbf{x})$ is the auctioneer's revenue at the symmetric equilibrium. Also, it is easy to show that if $h_a(x) \geq h_b(x) \geq x$, $\forall x$, $h_a(\cdot) \not\equiv h_b(\cdot)$, then $R(h_a) > R(h_b)$.

Next we establish that under assumptions similar to those made in Proposition 3, there exists a continuum of asymmetric equilibria in an open–exit auction with three or more bidders in which the expected selling price is less than the expected selling price at the symmetric equilibrium.

Proposition 4: *Suppose that in an open–exit auction with $n \geq 3$ bidders, the players' signals are independently distributed conditional on the true value. If the decay rate*

$$\frac{f(x_i|v)}{1 - F(x_i|v)}$$

is non–decreasing in x_i and if $V(x_1, x_2, ..., x_n) \equiv \mathbf{E}[\tilde{V}|\tilde{X}_1 = x_1, \tilde{X}_2 = x_2, ..., \tilde{X}_n = x_n]$ is quasiconcave, then the expected revenue at the symmetric equilibrium is greater than the expected revenue at a continuum of asymmetric equilibria.

Proof: Consider any asymmetric equilibrium in which all players play the symmetric equilibrium strategy until there are only two players left, who then switch to asymmetric strategies. It is easy to show that the symmetric equilibrium generates higher expected revenues than the above asymmetric equilibrium.

Suppose that in equilibrium the first $n-2$ bidders to drop out do so at price levels which imply that their signal realizations are $y_1, y_2, ..., y_{n-2}$. The rest of the proof is identical to the proof of Proposition 3 with $V(\cdot,\cdot,y_1,y_2,...,y_{n-2})$ playing the role of $V(\cdot,\cdot)$. ∎

Proposition 4 suggests that there might exist asymmetric equilibria in an open–exit auction with $n \geq 3$ bidders at which the auctioneer's revenue is less than at the symmetric equilibrium of the second–price auction. We establish by example that this is indeed possible.

Consider the example introduced in section 2 with the bidders' signals, $\tilde{X}_i$, drawn independently from the uniform distribution on $[0,1]$, $F(x) = x$. The true value is $\tilde{V} =$

$\tilde{X}_1 + \tilde{X}_2 + ... + \tilde{X}_n$. Since $\frac{f(x)}{F(x)} = \frac{1}{x}$ is decreasing, Lemma 3 implies that Assumption 1 is satisfied. Therefore, for $n \geq 3$, there do not exist any asymmetric equilibria in continuous and increasing strategies in the second–price auction. It is easily verified that the symmetric equilibrium strategy is

$$b(x) = \frac{n+2}{2}x$$

Thus the selling price at the symmetric equilibrium of the second–price auction is

$$\tilde{P}_2 = \frac{n+2}{2}\tilde{Y}_2$$

where $\tilde{Y}_k$ denotes the k–th order statistic of $(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)$. Also, the selling price at the symmetric equilibrium of the open–exit auction is

$$\tilde{P}_o = \tilde{Y}_2 + \tilde{Y}_2 + \tilde{Y}_3 + \tilde{Y}_4 + ... + \tilde{Y}_n$$

and thus

$$\mathbf{E}[\tilde{P}_o | \tilde{Y}_2] = \frac{n+2}{2}\tilde{Y}_2$$

Hence the expected seller revenue at the symmetric equilibrium in the two auctions is equal.

Next note that $V(x_1, x_2, ..., x_n) = x_1 + x_2 + ... + x_n$ is quasiconcave and $\frac{f(x)}{1-F(x)} = \frac{1}{1-x}$ is strictly increasing. Therefore Proposition 4 applies and there exists a continuum of asymmetric equilibria in the open–exit auction which yield strictly lower[6] expected revenue than the expected revenue at the symmetric equilibrium in the open–exit or the second–price auctions.

Thus Theorem 11 in Milgrom and Weber (1982), which states that the auctioneer's revenue at the symmetric equilibrium of an open-exit auction is not less than at the symmetric equilibrium of a second–price auction, cannot be extended to asymmetric equilibria. It is not true that the auctioneer's revenue at every equilibrium of an open–exit auction is at least as large as his revenue at the symmetric equilibrium of a second–price auction.

6 Concluding Remarks

In this paper, we have analyzed bidding when the item for sale has the same value, ex post, to all the buyers. In contrast to earlier papers which examine only the symmetric equilibria of common value auctions, we characterize the set of equilibrium strategies for two forms of the open ascending bid auction.

The first message is that, in general, there can be a continuum of equilibria. Whenever asymmetric equilibria are feasible, one such equilibrium involves more aggressive bidding behavior by a single buyer, relative to the symmetric equilibrium, and more passive behavior by the other buyers. That is, one buyer stays in the bidding longer and the others for a shorter time than in the symmetric equilibrium.

Intuitively, it is in the interest of any one buyer to establish a reputation for being aggressive, as in Bikhchandani (1988). For the more passive equilibrium response of his opponents increases his probability of winning, and decreases the price that he pays upon winning. Intuitively, also, the more passive behavior of all but one buyer is not in the interest of the seller since his revenue is just the expectation of the second highest price. With all but one buyer behaving relatively passively, this expectation is reduced.

While the paper shows that seller revenue does not necessarily fall under asymmetric bidding, the basic intuition holds under fairly mild additional restrictions. The second message, then, is that the unambiguous revenue ranking of the sealed high bid auction and the two open auctions does not hold for environments in which a buyer has been able to establish a reputation for bidding aggressively (in previous auctions or otherwise).

Thirdly, we have established that the multiple equilibrium problem is much more severe for the "open–exit" auction than the English ascending bid auction used so commonly in auction houses. To be precise, while there is always a continuum of equilibria for the two buyer case, only the "open–exit" auction has asymmetric equilibria with three or more bidders, as long as the underlying distribution of signals satisfies an additional restriction that seems likely to be met in many practical applications.

Having summarized the key results of the paper, it should be noted that they do depend critically upon the assumption that private valuations differ only because of informational differences. That is, if all the private signals were to be made public, buyers would agree on the value of the item for sale. Suppose, instead, that each individual i observes two signals, $\tilde{X}_i$ and $\tilde{Z}_i$, and his valuation, $\tilde{V}_i$, is

$$\tilde{V}_i = V(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n) + \tilde{Z}_i \tag{6.0.1}$$

That is, in addition to the common element $V(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)$ there is a second private element, $\tilde{Z}_i$, which is an independent draw from some distribution. It is our strong conjecture that, under moderate assumptions, there are no asymmetric equilibria for either of the open auctions analyzed here. The following example illustrates the point.

Consider the two buyer case and suppose that $\tilde{X}_1$ and $\tilde{X}_2$ are also independent draws from some distribution. Define

$$h(y) \equiv \mathbf{E}[\tilde{X}_i | \tilde{X}_i + \tilde{Z}_i = y] \tag{6.0.2}$$

We introduce the further mild restriction that

$$0 < h'(y) < 1, \qquad \forall y \tag{6.0.3}$$

Finally, we assume that

$$V(\tilde{X}_1, \tilde{X}_2) = \tilde{X}_1 + \tilde{X}_2 \tag{6.0.4}$$

To solve for the equilibrium, $(B_1(\tilde{X}_1, \tilde{Z}_1), B_2(\tilde{X}_2, \tilde{Z}_2))$, suppose that the asking price has reached b. Buyer 1 is better off remaining in the auction if and only if he will make a positive expected profit given that buyer 2 drops out at this price. Thus buyer 1 stays in if and only if

$$\mathbf{E}[\tilde{V}_1 | \tilde{X}_1, \tilde{Z}_1] - b = Z_1 + X_1 + \mathbf{E}[\tilde{X}_2 | B_2(\tilde{X}_2, \tilde{Z}_2) = b] - b > 0$$

Define

$$\phi_i(b) \equiv b - \mathbf{E}[\tilde{X}_j | B_j(\tilde{X}_j, \tilde{Z}_j) = b, \ j \neq i] \tag{6.0.5}$$

Then buyer $1's$ equilibrium bid function must satisfy

$$\phi_1(B_1(\tilde{X}_1, \tilde{Z}_1)) = \tilde{X}_1 + \tilde{Z}_1$$

or

$$B_1(\tilde{X}_1, \tilde{Z}_1) = \phi_1^{-1}(\tilde{X}_1 + \tilde{Z}_1) \tag{6.0.6}$$

Since exactly the same argument holds for buyer 2 we must have

$$B_2(\tilde{X}_2, \tilde{Z}_2) = \phi_2^{-1}(\tilde{X}_2 + \tilde{Z}_2)$$

Then

$$\begin{aligned} \mathbf{E}[\tilde{X}_2 | B_2(\tilde{X}_2, \tilde{Z}_2) = b] &= \mathbf{E}[\tilde{X}_2 | \tilde{X}_2 + \tilde{Z}_2 = \phi_2(b)] \\ &= h(\phi_2(b)) \end{aligned}$$

Substituting this in (6.0.5) we obtain

$$\phi_1(b) = b - h(\phi_2(b)) \tag{6.0.7}$$

Arguing symmetrically for buyer 2, we have

$$\phi_2(b) = b - h(\phi_1(b)) \tag{6.0.8}$$

Subtracting (6.0.8) from (6.0.7) yields

$$\phi_1(b) - \phi_2(b) = h(\phi_1(b)) - h(\phi_2(b)) \tag{6.0.9}$$

Given (6.0.3) it immediately follows that the only solution to (6.0.9) is the symmetric solution

$$\phi_1(b) = \phi_2(b) = \phi(b)$$

Finally, from (6.0.6) and (6.0.7), the unique monotonic equilibrium function is

$$B_i(\tilde{X}_i, \tilde{Z}_i) = \tilde{X}_i + \tilde{Z}_i + h(\tilde{X}_i + \tilde{Z}_i)$$

Of course the example is somewhat special. In particular the solution is greatly simplified by the assumption that all signals are independent. It is therefore by no means a trivial matter to go from the example to the general case of correlated signals.

Assuming our conjecture is correct, that the addition of a private element to the common element of each buyer's valuation implies uniqueness of the symmetric equilibrium in the two open auctions, the natural question that arises is whether it will be possible to rank expected revenue from the two open auctions and the sealed high bid auction. While this appears to be a problem which is mathematically difficult, we believe that the insights gained from analysis of the one signal per buyer model are capable of generalization. That is, the greater the amount of information revealed in the auction the higher the expected revenue, given that bidders adopt a symmetric equilibrium bidding strategy.

Given the extent of uncertainty in the actual bidding process, it might be argued that there would always be some private element in addition to a common value element in any valuation. But if the private element is small, it is far from clear that it would of relevance empirically. We believe that the pure common value model remains the appropriate approximation for applications such as oil lease bidding, except in those cases where it can be argued that private elements are an economically significant part of each buyer's total valuation.

Appendix I
Proofs of Lemmas 1 and 3

Proof of Lemma 1:

Consider the following assumption

Assumption 1*: For all α, β

$$f_{\tilde{V}|\tilde{X}}(v|\tilde{X}_1 = \alpha,\ \tilde{X}_2 \leq \beta,\ \ c_i \leq \tilde{X}_i \leq d_i,\ i \geq 3) = f_{\tilde{V}|\tilde{X}}(v|\tilde{X}_1 \leq \alpha,\ \tilde{X}_2 = \beta,\ c_i \leq \tilde{X}_i \leq d_i,\ i \geq 3),\quad \forall c_i, d_i.$$

Assumption 1* implies that (3.1.3) is satisfied. It also implies that Assumption 1 (introduced later in section 3.2) is violated.

By hypothesis the signals are independent conditional on the true value. Thus $f(v,x)$ satisfies Assumption 1* if and only if

$$f_{\tilde{V}|\tilde{X}}(v|\tilde{X}_1 = \alpha,\ \ \tilde{X}_2 \leq \beta,\ c_i \leq \tilde{X}_i \leq d_i,\ i \geq 3)$$

$$= \frac{f_{\tilde{V}}(v) f_{\tilde{X}_1|\tilde{V}}(\alpha|v) F_{\tilde{X}_2|\tilde{V}}(\beta|v) \prod_{i=3}^{n} (F_{\tilde{X}_i|\tilde{V}}(d_i|v) - F_{\tilde{X}_i|\tilde{V}}(c_i|v))}{\int f_{\tilde{V}}(v) f_{\tilde{X}_1|\tilde{V}}(\alpha|v) F_{\tilde{X}_2|\tilde{V}}(\beta|v) \prod_{i=3}^{n} (F_{\tilde{X}_i|\tilde{V}}(d_i|v) - F_{\tilde{X}_i|\tilde{V}}(c_i|v)) dv}$$

$$= \frac{f_{\tilde{V}}(v) f_{\tilde{X}_2|\tilde{V}}(\beta|v) F_{\tilde{X}_1|\tilde{V}}(\alpha|v) \prod_{i=3}^{n} (F_{\tilde{X}_i|\tilde{V}}(d_i|v) - F_{\tilde{X}_i|\tilde{V}}(c_i|v))}{\int f_{\tilde{V}}(v) f_{\tilde{X}_2|\tilde{V}}(\beta|v) F_{\tilde{X}_1|\tilde{V}}(\alpha|v) \prod_{i=3}^{n} (F_{\tilde{X}_i|\tilde{V}}(d_i|v) - F_{\tilde{X}_i|\tilde{V}}(c_i|v)) dv}$$

$$= f_{\tilde{V}|\tilde{X}}(v|\tilde{X}_1 \leq \alpha,\ \tilde{X}_2 = \beta,\ c_i \leq \tilde{X}_i \leq d_i,\ i \geq 3).$$

Rearranging terms, we get

$$\frac{f_{\tilde{X}_1|\tilde{V}}(\alpha|v) F_{\tilde{X}_2|\tilde{V}}(\beta|v)}{f_{\tilde{X}_2|\tilde{V}}(\beta|v) F_{\tilde{X}_1|\tilde{V}}(\alpha|v)} = I(\alpha,\beta,c_i,d_i),\quad \forall \alpha,\beta,c_i,d_i,v,$$

for some function $I(\alpha,\beta,c_i,d_i)$. But the left-hand side is a function of α, β and v whereas the right-hand side is a function of α, β, c_i and d_i. Therefore, both sides are functions of α, β alone and we may write

$$\frac{f_{\tilde{X}_1|\tilde{V}}(\alpha|v) F_{\tilde{X}_2|\tilde{V}}(\beta|v)}{f_{\tilde{X}_2|\tilde{V}}(\beta|v) F_{\tilde{X}_1|\tilde{V}}(\alpha|v)} = J(\alpha,\beta),\quad \forall \alpha,\beta,$$

where $J(\alpha, \beta)$ is some function. Since the signals are identically distributed, we may rewrite this as

$$\frac{f_{\tilde{X}_1|\tilde{V}}(\alpha|v)}{F_{\tilde{X}_1|\tilde{V}}(\alpha|v)} = \frac{f_{\tilde{X}_1|\tilde{V}}(\beta|v)}{F_{\tilde{X}_1|\tilde{V}}(\beta|v)} J(\alpha, \beta), \quad \forall \alpha, \beta.$$

Hence there exist functions $K_1(\alpha)$ and $K_2(v)$ such that

$$\begin{aligned} \frac{f_{\tilde{X}_1|\tilde{V}}(\alpha|v)}{F_{\tilde{X}_1|\tilde{V}}(\alpha|v)} &\equiv \frac{\partial \ln F_{\tilde{X}_1|\tilde{V}}(\alpha|v)}{\partial \alpha} \\ &= K_1(\alpha) K_2(v), \quad \forall \alpha, v. \end{aligned}$$

∎

Proof of Lemma 3: First we show that if $\ln f(x)$ is strictly concave, then so is $\ln F(x)$. That is, if $\frac{d \ln f(x)}{dx}$ is strictly decreasing so must be $\frac{d \ln F(x)}{dx}$.

Let

$$G(x) \equiv \frac{f(x)}{F(x)} = \frac{d \ln F(x)}{dx} \tag{A.1.1}$$

If $f(\underline{X}) > 0$ then $G(\underline{X}) = \infty$. If $f(\underline{X}) = 0$, then $f(x)$ is strictly increasing in some interval $(\underline{X}, \underline{X} + \epsilon)$. Then $F(x) \leq f(x)(x - \underline{X})$ for all x in $(\underline{X}, \underline{X} + \epsilon)$ and so again

$$\lim_{x \downarrow \underline{X}} G(x) \geq \lim_{x \downarrow \underline{X}} \frac{1}{x - \underline{X}} = \infty$$

The first part of the proof is then complete if we can show that $\frac{d}{dx}(\frac{f'(x)}{f(x)}) < 0$ implies that there exists no $\hat{x} > \underline{X}$ such that $G(\hat{x})$ is a local minimum. Differentiating $(A.1.1)$

$$G'(x) = \frac{f'(x)}{f(x)} G(x) - G^2(x)$$

Therefore, at $\hat{x}$, with $G'(\hat{x}) = 0$

$$\begin{aligned} G''(\hat{x}) &= G(\hat{x}) \frac{d}{dx} (\frac{f'}{f}) \Big|_{x=\hat{x}} \\ &< 0 \end{aligned}$$

by hypothesis. It follows that $G(x)$ does not, after all, have a local minimum at $x = \hat{x}$ and so $G(x)$ is everywhere decreasing, as desired.

Integrating by parts

$$D(x) = \int_{\underline{X}}^{x} \frac{F(y)}{F(x)} dy \tag{A.1.2}$$

Since $F(\cdot)$ is increasing, for all $x > \underline{X}$

$$0 < D(x) < \frac{F(x)(x - \underline{X})}{F(x)}$$

Taking the limit as $x \downarrow \underline{X}$, it follows that $D(\underline{X})$ is strictly increasing at $x = \underline{X}$. Suppose $D(x)$ has a local maximum at $\hat{x} > \underline{X}$. Differentiating $(A.1.2)$

$$D'(x) = 1 - D(x)\frac{F'(x)}{F(x)}$$

At $\hat{x}$, with $D'(\hat{x}) = 0$, we have

$$\begin{aligned} D''(\hat{x}) &= -D(\hat{x})\frac{d}{dx}\left(\frac{f}{F}\right)\Big|_{x=\hat{x}} \\ &> 0 \end{aligned}$$

by hypothesis. But then $D(x)$ cannot, after all, have a maximum at $x = \hat{x}$ and thus $D(x)$ is everywhere increasing as claimed. ∎

Appendix II
Partially Pooling Equilibria in Second–Price Auctions

In section 3.2 we proved that if there are more than two bidders and Assumption 1 is satisfied, there do not exist any asymmetric equilibria in increasing and continuous strategies. We now turn to strategies which are discontinuous and non–decreasing. We restrict our analysis to non–decreasing, step–function strategies, that is partially pooling strategies, and show that there do not exist equilibria where all players use such strategies, if Assumption 1 is satisfied. However, there exists a continuum of such equilibria for the example in section 3.1 and other distributions which satisfy (3.1.3).

We assume that the number of players, n, is three. The same argument can be extended to $n \geq 4$. Let $x_o \equiv \underline{X}$, and let $x_k \in (\underline{X}, \overline{X}]$, $k = 1,2,3,...$, be such that $x_k < x_{k+1}$. y_k, $k = 0,1,2,...$, and z_k, $k = 0,1,2,...$, are similarly defined. Let

$$\hat{b}_1(\tilde{X}_1) = b^{3k}, \quad \text{if } \tilde{X}_1 \in [x_k, x_{k+1}), \quad k = 0,1,2,....$$

$$\hat{b}_2(\tilde{X}_2) = b^{3k+1}, \quad \text{if } \tilde{X}_2 \in [y_k, y_{k+1}), \quad k = 0,1,2,....$$

$$\hat{b}_3(\tilde{X}_3) = b^{3k+2}, \quad \text{if } \tilde{X}_3 \in [z_k, z_{k+1}), \quad k = 0,1,2,....$$

where $...b^{3k+2} > b^{3k+1} > b^{3k} > b^{3k-1}...$. The following are necessary conditions for $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ to be an equilibrium

$$b^{3k} \leq \mathbf{E}[\tilde{V} | \tilde{X}_1 \in [x_k, x_{k+1}),\ \tilde{X}_2 = y_k,\ \tilde{X}_3 \in [z_o, z_k)], \quad \forall k \tag{A.2.1}$$

$$b^{3k} \geq \mathbf{E}[\tilde{V} | \tilde{X}_1 \in [x_k, x_{k+1}),\ \tilde{X}_2 \in [y_o, y_k),\ \tilde{X}_3 = z_k], \quad \forall k \tag{A.2.2}$$

$$b^{3k+1} \leq \mathbf{E}[\tilde{V} | \tilde{X}_1 \in [x_o, x_{k+1}),\ \tilde{X}_2 \in [y_k, y_{k+1}),\ \tilde{X}_3 = z_k], \quad \forall k \tag{A.2.3}$$

$$b^{3k+1} \geq \mathbf{E}[\tilde{V} | \tilde{X}_1 = x_{k+1},\ \tilde{X}_2 \in [y_k, y_{k+1}),\ \tilde{X}_3 \in [z_o, z_k)], \quad \forall k \tag{A.2.4}$$

$$b^{3k+2} \leq \mathbf{E}[\tilde{V} | \tilde{X}_1 = x_{k+1},\ \tilde{X}_2 \in [y_o, y_{k+1}),\ \tilde{X}_3 \in [z_k, z_{k+1})], \quad \forall k \tag{A.2.5}$$

$$b^{3k+2} \geq \mathbf{E}[\tilde{V} | \tilde{X}_1 \in [x_o, x_{k+1}),\ \tilde{X}_2 = y_{k+1},\ \tilde{X}_3 \in [z_k, z_{k+1}], \quad \forall k \tag{A.2.6}$$

If (A.2.1) is not satisfied for some k, player 2 would prefer to deviate from his strategy and bid slightly less than b^{3k}, when $\tilde{X}_2 \in [y_k, y_k + \epsilon)$, for some $\epsilon > 0$. If (A.2.2) is not satisfied for some k, player 3 would prefer to deviate from his strategy and bid slightly more than b^{3k}, when $\tilde{X}_3 \in [z_k - \epsilon, z_k)$, for some $\epsilon > 0$. Inequalities (A.2.3)–(A.2.6) are similarly derived.

Suppose Assumption 1 is satisfied. Then (A.2.1) and (A.2.2) imply that $y_k \geq z_k$, (A.2.3) and (A.2.4) imply $z_k \geq x_{k+1}$ and (A.2.5) and (A.2.6) imply $x_{k+1} \geq y_{k+1}$. This implies $y_{k+1} \leq y_k$, $\forall k$, which contradicts our assumption that $y_{k+1} > y_k$. Thus Assumption 1 implies that the above necessary conditions cannot be satisfied. Since x_k, y_k, and z_k, $k = 1, 2, 3, \ldots$ were arbitrary, we have proved that there does not exist an equilibrium with step–function strategies if Assumption 1 is satisfied.[7]

Sufficient conditions for an equilibrium are obtained by replacing the inequalities in (A.2.1)–(A.2.6) with equalities. If we define b^{3k}, b^{3k+1}, b^{3k+2} equal to the right-hand expressions in (A.2.1), (A.2.3), and (A.2.5) respectively, then (A.2.2), (A.2.4), and (A.2.6) are satisfied provided (3.1.3) holds. Since x_k, y_k, and z_k, $k = 1, 2, 3, \ldots$ are arbitrary increasing sequences, we have proved that if (3.1.3) holds there exists a continuum of equilibria with step–function strategies.

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Footnotes

[1] Matthews (1987), in which several auctions are ranked from the buyer's point of view, is an exception.

[2] Under private, independent values, second–price and open–exit auctions are equivalent.

[3] In a common value model with two bidders, an open–exit auction is equivalent to a second–price auction.

[4] Assuming that $b_i(x)$ are increasing and continuous functions, $\phi_i : [b_i(\underline{X}), b_i(\overline{X})] \rightarrow [\underline{X}, \overline{X}]$, are well-defined, increasing and continuous.

[5] Actually there are only $n - 1$ independent functions $h_{j1}(\cdot)$ mapping ϕ_1 into ϕ_j, $j = 1, 2, ..., n$. Once these functions are defined we have $h_{ji}(\cdot) = h_{j1}(h_{i1}^{-1}(\cdot))$.

[6] Since the $\frac{f(x)}{1-F(x)}$ is strictly increasing, the revenue ranking is strict.

[7] There are other types of step–function equilibria that we have not considered. For instance, the players need not bid successively higher, in turn. Using a similar argument these can all be ruled by Assumption 1.