MULTIPERSON UTILITY

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DRAFT, comments are welcomed

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A Theory of Multiperson Utility

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Abstract

The paper presents a theory of multiperson preference based on the ability of some pairs of individuals to agree on a joint preference. If such pairs form a spanning tree of the set of players, then the acceptance of the Pareto rule is enough to determine a unique preference for the entire group. Moreover, there is a set of positive weights, unique up to a positive factor of proportionality, such that the utility for the group can be expressed as the weighted sum of the individual utilities. Those ideas shed light in the theory of transferrable utility and the concept of interpersonal comparisons of utility.

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[1]This section is entirely based, except for some minor changes, on an unpublished paper of L. Shapley written some years ago.

1 The Theory of Incomplete Preferences

1.1 The prospect space

The underlying domain of *prospects* over which the preferences are given will be denoted by M. It will be assumed that M is a "mixture space" (see below), which means intuitively that any two prospects can be combined into a third prospect depending on a variable "mixing parameter" drawn from $[0,1]$. Formally, a *mixture space* is a set M equipped with a *mixing operation* $(x, y, \alpha) \longrightarrow x\alpha y$ from $M \times M \times [0,1]$ to M, such that for all $x, y, z \in M$ and $\alpha, \beta \in [0,1]$:

(M1) *If* $\alpha + \beta = 1$, $x\alpha y = y\beta x$

(M2) *If* $\gamma = \alpha + \beta - \alpha\beta > 0$, *then* $x\alpha(y\beta z) = (x\frac{\alpha}{\gamma}y)\gamma z$

(M3) $x\alpha x = x$

(M4) *If* $x\alpha z = y\alpha z$ *and* $\alpha \neq 0$, *then* $x = y$

We observe that $x1y = x$ and $x0y = y$. To see this, we have

$$x1y = x1(y\alpha y) = (x1y)1y \tag{1}$$

by **(M3)** and **(M2)**. Then, $x1y = x$ follows from applying **(M4)** to both sides of the expression (1). A trivial use of **(M1)** produces $x0y = y$.

One easily verifies that any convex subset of a vector space is a mixture space, with the mixing operation interpreted in the obvious way as weighted average, thus:

$$\textit{For all } x, y \in M \textit{ and all } \alpha \in [0,1],\ x\alpha y = \alpha x + (1-\alpha)y \tag{2}$$

Conversely, the following unsurprising but very useful "embedding" theorem, due originally to Marshall Stone [16], shows that every mixture space can be represented in this way:

Theorem 1.1 *The elements of any mixture space M can be identified with the points of a convex subset of a suitably chosen real vector space V in such a way that (2) holds. One can further stipulate that the origin of V is in M and that no proper subspace of V contains M. Moreover, with these stipulations the embedding is unique up to a linear transformation.*

A consequence of this result is that M and V can be made to have the same cardinal dimension. We shall call such embeddings *efficient.*

A basic example of a mixture space is the set of probability distributions over a set P of "pure" prospects, giving M the form of a probability simplex if P happens to be finite. But there are other possibilities. For example, the pure prospects might be a set of physical states, described in terms of numerical parameters, and we might be willing to assume that only the expected values of these parameters are significant. Then, M would take the form of a more or less arbitrary convex set in the parameter space, which could have significantly lower dimension than the set of probability measures on the physical alternatives themselves. Or in another example, there might be no pure prospects at all, M being an open set.

1.2 Incomplete preference relations

In this section we shall axiomatize the preference relation so as to obtain a mapping of M into a risk-neutral utility scale which is "cardinal" yet only partially ordered or incomplete in general. We shall write "$x \gtrsim y$" to mean that prospect x is preferred or indifferent to prospect y, and denote the relation itself by the distinct symbol $\succcurlyeq$.

Here are the axioms for an *incomplete preference* $\succcurlyeq$ —they are asserted for all $x, y, z, w \in M$ and all $\alpha \in [0, 1]$:

(P1) $x \gtrsim x$ *(Reflexivity)*

(P2) *If* $x \gtrsim y$ *and* $y \gtrsim z$, *then* $x \gtrsim z$ *(Transitivity)*

(P3) *If* $x \gtrsim y$, *then* $x\alpha z \gtrsim y\alpha z$ *(Independence)*

(P4) *The set* $\{\alpha : x\alpha y \gtrsim z\alpha w\}$ *is closed.* *(Continuity)*

The first two of these axioms establish $\succcurlyeq$ as a *partial ordering*, while the other two relate it to the mixing operation. If both $x \gtrsim y$ and $y \gtrsim x$ we say that x and y are *indifferent* and write $x \sim y$. If neither $x \gtrsim y$ nor $y \gtrsim x$ we say that x and y are *incomparable* and write $x|y$. Finally, if $x \gtrsim y$ but not $y \gtrsim x$, we say that x is *strictly preferred* to y and write $x \succ y$. If M has no incomparable pairs then $\succcurlyeq$ is said to be a *complete preference*; this can be expressed axiomatically by strengthening **(P1)** to

(P1′) For all $x, y \in M$, either $x \gtrsim y$ or $y \gtrsim x$ *(Completeness)*

The independence axiom (**P3**) asserts the "conservation of preference" under the equal admixture of a third prospect. The continuity axiom (**P4**) eliminates the "non-Archimedean" phenomenon of two prospects that are so apart in the scheme of preferences that the gap cannot be bridged continuously by probability mixes. Specifically, the possibility that for some $x, y \in M$, $x \gtrsim x\alpha y$ might hold for all $\alpha > 0$ but not for $\alpha = 0$ is eliminated. The continuity axiom used by von Neumann and Morgenstern [18], as well as by Marschak [9] and Herstein and Milnor [7], required merely that the set $\{\alpha : x \gtrsim z\alpha w\}$ be closed. The added strength of our (**P4**) allows us to state[1],

Lemma 1.2 *If $\alpha > 0$ and $p\alpha r \gtrsim q\alpha r$, then $p \gtrsim q$*

Proof. Let $\tau = \sup\{\alpha \in [0,1] : x\alpha z \gtrsim y\alpha z\}$, that is well defined since we always have $x0z \gtrsim y0z$, by $x0y = y$ and (**P1**). By (**P4**), we have $x\tau z \gtrsim y\tau z$, and by (**P3**) we have $(x\tau z)\beta x \gtrsim (y\tau z)\beta x$ and $(x\tau z)\beta y \gtrsim (y\tau z)\beta y$ for all $\beta \in [0,1]$. Setting $\beta = \frac{1}{1+\tau}$ and using (**M1**) and (**M2**), we calculate

$$(x\tau z)\beta y = y\bar{\beta}(z\bar{\tau}x) = (y\frac{\bar{\beta}}{\beta}z)\beta x = (y\tau z)\beta x \tag{4}$$

writing $\bar{\beta}$ for $1-\beta$, etc. and using the fact that $\bar{\beta}+\bar{\tau}-\bar{\beta}\bar{\tau} = \beta > 0$. By (**P2**) it follows that $(x\tau z)\beta x \gtrsim (y\tau z)\beta y$, which reduces by (**M1**), (**M2**), (**M3**) to

$$x\frac{2\tau}{1+\tau}z \gtrsim y\frac{2\tau}{1+\tau}z \tag{5}$$

By the definition of τ, this implies that $\tau \geq \frac{2\tau}{1+\tau}$, which is equivalent to the inequality $\tau^2 \geq \tau$. Thus, τ is either 0 or 1. But $\tau = 0$ is excluded by the hypotheses that $\alpha > 0$, so τ is 1 and (5) reduces to $x \gtrsim y$ by $x1z = x$ and $y1z = y$, as was to be shown. ∎

Observe that the foregoing proof made no use of Theorem 1.1, nor of the cancellation law (**M4**). Nevertheless, Lemma 1.2 implies the weaker cancellation law

$$If\ x\alpha z \sim y\alpha z\ and\ \alpha > 0,\ then\ x \sim y \tag{6}$$

[1] An alternative route is to adopt the continuity axiom in this second form and strenghten the independence axiom to

$$x \gtrsim y\ if\ and\ only\ if\ x\alpha z \gtrsim y\alpha z,\ if\ \alpha \neq 0 \tag{3}$$

thus avoiding the lemma altogether.

This would allow us to carry out the embedding without benefit of (**M4**) if we were willing to collapse M into its indifference classes. But it is better to retain (**M4**) in order to avoid having to make the embedding of M in V dependent on the preference relation, since in our application we shall be defining many different preference relations on the same mixture space.

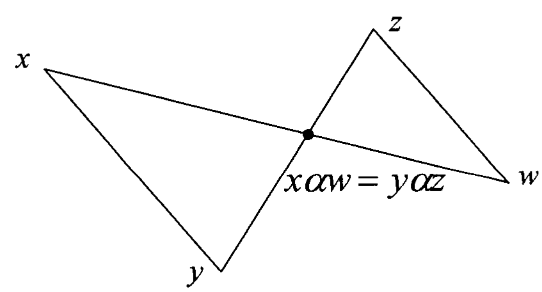
1.3 Representation of incomplete preferences by domination cones

From this point forward we shall omit citation of the individual axioms (**M1**) – (**M4**) in our proofs and take the efficient embedding of M in V for granted, using vector-space terminology freely, with the use of $x\alpha y$ or $\alpha x + (1-\alpha)y$ interchangeably. We avoid possible confusion by using only Greek letters to indicate the mixing parameters.

Lemma 1.3 *For all $x, y, z, w \in M$, if $x - y = \lambda(z - w)$ for some scalar $\lambda > 0$, then*

$$x \gtrsim y \; iff \; z \gtrsim w \tag{7}$$

Proof. (See Figure below) Setting $\alpha = \frac{1}{1+\lambda}$, we have $x\alpha w = y\alpha z$ by simple calculation (note the similar triangles). By independence and Lemma 1.2 we then have $x \gtrsim y \;\; iff \;\; x\alpha z \gtrsim y\alpha z \;\; iff \;\; x\alpha z \gtrsim x\alpha w$, as was to be shown. ∎



Geometrically, $x - y = \lambda(z - w)$ with $\lambda > 0$, means that the directed segments $\overrightarrow{xy}$ and $\overrightarrow{zw}$ are parallel and similarly oriented. Lemma 1.3 therefore asserts that the "directions of preference" are constant throughout M.

The next lemmas establish further geometric properties of the relation $\succcurlyeq$ and prepare the way for the first representation theorem. First, we define the *dominion* of an element $x \in M$ by $D(x) = \{y \in M : x \gtrsim y\}$. It can easily be verified by (**P1**) and (**P2**) that $D(x) = D(y)$ if and only if $x \sim y$.

Lemma 1.4 *For each $x \in M$, $D(x)$ is a convex cone in M with vertex x.*

Proof. (*a*) *Convexity.* Suppose that y and z are in $D(x)$. Then, using independence (twice) and transitivity, we have

$$x = x\alpha x \gtrsim x\alpha z \gtrsim y\alpha z \text{ for all } \alpha \in [0,1] \tag{8}$$

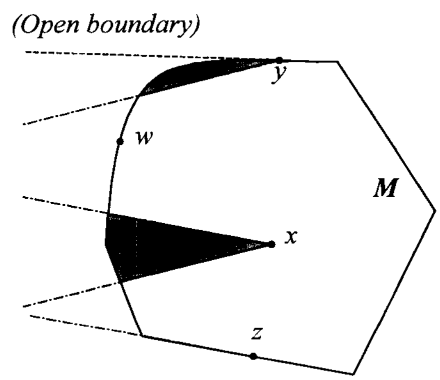
In other words, $y\alpha z \in D(x)$. It follows that $D(x)$ is convex.

(*b*) *Cone property.* It will suffice to show that $y \in D(x)$ if and only if $y\alpha x \in D(x)$, for arbitrary $\alpha \in (0,1)$. But this is immediate from independence and its converse, Lemma 1.2, taking $z = x$. ■

Although a cone in M is not ordinarily a cone in V, it becomes one if we extend it by scalar multiplication. Define the *extended cone* of $D(x)$:

$$\bar{D}(x) = \{r \in V : r - x = \lambda(y - x) \textit{ for some } y \in D(x) \textit{ and } \lambda > 0\} \tag{9}$$

However, despite the constant directions of preference promised by Lemma 1.3, even the $\bar{D}(x)$ sets may take on quite a variety of different shapes when x is in the boundary of M (see figure below).



But there is a remedy. In the finite-dimensional case we can start with any x in the interior of M (which is nonempty because we assumed an efficient embedding), take its extended cone $\bar{D}(x)$, translate it to the origin by subtracting x, and define thereby a cone, D, which we call the *domination cone* of $\succcurlyeq$. By Lemma 1.3 D does not depend on which interior point of M we started from. Moreover, for all $x \in M$, we have

$$D(x) = (x + D) \cap M \tag{10}$$

and so, for all $x, y \in M$,

$$x \succsim y \; iff \; y - x \in D \tag{11}$$

Thus the cone D completely characterizes the relation $\succcurlyeq$.

More generally, in the infinite-dimensional case where the existence of interior points is not assured, we may define

$$D = \cup_{x \in \mathbf{M}} (\bar{D}(x) - x) \tag{12}$$

and then (10) and (11) will still be valid.

A convex set in a vector space whose intersection with every line is a closed interval (possibly empty or unbounded) will be called *linearly closed.* It can be shown that this property implies that the intersection with any finite-dimensional linear (affine) subspace is also a closed set in the ordinary Euclidean topology.

Lemma 1.5 *D is a linearly closed, convex cone in V with vertex* 0.

Proof. (a) *Convexity*: Given $p, q \in D$, we can find $x, y \in M$ such that $p \in \bar{D}(x)$ and $q \in \bar{D}(y)$. We claim that for any $\alpha \in (0,1)$, both p and q are in $\bar{D}(x\alpha y)$. Since $\bar{D}(x\alpha y)$ is convex and a subset of D, the convexity of D follows.

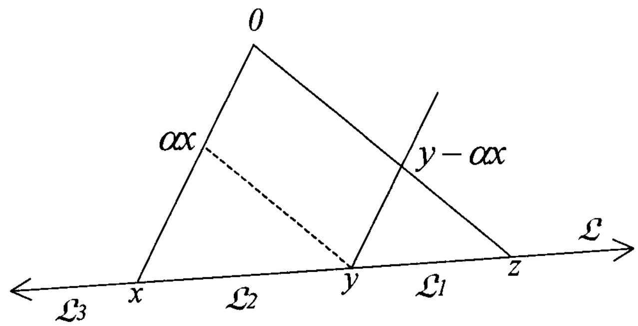
To prove the claim, let $z \in M$ be such that $\lambda p = z - x$ for some positive λ. Then $z\alpha y \in M$, by (**M1**), and we have

$$z\alpha y - x\alpha y = \alpha z + (1-\alpha)y - \alpha x - (1-\alpha)y = \alpha(z - x) = \lambda\alpha p \tag{13}$$

so $p \in \bar{D}(x\alpha y)$. Similarly, $q \in \bar{D}(x\alpha y)$.

(b) *Cone property*: This is obvious from the fact that, by definition, $p \in D$ if and only if there are $x \in M$ and $\lambda > 0$ such that $x + \lambda p \in M$.

(c) *Linear closure*: Let $\mathfrak{L}$ be any line contained in V. If $\mathfrak{L} \cap D$ is empty or a single point there is no problem, so we may assume that we have two distinct points $x, y \in \mathfrak{L} \cap D$. If $0 \in \mathfrak{L}$ there is again no problem, so we may assume that x and y are linearly independent. As shown in (a), there is an element of M having both x and y in its extended domination cone, and by Theorem 1.1 we may take this element to be 0. By (b), there are positive numbers λ_1 and λ_2 such that $\lambda_1 x$ and $\lambda_2 y$ are both in M. By convexity, this remains true if λ_1 and λ_2 are replaced by $\lambda = \min(\lambda_1, \lambda_2)$. Moreover, since D is a cone, $\mathfrak{L} \cap D$ is a closed interval if and only if $\lambda\mathfrak{L} \cap D$ is a closed interval. So we might as well take $\lambda = 1$, thereby reducing the problem to the case of three, non-collinear points, 0, x and y, all contained in M.



Now for the proof itself. We have

$$\text{For all } \alpha \in [0,1],\ \alpha x \succsim y \ \textit{if and only if}\ y - \alpha x \in D \tag{14}$$

If $\alpha \neq 1$, define $\gamma(\alpha) = \alpha/(1-\alpha)$ and $z(\gamma) = y + \gamma(y - x)$, where the domain of γ is $[0, \infty)$. We then have

$$y - \alpha x = (1-\alpha)y + \alpha(y - x) = (1-\alpha)z(\gamma(\alpha)), \tag{15}$$

for $\alpha \in [0, 1)$, and so

$$\alpha x \succsim y \text{ if and only if } z(\gamma(\alpha)) \in D. \tag{16}$$

Now consider two sets of real numbers, namely, $\Lambda = \{\alpha \in 0, 1] : \alpha x \succsim y\}$ and $\Gamma = \{\gamma \in [0, \infty) : z(\gamma) \in D\}$. The set Λ is closed, by continuity, and Γ is convex, by (a). Also, both sets contain 0, by definition. If Γ is bounded then Λ is bounded away from 1, so Γ is the continuous pre-image of a closed set and hence closed. But if Γ is not bounded, then it is exactly the set $[0, \infty)$, by convexity, and hence again is closed. Since $z(\gamma) = y + \gamma(y - x)$ sweeps out that portion of $\mathfrak{L}$ marked $\mathfrak{L}_1$ in Figure above, we have shown that $\mathfrak{L}_1$ meets D in a nonempty closed set, including y. Similarly, $\mathfrak{L}_3$ meets D in a nonempty closed set including x. As the intervening points between x and y are all in D by convexity, we conclude that $\mathfrak{L}$ itself meets D in a closed set. This completes the proof. ■

We have now arrived at the main result of this section, the First Representation Theorem for incomplete preference relations.

Theorem 1.6 (First Representation Theorem) *Let $\succcurlyeq$ be an incomplete preference defined on a mixture space M efficiently embedded in a vector space*

V with $0 \in M$. Then there exists a linearly closed convex cone D in V, with vertex 0, such that for all $x, y \in M$

$$x \gtrsim y \; iff \; y - x \in D \tag{17}$$

Conversely, let D be any linearly closed convex cone in V and let M be any mixture space efficiently embedded in V. Then the relation $\succcurlyeq$ that 17 defines is an incomplete preference.

Proof. The first part is covered by the preceding lemmas. The converse can be verified by direct reference to the axioms and definitions. ∎

1.4 A generalization of utility functions

If our theory were concerned only with complete preferences, the next step would be the introduction of what is sometimes called a "measurable utility." (Compare Herstein and Milnor [7].) This is a linear function u from M to the reals that characterizes the preference relation $\succcurlyeq$ by means of the rule:

$$x \gtrsim y \; iff \; u(x) \geq u(y). \tag{18}$$

To correspond to this device, in our incomplete system, we shall now develop the notation of a "utility set" U, which is a family of utilities which represent $\succcurlyeq$ by means of the rule:

$$x \gtrsim y \; iff \; u(x) \geq u(y) \; for \; all \; u \in U. \tag{19}$$

Proceeding with the formal development, let us denote by M^* the set of all real-valued functions on M that are both *linear*, in the sense that always

$$u(x\alpha y) = \alpha u(x) + (1 - \alpha)u(y) \tag{20}$$

and *homogeneous*, in the sense that they vanish at some prescribed point $0 \in M$. It is convenient to take 0 as the origin of an efficient embedding space V. Then, M^* coincides with V^*, the space of *homogeneous* linear functions on V. In any case, M^* is a linear space in its own right, of dimension equal to that of M in the finite-dimensional case[2].

[2] In the infinite-dimensional case, M^* is bigger than the ordinary dual space because we have not required that its elements be continuous with respect to some stated topology

Let D be the domination cone of some $\succcurlyeq$, and let D^* be its *polar cone* in M^* defined by

$$u \in D^* \; iff \; u(x) \leq 0 \; for \; all \; x \in D \tag{22}$$

Similarly, the "polar" of D^* may be defined:

$$x \in D^{**} \; iff \; u(x) \leq 0 \; for \; all \; u \in D^* \tag{23}$$

The weak inequalities in these definitions ensure that D^* and D^{**} are both linearly closed, convex cones with vertex 0; they are of course both convex. It is evident that $D \subseteq D^{**}$. If equality were guaranteed here, the two polar mappings would be mutual inverses, and D^* could be used unambiguously to represent the incomplete preference $\succcurlyeq$. In the finite-dimensional case equality is indeed always assured, from the fact that D is linearly closed, and hence closed. In the infinite-dimensional case, however, linear closure is not enough by itself; some further condition must be fulfilled. The one presented in the following lemma has the virtue of not involving any topological structure in M, beyond what is already implied by the idea of linear closure.

A point x in a linear space V is called *internal* to a convex set E if, for every $y \in V$, there is a $y' \in E$ with $x = y\alpha y'$ for some $\alpha \in (0, 1)$. x is called *relatively internal* if the above holds with y restricted to E. Every finite-dimensional convex set has a relatively internal point.

Lemma 1.7 *Let D be a linearly closed, convex cone in an arbitrary linear space V. Then, if D has a relatively internal point, $D = D^{**}$.*

Proof. As already remarked, we have $D \subseteq D^{**}$. To show that $D^{**} \subseteq D$, let V_1 be the smallest subspace of V containing D and let y be any point in $V \backslash V_1$. A linear function u can be found that vanishes on V_1 and has the value 1 at y. By (22), $u \in D^*$, and by (23), $y \notin D^{**}$. Hence $D^{**} \subseteq V_1$, and we can confine our attention to that subspace. Now let z be any point in $V_1 \backslash V$, and let x be a relatively internal point of D. Since D is linearly closed,

on M. This extra discriminatory power ensures that any two distinct points in M can be separated, i.e.,

$$If \; x \neq y, \; then \; u(x) \neq u(y) \; for \; some \; u \in M^* \tag{21}$$

and will enable us to extend linear functions defined on lower-dimensional subsets of M (or V) to the whole space.

we can find on the segment xy a point $z' \neq z$ that is not in D. Since x is actually internal to D in the subspace V_1, a standard separation theorem can be applied that asserts the existence of a nontrivial linear function u_1 on V_1 that is non-positive on D and nonnegative at z'. But $u_1(x)$ is easily seen to be strictly negative, from the internal situation of x, and consequently $u_1(z)$ must be strictly positive. If we now extend u_1 to the full space V, we see that the extended function is an element of D^*, by (22). Hence $z \notin D^{**}$, by (23) and we conclude that $D^{**} \subseteq D$. ∎

Assuming that the internal-point condition is met, we have in a sense achieved our announced aim of producing a "utility set" D^*, which is capable of reproducing the ordering $\succcurlyeq$ via the rule (19). However, the number of inequalities to be considered —the cardinality of D^*— is far greater than necessary. For example, we clearly do not need to verify $u(x) \geq u(y)$ for functions u that are nonnegative multiples, or convex combinations, of functions that have already passed the test. In some cases, at least, we should expect that a few extremal elements of D^* would be sufficient.

Accordingly, let us make the following formal definition: A *utility set* of an incomplete preference $\succcurlyeq$ with domination cone D, is any nonempty subset U of M^* such that (i) the polar of U (in the sense of 23) is precisely D, and (ii) the origin 0 of M^* is not in U, unless $U = \{0\}$.

For want of a better term, let us call $\succcurlyeq$ *proper* if its domination cone contains a relatively internal point. Lemma 1.7 then assures us that every proper incomplete preference has at least one utility set —namely, either $D^*\backslash\{0\}$ or $\{0\}$. We emphasize that all incomplete preferences on a finite-dimensional prospect space are proper.

In order for a set U (not containing the origin) to be a utility set for a proper incomplete preference, it is necessary and sufficient that the linear closure of the set of positive multiples of convex combinations of elements of U be equal to D^*. A great many preference relations —they might be called "polyhedral"— possess *finite* utility sets. If M is finite-dimensional, the utility set need be at most countable infinite. But note that even in a minimal utility set, there may have to be included some non-extremal elements. For example, if U is a half space, then three points are required, one of them internal. A one-point utility set of course implies a complete preference, and vice versa.

We sum up the principal result of this section:

Theorem 1.8 (Second Representation Theorem) *Let $\succcurlyeq$ be a proper in-*

complete preference defined on a mixture space M efficiently embedded in a linear space V with $0 \in M$. Let M^* be the space of all homogeneous, linear functions from M to the reals. Then, there exists a nonempty subset $U \subseteq \mathbf{M}^*$, not containing 0 unless $U = \{0\}$, such that for all $x, y \in M$,

$$x \succsim y \; iff \; u(x) \geq u(y) \; for \; all \; u \in U \tag{24}$$

Conversely, given any such set U, the relation $\succcurlyeq$ defined by (24) is a proper incomplete preference, in the sense of axioms (**P1**) – (**P4**).

Proof. Only the converse remains to be established. Since U^* is easily seen to be linearly closed, convex cone in V with vertex 0, it follows from the first representation theorem that $\succcurlyeq$ is an incomplete preference in the sense of (**P1**) – (**P4**). To show that it is proper, we must show that $U^* = U^{***}$, since these are the "D" and "D^{**}" of the ordering. We have $U \subseteq U^{**}$ and $U^* \subseteq U^{***}$ immediately from the definitions of "polar," (22) and (23). But it is obvious from (23) that "polar" is a monotonic decreasing function from subsets of M^* to subsets of M; hence $U \subseteq U^{**}$ implies $U^* \supseteq U^{***}$. ∎

1.5 Examples of incomplete preferences

We close this section by describing the preference relations associated with certain special types of domination cones D and their corresponding polar cones D^* and utility sets U. The following are equivalent ways to characterize several incomplete orderings.

1. When D is a half space, D^* consist of a ray defined by nonnegative multiples of some $u \neq 0$ and $U = \{u\}$. In this case, $\succcurlyeq$ is a nontrivial complete preference.
2. When D is a subspace of V, D^* is a subspace of M^* and U is any base for such subspace. In this case, $\succcurlyeq$ has no strict preferences and M decomposes into a collection of mutually incomparable indifference classes. Two extreme sub-examples are:
   (a) $D = V$, $U = \{0\}$ and $\succcurlyeq$ is trivial; *for all* $x, y \in M$, $x \sim y$.
   (b) $D = \{0\}$, $D^* = M^*$ and $\succcurlyeq$ is the empty relation; *for all* $x, y \in M$, $x \neq y$, $x|y$.
3. When D contains complete lines, D^* is less than full-dimensional and $\succcurlyeq$ contains segments of indifference.

2 Multiperson Utility

Having developed a theory of incomplete preferences, we now proceed to utilize it to establish preferences for coalitions of agents. Before proceeding, a word on notation.

We will consider a finite set of individuals $N = \{1, \ldots, n\}$. Nonempty subsets S of N are called *coalitions*. Since we will working with coalitions, we will usually denote an individual $i \in N$ as $\{i\}$. Collections of coalitions will be written with calligraphic characters, like the set of all coalitions $\mathcal{P}$ or the set of single member coalitions $\mathcal{A} = \{\{i\} \subseteq N : i \in N\}$. Finally, we shall reserve "$\subset$" for strict inclusion.

2.1 The Pareto rule and its representation

We begin by assuming a set of preferences $\succsim_{\{i\}}$, possibly incomplete, for the individuals in N. How are we to define preferences for coalitions? We might want to say that a coalition S prefers x to y if and only if all of its members do. However, this preference will in general not be complete unless all the individuals in S have identical complete preferences. Therefore, we want to require that

$$If\ x \succsim_{\{i\}} y\ for\ all\ i \in S,\ then\ x \succsim_S y \tag{25}$$

We will impose a more general condition that we call the Pareto rule. Given a collection of incomplete preferences $\succsim_S$ for each coalition S, let A and B be any two *disjoint* coalitions. Then,

(P5) *For all $x, y \in M$, if $x \succsim_A y$ and $x \succsim_B y$, then $x \succsim_{A \cup B} y$ (Pareto rule)*

If $\mathcal{Q}$ is a partition of S, the Pareto rule requires that if $x \succsim_A y$ for all $A \in \mathcal{Q}$, then $x \succsim_S y$. In particular, (25) holds. By the first representation theorem, we see that the Pareto rule also imposes the following relations on the domination cones and the polar cones. For any two disjoint coalitions A and B, we have:

$$D_A \cap D_B \subseteq D_{A \cup B} \tag{26}$$

Since $(D_A \cap D_B)^* = Co(D_A^* \cap D_B^*)$[3] we also have that,

$$D_{A\cup B}^* \subseteq Co\,(D_A^* \cup D_B^*) \tag{27}$$

For (27) to have some power we need to have some control on the size of $Co\,(D_A^* \cup D_B^*)$. Recall that $D_{\{i\}}^*$ are cones in M^*. If the individual preferences where extremely different and spring in all directions, then $Co\,(D_A^* \cup D_B^*)$ would grow and even be identical to M^*. On the other extreme, if an individual i is totally indifferent, then $D_{\{i\}}^* = \{0\}$ and he does not have any influence in (27). We can therefore set those individuals apart and suppose that individuals have non-trivial preferences.

We will impose two conditions to control the size of the convex hulls. First, we assume that individuals have non-trivial *complete* preferences so that the $D_{\{i\}}^*$ are rays. Second, we assume the existence of at least one direction of common preference. Formally, we say that the members of N *minimally agree* if there are two prospects $x_0, x_1 \in M$ such that

(P6) *For all* $i \in N$, $x_1 \succ_{\{i\}} x_0$ *(Minimal agreement)*

Without loss of generality, we can set the origin of M to be x_0 and define the following projection hyperplane in M^*

$$W = \{u \in M^* : u(x_1) = 1\} \tag{28}$$

Since $u(x_1) > 0$ for all $u \in D_{\{i\}}^* \backslash 0$ all the rays $D_{\{i\}}^*$ intersect W in a single point that we can take as the utility set. Thus, for all $i \in N$, we let

$$U_{\{i\}}^* = D_{\{i\}}^* \cap W = \{u_{\{i\}}\} \tag{29}$$

where $u_{\{i\}}(x_1) = 1$.

To summarize the previous discussion, we present the following representation theorem for the Pareto rule.

Theorem 2.1 (Third Representation Theorem) *Given a collection of incomplete preferences* $\succsim_S$ *for each coalition* S*, let* A *and* B *be any two disjoint coalitions. Then, the following are equivalent to the Pareto rule:*

1. $D_A \cap D_B \subseteq D_{A\cup B}$

[3] See, for example, Rockafellar, page 149-151.

2. $D^*_{A \cup B} \subseteq Co\,(D^*_A \cup D^*_B)$

   Moreover, if we have non-trivial individual preferences and minimal agreement we may define for all coalitions S, $W_S = D^*_S \cap W$, and the equivalence extends to

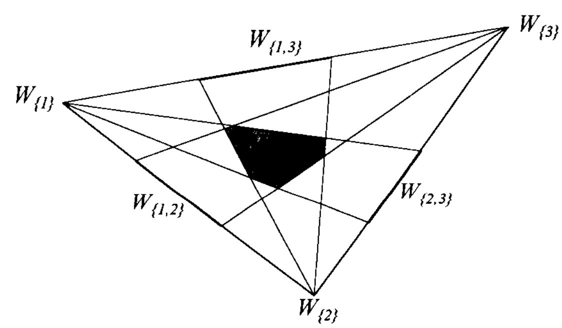
3. $W_{A \cup B} \subseteq Co\,(W_A \cup W_B)$

Proof. The first two equivalences follow from previous remarks. For the third, note that the individual preferences are characterized by $u_{\{i\}}$ given in (29). From (3) it easily follows that $D^*_S \subseteq Co(\cup_{i \in S} D^*_{\{i\}})$. Since $D^*_{\{i\}}$ is characterized by $u_{\{i\}}$ it follows that D^*_S is also characterized by W_S and we conclude that (3) is equivalent to (4). ■

From the previous section, we recall that the "size" of W_S reflect the incompleteness of $\succcurlyeq_S$. The coalition preferences are the more incomplete possible when $W_{A \cup B} = Co\,(W_A \cup W_B)$ for all disjoint coalitions A and B. This is precisely the case when all the $\succcurlyeq_S$ are defined as:

$$x \succsim_S y \; iff \; x \succsim_{\{i\}} y \; for \; all \; i \in S \tag{30}$$

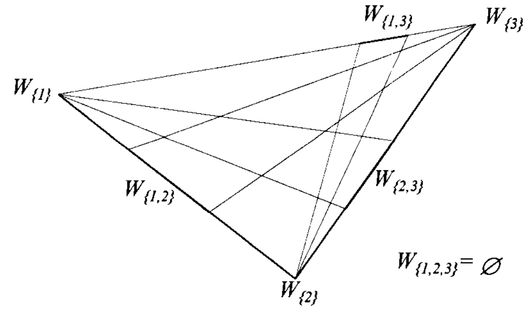
For a given coalition S, there are $2^{|S|-1} - 1$ ways to divide S in two coalitions. If if $W_{A \cup B} \subset Co\,(W_A \cup W_B)$ for some disjoint coalitions A and B with $A \cup B = S$, then we can control the size of W_S (See figure below).



Note in the figure that since $W_{\{1\}} = \{u_{\{1\}}\}$ and $W_{\{2\}} = \{u_B\}$ then

$$W_{\{1,2\}} \subseteq Co\left(W_{\{1\}} \cup W_{\{2\}}\right) = \{u_A \alpha u_B : 0 \leq \alpha \leq 1\} \tag{31}$$

The restrictions in W_S can be such that it forces W_S to be empty (See figure below). We then conclude that there is no preference for the coalition S that is consistent with the Pareto rule, something that we want to avoid.



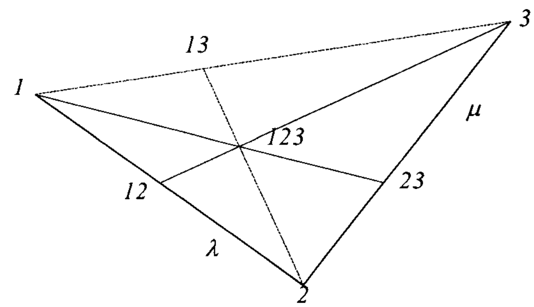
2.2 Building preferences for coalitions based on bilateral agreements

Consider $N = \{1, 2\}$ and utilities $u_{\{1\}}, u_{\{2\}} \in W$. The only restriction that the Pareto Rule imposes on $W_{\{1,2\}}$ is $W_{\{1,2\}} \subseteq \overleftrightarrow{u_{\{1\}}u_{\{2\}}}$. Since our aim is to build complete preferences for the group, we have to require that individuals 1 and 2 agree on a common preference represented by some utility $u_{\{1,2\}} \in \overleftrightarrow{u_{\{1\}}u_{\{2\}}}$. We then say that the pair $\{1, 2\}$ has reached a *bilateral agreement.* In this case, there is a unique $\lambda \in [0, 1]$ such that $u_{\{i,j\}} = u_{\{i\}}\lambda u_{\{j\}}$. If $\lambda \in (0, 1)$ we say that the pair $\{1, 2\}$ has reached a *compromising agreement.*

At this point, an important conceptual or "philosophical" point arises: the notion of transferrable utility depends essentially on the ability of two persons to communicate and agree on a common preference. In other words, we have an intrinsic relational aspect in building a joint preference.

For $x, y \in W$, $x \neq y$, we denote by $\overleftrightarrow{xy}$ the open segment $\{x\alpha y : 0 < \alpha < 1\}$, by $\overrightarrow{xy}$ the half line starting at x and passing through y not containing x and by $\overline{xy}$ the line that connects x and y. If $x = y$ then $\overleftrightarrow{xy} = \overrightarrow{xy} = \overline{xy} = x$.

Now, consider $N = \{1, 2, 3\}$ with utilities $u_{\{1\}}, u_{\{2\}}, u_{\{3\}} \in W$. Suppose those utilities are *non-collinear*, i.e., there is no line that contains them. Assume that $\{1, 2\}$ and $\{2, 3\}$ have both reached compromising agreements represented by $u_{\{1,2\}} \in \overleftrightarrow{u_{\{1\}}u_{\{2\}}}$ and $u_{\{2,3\}} \in \overleftrightarrow{u_{\{2\}}u_{\{3\}}}$ respectively (see figure below where "12", for example, indicates $u_{\{1,2\}}$). By the third representation theorem, $W_{\{1,2,3\}}$ is contained in both $\overleftrightarrow{u_{\{1\}}u_{\{2,3\}}}$ and $\overleftrightarrow{u_{\{3\}}u_{\{1,2\}}}$. If these two segments intersect in a single point $u_{\{1,2,3,\}}$, then $U_{\{1,2,3\}}$ will be a singleton. In other words, the group has a complete preference!



$$U_{\{1,2,3\}} \subseteq \overleftrightarrow{u_{\{1\}}u_{\{2,3\}}} \cap \overleftrightarrow{u_{\{3\}}u_{\{1,2\}}} = \{u_{\{1,2,3\}}\}$$

If λ and μ are such that $u_{\{1,2\}} = u_{\{1\}}\lambda u_{\{2\}}$ and $u_{\{2,3\}} = u_{\{2\}}\mu u_{\{3\}}$, then we obtain by calculation that

$$u_{\{1,2,3\}} = \tfrac{1}{1-\lambda(1-\mu)}[\lambda\mu u_{\{1\}} + (1-\lambda)\mu u_{\{2\}} + (1-\lambda)(1-\mu)u_{\{3\}}] \tag{32}$$

The representation theorem also imposes that $W_{\{1,2,3\}} \subseteq Co(u_{\{2\}}, W_{\{1,3\}})$, or

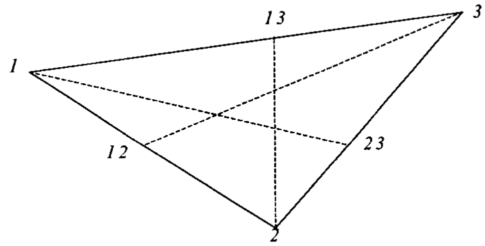
$$u_{\{1,2,3\}} \in Co(u_{\{2\}}, W_{\{1,3\}}) \tag{33}$$

therefore, if we denote by $u_{\{1,3\}}$ the intersection point of $\overrightarrow{u_{\{2\}}u_{\{1,2,3\}}}$ and $\overleftrightarrow{u_{\{1\}}u_{\{3\}}}$, then we conclude that for (33) to hold we require $u_{\{1,3\}} \in W_{\{1,2,3\}}$. Calculation shows that

$$u_{\{1,3\}} = \tfrac{1}{\lambda\mu+(1-\lambda)(1-\mu)}[\lambda\mu u_{\{1\}} + (1-\lambda)(1-\mu)u_{\{3\}}] \tag{34}$$

If $u_{\{1,3\}} \notin W_{\{1,2,3\}}$ then $W_{\{1,2,3\}} = \emptyset$, as illustrated before. Also, observe that the fact that $\lambda, \mu \in (0,1)$ guarantees that the denominator $1-\lambda(1-\mu)$ in (32) and $\lambda\mu + (1-\lambda)(1-\mu)$ in (34) do not vanish.

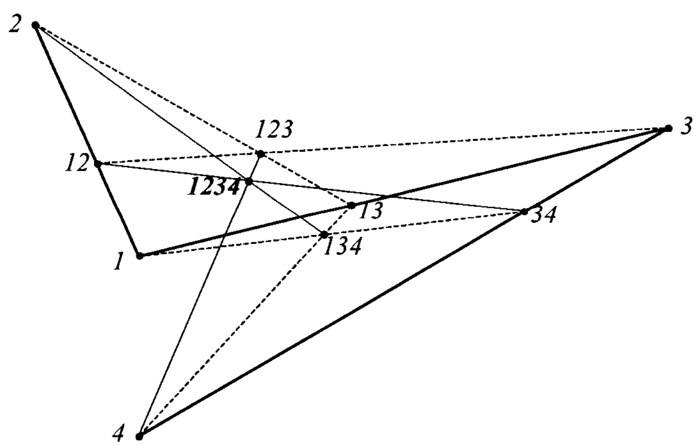
Continuing with our presentation, we have seen that with two bilateral agreements is possible to build complete preferences for a set of three agents. How are we to generalize the previous analysis to the case of any N? First, it is clear that if bilateral agreements are the main ingredient, then all the individual agents have to be involved in some agreement. Otherwise, there is no way to construct utilities for coalitions involving a "disconnected" individual. Second, we cannot allow the bilateral agreements to form cycles, otherwise we encounter inconsistencies. For example, the figure below illustrates how the bilateral agreements $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ lead to $W_{\{1,2,3\}} = \emptyset$.



$$W_{\{1,2,3\}} \subseteq \overleftrightarrow{u_{\{1\}}u_{\{2,3\}}} \cap \overleftrightarrow{u_{\{2\}}u_{\{1,3\}}} \cap \overleftrightarrow{u_{\{3\}}u_{\{1,2\}}} = \emptyset$$

We can think of the individuals and the agreements as nodes and edges of a graph, respectively. Our conjecture is that he need to form a connected and acyclic graph, i.e., a *spanning tree of compromising agreements.*

To test the conjecture, consider the case of $N = \{1,2,3,4\}$, with the spanning tree of compromising agreements being $\{1,2\}$, $\{1,3\}$ and $\{3,4\}$. Assume that the triples $u_{\{1\}}, u_{\{2\}}, u_{\{3\}}$ and that triple $u_{\{1\}}, u_{\{3\}}, u_{\{4\}}$ are both not-collinear. We can repeat the previous analysis to obtain $u_{\{1,2,3\}}$ and $u_{\{2,3,4\}}$. Now, we can use any two of three segments $\overleftrightarrow{u_{\{1,2,3\}}u_{\{4\}}}$, $\overleftrightarrow{u_{\{1\}}u_{\{2,3,4\}}}$ and $\overleftrightarrow{u_{\{1,2\}}u_{\{3,4\}}}$ to obtain u_N. The obvious question is whether or not these three line segments have a point in common (see Figure below). Otherwise, we should conclude that $W_N = \emptyset$, a fact that would frustrate our theoretical development. The problem is manifestly non-trivial as the size of N increases, since the number of segments passing through u_N is $2^{|N|-1} - 1$.



Are the segments $\overleftrightarrow{u_{\{12\}}u_{\{3,4\}}}$, $\overleftrightarrow{u_{\{1\}}u_{\{2,3,4\}}}$ and $\overleftrightarrow{u_{\{1,2,3,\}}u_{\{4\}}}$ concurrent?

2.3 A geometrical theorem

Fortunately, the *Desargues Theorem*, a result due to the $17th$ century French mathematician Gerard Desargues, will dissolve this difficulty. We use the theorem to prove the following geometric result.

Lemma 2.2 *Suppose that* p_1, p_2, p_3 *and* q_1, q_2, q_3*, are two sets of non-collinear points in a vector space* W *satisfying* $p_i \neq q_i$ $(i = 1, 2, 3)$*. Assume that the intersection of the segments* $\overleftrightarrow{p_1q_1}$ *and* $\overleftrightarrow{p_3q_3}$ *define a point* w*. Also, consider the three points* $s_{12} = \overline{p_1p_2} \cap \overline{q_1q_2}$*,* $s_{13} = \overline{p_1p_3} \cap \overline{q_1q_3}$ *and* $s_{23} = \overline{p_2p_3} \cap \overline{q_2q_3}$*. Then,*

$$\text{if } s_{13} \in \overleftrightarrow{s_{12}s_{23}}, \text{ then } w \in \overleftrightarrow{p_2q_2} \tag{35}$$

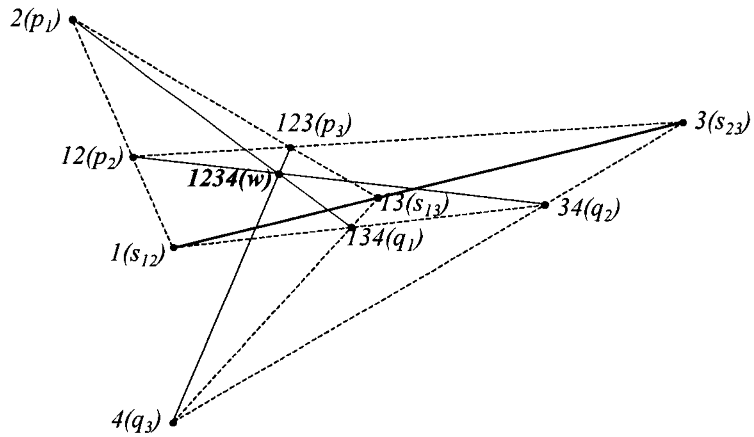
Proof. The Desargues Theorem involves lines and asserts that the three lines $\overline{p_iq_i}$ $(i = 1, 2, 3)$ are concurrent at some point w if the three points s_{12}, s_{13} and s_{23} are collinear. In the proof of the theorem we obtain $w = p_1\alpha_1q_1 = p_3\alpha_3q_3 = p_2\alpha_2q_2$, for some unrestricted α_i. Those values are algebraically linked by $\lambda = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}$ or

$$\alpha_2 = \lambda\alpha_3 + (1 - \lambda)\alpha_1 \tag{36}$$

where λ comes from $s_{13} = s_{12}\lambda s_{23}$. If $w = \overleftrightarrow{p_1q_1} \cap \overleftrightarrow{p_3q_3}$, then both α_1 and α_3 are in $(0, 1)$. If $s_{13} \in \overleftrightarrow{s_{12}s_{23}}$ then $\lambda \in (0, 1)$ and we conclude that $\alpha_2 \in (0, 1)$, which implies $w \in \overleftrightarrow{p_2q_2}$.

For the proof of the Desargues Theorem, see Appendix A. ■

We apply the result to the previous example. Let $p_1 = u_{\{2\}}$, $q_1 = u_{\{1,3,4\}}$, $p_2 = u_{\{1,2\}}$, $q_2 = u_{\{3,4\}}$, $p_3 = u_{\{1,2,3\}}$ and $q_3 = u_{\{4\}}$. By definition, $w = \overleftrightarrow{p_1q_1} \cap \overleftrightarrow{p_3q_3}$. From the Figure below we also observe that $s_{12} = \overline{p_1p_2} \cap \overline{q_1q_2} = u_{\{1,2\}}$, $s_{13} = \overline{p_1p_3} \cap \overline{q_1q_3} = u_{\{1,3\}}$ and $s_{23} = \overline{p_2p_3} \cap \overline{q_2q_3} = u_{\{3\}}$. Clearly, $s_{13} \in \overleftrightarrow{s_{12}s_{23}}$ since $u_{\{2,3\}} \in \overleftrightarrow{u_{\{2\}}u_{\{3\}}}$. Therefore, we conclude that $w \in \overleftrightarrow{u_{\{1,2\}}u_{\{3,4\}}}$ and define $u_{\{1,2,3,4\}} = w$.



The Desargues Theorem shows that $u_{\{1,2,3,4\}} \in \overleftrightarrow{u_{\{1,2\}}u_{\{3,4\}}}$.

The location of $u_{\{1,2,3,4\}}$ imposes restrictions on the utility sets of other coalitions. For example, since $u_{\{1,2,3,4\}} \in Co(u_{\{3\}}, W_{\{1,2,4\}})$, it must be the case that

$$u_{\{1,2,4\}} = \overleftrightarrow{u_{\{1,2\}}u_{\{4\}}} \cap \overrightarrow{u_{\{3\}}u_{\{1,2,3,4\}}} \in W_{\{1,2,4\}} \tag{37}$$

Again, we have to use Desargues Theorem to check that $u_{\{1,2,4\}}$ belongs also to the segments $\overleftrightarrow{u_{\{1\}}u_{\{2,4\}}}$ and $\overleftrightarrow{u_{\{1,4\}}u_{\{2\}}}$. A similar argument applies to the coalitions $\{2,3\}$, $\{1,4\}$, $\{2,4\}$. In the next section we will determine for which coalitions can find a u_S and for which we can find a u_S.

2.4 Spanning tree and connected coalitions

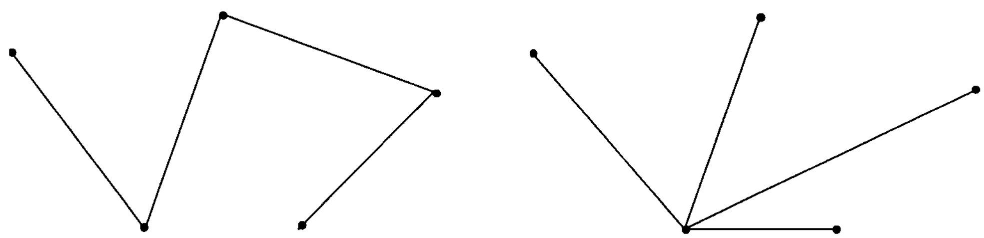
We start by formally defining the notion of a spanning tree. Given N, let $\mathcal{T}$ be some family of two-member coalitions. Two individuals $i, j \in N$ are adjacent if $\{i,j\} \in \mathcal{T}$. Given two individuals $i, j \in N$, a *path* from i to j is a sequence $(i_\ell)_{\ell=0}^{k}$ with $i_0 = i$, $i_k = j$ and such that the pairs $\{i_{\ell-1}, i_\ell\}_{\ell=1}^{k}$ are distinct and belong to $\mathcal{T}$. If such path exits, we say that i and j are *connected* in $\mathcal{T}$. $\mathcal{T}$ is a *spanning tree* of N if any two *distinct* individuals are connected and no individual is connected to himself. It follows that $\mathcal{T}$ contains precisely $n-1$ coalitions.

A coalition S is *connected* in a spanning tree $\mathcal{T}$ if the sub-tree

$$\mathcal{T}_S = \{\{i,j\} \in \mathcal{T} : i, j \in S\} \tag{38}$$

is a spanning tree of S. By vacuous implication, we consider that the individual coalitions are connected. The only connected coalitions of size two are the elements of $\mathcal{T}$ and N is always connected for any $\mathcal{T}$. We denote by $\mathcal{C}$ the collection of connected coalitions. In the previous example, besides the individual coalitions and the pairs in $\mathcal{T}$, the other connected coalitions where $\{1,2,3\},\{2,3,4\}$ and $\{1,2,3,4\}$. Observe that we were able to assign definite utilities for precisely those coalitions. As the main theorem of this section will show, we will be able to obtain complete utilities for precisely the connected coalitions.

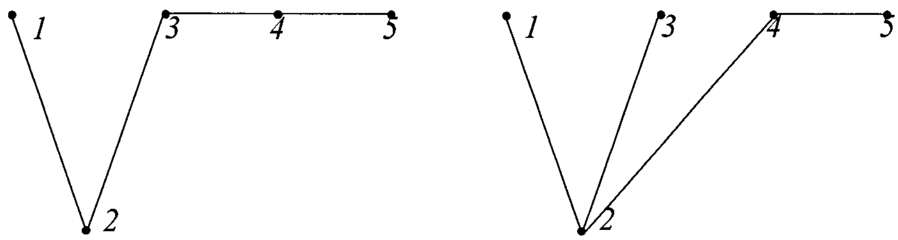
The number of connected coalitions will vary according to the shape of the tree. The two extreme cases are



A *connecting line* and a *full centralization* spanning tree, respectively.

In the case of a *connecting line*, there are $n-k+1$ connected coalitions of size k that amount for a total of $\frac{3}{2}n(n+1)$ connected coalitions. In the full centralization case, there are n connected coalitions of size 1, $\binom{n-1}{k-1}$ connected coalitions of size $k>1$. Since $\sum_{k=0}^{n}\binom{n}{k}=2^n$, we conclude that the total number of connected coalitions is $2^{n-1}+n-1$. Besides the The connected coalitions are all those that contain

Before presenting the theorem, it is convenient to comment on collinearities. In the previous geometric examples, it is clear that if the three utilities forming a triangle in the tree are collinear, then we cannot use the representation theorem. Fortunately, that is all we need, i.e., that for any connected coalitions of size three, the utilities of its members $u_{\{i\}}$, $u_{\{j\}}$ and $u_{\{k\}}$ be non-collinear. In that case, we say that the *avoid collinearities.* If some utilities are collinear (See figure below), we have to be careful in defining a spanning tree that avoids collinearities. It is part of the current research to find necessary and sufficient conditions to determine the existence of a tree that allows the construction of utilities for all the connected coalitions. Obviously, if any three utilities are non-collinear, then any spanning tree of N avoids collinearities.



The coalition $\{3,4,5\}$ on the left is collinear. On the right, all connected coalitions of size three are non-collinear.

Theorem 2.3 *For $i \in N$, let $\succcurlyeq_{\{i\}}$ be a family of non-trivial complete preferences satisfying minimal agreement. Let $\mathcal{T}$ be a spanning tree of N that avoids collinearities and assume that each pair in $\mathcal{T}$ reaches a compromising agreement. Assume the existence of incomplete preferences $\succcurlyeq_S$ for each coalition S in N satisfying the Pareto rule. Then,*

1. *If S is connected, $\succcurlyeq_S$ exists and it is a uniquely determined, complete preference.*

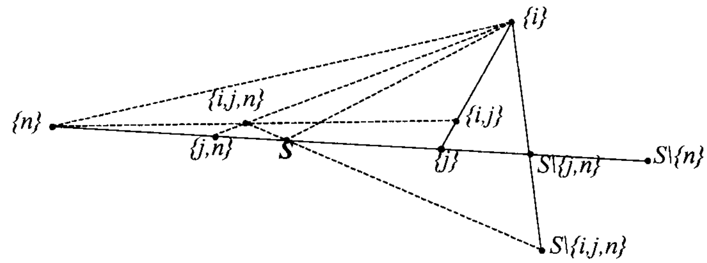
2. *If S is disconnected, then $\succcurlyeq_S$ exists, although it may be incomplete, and it contains a uniquely determined complete preference*

Proof. We will prove by induction that there is a unique set of utilities u_S such that for any two disjoint coalitions A and B, $u_S \in \overleftrightarrow{u_A u_B}$. Moreover, $W_S = \{u_S\}$ for the connected coalitions and $u_S \in W_S$ for the disconnected coalitions. For the induction, we start by sequentially labelling the individuals and making sure that the next member to be labelled n is adjacent to one and only one *already labelled* member $k < n$. The theorem is trivially true for $N = \{1\}$. For $N = \{1,2\}$, (1) holds since $u_{\{1,2\}} \in \overleftrightarrow{u_{\{1\}}u_{\{2\}}}$, all the coalitions are connected and their utilities uniquely given and there are no disconnected coalitions.

Consider an arbitrary N with $|N| \geq 3$ and suppose that the result is true for $N \backslash \{n\}$. Let u_S be the collection of utilities for coalitions with $n \notin S$ given by the induction hypothesis. Let j be the unique adjacent of n and i the unique adjacent preceding j. We are given $u_{\{n\}}$, $u_{\{j,n\}} \in \overleftrightarrow{u_{\{j\}}u_{\{n\}}}$ and $u_{\{i,j\}} \in \overleftrightarrow{u_{\{i\}}u_{\{j\}}}$.

Since we assume that $u_{\{i\}}$, $u_{\{j\}}$ and $u_{\{n\}}$ are not collinear, we can use (32) to obtain $u_{\{i,j,n\}} = \overleftrightarrow{u_{\{i,j\}}u_{\{n\}}} \cap \overleftrightarrow{u_{\{i\}}u_{\{j,n\}}}$ and $u_{\{i,n\}} = \overleftrightarrow{u_{\{i\}}u_{\{n\}}} \cap \overline{u_{\{j\}}u_{\{i,j,n\}}}$. Consider the line $\overline{u_{\{j\}}u_{\{n\}}}$. For a connected coalition, note that $j \in S$ and

$u_{S\setminus\{j,n\}} \in \overleftrightarrow{u_{\{i\}}u_{S\setminus\{i,j,n\}}}$. If it happens that $u_{S\setminus\{j,n\}} \in \overline{u_{\{j\}}u_{\{n\}}}$ then it is the case that $u_{S\setminus\{i,j,n\}} \notin \overline{u_{\{n\}}u_{S\setminus\{n\}}}$, otherwise $u_{\{i\}} = u_{\{j\}}$ (See figure below).



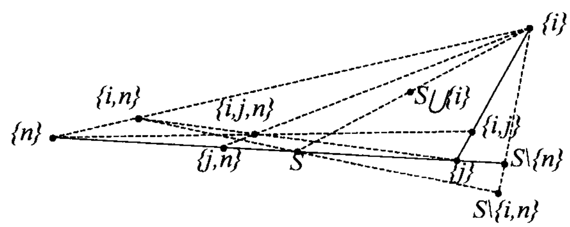
We then define for the "new" connected coalitions S:

1. $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{j,n\}}u_{S\setminus\{j,n\}}}$ if $u_{\{j\}}$, $u_{\{j\}}$ and $u_{S\setminus\{j,n\}}$ are non-collinear.

2. $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{i,j,n\}}u_{S\setminus\{i,j,n\}}}$ otherwise.

Note that all the utilities involved are known by the induction hypotheses. Also, observe that $S\setminus\{n\}$ and $S\setminus\{j,n\}$ is correct since all the "new" connected coalitions will contain both n and j.

For a coalition S, consider the line $\overline{u_{\{j\}}u_{\{n\}}}$ and suppose that $u_{S\setminus\{j,n\}}, u_{S\cup\{j\}} \in \overline{u_{\{j\}}u_{\{n\}}}$. If $i \in S$, since $u_{S\setminus\{i,n\}} \in \overleftrightarrow{u_{\{i,n\}}u_S}$, we have that $u_{S\setminus\{j,n\}} \in \overline{u_{\{j\}}u_{\{n\}}}$ and $u_{S\setminus\{i,n\}} \notin \overline{u_{\{n\}}u_{S\setminus\{n\}}}$, otherwise $u_{\{i\}} = u_{\{n\}}$ (See figure below). If $i \notin S$ then we are guaranteed that $u_{S\cup\{i\}} \in \overleftrightarrow{u_S u_{\{i\}}}$ and $u_{S\cup\{i\}} \notin \overline{u_{\{n\}}u_{S\setminus\{n\}}}$.



We then apply the following definitions in descending order of the number of members of S:

1. $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{j,n\}}u_{S\setminus\{j,n\}}}$ if $j \in S$ and $u_{\{n\}}$, $u_{\{j\}}$ and $u_{S\setminus\{j,n\}}$ are non-collinear.

2. $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overline{u_{\{j\}}u_{S\cup\{j\}}}$ if $j \notin S$ and $u_{\{n\}}$, $u_{\{j\}}$ and $u_{S\setminus\{j,n\}}$ are non-collinear.

   If the previous definitions fail, we can apply:

3. $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{i,n\}}u_{S\setminus\{i,n\}}}$ if $i \in S$.
4. $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overrightarrow{u_{\{i\}}u_{S\cup\{i\}}}$ if $i \notin S$.

The descending order guarantees that $S \cup \{j\}$ or $S \cup \{i\}$ has been defined.

We have already defined utilities for all the coalitions. We have to check that the Pareto rule works. Let A and B be any two disjoint coalitions such that $A \cup B = S$. By induction hypothesis, if $n \notin S$ then we already have that $u_S \in \overleftrightarrow{u_A u_B}$. To prove that this also hold for $n \in S$, assume *wlog* that $n \in A$. Denote by k the element used to define S, i.e., k is either i or j. For the case of connected coalitions defined with $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{i,j,n\}}u_{S\setminus\{i,j,n\}}}$, we provisionally redefine them using $u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{i,n\}}u_{S\setminus\{i,n\}}}$. The application of the Pareto rule will imply that both definitions are equivalent. Then, depending on the case, define the following points:

$$\begin{array}{ccccccc} \text{Case} & p_1 & p_2 & p_3 & q_1 & q_2 & q_3 \\ k \in S \text{ and } k \in A & u_{\{n\}} & u_{\{k,n\}} & u_A & u_{S\setminus\{n\}} & u_{S\setminus\{k,n\}} & u_B \\ k \in S \text{ and } k \notin A & u_{\{n\}} & u_A & u_{S\setminus\{k\}} & u_{S\setminus\{n\}} & u_B & u_{\{k\}} \\ k \notin S \text{ and } k \notin A & u_{\{k\}} & u_{\{n\}} & u_A & u_{S\cup\{k\}} & u_{S\setminus\{n\}} & u_B \end{array} \tag{39}$$

The case $k \notin S$ and $k \in A$ does not occur since $A \subset S$. We are guaranteed that $u_{S\setminus\{k,n\}} \neq u_{\{k,n\}}$ and $u_{S\cup\{k\}} \neq u_{\{k\}}$. The case $u_{\{n\}} = u_{S\setminus\{n\}}$ and $u_A = u_B$ can be solved by geometrical consideration. Also, if the $p_i's$ or the $q_i's$ are collinear, then there is nothing to prove. The s_{ij} of the Desargues theorem are

$$\begin{array}{cccc} \text{Case} & s_{12} & s_{13} & s_{23} \\ k \in S \text{ and } k \in A & u_{\{k\}} & u_{A\setminus\{n\}} & u_{A\setminus\{k,n\}} \\ k \in S \text{ and } k \notin A & u_{A\setminus\{n\}} & u_{S\setminus\{k,n\}} & u_{S\setminus(A\cup\{k\})} \\ k \notin S \text{ and } k \notin A & u_{\{k,n\}} & u_{A\cup\{k\}} & u_{A\setminus\{n\}} \end{array} \tag{40}$$

In case 1 and 2 $u_{A\setminus\{n\}} \in \overleftrightarrow{u_{\{k\}}u_{A\setminus\{k,n\}}}$ and $u_{S\setminus\{k,n\}} \in \overleftrightarrow{u_{A\setminus\{n\}}u_{S\setminus(A\cup\{k\})}}$ and none of the coalitions contain n. The induction hypotheses implies that $s_{13} \in \overleftrightarrow{s_{12}s_{23}}$. In the third case, n belongs to the three coalitions involved, but by definition $u_{A\cup\{k\}} \in \overleftrightarrow{u_{\{k,n\}}u_{A\setminus\{n\}}}$ since $k \in A \cup \{k\}$. We then conclude that $u_S \in \overleftrightarrow{u_A u_B}$.

Now that we are guaranteed that the u_S are defined in a consistent way, we just check that for the connected coalitions we have $W_S = \{u_S\}$. This is the case since they have been defined with sets of the form

$$u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{j,n\}}u_{S\setminus\{j,n\}}} \textit{ or } u_S = \overleftrightarrow{u_{\{n\}}u_{S\setminus\{n\}}} \cap \overleftrightarrow{u_{\{i,j,n\}}u_{S\setminus\{i,j,n\}}} \tag{41}$$

For the coalitions not connected, to see that $u_S \in W_S$ we distinguish two cases. If $S = N\backslash\{i\}$ for some i, then it divides in two disjoint connected coalitions and we have

$$u_S = \overleftrightarrow{u_A u_B} \cap \overrightarrow{u_{\{i\}} u_{S\cup\{i\}}} \tag{42}$$

which forces $u_S \in W_S$. If S has two missing elements with respect to N, then let

$$u_S = \overrightarrow{u_{\{i\}} u_{S\cup\{i\}}} \cap \overrightarrow{u_{\{j\}} u_{S\cup\{j\}}} \tag{43}$$

which again forces $u_S \in W_S$. ■

2.5 Cardinal measurable and full comparable utility

We have build a theory that allows us to construct utilities for a group if only some pairs agree on a joint preference. If such pairs form a spanning tree, then the sole acceptance of the Pareto rule implies a unique preference for the entire group. In this section we will prove the existence of a set of positive weights, unique up to a positive factor of proportionality, such that the utility for the group can be expressed as the weighted sum of the individual utilities.

We recall from the first section that we where able to choose a origin in the prospect space. This means that we can translate the origin of our utilities by a common factor. Similarly, we can multiply them by any positive constant. Formally, if $(u_S)_{S\in\mathcal{P}}$ is a profile obtained in the previous theory, so is $(\hat{u}_S)_{S\in\mathcal{P}}$ where $\hat{u}_S = a + bu_S$ for some a and $b > 0$. Technically, our model belongs to the group of cardinally measurable and full comparable utilities.

A different approach to obtain a representation of a utility for such group is presented in Maskin [10]. His is a "top down" approach. By defining properties of a desired welfare functional $f : R^n \to R$ applied to the vector of individual utilities he concludes that $f = \sum_{i\in N} u_i$. In his case, the informational requirement in utilities is also cardinally measurable and full comparable. In our case, we follow a "bottom up" approach: we derive the social functional starting from the individual utilities and joint agreements between members of the group.

If those agreements can be interpreted as a measure to compare utility between the pair, then we have shown that the interpersonal comparisons of utility extend to the whole group.

For the theorem:

Theorem 2.4 *Let u_S be the utilities obtained in Theorem 2.3 for all coalitions S. Then, there is a vector $\lambda = (\lambda_1, \ldots \lambda_n)$, unique up to a positive factor of proportionality, with $\lambda_i > 0$ and such that if $u'_S = \left(\sum_{i \in S} \lambda_i\right) u_S$ then,*

$$u'_S = \sum_{i \in S} u'_{\{i\}} \text{ for any coalition } S \tag{44}$$

Proof. (Sketch) In the proof we normalize $\sum_{i \in N} \lambda_i = 1$ so that $u'_N = u_N$. We will use an induction with the same labels as in the proof of Theorem (2.3). In the step k, when $N = \{1, ..., k\}$, we denote by λ_i^k the weight for agent i, with $i = 1, \ldots, k$. By defining $\lambda_1^1 = 1$, the theorem is trivial for $N = \{1\}$. For $N = \{1, 2\}$, we set λ_1^2 and λ_2^2 such that $u_{\{1,2\}} = u_{\{1\}} \lambda_1^2 u_{\{2\}}$, with $0 < \lambda_1^2 < 1$ and $\lambda_2^2 \equiv 1 - \lambda_1^2 > 0$. Suppose it holds for $N \backslash \{n\} = \{1, \ldots, n-1\}$ with $n \geq 3$. Since $u_N \in \overleftrightarrow{u_{N \backslash \{n\}} u_{\{n\}}}$ we set $0 < \lambda_n^n < 1$ so that $u_N = u_{\{n\}} \lambda_n^n u_{N \backslash \{n\}}$ and let $\lambda_i^n \equiv (1 - \lambda_n^n) \lambda_i^{n-1}$ for $i = 1, \ldots, n-1$. By induction hypothesis, $u_{N \backslash \{n\}} = \sum_{i \in N \backslash \{n\}} \lambda_i^{n-1} u_i^{n-1}$ and we obtain

$$u_N = \lambda_n^n u_{\{n\}} + (1 - \lambda_n^n) \sum_{i \in N \backslash \{n\}} \lambda_i^{n-1} u_i = \sum_{i \in N} \lambda_i^n u_i \tag{45}$$

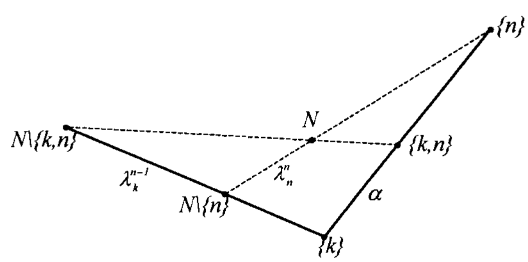
Next, for S with $n \notin S$, we just observe that

$$u_S = \frac{\sum_{i \in S} \lambda_i^{n-1} u_i}{\sum_{i \in S} \lambda_i^{n-1}} = \frac{\sum_{i \in S} (1 - \lambda_n^n) \lambda_i^{n-1} u_i}{\sum_{i \in S} (1 - \lambda_n^n) \lambda_i^{n-1}} = \frac{\sum_{i \in S} \lambda_i^n u_i}{\sum_{i \in S} \lambda_i^n} \tag{46}$$

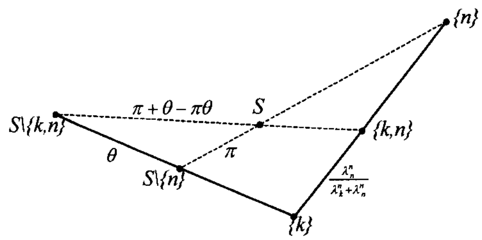
If k is the pivotal agent of coalition N, since $u_{\{k,n\}} = u_{\{n\}} \alpha u_{\{k\}}$ and $u_N \in \overleftrightarrow{u_{N \backslash \{n\}} u_{\{n\}}} \cap \overleftrightarrow{u_{N \backslash \{k,n\}} u_{\{k,n\}}}$, we have

$$\alpha = \frac{\lambda_n^n}{\lambda_n^n + \lambda_k^{n-1} - \lambda_n^n \lambda_k^{n-1}} = \frac{\lambda_n^n}{\lambda_n^n + (1 - \lambda_n^n) \lambda_k^{n-1}} = \frac{\lambda_n^n}{\lambda_n^n + \lambda_k^n} \tag{47}$$

and conclude that the theorem holds for $S = \{k, n\}$ (See Figure below)



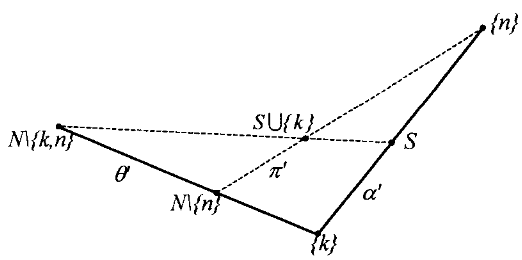
Now, in descending order of the size of the coalitions, let S be such that $u_S \in \overleftrightarrow{u_{S\setminus\{n\}}u_{\{n\}}} \cap \overleftrightarrow{u_{S\setminus\{k,n\}}u_{\{k,n\}}}$. If $\theta = \frac{\lambda_k^{n-1}}{\sum_{i\in S\setminus\{n\}} \lambda_i^{(n-1)}}$, by induction hypotheses, $u_{S\setminus\{n\}} = u_{\{k\}}\theta u_{S\setminus\{k,n\}}$ and the π such that $u_S = u_{\{n\}}\pi u_{S\setminus\{n\}}$ is given by $\frac{\lambda_n^n}{\lambda_n^n+\lambda_k^n} = \frac{\pi}{\pi+\theta-\pi\theta}$ (See Figure below).



After solving this equation we obtain $\pi = \frac{\lambda_n^n}{\sum_{i\in S}\lambda_i^n}$. Since $u_{S\setminus\{n\}} = \frac{\sum_{i\in S\setminus\{n\}}\lambda_i^n u_i}{\sum_{i\in S\setminus\{n\}}\lambda_i^n}$, we substitute both expressions and obtain

$$u_S = \frac{\lambda_n^n}{\sum_{i\in S}\lambda_i^n}u_{\{n\}} + (1 - \frac{\lambda_n^n}{\sum_{i\in S}\lambda_i^n})\frac{\sum_{i\in S\setminus\{n\}}\lambda_i^n u_i}{\sum_{i\in S\setminus\{n\}}\lambda_i^n} = \frac{\sum_{i\in S}\lambda_i^n u_i}{\sum_{i\in S}\lambda_i^n} \tag{48}$$

Finally if $u_S \in \overleftrightarrow{u_{S\setminus\{n\}}u_{\{n\}}} \cap \overleftrightarrow{u_{S\cup\{k\}}u_{\{k\}}}$, let $\delta = \sum_{i\in S\setminus\{n\}}\lambda_i^n$, $\theta' = \frac{\delta}{\delta+\lambda_k^{(n)}}$ and $\pi' = \frac{\lambda_n^n}{\delta+\lambda_n^n+\lambda_k^n}$. Then, by induction hypotheses, $u_{(S\cup\{k\})\setminus\{n\}} = u_{\{k\}}\theta' u_{S\setminus\{n\}}$. Also, by previous definition, $u_{S\cup\{k\}} = u_{\{n\}}\pi' u_{(S\cup\{k\})\setminus\{n\}}$. The α' such that $u_S = u_{\{n\}}\alpha' u_{S\setminus\{n\}}$ is given by $\alpha' = \frac{\pi'}{\pi'+\theta'-\pi'\theta'}$ (See Figure below)



After solving this equation we obtain $\alpha' = \frac{\lambda_n^n}{\sum_{i\in S}\lambda_i^n}$ and, as before, conclude that $u_S = \frac{\sum_{i\in S}\lambda_i^n u_i}{\sum_{i\in S}\lambda_i^n}$. ■

A Desargues Theorem

Theorem A.1 (Desargues) *Let W be a vector space. Suppose that p_1, p_2, p_3 and q_1, q_2, q_3, are two sets of non-collinear points in W satisfying $p_i \neq q_i$ ($i = 1,2,3$). Then, the three lines $\overline{p_iq_i}$ ($i = 1,2,3$) are concurrent at some point w if the three points $s_{12} = \overline{p_1p_2} \cap \overline{q_1q_2}$, $s_{13} = \overline{p_1p_3} \cap \overline{q_1q_3}$ and $s_{23} = \overline{p_2p_3} \cap \overline{q_2q_3}$ are collinear.*

Proof. First, observe that if for some $i < j$ the lines $\overline{p_ip_j}$ and $\overline{q_iq_j}$ are parallel, then the theorem is obvious since the line defined by the other s_{kl} always intersects with ∞. Therefore we can assume *wlog* that for $i < j$, the lines $\overline{p_ip_j}$ and $\overline{q_iq_j}$ are not parallel and $s_{ij} \neq \infty$ for $i < j$. In that case, the $(s_{ij})_{i<j}$ are different.

For $i < j$, let $z_{ij} = \overline{p_iq_i} \cap \overline{p_jq_j}$ and define $\alpha_1, \alpha_2, \alpha_3$ and $\alpha'_1, \alpha'_2, \alpha'_3$ such that $z_{12} = p_1\alpha_1 q_1 = p_2\alpha_2 q_2$, $z_{13} = p_1\alpha'_1 q_1 = p_3\alpha_3 q_3$ and $z_{23} = p_2\alpha'_2 q_2 = p_3\alpha'_3 q_3$. By calculation, we have that

$$s_{12} = p_1 \tfrac{\alpha_1}{\alpha_1-\alpha_2} p_2 = q_1 \tfrac{1-\alpha_1}{\alpha_2-\alpha_1} q_2 \quad (49)$$

$$s_{13} = p_1 \tfrac{\alpha'_1}{\alpha'_1-\alpha_3} p_3 = q_1 \tfrac{1-\alpha'_1}{\alpha_3-\alpha'_1} q_3 \quad (50)$$

$$s_{23} = p_2 \tfrac{\alpha'_2}{\alpha'_2-\alpha'_3} p_3 = q_2 \tfrac{1-\alpha'_2}{\alpha'_3-\alpha'_2} q_3 \quad (51)$$

If the $(s_{ij})_{i<j}$ are collinear, then $s_{13} = s_{12}\lambda s_{23}$ for some λ. By the previous considerations, we can suppose $\lambda \neq \{0,1\}$. Using this fact in (49) and letting $\gamma_1 = \lambda + \frac{\alpha'_2}{\alpha'_2-\alpha'_3} - \lambda\frac{\alpha'_2}{\alpha'_2-\alpha'_3}$ we obtain:

$$p_1 \tfrac{\alpha'_1}{\alpha'_1-\alpha_3} p_3 = \left(p_1 \tfrac{\alpha_1}{\alpha_1-\alpha_2} p_2\right) \lambda \left(p_2 \tfrac{\alpha'_2}{\alpha'_2-\alpha'_3} p_3\right) = \left(p_1 \tfrac{\lambda\alpha_1}{\gamma_1(\alpha_1-\alpha_2)} p_2\right) \gamma_1 p_3 \quad (52)$$

Similarly, if $\gamma_2 = \lambda + \frac{1-\alpha'_2}{\alpha'_3-\alpha'_2} - \lambda\frac{1-\alpha'_2}{\alpha'_3-\alpha'_2}$ we get,

$$q_1 \tfrac{1-\alpha'_1}{\alpha_3-\alpha'_1} q_3 = \left(q_1 \tfrac{1-\alpha_1}{\alpha_2-\alpha_1} q_2\right) \lambda \left(q_2 \tfrac{1-\alpha'_2}{\alpha'_3-\alpha'_2} q_3\right) = \left(q_1 \tfrac{\lambda(1-\alpha_1)}{\gamma_2(\alpha_2-\alpha_1)} q_2\right) \gamma_2 q_3 \quad (53)$$

For (52) to hold we require $\frac{\lambda\alpha_1}{\gamma_1(\alpha_1-\alpha_2)} = 1$ and $\gamma_1 = \frac{\alpha'_1}{\alpha'_1-\alpha_3}$. Similarly, for (53) to hold we need $\frac{\lambda(1-\alpha_1)}{\gamma_2(\alpha_2-\alpha_1)} = 1$ and $\gamma_2 = \frac{1-\alpha'_1}{\alpha_3-\alpha'_1}$. Using the first two equations we obtain $\alpha'_3 = \frac{\alpha_1\alpha'_2}{\alpha'_1\alpha_2}\alpha_3$, that can be substituted in the first and third equation to obtain $\alpha_2 = \alpha'_2$. Substituting those results in the first and fourth equation gives $(\alpha_1 - \alpha_1')(\alpha_1\alpha_3 - \alpha_2\alpha_1') = 0$. If $\alpha_1' = \frac{\alpha_3}{\alpha_2}\alpha_3$ we obtain $\lambda = 1$, that is ruled out. Therefore, we easily conclude that $\alpha_1 = \alpha_1', \alpha_2 = \alpha_2'$ and $\alpha_3 = \alpha_3'$. Furthermore, $\lambda = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}$ and conclude that the three lines $\overline{x_i y_i}$ $(i = 1, 2, 3)$ are concurrent. ■

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