Efficiency in Repeated Games Revisited: The Role of Private Strategies*

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Abstract

Most theoretical or applied research on repeated games with imperfect monitoring has restricted attention to *public strategies*; strategies that only depend on history of publicly observable signals, and *perfect public equilibrium* (PPE); sequential equilibrium in public strategies. Although public strategies are attractive due to their simplicity and tractability, they are restrictive. The present paper sheds light on the role of *private strategies*; strategies that depend on players' own actions in the past as well as observed public signals. Our main finding is that players can sometimes make better use of information by using private strategies and *efficiency in repeated games can often be drastically improved.* We first study a simple repeated partnership game with two public signal, for which Radner, Myerson, and Maskin (1986)'s anti-folk theorem holds. For this game, we explicitly construct a symmetric sequential equilibrium using private strategies, whose equilibrium payoff lies outside of the set of PPE payoffs and Pareto-dominates the best symmetric PPE payoff. This equilibrium based on private strategies, which we call *private equilibrium (PE)*, sometimes even achieves full efficiency. Then, we extend our construction of private equilibrium to more general stage games.

We also offer several examples to emphasize the importance of private strategies. In the first example with two public signals, we show that a private equilibrium achieves almost efficiency as players become patient while the only PPE is the repetition of the one-shot Nash equilibrium. This example suggests that the difference between the PPE payoff set and the sequential equilibrium payoff set is potentially quite large. We also provide an example with a richer information structure where the folk

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theorem holds. We show that whenever ther exists a nontrivial PPE, *there exists a PE which is strictly more efficient than any PPE.* In another word, PE approximates the efficent payoff profile faster than PPE as players become patient.

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1 Introduction

The theory of repeated games under imperfect monitoring provides a formal framework to explore the possibility of cooperation in long term relationships, where each agent's action is not directly observable. It has been successfully applied to a number of economic problems; cartel enforcement, internal labor market, and international policy coordination, to name a few. However, almost all existing works focus on a simple class of strategies, known as public strategies. In the present paper, we illustrate that players can make better use of information by using non-public strategies, which we call *private strategies*, and show that *efficiency in repeated games can often be drastically improved.*

In public strategies, each player's current action only depends on the history of publicly observable signal, such as market price. In other words, the players disregard their past actions. In contrast, we allow the possibility that the players condition their actions on both the public signal *and* their past actions.

A rough intuition for the improved efficiency comes from the following observation. It is often the case that a player has a costly action that helps to monitor the other players' behavior more accurately. For example, under decreasing returns to scale, observable output becomes more sensitive to an opponent's effort when a player's effort is low. Hence, if the costly "monitoring" action is played with a small probability and the opponents are rewarded/punished *only after* such an action is taken, opponents' moral hazard problem can be resolved in a more efficient way. Note that in this story it is vital that the players' future behavior (punishment/reward) depends on their past actions.

Let us explain our point in more detail with an explicit example. Consider a simple repeated partnership game with two actions $\{C, D\}$, two public signals $\{$"*good*", "*bad*"$\}$, where the stage game (expected) payoff matrix has the same structure as in the standard prisoners' dilemma. We assume that "*good*" is more likely to be observed when one player plays C than when no one plays C, but another C does not increase the probability of "*good*" that much (decreasing returns to scale). In another word, $\Pr(\text{"}bad\text{"}|C,C)+\varepsilon=\Pr(\text{"}bad\text{"}|C,D)=\Pr(\text{"}bad\text{"}|D,C)<<\Pr(\text{"}bad\text{"}|D,D)$. We restrict our attention to strongly symmetric strategies and perfect public equilibrium (PPE).[1,2]

[1] A public strategy profile is strongly symmetric if both player plays the same behavior strategy after any public history. Perfect public equilibrium is essentially a sequential equilibrium in public strategies.

[2] This restriction to strongly symmetric strategies can be formally justified if the sum of

First, note that any level of cooperation cannot be sustained in strongly symmetric pure strategies when ε is very small. The public signal is insensitive to a deviation when (C, C) is played. Now let players play the inefficient "monitoring" action D with small probability. Although the stage game payoff is less than the efficient level, now the public signal becomes more informative about players' actions. This allows players to use a mutual punishment after the public signal "*bad*" to sustain a certain level of cooperation.[3] The level of punishment can be adjusted so that players are actually indifferent between choosing C and choosing D. This construction works, for example, when $\Pr(\text{"}bad\text{"}|CC)$ is small and players are patient enough. Note that the efficiency is improved *even within the class of strongly symmetric (public) strategy profiles* by mixing D to improve the quality of monitoring.

However, we can improve efficiency further by allowing more flexible strategies. In the above strategy, the observed public signal conveys almost no information about the opponent's action when a player is playing C. Whether a player starts punishment or not after playing C and observing "*bad*" does not matter much in terms of keeping the other player's incentive to be cooperative. Given that the punishment occurs with positive probability, this is a pure waste of efficiency. Hence, players potentially improve efficiency by punishing the opponent *only after playing D and observing "bad"*.[4] This explains why private strategies can help to improve efficiency.[5]

In the present paper, we take as a starting point a simple repeated partnership game with two public signals. As shown by Radner, Myerson, and Maskin [16], the folk theorem does not hold for this game with public strategies. In particular, the set of PPE payoffs is bounded away from the efficient frontier. For this game, using private strategies, we explicitly construct a symmetric sequential equilibrium with a payoff that Pareto-dominates the best symmetric PPE payoff. This equilibrium based on private strategies, which we call *private equilibrium* (PE), sometimes even achieves full efficiency.

In order to emphasize the importance of private strategies, we offer several examples. In the first example with two public signal, we show that a private equilibrium achieves almost efficiency as players become patient while the only PPE is the repetition of the one-shot Nash equilibrium. This example suggests that the difference between the PPE payoff set and the sequential equilibrium payoff set is potentially quite large. We also provide an example with three public signals where each player's deviation from (C, C) is statistically distin-

the payoffs from the asymmetric profile is very low. See Proposition 3 and Lemma ?? in the appendix A.

[3] We allow players to use a public randomization device for PPE. So, the mutual punishment would be to play the one-shot Nash equilibrium forever with some probability.

[4] The equilibrium strategy will be still symmetric with respect to private history; history of past public signals and actions.

[5] At this point, it is worth recalling that any pure strategy is realization equivalent to a public strategy with full support monitoring (cf. Abreu, Pearce, and Stacchetti [2]). This implies that, to support any payoff profile which cannot be supported by PPE, any such private equilibrium must be based on mixed strategies as in this example.

guished and the folk theorem holds.[6] While the PPE payoff set approximates the individually rational and feasible set as players become patient, *a private equilibrium is always (strictly) more efficient than any PPE for any (large) discount factor* in this example. In another word, PE approximates the efficient payoff faster than PPE as players become patient. This example clearly illustrates that our main message remains valid even under a richer information structure.

It is not an easy task to construct an equilibrium based on the idea described above. Players' continuation strategies are not common knowledge after the signal "*bad*" is observed. Since players continuation strategies depend on their private information once "*bad*" is observed, the continuation game after "*bad*" is equivalent to an incomplete information repeated game, where players "type" is either "*played C*" or "*played D*" in the previous period. This implies that each player has to update his/her belief about the opponent's continuation strategy over time. It is a highly complicated task to find an equilibrium profile which is sequentially rational with respect to the dynamics of belief, which in turn is generated by itself. This observation in fact explains why there has been few works on private strategies in repeated games with imperfect monitoring.

Another contribution of the present paper then is the construction of equilibrium itself. Private strategies are constructed in such a way that players are indifferent among all the repeated game strategies. This makes a player's belief about the opponent's continuation strategy irrelevant. This construction provides a way to deal with the endogenously generated private information described above.

The idea of this strategy is indeed very powerful in dealing with private information. It can deal with not only private information about past actions but also private information about private signals if any. A similar idea was first applied by Piccione [14] in the framework of repeated games with private monitoring.[7]

The structure of the paper is as follows. We give a simple efficiency result in Section 2. In a repeated partnership game in which the public signal distributions have moving supports, we construct a PE which approximates efficiency as δ tends to 1, while any PPE payoff is bounded away from the efficient frontier independent of the discount factor. In Section 3, we take a monitoring structure with full support which is more natural for partnership games and was studied by Radner, Myerson, and Maskin [16]. We first derive the upper bound of all the PPE payoffs, including the mixed strategy PPE payoffs. Then, we construct a symmetric PE based on a two state machine, which Pareto-dominates the best symmetric PPE payoff. Section 4 provides an example to emphasize a potential

[6] To be precise, this is the case where the distributions of the public signal given (C,C), (C,D) and (D,C) are linearly independent, that is, the pairwise full rank condition is satisfied at (C,C). This condition guarantees that the folk theorem holds for this case. (cf. Fudenberg, Levine and Maskin [7]).

[7] Two-state machine strategies used in this paper were first independently found by Ely and Välimäki [6] in repeated games with private monitoring and Obara [13] for private equilibria in repeated games with imperfect public monitoring.

difference in efficiency implications by PPE and PE. We also discuss an example with three public signals to argue that our substantial insight does not depend on the assumption of two public signals and extends to a richer information structure. The construction of PE based on two state machine is extended to more general games in Section 5. We first show that adding more states does not help to improve efficiency. Thus, restricting our attention to two state machines is without loss of generality. Then, we characterize such PE based on two state machines, discuss when they can be constructed in general, and, finally, provide a sufficient condition for the existence of PE which Pareto-dominates PPE. Section 6 discusses related literature and concludes.

2 An Efficiency Result

In this section we look at a particular type of repeated prisoners' dilemma game with imperfect public monitoring. There are two players, and two actions, C and D, are available for each player. Actions are not observable, but there is a publicly observable signal $\omega \in \Omega$ which takes on two values, X or Y. The expected stage game payoff profiles are summarized by the following table.

	C	D
C	$1,1$	$-h, 1+d$
D	$1+d, -h$	$0,0$

Each entry of the table denotes the row player's payoff followed by the column player's. We assume that this is a prisoners' dilemma game; $d, h > 0$ (D is dominant) and $d - h < 1$ ((C, C) is efficient, that is, it Pareto-dominates the equal (public) randomization between (C, D) and (D, C)). This is a simplified version of the model examined by Radner, Myerson and Maskin [16].[8]

Let $p(\omega|a_1, a_2)$ be the probability to observe ω given the action profile (a_1, a_2), and assume the following information structure; $0 < p(X|C, C) < 1$, $0 < p(X|D, D) < 1$, and $p(X|C, D) = p(X|D, C) = 0$. The last equalities represent a "moving support" assumption, but note that this does not help to support the efficient payoff profile $(1, 1)$ by public strategies. Also note that the prisoners' dilemma payoffs in the above table can be generated by suitable choices of realized payoffs $u_i(a_i, \omega)$ so as to satisfy;

$$\begin{aligned} 1 &= u_i(C, X)p(X|C, C) + u_i(C, Y)p(Y|C, C) \\ -h &= u_i(C, Y) \\ 1 + d &= u_i(D, Y) \\ 0 &= u_i(D, X)p(X|D, D) + u_i(D, Y)p(Y|D, D) \end{aligned}$$

We first examine roughly what could be supported by PPE. A more detailed analysis of PPE will be given in Section 4. It is not difficult to show that the best strongly symmetric PPE can be achieved by the following simple strategy:

[8] The action set is continuum in Radner, Myerson, and Maskin [16].

$$(\#)\begin{cases} (1): & \text{Play } (C,C) \text{ in the stage game.} \\ (2): & \text{If } X \text{ is observed, go back to (1)} \\ & \text{If } Y \text{ is observed, start playing } (D,D) \text{ forever with probability } \rho \text{ and} \\ & \text{go back to (1) with probability } 1-\rho. \end{cases}$$

Note that Y is the "bad" signal given that players are playing (C,C). Since the punishment occurs with positive probability, ρ is reduced up to the level where players' incentive constraints are binding. Here we allow players to use a public correlation device to coordinate their behavior to minimize the punishment.[9]

The equilibrium conditions are

$$v = (1-\delta) + \delta\{1 - \rho p(X|C,C)\}v$$

$$v = (1-\delta)(1+d) + \delta\{1 - \rho p(X|D,C)\}v$$

Then, the equilibrium payoff is given by the following formula, which is first derived in Abreu, Milgrom, and Pearce [1]:

$$1_{(\text{cooperative payoff})} - \frac{d_{(\text{deviation gain})}}{\frac{p(y|D,C)}{p(y|C,C)}_{(\text{likelihood ratio})} - 1}.$$

Note that the discount factor and ρ do not appear in the formula.[10] This formula is interesting because its second term provides a clear expression of the efficiency loss in terms of the primitives of the stage game. In particular, we call your attention to the likelihood ratio in the denominator. The quality of signal about the opponent's defection is a crucial factor to determine the upper bound of strongly symmetric PPE payoffs. Although there are other asymmetric equilibria such as alternating (C,D) and (D,C), their payoffs are also bounded away from the efficient frontier as we will show in Section 4.

Now we show that the efficient payoff profile can be approximated by a private strategy. Consider the following strategy. In the initial period, each player mixes between C and D. Action D is chosen with a (small) probability $q \in (0,1)$. If the realization of the signal at the end of the current period is X and she played D, then she will play D forever. Otherwise, the player repeats the same action plan as in the initial period. Note well that (i) the players' future actions partly depend on their current actions and (ii) thanks to the assumption $p(X|C,D) = p(X|D,C) = 0$, when a player chose D and observes X, it is common knowledge that the other player also chose D (and

[9]This is an innocuous assumption because our purpose is to show that there exists a private equilibrium (PE) which Pareto-dominates the best symmetric PPE payoff profile, and we construct our PE *without any public randomization device.* This assumption is rather an additional burden in deriving our result.

[10]This formula of the best strongly symmetric PPE payoff is valid for any discount factor above some critical dicount factor $\underline{\delta} \in (0,1)$.

of course observs X). A strategy such as this, which depends on one's past action in a nontrivial way, is called *private strategy.* Formally, a strategy σ_i is private if there exists $h_i'^t$ and $h_i''^t$ such that $\sigma_i(h_i'^t) \neq \sigma_i(h_i''^t)$ while the public history of $h_i'^t$ and $h_i''^t$ being the same.[11] *Private equilibrium* (PE) is a sequential equilibrium in private strategies. The equilibrium conditions are

$$v = (1-\delta)(1-q-qh) + \delta v \tag{1}$$

$$v = (1-\delta)(1-q)(1+d) + \delta\{1 - qp(X|D,D)\}v \tag{2}$$

Equation (1) represents the average payoff at the beginning of the initial period when the player under consideration plays C (while the opponent is employing the above strategy). In this case, the current payoff is either 1 or $-h$ depending on the opponent's action, and punishment is surely avoided (as defection is triggered if and only if both players play D and the signal is X). On the other hand, equation (2) shows the average payoff when the player chooses D. The current payoff is either $1+d$ or 0, and the punishment will be triggered when the opponent also chooses D and the signal is X. This happens with probability $qp(X|D,D)$, so with the complementary probability, the player will enjoy the original average payoff v at the beginning of the following period. Equation (1) and (2), taken together, imply that the player is just indifferent between choosing C and D (the condition for a mixed strategy equilibrium).

From (1), we have

$$v = 1 - q - qh \tag{3}$$

Also, by (1) and (2) we get

$$(1-\delta)\{(1-q)d + qh\} = \delta qp(X|D,D)v. \tag{4}$$

This and (3) result in a quadratic equation in q;

$$(1-\delta)\{(h-d)q + d\} = \delta qp(X|D,D)(1-q-qh) \tag{5}$$

It is easy to show that there is a root of this equation in $(0,1)$ which tends to 0 as $\delta \to 1$. Equation (3) then implies that, as q tends to 0, the average payoff tends to 1, the payoff from full cooperation. This leads us to the following result.

Proposition 1 *In the repeated prisoners' dilemma game defined above, there is a private equilibrium that approximately attains the fully efficient average payoff $(= 1)$ as the discount factor tends to unity, while any perfect public equilibrium average payoff is bounded away from 1 independent of the discount factor.*

[11] Note that there always exists a realization equivalent public strategy to any strategy (see Amarante [3]). So, any pure private strategy is not essentially different from a public strategy. However, for some mixed strategies, there does not exist a realization equivalent public behavior strategies (although there does exist an realization equivalent mixed public strategy). Hence any interesting private strategy is necessary a mixed strategy, and our definition of private strategies is based on behavior strategy representation of mixed strategies.

Proof. To show the efficiency of the private equilibrium given above, we need to prove that a root of equation (5) lies in $(0,1)$ and tends to unity as δ tends to 1. At $q=0$, the left hand side of (4) is strictly positive but the right hand side is equal to zero. Now let q be any number $q' \in (0, \frac{1}{1+h})$ and let δ tends to 1. The left hand side of (4) tends to zero, while the right hand side tends to

$$q'p(X|D,D)\{1-q'(1+h)\}>0 \tag{6}$$

Thus equation (4) has a solution in $(0,q')$ as δ tends to 1, where q' is any number close to 0. It can be shown that any perfect public equilibrium payoff is bounded away from 1. The details are similar to Radner, Myerson and Maskin [16] and therefore omitted. ■

Since it is much easier to detect the other player's defection when one defects herself, it is more efficient to trigger a punishment only after such a (private) history. Private strategies allow players to condition their strategies on the combination of action and public signal. Each player punish the opponent only at the state where the likelihood ratio with respect to the opponent's defection is maximized. This efficiency result is based on a simple principle: the efficient use of information, which is one of the main theme underlying in any moral hazard model. With public strategies, a player can use this high likelihood ratio to punish the opponent only if she is playing D with high probability at the cost of reducing efficiency. Note that basically we avoid this trade-off between efficiency and detectability by utilizing private strategies.

You might wonder why these private strategies, based on a simple intuition, have rarely appeared in literature. The answer hinges on the assumption of the moving support in this particular example. It allows players to coordinate their punishments after playing D and observing X. However, this cannot be the case for a monitoing with full support. Suppose that X is observed with probability ε when (C,D) or (D,C) is played. Then the strategy described above is not an equilibrium. The problem is that, once X is observed, a player is not sure about the opponent's continuation strategy any more. Since a player is indifferent between playing C and playing D only if she believes that her opponent is cooperating with probability 1, she cannot stay in the cooperative phase after she chose C and observes X. Indeed, it is not difficult to show that any strategy which is close to the above strategy fails to be an equilibrium with this perturbation.

This endogenous private information and uncertainty is the most difficult problem associated with private strategies. We address this issue in the next section.

3 Repeated Partnership Games

In this section, we study a partnership game with a more typical information structure. We assume that

$$0<p(X|CC)<p(X|DC)=p(X|CD)<p(X|DD)$$

The idea is that the "bad" signal X is more likely to realize as more players defect. This is similar to the information structure in RMM [16].

There are two important points which are worth emphasizing here. First, we do not expect to approximate full efficiency in this setting, at least with our private strategies. Likelihood ratio is a very important factor to determine the efficiency loss as seen in the last sections. Since the likelihood ratio is bounded above here, there should remains a significant efficiency loss. This results in more subtle comparison between PPE and PE because both equilibrium payoffs are bounded away from the efficient frontier. This calls upon a careful characterization of the PPE payoff set. Second, we need an assumption on the information structure to make our private strategies more efficient than public strategies. The advantage of private strategies we saw was that they made it easier for a player to detect the opponent's defection when a player defects herself. We impose the following condition to apply this insight in the current setting.

Assumption

$$\frac{p(X|D,C)}{p(X|C,C)} < \frac{p(X|D,D)}{p(X|C,D)}$$

Note that this implies decreasing returns to scale: $p(Y|C,C) - p(Y|D,C) < p(Y|C,D) - p(Y|DD)$. Let us denote $p(X|CC) = p_0$, $p(X|CD) = p(X|DC) = p_1$, and $p(X|DD) = p_2$ in this section. Let $L^q = \frac{(1-q)p_1 + qp_2}{(1-q)p_0 + qp_1}$ be the likelihood ratio of the signal X with respect to the defection when a player is playing C and D with probability $1-q$ and q. We first derive the upper bound of the PPE payoff set in the next subsection. It is an analogue of RMM inefficiency result for this discrete version of partnership games.

3.1 Upper Bound of PPE

The upper bound of the pure strategy strongly symmetric PPE payoff is easy to obtain. Let $\overline{v}_{ps}$ be the best pure strategy strongly symmetric PPE payoff. Since there are only two signals available, it is not possible to "reward" one player when the other player is "punished". Both player has to be punished at the same time when the signal X is observed. So it is efficient to set the punishment level as small as the level where players are indifferent between C and D. When the signal Y is observed, it is efficient to use $\overline{v}_{ps}$ again. These observations lead to the following recursive equation:

$$\overline{v}_{ps} = (1-\delta) + \delta(1-\rho p_0)\overline{v}_{ps} \tag{7}$$

where ρ is a probability to trigger punishment given that X is observed. The indifference condition is given by:

$$\overline{v}_{ps} = (1-\delta)(1+d) + \delta(1-\rho p_1)\overline{v}_{ps} \tag{8}$$

Solving equations (7) and (8) for $\overline{v}_{ps}$ and ρ, the following formula is obtained.

$$\overline{v}_{ps} = 1 - \frac{d}{L^0 - 1} \tag{9}$$

as described in the introduction.

The best strongly symmetric PPE can be a mixed one. It is sometimes necessary to use mixed strategies to achieve the maximum efficient payoff even within the class of strongly symmetric public strategies.[12] In such a case, the best mixed strategy symmetric PPE is obtained just by mixing C and D with probability $1-q$ and q instead of using the profile (C, C) in $(\#)$. Solving a recursive equation and an indifference condition similar to (7) and (8), we can obtain the equilibrium payoff of such a mixed PPE:

$$1 - q - qh - \frac{(1-q)\,d + qh}{L^q - 1}$$

The interpretation of this formula is the same as before. Note that if $q = 0$, then this is equivalent to (9). Why can mixing help to improve the best strongly symmetric PPE payoff even though it reduces the efficiency in the stage game? It is because the efficiency loss associated with the punishment might decrease. We can see from the above formula that (1): deviation gain can become small if $d > h$ or/and (2): the likelihood ratio may increase as q increases. Let $q^* = \arg\max_{q\in[0,1]} 1 - q - qh - \frac{(1-q)d+qh}{L^q-1}$. The following is the formal statement with respect to the bound of strongly symmetric PPE payoffs including mixed ones.

Proposition 2 *The bound of the strongly symmetric PPE payoffs of this repeated partnership game is given by* $\overline{v}_{ss} = \max\left\{1 - q^* - q^*h - \frac{(1-q^*)d+q^*h}{L^{q^*}-1}, 0\right\}$.

Proof. See Appendix. ∎

This bound is a tight one. Either the stationary strategy described above obtains $1 - q^* - q^*h - \frac{(1-q^*)d+q^*h}{L^{q^*}-1}$ or any level of cooperation cannot be sustained.

In order to get the bound of all the symmetric PPE payoff, we also need to take care of the cases where the optimal strategy pair is asymmetric. If that possibility is taken account, the upper bound has to be modified in the following way:

Proposition 3 *The bound of the symmetric PPE payoff of this repeated partnership game is given by* $\overline{v}_s = \max\left\{1 - q^* - q^*h - \frac{(1-q^*)d+q^*h}{L^{q^*}-1}, \frac{1+d-h}{2}, 0\right\}$, *and* $\overline{v}_s \geqq \frac{V_1^*+V_2^*}{2}$ *for any PPE payoff profile* (V_1^*, V_2^*).

[12] Mixed public strategies have not been investigated systematically in Radner, Myerson, and Maskin [16] or Abreu, Milgrom, and Pearce [1].

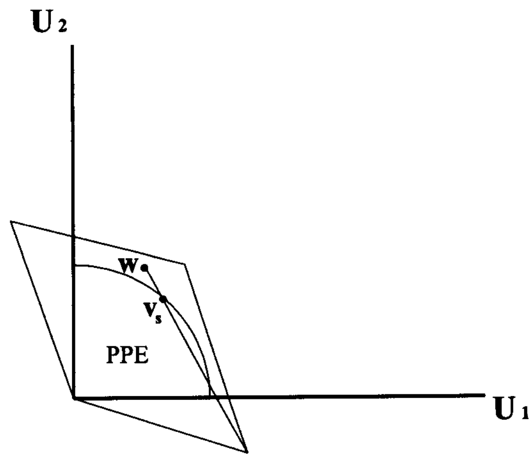


Figure 1:

Proof. See Appendix. ■

Interestingly, it turns out that when an asymmetric strategy achieves the best symmetric payoff, at least one player has to play D with probability 1 in the first period. The equilibrium in which each player uses a different nondegenerate behavior strategy in the first period is not an efficient one. Given this, it is clear that $\frac{1+d-h}{2}$ is the upper bound for PPE when it is achieved by asymmetric public strategies. Figure 1 illustrates that if the best symmetric PPE (denoted by V_s) with (D, C) played in the initial period achieves more than $\frac{1+d-h}{2}$, then the continuation payoff profile (denoted by W) cannot stay in the PPE payoff set.[13]

It is easy to find a set of parameter profiles for which $\frac{1+d-h}{2}$ is really the upper bound obtained by an asymmetric PPE where players play (C, D) (D, C) alternatively. However, this bound may not be always tight. When $p(X|a)$ is linearly increasing in the number of defections, which is the case analyzed in detail by Fudenberg and Levine [8], the bound in Proposition 3 is tight in the sense that one of the three numbers $\left\{1 - q^* - q^*h - \frac{(1-q^*)d+q^*h}{L^{q^*}-1}, \frac{1+d-h}{2}, 0\right\}$ is the upper bound and there exists a strategy which achieves this upper bound. Otherwise, there may exist the best symmetric PPE payoff achieved by some asymmetric strategy, which does not hit the above bound.

[13]Note that the PPE payoff set is convex because players have an acesss to a public correlation device.

3.2 Constructing Private Equilibrium

In this subsection, a PE is constructed and compared to the bound of the symmetric PPE payoffs obtained in the last subsection. Let us introduce a formal description of machines here because every private strategy in this paper can be described as a simple machine. A machine M_i is a quadruple $\langle \Theta_i, \theta_{i,0}, q_i, \mu_i \rangle$. For this quadruple, $\Theta_i = \{\theta_{i,n}\}_{n=0}^{l_i}$ is the set of states of the machine with $\theta_{i,0}$ being the initial state. The level of mixture between C and D at each state is determined by a function $q_i : \Theta_i \to [0,1]$. For example, $q_i(\theta_{i,n})$ is the probability of playing D when player i is in the state $\theta_{i,n}$. The transition function is $\mu_i : \Theta_i \times A_i \times \Omega \times \Omega_i \to \Theta_i$, where $A_i = \{C, D\}$, $\Omega = \{X, Y\}$, and $\Omega_i = [0,1]$. The last coordinate $\omega_i \in \Omega_i$ is introduced to allow random transitions over states given $(\theta_{i,n}, a_i, \omega)$. Assume a unifrm distribution on $\Omega_i = [0,1]$ without loss of generality. Each machine M_i induces a mixed strategy, which may or may not be a behavior strategy when the transition is random given $(\theta_{i,n}, a_i, \omega)$. We denote by $\sigma_i(M_i)$ the behavior strategy corresponding to the mixed strategy generated by a machine M_i.[14,15]

The (symmetric) private strategy we employ in this section is as follows:

- State R (Reward State):

  Choose D with probability q_R (a small number). Go to state P if D was taken and X was observed (otherwise, stay in State R).

- State P (Punishment State):

  Choose D with probability q_P (a large number). Go to state R with probability $\rho \in (0,1)$ if D was taken and Y was observed (otherwise, stay in State P).

This is just a machine with two states $\Theta_i = \{R, P\}$ with R being the initial state, and the transition function for this machine is formally given by

$$\mu_i(R, a_i, \omega, \omega_i) = \begin{cases} R \text{ if } (a_i, \omega) \neq (D, X) \\ P \text{ if } (a_i, \omega) = (D, X) \end{cases}$$

$$\mu_i(P, a_i, \omega, \omega_i) = \begin{cases} P & \text{if } (a_i, \omega) \neq (D, Y) \\ P & \text{if } (a_i, \omega) = (D, Y) \text{ and } \omega_i \in (\rho, 1] \\ R & \text{if } (a_i, \omega) = (D, Y) \text{ and } \omega_i \in [0, \rho_i] \end{cases}$$

Figure 2 describes this machine graphically.

First note that this private strategy has the same feature as the one in Section 2. A player moves to state P (Punishement State) only after (D, X); the most informative action-signal pair. Second, note that there is a strategic undertainty we described before. A player is not sure whether the other player is

[14] Aumann (1964) [4]

[15] This machine can be "purified" by introducing more (private) inputs and expanding the state space with an appropriate transition function.

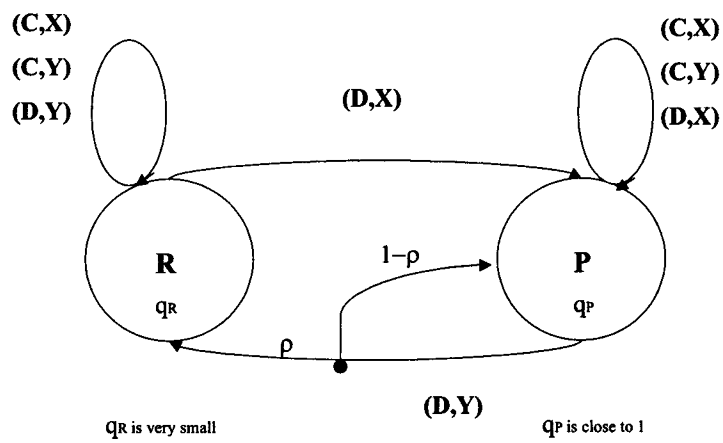


Figure 2:

state R or state P after X is observed (and never will). How can we check if this machine is playing a best reponse strategy at every history given such a ever-changing belief? To resolve this problem, we choose (q_R, q_P, ρ) in such a way that *no matter which state player 2 is in, player 1 is always indifferent between choosing C and choosing D.* This means that any repeated game strategy is a best response to the machine, hence so is the machine itself.

Since the strategy is symmetric, subscript i is omitted after here as long as it does not cause any confusion. A set of parameters (q_R, q_P, ρ) is chosen to satisfy the following four equations. When player 2 is in state R , the equilibrium conditions for player 1 are

- (player 1 plays C today)

$$V_R = (1-\delta)(1-q_R-q_Rh)+\delta\{(1-q_Rp_1)V_R+q_Rp_1V_P\} \tag{10}$$

and

- (player 1 plays D today).

$$V_R = (1-\delta)(1-q_R)(1+d)+\delta\{(1-q_Rp_2)V_R+q_Rp_2V_P\} \tag{11}$$

When player 2 is in state P , the equilibrium conditions for player 1 are

- (player 1 plays D today)

$$V_P = (1-\delta)(1-q_P-q_P\beta)+\delta\left[\begin{array}{c} q_p(1-p_1)\rho V_R \\ +\{1-q_p(1-p_1)\rho\}V_P \end{array}\right] \quad (12)$$

, and

- (player 1 plays C today)

$$V_P = (1-\delta)(1-q_P)(1+d)+\delta\left[\begin{array}{c} q_p(1-p_2)\rho V_R \\ +\{1-q_p(1-p_2)\rho\}V_P \end{array}\right] \quad (13)$$

where V_s can be interpreted as the discounted average payoff for player 1 when player 2 is in state $s = R, P$.

Equation (10) and (11) imply that player 1 is indifferent between C and D when player 2 is in state R and if her continuation payoff is completely determined by her opponent's state. Similarly, (12) and (13) imply that player 1 is indifferent between C and D when player 2 is in state P. A system of these equations indeed implies that player 1 is completely indifferent among all the repeated game strategies and player 2's state determines player 1's continuation payoff completely as we assumed. Any payoff difference one can make in the current period is exactly offset by the difference of the continuation payoffs caused by the change of the other player's transition probability. Let us emphasize again that a player never knows what is the opponent's continuation strategy or which state the opponent is in during the game. However, players do not have to know them because their expected payoffs cannot be affected by their own strategies. Note that this logic is somewhat similar to the one for a totally mixed strategy equilibrium in a finite normal form game. What is interesting here is that the same thing is done for an infinite game with only a finite number of equations and some value functions.

If the solution (q_R^*, q_P^*, ρ^*) of these equations are in $[0,1]$, then these numbers can be used for the function f and the transition function μ, generating a behavior strategy $\sigma(M)$ which is a sequential equilibrium. The following main proposition shows that for δ close to 1, we can find a solution $(q_R^*(\delta), q_P^*(\delta), \rho^*(\delta), V_R^*(\delta), V_P^*(\delta))$ parameterized by δ for the above equations (10) - (13), where $q_R^*(\delta)(>0) \to 0$ as $\delta \to 1$ and $q_P^*(\delta) = 1$ with an appropriately chosen $\rho^*(\delta) \in [0,1]$.[16] Since $V_R^*(\delta) = 1 - q_R^*(\delta) - q_R^*(\delta)h - \frac{(1-q_R^*(\delta))d + q_R^*(\delta)h}{L^1-1}$, the payoff arbitrary close to $1 - \frac{d}{L^1-1}$ is achieved as a PE as $\delta \to 1$. Note that this formula uses the likelihood ratio L^1 $(> L^0)$ instead of L^0, but otherwise it looks exactly like the best strongly symmetric PPE payoff.

Proposition 4 *Suppose that* $p_2 - p_1 > p_1 d + (1-p_2)h$.[17] *Then for any* $\eta > 0$, *there exists a* $\underline{\delta}$ *such that for all* $\delta \in (\underline{\delta}, 1)$, *there exists a symmetric private strategy pair* $(\sigma(M(\delta)), \sigma(M(\delta)))$ *parameterized by* δ, *which is a sequential*

[16] Since there are five unknowns in four equations, we can choose a value of one variable.

[17] This assumption is equivalent to $V_R(\delta) > V_P(\delta)$, where $V_R(\delta)$ and $V_P(\delta)$ are derived from the equations (10) - (13).

equilibrium with a compatible belief system and generates the symmetric equilibrium payoff $(V(\delta), V(\delta))$ *such that* $V(\delta) > 1 - \frac{d}{L^1-1} - \eta$.

Proof. Given that $0 < \delta < 1$, we can derive the following system of equations equivalent to (10) - (13).

$$V_R = 1 - q_R - q_R h - \frac{(1-q_R)d + q_R h}{L^1 - 1} \tag{14}$$

$$V_P = 1 - q_P - q_P h + \frac{(1-q_P)d + q_P h}{L^1 - 1}\frac{1-p_1}{p_1} \tag{15}$$

$$(1-\delta)\{(1-q_R)d + q_R h\} = \delta q_R (p_2 - p_1)(V_R - V_P) \tag{16}$$

$$q_P = \frac{q_R d}{q_R(d-h) + \rho\{(1-q_R)d + q_R h\}} \tag{17}$$

Once q_R is obtained, then V_R and V_P can be obtained from (14) and (15) respectively. Since ρ can be an arbitrary number between 0 and 1, it is set to be $\frac{q_R h}{(1-q_R)d+q_R h} \in [0,1]$ so that $q_P = 1$. Substituting (14), (15) and $q_P = 1$ for V_R, V_P and q_P in (16), we get a quadratic equation, whose solution can be used for q_R :

$$F(x,\delta) = c_2(\delta)x^2 + c_1(\delta)x + c_0(\delta) = 0$$

with

$$c_2(\delta) = \delta\{p_2(1+h) - p_1(1+d)\}$$

$$c_1(\delta) = (1-\delta)(h-d) + \delta\{p_1 d + (1-p_2)h - (p_2-p_1)\}$$

$$c_0(\delta) = (1-\delta)d$$

One root of this quadratic equation is clearly $(x,\delta) = (0,1)$. Since $\frac{\partial F}{\partial x}|_{(x,\delta)=(0,1)} \neq 0$ by the assumption $p_2 - p_1 > p_1 d + (1-p_2)h$, the implicit function theorem can be applied to get a C^1 function $q_R(\delta)$ around $\delta = 1$ with $\frac{dq_R(1)}{d\delta} = -\frac{\frac{\partial F}{\partial \delta}|_{(x,\delta)=(0,1)}}{\frac{\partial F}{\partial x}|_{(x,\delta)=(0,1)}} = \frac{d}{p_1 d + (1-p_2)h - (p_2-p_1)}$, which is negative by assumption. So, there exists a $q_R(\delta) \in (0,1)$ for large enough δ such that $q_R(\delta) \to 0$ as $\delta \to 1$. Hence we get a parametrized solution $\left(q_R(\delta), 1, \frac{q_R(\delta)h}{(1-q_R(\delta))d+q_R(\delta)h}, V_R(\delta), V_P(\delta)\right)$ for (14) - (17). Now $(\sigma(M(\delta)), \sigma(M(\delta)))$ with $f(R) = q_R(\delta)$, $f(P) = 1$, and $\rho(\delta) = \frac{q_R(\delta)h}{(1-q_R(\delta))d+q_R(\delta)h}$ is a sequential equilibrium with a compatible belief.[18] The equilibrium payoff is $V_R(\delta)$, which converges to $1 - \frac{d}{L^1-1}$ as $\delta \to 1$. For any $\eta > 0$, we can pick $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$, $(\sigma(M(\delta)), \sigma(M(\delta)))$ generates the equilibrium payoff $V(\delta)$ more than $1 - \frac{\alpha}{L^1-1} - \eta$. ■

With more conditions on the parameters in addition to $L^1 > L^0$ and $p_2 - p_1 > p_1 d + (1-p_2)h$, we can actually show that the PE Pareto-dominates the best symmetric PPE obtained in the last subsection.

[18] Belief can be simply derived by Bayes rule at any history. Since any deviation is not observable to the opponent, a player always updates her belief assuming that the opponent has never deviated.

Proposition 5 *If $L^1 > L^0$, $p_2 - p_1 > p_1 d + (1-p_2) h$, $h > d$, and $1 - \frac{d}{L^1 - 1} > \frac{1+d-h}{2}$, then there exists a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$, the equilibrium payoff generated by $(\sigma(M(\delta)), \sigma(M(\delta)))$ is larger than $\overline{v}_s$.*

Proof. See Appendix. ∎

Although many restrictions are imposed on the structure of the stage game to get this result, there still exists an open set of parameters which satisfies all these restrictions. The first example in the next section satisfies this restriction.

4 Examples

We provide two examples in this section. In the first example with two public signals, a PE is shown to be much more efficient than any PPE. The second example suggests that our insight about private strategies is also valid in cases where the public signal takes on more than two values.

Example 1:

It is assumed that $d = \kappa > 0$, $h = 1 + \kappa > 0$, and

$$\begin{cases} p(X|CC) = \frac{1}{2} \\ p(X|CD) = p(X|DC) = \frac{1}{2} + \epsilon \\ p(X|DD) = 1 - \epsilon \end{cases}$$

where ϵ is a small positive number.[19]

Note that the assumptions for Proposition 4 or 5 are satisfied for small ε if $\kappa < 1$. As ε becomes small, it becomes more difficult to detect the opponent's deviation when (C, C) is played.

It is easy to see that any strongly symmetric PPE does not work. A player has to mix D to monitor the opponent effectively, but the stage game payoff decreases significantly if both players do so. This negative effect overcomes the positive effect which comes from the improved monitoring, hence any cooperation is not sustainable in strongly symmetric strategies. This is clear from the formula for the best (nontrivial) strongly symmetric PPE payoff. When ϵ is small, the formula $1 - q - qh - \frac{(1-q)d+qh}{L^q - 1}$ is approximately $1 - q(2+\kappa) - \frac{(1-q)\kappa + q(1+\kappa)}{q} = -q(2+\kappa) - \frac{\kappa}{q}$, which is a negative number.

Another candidate of the upper bound for symmetric PPE payoffs is simply $\frac{1+d-h}{2} = 0$ by Proposition 3. So there exists a $\overline{\epsilon}$ such that for $\epsilon \in (0, \overline{\epsilon})$ the only

[19]When a player is playing C, the distribution of the public signal is not so sensitive to the other player's action. This implies that the realized payoffs have to vary large to generate the fixed expected payoff matrix as ϵ becomes small. In particular, $u(C, X) \to -\infty$ and $u(C, Y) \to \infty$ as $\epsilon \to 0$.

Also note that we need ϵ to be strictly positive. Otherwise, we cannot recover the expected payoff matrix assumed here.

PPE is the repetition of the one shot Nash equilibrium independent of discount factor.

On the other hand, the private equilibrium payoff approximates $1 - \frac{d}{L^1 - 1} = 1 - \kappa$ as $\delta \to 1$ when ϵ is small. Since κ is an arbitrary small positive number, we can construct an example where the PE approximates the efficient outcome arbitrarily closely and the only PPE is the repetition of the one-shot Nash equilibrium.

Example 2:

The next example shows that even when the folk theorem holds, a PE does better than any PPE for any discount factor $\delta < 1$. It is a version of the prisoners' dilemma, whose expected stage game payoffs are given by the following table.

	C	D
C	$1, 1$	$-6, 2$
D	$2, -6$	$0, 0$

The public signal ω takes on three values, X, Y_1, and Y_2, and the probability distributions are given below.

	X	Y_1	Y_2
(C, C)	$1/3$	$1/3$	$1/3$
(D, C)	0	$1/2 + \epsilon$	$1/2 - \epsilon$
(C, D)	0	$1/2 - \epsilon$	$1/2 + \epsilon$
(D, D)	$1/3$	$1/3$	$1/3$

Note that, as long as $\epsilon > 0$, the pairwise full rank condition (PFR) is satisfied at (C, C), that is, the first three rows are linearly independent.[20] This means that each player's defection at (C, C) is statistically discriminated (player i's deviation makes signal Y_i more likely, $i = 1, 2$). So Fudenberg-Levine-Maskin Folk Theorem applies, and the efficient payoff $(1, 1)$ can be approximately achieved by a PPE as $\delta \to 1$. Also note that this model is similar to the model in Section 2, where signal X arises only when both players take the same action. Therefore, the efficient payoff $(1, 1)$ can also be approximately achieved by a PE as $\delta \to 1$ as in Section 2. In summary, both PPE and PE asymptotically achieves efficiency as $\delta \to 1$ in this example. We can show, however, that the PE in Section 2 does better than *any* PPE for any $\delta < 1$ for small enough ϵ.

Formally, we derive the following upper bound of the best symmetric PPE payoffs.

Proposition 6 *For any (large) $H > 0$, there is a (small enough) value of the signal distribution parameter $\epsilon > 0$ such that*

$$\max\left\{1 - \left(\frac{1-\delta}{\delta}\right)H, 0\right\}$$

is an upper bound of the best symmetric PPE payoffs under δ.

[20]When $\epsilon = 0$, PFR fails at any (possibly mixed) action profile, because at most two rows in the above table are linearly independent.

Note that, when H is large, the upper bound is a steep (almost linear) curve for δ sufficiently close to 1 (and otherwise it is 0). The proof is given in Appendix B. Intuitively, this bound is derived by the following observation. It turns out that in our example positive payoffs cannot be sustained if we punish the players *simultaneously*. However, as long as $\epsilon > 0$, we can utilize an *asymmetric* punishment where we "transfer" player i's future payoff to player j, when player i's defection is suspected (i.e., when Y_i arises). Hence to support a payoff profile by a PPE, we must require the future payoffs to vary in the northwest/southeast directions around the payoff profile to be supported. As the players' defections become indistinguishable ($\epsilon \to 0$), however, we need huge payoff transfers to support cooperation, and for those transfers to be in the equilibrium payoff set, the discount factor should be sufficiently large. This observation provides a lower bound of δ to support the given payoff profile, which in turn provides the upper bound of the PPE payoffs in Proposition 6.

On the other hand, our private equilibrium relies onlly on the assumption $p(X|D,D) > 0 = p(X|D,C) = p(X|C,D)$, not the level of ϵ. As in Section 2, we can derive the equilibrium probability q_i of defection for each player i by solving the following quadratic equation in q;

$$(1-\delta)\{(h-d)q+d\} = \delta q p(X|D,D)(1-q-qh) \tag{18}$$

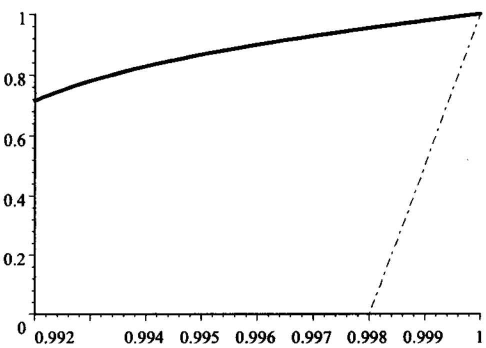
Note that, in the current example, we have $h=6$, $d=1$ and $p(X|DD)=1/3$. Hence (18) becomes

$$f(q) \equiv 7\delta q^2 + (15-16\delta)q + 3(1-\delta) = 0.$$

As $f(0) = 3(1-\delta) > 0$ and $f(1) = 18 - 12\delta > 0$, if we have real solutions they both lie in $[0,1]$. As we are interested in the most efficient equilibrium (hence the one with the smallest q), we choose the smaller root

$$q(\delta) = \frac{-15+16\delta-\sqrt{225-564\delta+340\delta^2}}{14\delta}$$

Computation shows that this solution is real when $\delta \geq 0.992$. The associated symmetric private equilibrium payoff for each player is $v(\delta) = 1 - 7q(\delta)$. Figure 4 plots this and the upper bound of symmetric PPE payoffs in Proposition 6: $1 - (\frac{1-\delta}{\delta})H$, where H is set to be 500 by choosing a suitable small ϵ. The horizontal axis represents the discount factor δ. The solid curve represents the private equilibrium payoff, while the thin dotted line is an upper bound of all PPE payoffs.



Hence a PE does better than any PPE for each $\delta < 1$ when information is close to symmetric ($\epsilon > 0$ is small).

5 Generalization

5.1 Two State Is Enough

We generalize our construction of private strategy equilibria to more general two player games. First, we introduce general machines with many states, which share the same property with the simple two state machines in the previous sections, and show that they can be reduced to two state machines. This implies that we can focus on two state machines without loss of generality.

Let $a_i \in A_i$ be a pure action of player i, and $g_i(a)$ be player i's payoff associated with the action profile $a \in A = A_1 \times A_2$. Denote i's mixed action by $\alpha_i \in \Delta_i$, and with an abuse of notation, let $g_i(\alpha)$ be player i's expected payoff associated with mixed action profile $\alpha \in \Delta_1 \times \Delta_2$. Let $p(\cdot|\alpha)$ be probability distribution on public signals given $\alpha \in \Delta_1 \times \Delta_2$.

Now we formally define generalized machine with many states. A machine M_i for player i is $\left\{\{\theta_i^n\}_{n=0}^{l_i}, \alpha_i, \mu_i\right\}$ ($l_i, i = 1, 2$ can be ∞), where $\{\theta_i^n\}_{n=0}^{l_i}$ is the set of player $i's$ states with θ_i^0 being the initial state. Player $i's$ behavior strategy at the state θ_i^n is $\alpha_i(\theta_i^n) \in \Delta_i$, and $\mu_i(\theta_i^m | (a_i, \omega), \theta_i^n)$ is the probability to transit from θ_i^n to θ_i^m when a_i is played and ω is observed.

Let supp(α_i^n) be the support of α_i^n. Suppose that (M_1, M_2) satisfies the following conditions for some bounded sequence of real numbers $V = \left(\{V_1^n\}_{n=0}^{l_2}, \{V_2^n\}_{n=0}^{l_1}\right)$.

$$\begin{aligned} For\ n &= 1, ..., l_2 \qquad (19) \\ \forall a_1 &\in A_1^*,\ V_1^n = (1-\delta)\, g_1(a_1, \alpha_2^n) + \\ &\delta \sum_{a_2 \in supp(\alpha_2^n)} \sum_{\omega \in \Omega} \sum_{m=1}^{l_2} \alpha_2^n(a_2)\, p(\omega | a_1, a_2)\, \mu_2(\theta_2^m | (a_2, \omega), \theta_2^n)\, V_1^m \end{aligned}$$

$$
\begin{aligned}
\forall a_1 &\notin A_1^*,\ V_1^n \geq (1-\delta)\, g_1(a_1, \alpha_2^n) + \\
&\delta \sum_{a_2 \in supp(\alpha_2^n)} \sum_{\omega \in \Omega} \sum_{m=1}^{l_2} \alpha_2^n(a_2)\, p(\omega | a_1, a_2)\, \mu_2(\theta_2^m | (a_2, \omega), \theta_2^n)\, V_1^m \\
For\ n &= 1, ..., l_1 \\
\forall a_2 &\in A_2^*,\ V_2^n = (1-\delta)\, g_2(\alpha_1^n, a_2) + \\
&\delta \sum_{a_1 \in supp(\alpha_1^n)} \sum_{\omega \in \Omega} \sum_{m=1}^{l_1} \alpha_1^n(a_1)\, p(\omega | a_1, a_2)\, \mu_1(\theta_1^m | (a_1, \omega), \theta_1^n)\, V_2^m \\
\forall a_2 &\notin A_2^*,\ V_2^n \geq (1-\delta)\, g_2(a_2, \alpha_1^n) + \\
&\delta \sum_{a_1 \in supp(\alpha_1^n)} \sum_{\omega \in \Omega} \sum_{m=1}^{l_1} \alpha_1^n(a_1)\, p(\omega | a_1, a_2)\, \mu_1(\theta_1^m | (a_1, \omega), \theta_1^n)\, V_2^m \\
A_i^* &= \cup_{n=1}^{l_i} supp(\alpha_i^n), i = 1, 2
\end{aligned}
$$

It is not difficult to see that (M_1, M_2) constitutes a sequential equilibrium with payoff (V_1^0, V_2^0) as in the two state case if the above conditions are satisfied. This machine is basically an $n-$state analogue of the two state machine in previous sections. We can show that, when a sequential equilibrium consists of a pair of machines which satisfies the above equations, there exists a sequential equilibrium with a two state machine which is payoff equivalent to such equilibrium.[21]

Proposition 7 *If a pair of machines (M_1, M_2) with many states $(l_1, l_2 \geq 2)$ satisfies (19), there exists a pair of two state machines which constitute a sequential equilibrium with the payoff profile* $(\overline{V}_1, \overline{V}_2) = (\sup_{n=0,...,l_2} \{V_1^n\}, \sup_{n=0,...,l_1} \{V_2^n\})$.

Proof. See Appendix. ■

The intuition of the proof is very simple. Player i's state θ_i^n determines player j's continuation payoff completely. If the number of player i's states is finite, then there exists player i's state $\overline{\theta}_i$ which maximizes player j's continuation payoff and $\underline{\theta}_i$ which minimizes player j's continuation payoff. Then, player i can always generate player j's payoff at any other state θ_i^n by randomly moving to $\overline{\theta}_i$ and $\underline{\theta}_i$ when she is supposed to move to θ_i^n. Hence, she needs only two states to generate any payoff of player j associated with her states. When the number of the states is not finite, we may not be able to find such $\overline{\theta}_i$ and $\underline{\theta}_i$. However, we can still find a sequence of the states (and mixed actions associated with them) to approximate $\overline{V}_j = \sup_{n=0,...,l_i} \{V_j^n\}$ and $\underline{V}_j = \inf_{n=0,...,l_i} \{V_j^n\}$, and we can

[21] Remember that Piccione [14] used such a machine with countable states in the context of repeated games with private monitoring, and Ely and Valimaki [6] succeeded to simplify it to a two state machine. The following result provides an algorithm to reduce the number of states to two in more general settings.

construct a two state machine whose states correspond to $\sup_{n=0,\ldots,l_i}\{V_j^n\}$ and $\inf_{n=0,\ldots,l_i}\{V_j^n\}$ $i,j=1,2$, by choosing a convergent subsequence.

Remark 8 *Indeed any payoff profile* $(V_1,V_2)\in[\underline{V}_1,\overline{V}_1]\times[\underline{V}_2,\overline{V}_2]$ *can be supported by using the two state machine we constructed. For example, if player* i *chooses* $\overline{\theta}_i$ *and* $\underline{\theta}_i$ *with probability* $(1-\lambda_i,\lambda_i)$ *as an initial state, then this pair of machines still constitutes a sequential equilibrium and player* j*'s expected average payoff is* $(1-\lambda_i)\overline{V}_j+\lambda_i\underline{V}_j$, $j\neq i$.

5.2 General Two State Machine

5.2.1 Characterization

Now we can focus on two state machines. We use R and P to denote the two states as before. Let $A_i^{Z*}=supp\left(\alpha_i^Z\right)$ for $Z=R,P$, and F be the set of (α,x,V,A^*) $\left(\alpha=\left(\alpha_1^R,\alpha_1^P,\alpha_2^R\alpha_2^P\right)\text{ and so on}\right)$, that satisfies the following conditions for each player i and her opponent j.

$$\forall a_i\in A_i^*\quad V_i^R=g_i(a_i,\alpha_j^R)-E[x_i^R(\omega,a_j)|a_i,\alpha_j^R] \tag{20}$$

$$\forall a_i\notin A_i^*\quad V_i^R\geqq g_i(a_i,\alpha_j^R)-E[x_i^R(\omega,a_j)|a_i,\alpha_j^R] \tag{21}$$

$$\forall(\omega,a_j)\quad x_i^R(\omega,a_j)\geq 0 \tag{22}$$

$$\forall a_i\in A_i^*\quad V_i^P=g_i(a_i,\alpha_j^P)+E[x_i^P(\omega,a_j)|a_i,\alpha_j^P] \tag{23}$$

$$\forall a_i\notin A_i^*\quad V_i^P\geqq g_i(a_i,\alpha_j^p)+E[x_i^P(\omega,a_j)|a_i,\alpha_j^P] \tag{24}$$

$$\forall(\omega,a_j)\quad x_i^P(\omega,a_j)\geq 0 \tag{25}$$

$$A_i^*=A_i^{R*}\cup A_i^{P*} \tag{26}$$

$$V_i^R>V_i^P \tag{27}$$

This system of (in)equalities turns out to be equivalent to (19) with two states, hence characterizes two state machine equilibria which satisfy it.

Proposition 9 *(i) If there is a two-state machine equilibrium which satisfies (19), then* $(\alpha,x,V,A^*)\in F$. *Conversely, if* $(\alpha,x,V,A^*)\in F$*, then there is a two-state machine equilibrium where (19) is satisfied for* V*, provided that discount factor* δ *is close enough to unity. (ii) The (constrained) Pareto efficient asymptotic values (as* $\delta\to 1$*) that can be supported by such two-state machines are found by, for each welfare weight vector* $\gamma\in\Re_+^2$,

$$W(\gamma)=\sup_{(\alpha,x,V,A^*)\in F}(\gamma_1V_1^R+\gamma_2V_2^R).$$

Let us introduce some more notations. Let $g_i(\mathbf{a}, \alpha_j^R)$ be the vector of player's i's expected payoffs given α_j^R. Let $P(\alpha_j)$ be a positive $|A_i| \times |\Omega|$ matrix whose k, l element is $p(\omega^l | a_i^k, \alpha_j)$ and define $P(A_j')$ by $P(A_j') = \left(P(a_j^1), ..., P\left(a_j^{|A_j'|}\right)\right)$ for $A_j' = \left\{a_j^1, ..., a_j^{|A_j'|}\right\} \subset A_j$ ($|A_i| \times |\Omega| \, |A_j'|$ matrix). Finally, let $\mathbf{x}_i^Z(a_j) = \alpha_j^Z(a_j) \cdot (x_i^Z(\omega^1, a_j), ..., x_i^Z(\omega^{|\Omega|}, a_j))$ for $Z = R, P$ and $\mathbf{x}_i(A_j') = \left(\mathbf{x}_i(a_j^1), ..., \mathbf{x}_i(a_j^{|A_j'|})\right)'$. Then, conditions (20)$-$(21) and (23)$-$(24) can be compactly expressed as

$$\begin{aligned} g_i(\mathbf{a}, \alpha_j^R) - V_i^R \cdot \mathbf{I} + \mathbf{h}_i^R &= P(A_j^*) \cdot \mathbf{x}_i^R(A_j^*) \\ g_i(\mathbf{a}, \alpha_j^P) - V_i^P \cdot \mathbf{I} + \mathbf{h}_i^P &= -P(A_j^*) \cdot \mathbf{x}_i^P(A_j^*) \end{aligned} \tag{28}$$

where $\mathbf{I} = (1, ..., 1)' \in \Re^{|A_i|}$ and $\mathbf{h}_i^Z \geq \mathbf{0}$ denotes non-negative slack variables, which correspond to the difference between the left and right hand sides of incentive constraints (20) - (24) (hence h_i^k is 0 if the corresponding action profile a_i^k is in A_i^*). Geometrically, this means that the left hand side is contained in the cone generated by the column vectors of $P(A_j^*)$ (or $-P(A_j^*)$), which we denote by *cone* $(P(A_j^*))$ (*cone* $(-P(A_j^*))$).

This system of equations is more than just another representation of two state machines. It provides us with a deeper insight into their nature. We first review the two state machine used for partnership games in light of this new representation and discuss its geometric interpretation.

5.2.2 Review of Partnership Game

First, let us write down the first equations of (28) for the partnership game.

$$\begin{pmatrix} g_i(C, q_j^R) \\ g_i(D, q_j^R) \end{pmatrix} - \begin{pmatrix} V_i^R \\ V_i^R \end{pmatrix} = \tag{29}$$

$$= \begin{pmatrix} 1-\pi_0 & \pi_0 & 1-\pi_1 & \pi_1 \\ 1-\pi_1 & \pi_1 & 1-\pi_2 & \pi_2 \end{pmatrix} \begin{pmatrix} (1-q_j^R)\, x_i^R(Y, C) \\ (1-q_j^R)\, x_i^R(X, C) \\ q_j^R x_i^R(Y, D) \\ q_j^R x_i^R(X, D) \end{pmatrix}$$

$$(x_i^z(\omega, a_j) \geq 0 \text{ for all } z, (\omega, a_j) \text{ and } i = 1, 2)$$

The best two state machine for player i is represented by a pair of (q_j^R, x_i^R) which maximizes V_i^R (ignoring the existence of V_i^P which satisfies the feasibility condition (27)). Note that the best (public) trigger strategy equilibrium can also be expressed in these equations with additional restrictions such as $x_i^R(\omega, C) = x_i^R(\omega, D)$ for $\omega = Y, X$. So, we can compare the best two state machine (private) equilibrium and the best (public) trigger strategy equilibrium using this system of equations. The best trigger strategy equilibrium payoff V_i^T is obtained by maximizing V_i^R with respect to (q_j^R, x_i^R) with constraints $x_i^R(\omega, C) = x_i^R(\omega, D)$ for $\omega = X, Y$. The best two state machine equilibrium payoff V_i^{R*} is obtained by maximizing V_i^R with respect to (q_j^R, x_i^R) without such constraints, so it is at least weakly larger than V_i^T. However, as an additional constraint, we need to find V_i^{P*} to satisfy the second equations of (28) and (27).

These two equations mean that $\binom{g_i(C,q_j^R)}{g_i(D,q_j^R)} - \binom{V_i^R}{V_i^R}$ is inside of a cone spanned by $\binom{1-\pi_0}{1-\pi_1}, \binom{\pi_0}{\pi_1}, \binom{1-\pi_1}{1-\pi_2}$ and $\binom{\pi_1}{\pi_2}$. Suppose that the assumption we employed $(\frac{\pi_1}{\pi_0} < \frac{\pi_2}{\pi 1})$ indeed holds. Then, we can verify that $\binom{1-\pi_1}{1-\pi_2}$ and $\binom{\pi_1}{\pi_2}$ are the extreme vectors which span the cone as in Figure 3. Similarly, if $\frac{\pi_1}{\pi_0} > \frac{\pi_2}{\pi 1}$, then $\binom{1-\pi_0}{1-\pi_1}$ and $\binom{\pi_0}{\pi_1}$ are the extreme vectors of the cone.

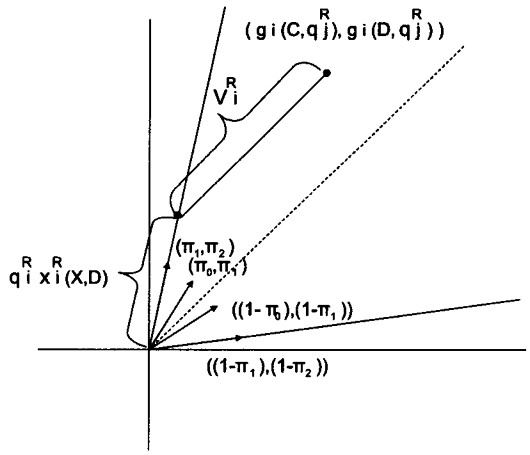


Figure 3

We have a couple of interesting observations.

1. The equations (29) means that V_i^R is equal to the length of the vector I from $\binom{g_i(C,q_j^R)}{g_i(D,q_j^R)}$ to a point in the cone. This clearly shows that *the right hand side of (29) has to be on the face of the cone when V_i^R is maximized* (Figure 3). First note that q_j^R needs to be strictly positive because otherwise we can only use a smaller cone generated by $\binom{1-\pi_0}{1-\pi_1}, \binom{\pi_0}{\pi_1}$ (although q_j^R should go to 0 as δ goes to unity as we see later). Second, $\binom{g_i(C,q_j^R)}{g_i(D,q_j^R)}$ is above the 45° line as shown in Figure 1 because $g_i(D, q_j^R) > g_i(C, q_j^R)$ for small q_j^R. So, only $x_i^R(X, D)$ should be strictly positive and all the other $x_i^R(Y, C), x_i^R(Y, D), x_i^R(X, C)$ have to be 0. This implies that, provided that $V_i^P < V_i^R$ exists, *the best two state machine which satisfies (29) has to be a private strategy.* Note that, for the public triger strategy

equilirbium, x_i^R is necesarily in the interior of the cone, hence does not maximize V^i.

2. Note that the only reason to play D is that it makes a larger cone available. Hence, if $\frac{\pi_1}{\pi_0} \geq \frac{\pi_2}{\pi 1}$, then there is no reason to play D anymore. So, only $x_i^R(X,C)$ should be strictly positive and $x_i^R(Y,C) = x_i^R(Y,D) = x_i^R(X,D) = 0$. Clearly, the usual (public) trigger strategy is (weakly) better than any two state machine in this case. Combined this observation with the last one, we can conclude that *we can construct a two state machine private equilibrium which satisfies (29) and is Pareto superior to the best trigger strategy equilibrium as* $\delta \to 1$ *if and only if* $\frac{\pi_1}{\pi_0} < \frac{\pi_2}{\pi 1}$, provided that V_i^P $(< V_i^R)$ exists.

3. We can interpret a wider cone as a better information structure. Fix the expected payoff structure and change the information structure to a better one associated with a wider cone. The left hand side of (29), which is contained in the cone generated by the current information structure, would be contained in the cone generated by the better information structure (Figure 4). Hence, when signal becomes more informative in this sense, a two state machine equilibrium continues to be a sequential equilibrium (and achieves a larger payoff) after with the new information structure.

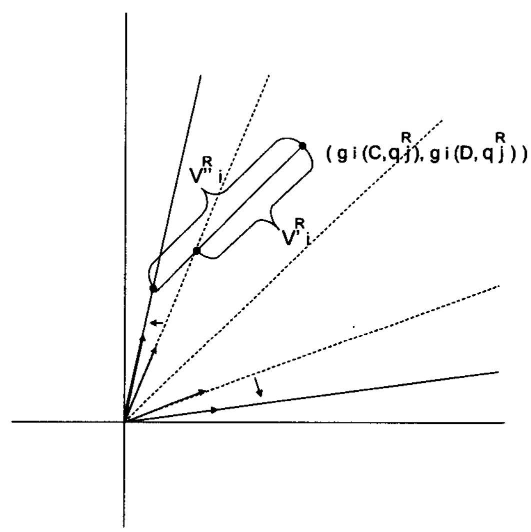


Figure 4

4. Since $\binom{g_i(C,q_j^R)}{g_i(D,q_j^R)}$ moves toward to $\binom{1}{1+d}$ as q_j^R decreases and V_i^R increases toward V_i^{R*} as shown in Figure 5, we need to minimize q_j^R for a given δ, (but not 0). Remember that $x_i^R(X,D) = \frac{\delta}{1-\delta}\rho_j^R(X,D)(V_i^R - V_i^P)$ (cf.

Proof of Proposition 9). We can derive two implications from this formula; (1) This formula suggests that q_j^R can converge to 0 as $\delta \to 1$ because x_i^R can take arbitrary large values as $\delta \to 1$. (2) Since q_j^R should be as small as possible at the optimal level, $\rho_j^R(X,D)$ has to be 1. Otherwise, we can choose smaller q_j^R and larger $x_i^R(X,D)$ to satisfy (29) by increasing $\rho_j^R(X,D)$. *This implies that a player has to move to the punishment state with probability 1 after* (X,D) *as we constructed.*

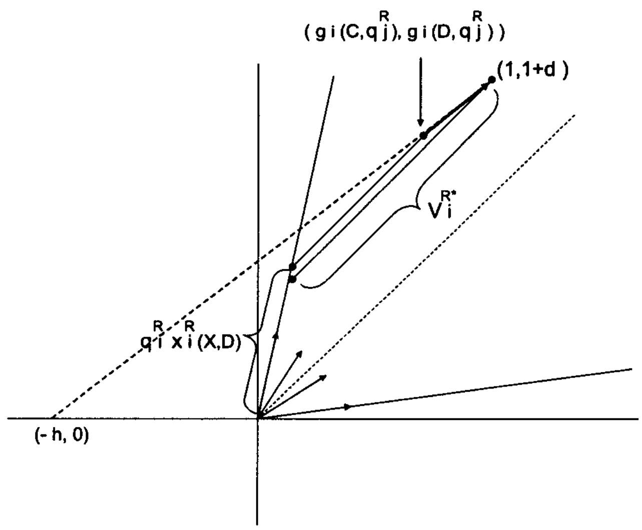


Figure 5

Some of these insights will be useful in the following sections.

5.2.3 Construction of Two State Machine

Next we examine when and how we can construct a two-state machine equilibrium. First, we show that a majority of the relevant constraints (20) - (27) (more precisely, all but the last condition (27)) can be satisfied under a mild condition.

Since the number of rows for each state in (28) is equal to $|A_i|$, for any (A_i^*, α_i^Z) and $\mathbf{h}_i^Z$ $(i = 1,2)$, (28) has a solution $\mathbf{x}_i^Z(A_j^*)$ if $P\left(A_j^*\right)$ has full row rank:

$$\text{rank} P\left(A_j^*\right) = |A_i|. \tag{30}$$

Note that if such $\mathbf{x}_i^Z(A_j^*)$ is nonnegative, then we can find $x_i^Z(\omega, a_j)$ to satisfy (20) - (26) by setting $\left(x_i^Z(\omega^1, a_j), ..., x_i^Z(\omega^{|\Omega|}, a_j)\right) = \mathbf{x}_i(a_j)/\alpha_j^Z(a_j)$ for $a_j \in A_j^{Z*}$ and, say, $x_i^Z(\omega, a_j) = 0$ for $a_j \notin A_j^{Z*}$. Hence, we only need to show that such $\mathbf{x}_i^Z(A_j^*)$ can be taken to be nonnegative to prove the following lemma.

Lemma 10 *(Full Rank): If* $(|\Omega|-1)|A_j^*|+1 \geq |A_i|$ *for* $i,j=1,2$*, then, for a generic choice of signal distribution* $p(\omega|a_1,a_2)$*, the following holds: for any* (A^*,α^R,α^P)*, which satisfies (26), conditions (20)–(25) can be satisfied for some* (V^R,V^P,x^R,x^P).

Proof. See Appendix ■

Remark 11 *A sufficient condition for* $(|\Omega|-1)|A_j^*|+1 \geq |A_i|$ *to be satisfied is that the signal space* Ω *contains sufficiently many outcomes. However, this is not necessary, when* $|A_j^*|$*, the number of actions taken on the equilibrium path, is large. For example,* $(|\Omega|-1)|A_j^*|+1 \geq |A_i|$ *is always satisfied if* $|A_1|=|A_2|$ *and all actions are taken with positive probability on the equilibrium path (as long as* Ω *is not a singleton set).*

Given Lemma 10, we now seek conditions under which the last condition (27) is also satisfied. From the non-negativity conditions (22) and (25) for the punishments x^R and rewards x^P, we clearly need

$$\min_{a_i \in A_i^*} g_i(a_i,\alpha_j^R) > \max_{a_i \in A_i} g_i(a_i,\alpha_j^P) \tag{31}$$

for (27) to be satisfied. This imposes a certain restriction on the actions that can be taken on the equilibrium path (i.e., the supports of α^R and α^P). To see this, let us introduce the notion of *separation*, which is a stronger version of domination:

Definition 12 *A mixed action* α_i *is* separated above *by another mixed action* α_i' *if*

$$\min_{\alpha_j} g_i(\alpha_i',\alpha_j) > \max_{\alpha_j} g_i(\alpha_i,\alpha_j).$$

Let us compare this definition with domination. If α_i is (strongly) dominated by α_i', we have

$$\forall \alpha_j \; g_i(\alpha_i',\alpha_j) > g_i(\alpha_i,\alpha_j).$$

In contrast, under separation, we have a number γ (that is independent of α_j) such that

$$\forall \alpha_j \; g_i(\alpha_i',\alpha_j) > \gamma > g_i(\alpha_i,\alpha_j). \tag{32}$$

Lemma 13 *If* a_i *is separated above by a mixed action, then it cannot be played on the path of play in a two-state machine equilibrium.*

Remark 14 *We can immediately see that* a_i *cannot be in* A_i^* *to satisfy (31). Below we provide more direct proof.*

Proof. Suppose, on the contrary, that a_i is separated above by a mixed action α_i' and the former is played on the equilibrium path. Then (32) is satisfied for $\alpha_i = a_i$. Since each player i is always indifferent between the actions played on the equilibrium path, she obtains the equilibrium payoff when she *always* plays a_i. By (32), the average equilibrium payoff is less than γ. On the other hand, if she always chooses α_i', the average payoff is greater than γ. This means she can profitably deviate, a contradiction. ■

Finally, we check when the last condition (27) is satisfied for a profile of mixed actions $(A^*, \alpha^R, \alpha^P)$ that satisfy condition (31). We define a partial order in the space of cones in $\Re^n_+$ by $A \preceq B$ if and only if cone A is included in cone B. Consider a sequence of information structure $P_n(A_i^*), n = 1, 2, ...$ such that $P_n(A_i^*) \preceq P_{n+1}(A_i^*)$ and $P_n \to R^n_+$ (in Hausdorff metric) for $i = 1, 2$, keeping expected payoffs fixed. Then we can find n^* such that there exists a solution for (28) for all $n \geqq n^*$(See figure 3 for the partnership game). This means that a two state machine can be constructed for arbitrary $(A^*, \alpha^R, \alpha^P)$ which satisfies (31) if information is close to perfect. Next, take two different information structures P' and P'' such that $P' \preceq P''$. If we can find $\mathbf{x}_j^Z(A_i^*)$ for (28) with $P'(A_i^*)$, we can also find the solution for (28) with $P''(A_i^*)$ as well. This means that, for each two state machine equilibrium with P', there exists a similar two state machine equilibrium (with the same $(A^*, \alpha^R, \alpha^P)$) as the signal becomes more informative (as the cone gets wider). The next proposition summarizes these results.

Proposition 15 *(i) Suppose that $(A^*, \alpha^R, \alpha^P)$ satisfies (31). Then a profile of two state machines (α, x, V, A^*) can be constructed if cone $(P(A_i^*))$ is close enough to $R_+^{|A_i|}$ for $i = 1, 2$. (ii) Suppose that a profile of two state machines (α, x, V, A^*) is an equilibrium for P'. If cone $(P'(A_i^*)) \preceq$ cone $(P''(A_i^*))$ for $i = 1, 2$ for another information structure P'', then there exist x', V' with which (α, x', V', A^*) is a two state machine equilibrium for P'' and achieves a (weekly) larger payoff when players are patient enough.*

Proof. See Appendix. ■

Corollary 16 *If P' is a garbling of P in the sense of Blackwell, then, for any two state machine equilibrium (α, x, V, A^*) for P, there exists a profile of two state machines (α, x', V', A^*) which is an equilibrium for P' and achieves a larger payoff when players are patient enough.*

Proof. Note that $p'(\omega|a) = \sum_{\omega'} q(\omega|\omega') p''(\omega'|a)$ where $q(\cdot|\omega')$ is a density function on Ω for each $\omega' \in \Omega$. This implies that *cone* $(P'(A_i')) \preceq$ *cone* $(P''(A_i'))$ for any $A_i' \subset A_i$ and any i. Hence, the result follows. ■

Remark 17 *Note that public signal itself does not have to be close to perfect information. We only require that combination of private action and public signal convey almost perfect information. This would be the case, for example, if one of available actions is some sort of monitoring activity.*

5.3 When Can We Find a PE Better Than a PPE?

5.3.1 Generalized Trigger Strategy Equilibrium

Let $\left(V_1^T, V_2^T\right)$ be a pair of real numbers which satisfy the following system of (in)equalities for some α^T and $x^T(\omega) \geq 0$;

$$\forall a_i \in A_i^{T*} \quad V_i^T = g_i(a_i, \alpha_j^T) - E[x_i^T(\omega)|a_i, \alpha_j^T] \tag{33}$$

$$\forall a_i \notin A_i^{T*} \quad V_i^T \geqq g_i(a_i, \alpha_j^T) - E[x_i^T(\omega)|a_i, \alpha_j^T] \tag{34}$$

$$\forall \omega \ \ x_i^T(\omega) \geq 0 \tag{35}$$

where A_i^{T*} is the support of α_i^T as before. Note that this condition corresponds to (20), (21), and (22) with additional restriction; $x_i(\omega, a_j') = x_i(\omega, a_j'')$ for $a_j', a_j'' \in A_j$. Indeed, if these conditions are satisfied and there exists an equilibrium whose payoff V_i' is lower than V_i^T for $i = 1, 2$, then there exists a trigger strategy equilibrium with payoff V_i^T, in which players play α^T in the cooperative phase and use V_i' as mutual punishments after certain realization of public signals. We call $\left(V_1^T, V_2^T\right)$ the efficient generalized trigger strategy equilibrium (EGTE) payoff if there does not exist (V_1', V_2') which satisfies the above condition and $(V_1', V_2') > \left(V_1^T, V_2^T\right)$. (One is strictly larger at least.)

In the following, we examine when we can construct a private two state machine equilibrium which Pareto-dominates EGTE, as we did for the simple partnership game.[22] Because of the similarity between (20), (21), (22) and (33), (34), (35), it is clear that there exists V_i^R along with $x_i(\omega, a_j)$ and α_j^R such that $V_i^R \geq V_i^T, i = 1, 2$ by definition. So, if (i) there exists V_i^P such that $V_i^R > V_i^P$ and (ii) $V_i^R > V_i^T$ for at least one player, we can construct a private equilibrium whose equilibrium payoff Pareto-dominates $\left(V_1^T, V_2^T\right)$.

5.3.2 Construction of a Pareto-Improving PE

We first write down (33), (34), and (35) in a simple form similar to (28);

$$g_i(\mathbf{a}, \alpha_j^T) - V_i^T \cdot \mathbf{I} - \mathbf{h}_i^T = P\left(\alpha_j^T\right) \cdot \mathbf{x}_i^T \tag{36}$$

where $\mathbf{x}_i^T = \left(x_i^T\left(\omega^1\right), ..., x_i^T\left(\omega^{|\Omega|}\right)\right)'$.

Let $\widehat{A}_j$ be the set of player j's indifferent actions for the best trigger strategy equilibrium, which might not be chosen with positive probability. (Hence, $A_j^* \subset \widehat{A}_j \subset A_j$). Remember that for the partnership game, we constructed a better PE by using an action (D) which a player does not play for the best PPE. The advantage of using such an action came from the fact that it provides a better information to detect the other player's deviation. Similarly, a player might

[22] Note that the best EGTE payoff is typically bounded away from the efficient frontier even if players are very patient. We don't consider the best PPE which is not a trigger strategy. Folk theorem (Fudenberg, Levine, and Maskin [7]) implies that the best PPE is unimprovable at the limit $\delta = 1$ in general. However, as the example in the last section suggests, private strategy may improve the best PPE for each level of δ, if not asymptotically as $\delta \to 1$.

benefit from using an action in $\widehat{A}_j$ but not in A_j^* because it allows player j to use $cone\left(P\left(\widehat{A}_j\right)\right)$ rather than $cone\left(P\left(A_j^*\right)\right)$.

Suppose that $P\left(\widehat{A}_j\right)$ is full row rank, then $cone\left(P\left(\widehat{A}_j\right)\right)$ has a nonempty interior in $\Re_+^{|A_i|}$. If the right hand side of (36) is in the interior of the cone generated by $P\left(\widehat{A}_j\right)$, that is, if

$$P\left(\alpha_j^T\right)\cdot \mathbf{x}_i^T \in int\left(cone\left(P\left(\widehat{A}_j\right)\right)\right) \tag{37}$$

holds, then we can find $\mathbf{x}_i^R\left(\widehat{A}_j\right) \in int\left(cone\left(P\left(\widehat{A}_j\right)\right)\right)$ and $V_i^R > V_i^T$ to satisfy

$$g_i(\mathbf{a},\alpha_j^T) - V_i^R\cdot \mathbf{I} - \mathbf{h}_i^T = P\left(\widehat{A}_j\right)\cdot \mathbf{x}_i^R\left(\widehat{A}_j\right)$$

Note that this is not quite the same as (28) yet, because the support of α_j^T may be strictly smaller than $\widehat{A}_j$. However, we can construct a sequence of α_j^n converging to α_j^T, whose support is $\widehat{A}_i$, associated with a sequence of V_i^n and $\mathbf{x}_i^n\left(\widehat{A}_j\right)$ which converges to V_i^R and $\mathbf{x}_i^R\left(\widehat{A}_j\right)$ respectively and satisfies

$$g_i(\mathbf{a},\alpha_j^n) - V_i^n\cdot \mathbf{I} - \mathbf{h}_i^T = P\left(\widehat{A}_j\right)\cdot \mathbf{x}_i^n\left(\widehat{A}_j\right)$$

This is because $\mathbf{x}_i^R\left(\widehat{A}_j\right)$ is chosen from the interior of the cone generated by P.

For each n, α_j^n, V_i^n, and $\mathbf{x}_i^n\left(\widehat{A}_j\right)$ satisfy (the first half of) (28), so for n large enough, a two state machine which corresponds to α_j^n achieves a higher payoff than V_i^T as long as there exist $\left(\alpha_j^P, \mathbf{x}_i^P, V_i^P\right)$ such that supp$\alpha_j^P \subset \widehat{A}_j$ and $V_i^R > V_i^P$ (feasibility (27)).

Proposition 18 *Let $\left(\alpha^T, \mathbf{x}^T, V^T, \widehat{A}\right)$ be an EGTE. Suppose that full row rank condition (30) is satisfied for $\widehat{A}$. Then, there exists a two state machine $\left(\alpha, \mathbf{x}, V, \widehat{A}\right) \in F$ which Pareto-dominates $\left(\alpha^T, \mathbf{x}^T, V^T, \widehat{A}\right)$ if (i) the condition (37) holds and (ii) there exists $\left(\alpha^P, \mathbf{x}^P, V^P\right)$ such that supp$\alpha_i^P \in \widehat{A}_i$ and $V_i^T \geq V_i^P$ for $i = 1, 2$.*

Remark 19 *We restricted our attention to the construction of a two state machine where the support of α_j is $\widehat{A}_j$. It might be possible to construct a Pareto improving two state machine whose support of mixed actions is not contained in $\widehat{A}$ even when (i) or/and (ii) is violated.*

Remark 20 *There are many conditions to guarantee that $P\left(\alpha_j^T\right)\cdot \mathbf{x}_i^T \in int\left(cone\left(P\left(\widehat{A}_j\right)\right)\right)$ holds. Let Ω^* be the set of ω such that $x_i^T(\omega) > 0$. Since $P\left(\alpha_j^T\right)\cdot \mathbf{x}_i^T =$*

$\sum_{a_j}\sum_{\omega} p(\omega|\cdot,a_j)\,\alpha_j^T(a_j)\,x_i^T(\omega)$, *if any* $p(\omega|\cdot,a_j)$ *for* $(\omega,a_j)\in\Omega^*\times A_j^*$ *is in the interior of cone* $\left(P\left(\widehat{A}_j\right)\right)$, *the interior condition (37) holds. Even if all such* $p(\omega|\cdot,a_j)$ *for* $(\omega,a_j)\in\Omega^*\times A_j^*$ *are on the boundary of cone* $\left(P\left(\widehat{A}_i\right)\right)$, *it can be shown that the interior condition (37) still holds if cone* $\left(\{p(\omega|\cdot,a_j)\mid(\omega,a_j)\in\Omega^*\times A_j^*\}\right)$ *has nonempty interior in* $\Re_+^{|A_i|}$.

6 Related Literature and Comments

There are few works on private strategies. As far as we know, our paper provides the first example of infinitely repeated games with discounting in which the use of private strategies makes a significant difference. Recently, Mailath, Matthews, and Sekiguchi [12] found examples of finitely repeated games for which there exists a PE which is better than any PPE. Lehrer [10] uses private strategies as endogenous correlation devices in repeated games without discounting.

There are a couple of comments on the robustness of the private equilibria. First, when the parameters such as $(d,h,p(X|CC),p(X|CD),p(X|DD))$ change slightly, there exists a PE close to the original PE. Secondly, suppose that each player can observe additional signals which are informative about the other player's current state. Our PE still continues to be a sequential equilibrium in that setting because a player does not have to know which state the other player is in. These facts suggest that our private equilibria is robust to some extent. Finally, note that our private strategy works even if there is no public signal at all. On the other hand, PPE does not have any bite by definition in such situation. To see this, suppose that the stage game is perturbed in the following way. The public signal has the same distribution as before, but it is not observable to players. Instead, each player observes a public signal plus a private noise. Players observe the true public signal most of the time, but observe the wrong one with a small probability. The private strategy works even in this setting. Since players do not have to know the other player's state, it is not important whether a player could observe the signal which her opponent receives.

Formally, this modified model belongs to repeated games with private monitoring. The method in this paper to deal with private information is indeed applicable to this wider class of model. Ely and Välimäki [6] independently found a similar two state machine strategy in repeated games with private monitoring. As in this paper, a player is indifferent among all the repeated game strategies whatever state the opponent is in.[23] However, there is a critical difference between our paper and Ely and Välimäki [6]. In Ely and Välimäki, a player plays a pure action at each state and, as in this paper, it does not matter whether a player knows the opponent's state (henceforth action) or not. On the other hand, it is important for us that a player does not know what

[23]The idea behind these strategies goes back to Piccione [14], where the equilibrium strategy is basically a machine with a countably infinite number of states.

action the opponent is choosing. If a player knows the opponent's action, she is more tempted to defect when C is being played and more likely to cooperate when the "monitoring" action D is being played. Since players need to use the action-signal pair without being noticed for the efficient punishment, they need to play a mixed action at the reward state in our paper. This efficient use of the signaling structure is the key to our efficient private equilibria. The idea of efficient monitoring is not new, rather a familiar one. It is an old and simple idea which lies at the heart of any moral hazard model. One contribution of this paper is to find a way to apply this idea to private information in repeated games or dynamic moral hazard models.

Another point we should make about the above model is that it is not a model with almost perfect monitoring, which has been the main focus of private monitoring literature (such as [5], [6], [14], [17]). The game we described above is a repeated game with almost public monitoring (Mailath and Morris [11]). Hence our PE can be regarded as one of the first example of sequential equilibria which works with private monitoring which is not almost perfect. Observe how the private strategy is related to the conditions suggested by Mailath and Morris [11]. They suggested the conditions under which a particular PPE remains a sequential equilibrium with an almost public monitoring when a public signal structure is perturbed slightly with private noise. Their conditions require players to have almost common knowledge about the other players' continuation strategies all the time. Our PE clearly does not satisfy this sufficient condition. On the contrary, its property is rather orthogonal to such requirement. Players do not have to have any additional knowledge about the opponent's continuation strategy through the course of the game.

There is one important open question left. Although we could show that a PE can be much more efficient than PPE, we have not characterized the best symmetric sequential equilibrium payoff yet. A further insight is needed to see whether a version of the inefficiency result by RMM extends to the whole set of sequential equilibria or some efficiency result stands out surprisingly.

Appendix A: proofs

Proof of Proposition 2 .

Let us first prove a useful lemma which generally holds for the best symmetric PPE payoff with public correlation devices. We are looking at symmetric payoffs, but in the lemma we do not restrict our attention to strongly symmetric equilibria. Also note that the best symmetric PPE can be found by maximizing the sum of two players' payoffs over the set of PPE payoff profiles.

Lemma 21 *Let (v^*, v^*) be the best symmetric PPE payoff in a repeated partnership game. Then, there exists a PPE which achieves the same total payoff $2v^*$ and do not use any public correlation device in the initial period. Furthermore, the sum of the expeceted stage payoffs in the initial period is no less than $2v^*$.*

Proof. When the best symmetric PPE payoff is achieved by public randomization over some PPE, each of them must obtain the same, non-negative *total* payoff $2v^*$ (otherwise, we can just pick up (v_1, v_2) with the highest total payoff and achieve a higher symmetric payoff by equally randomizing over (v_1, v_2) and (v_2, v_1), a contradiction). Pick up any one of those PPE. By definition, it does not use any public randomization in the first period, and therefore it is achieved by a current (possibly mixed) action profile α and continuation payoffs $(V_1(\omega), V_2(\omega))$. The sum of payoffs satisfies

$$2v^* = (1-\delta)(g_1(\alpha) + g_2(\alpha)) + \delta E\left[V_1(\omega) + V_2(\omega)|\alpha\right],$$

where g_i is player i's payoff function and $E[\ |\alpha]$ is the expectation under α. If $g_1(\alpha) + g_2(\alpha) < 2v^*$, the sum of payoffs associated with this PPE would be

$$E\left[V_1(\omega) + V_2(\omega)|\alpha\right] > 2v^*$$

This contradicts our assumption that (v^*, v^*) is the best symmetric PPE payoff profile. Hence $g_1(\alpha) + g_2(\alpha) \geq 2v^*$. ■

Note that this proof only relies on the fact that $(0, 0)$ is the mutual mixmax. In particular, the space of public signal can be arbitrary. We use this lemma later when we analyze a partenrship game with three public signals.

Let $\overline{v}_s$ (> 0) be the best symmetric PPE payoff in the current repeated partnership game. Lemma 21 implies that (i) there exists a PPE payoff profile $(\overline{v}_1, \overline{v}_2)$ such that $2\overline{v}_s = \overline{v}_1 + \overline{v}_2$, (ii) players do not use a public correlation device in the initial period, and (iii) at least one player is playing C in th einitial period. Suppose that both players play C with positive probability in the initial period for now. We come back to the case where one player plays D with probability 1 when we deal with all the asymmetric strategies in the next proposition. Let $q_i < 1$ be the probability for player i to play D in the initial period.

$\overline{v}_1$ and q_2 satisfy the following inequality derived from the recursive equation:

$$\begin{aligned} \overline{v}_1 &\leqq (1-\delta)(1-q_2-q_2h) \\ &+\delta\left[\begin{array}{c}(1-q_2)\{(1-p_0)v_1^*+p_0(1-\rho_1)v_1^*\} \\ +q_2\{(1-p_1)v_1^*+p_1(1-\rho_1)v_1^*\}\end{array}\right]\end{aligned} \tag{38}$$

where $\rho_1 \in [0,1]$, and the incentive constraint:

$$(1-\delta)\{(1-q_2)d+q_2h\} = \delta\{(1-q_2)(p_1-p_0)+q_2(p_2-p_1)\}\rho_1 v_1^* \tag{39}$$

v_1^* is player 1's continuation payoff after signal Y. Equation 39 means that if the continuation payoff after signal X decreased by $\rho_1 v_1^*$, then player 1 would be indifferent between C and D. Since the true punishment associated with $\overline{v}_1$ should be as harsh as this hypothetical punishement, we have the inequality 38.

Similar inequality and equation hold for player 2:

$$\begin{aligned} \overline{v}_2 &\leqq (1-\delta)(1-q_1-q_1h) \\ &+\delta\left[\begin{array}{c}(1-q_1)\{(1-p_0)v_2^*+p_0(1-\rho_2)v_2^*\} \\ +q_1\{(1-p_1)v_2^*+p_1(1-\rho_2)v_2^*\}\end{array}\right]\end{aligned} \tag{40}$$

$$(1-\delta)\{(1-q_1)d+q_1h\} = \delta\{(1-q_1)(p_1-p_0)+q_2(p_2-p_1)\}\rho_2 v_2^* \tag{41}$$

Adding (38) and (40) and using $v_1^*+v_2^* \leqq \overline{v}_1+\overline{v}_2$, we get

$$\begin{aligned} \overline{v}_1+\overline{v}_2 &\leqq 1-q_1-q_1h-\frac{\delta\{(1-q_1)p_0+q_1p_1\}\rho_2 v_2^*}{1-\delta} \\ &+1-q_2-q_2h-\frac{\delta\{(1-q_2)p_0+q_2p_1\}\rho_1 v_1^*}{1-\delta}\end{aligned}$$

Substituting (39) and (41) into this equation ,

$$\overline{v}_1+\overline{v}_2 \leqq 1-q_1-q_1h-\frac{(1-q_1)d+q_1h}{L^{q_1}-1}+1-q_2-q_2h-\frac{(1-q_2)d+q_2h}{L^{q_2}-1}$$

Note that the bound of player 1's (2's) payoff only depends on $q_2\,(q_1)$. Then, $q_1=q_2=q^*$ gives the optimal bound of $\overline{v}_1+\overline{v}_2$ and

$$\overline{v}_s = \frac{\overline{v}_1+\overline{v}_2}{2} \leqq 1-q^*-q^*h-\frac{(1-q^*)d+q^*h}{L^{q^*}-1}$$

It is clear that this bound is achieved by the strongly symmetric strategy PPE where mixing C and D with $(1-q^*,q^*)$ is used instead of (C,C) in (#) and that $\overline{v}_s=\overline{v}_1=\overline{v}_2$.■

Note that the above proof shows that the best strongly symmetric PPE payoff achieves the best symmetric payoff even among a large class of asymmetric strategies(i.e. both players play C with positive probability.). This implies that if any asymmetric profile is used to support $\overline{v}_s$, then one player has to play D

with probability 1 in the initial period. This fact makes the proof of the next proposition simple.

Proof of Proposition 3

We only need to consider the case where one player plays D with probability 1 in the initial period. Suppose that this player is player 2 without loss of generality. Then, Lemma 21 immediately implies that the best symmetric PPE payoff is bounded by $\frac{1+d-h}{2}$. Combining this observation with the previous proposition, the result is obtained.

Proof of Proposition ??

Proof. We just need to show that $1-\frac{d}{L^1-1} > \overline{v}_s$.

1. $1-\frac{d}{L^1-1} > 0$

   By $p_2 - p_1 > p_1 d + (1-p_2)h$,

$$1-\frac{d}{L^1-1} > \frac{(1-p_2)h}{p_2-p_1} > 0$$

2. $1-\frac{d}{L^1-1} > \frac{1+d-h}{2}$

   This holds by assumption.

3. $1-\frac{d}{L^1-1} > 1-q-qh-\frac{(1-q)d+qh}{L^q-1}$ for all $q \in [0,1]$.

   Let $g(q) = 1-q-qh-\frac{(1-q)d+qh}{L^q-1}$. Since it is easy to show that $g'(q) < 0$ for all $q \in [0,1]$ with $L^1 > L^0$ and $h > d$,

$$\begin{aligned} 1-\frac{d}{L^1-1} &> 1-\frac{d}{L^0-1} \\ &\geqq 1-q-qh-\frac{(1-q)d+qh}{L^q-1} \end{aligned}$$

   for all $q \in [0,1]$. These imply that $1-\frac{d}{L^1-1} > \overline{v}_s$.

■

Proof of Proposition 7

Proof. Suppose that both M_1 and M_2 has only a finite number of states. We first focus on the value functions of player 1; $\{V_1^n\}_{n=0}^{l_2}$. Then, there exists player 2's state which corresponds to the largest V_1^n. Suppose without loss of generality that $n=0$ is such state. Similarly, let $n=1$ be the state which minimizes the value function of player 1. We modify player $2's$ machine in the following way. When player 2 is supposed to move to θ_2^n from θ_2^0 or θ_2^1after some action and signal is observed, she instead move to θ_2^0 and θ_2^1 with some probability $1-\lambda_2^n$and λ_2^n where λ_2^n is the number between 0 and 1 to satisfy $V_1^n = (1-\lambda_2^n) V_1^0 + \lambda_2^n V_1^1$. Then, we obtain the following system of (in)equalities;

$$\begin{aligned}
For\ n &= 0,1, \\
\forall a_1 &\in A_1^*,\ V_1^n = (1-\delta) g_1(a_1, \alpha_2^n) + \\
&\quad \delta \sum_{a_2 \in \text{supp}(\alpha_2^n)} \sum_{\omega \in \Omega} \sum_{k=0}^{1} \alpha_2^n (a_2) p(\omega | a_1, a_2) \mu_2' \left(\theta_2^k | (a_2, \omega), \theta_2^n\right) V_1^k \\
\forall a_1 &\notin A_1^*,\ V_1^n \geq (1-\delta) g_1(a_1, \alpha_2^n) + \\
&\quad \delta \sum_{a_2 \in \text{supp}(\alpha_2^n)} \sum_{\omega \in \Omega} \sum_{k=0}^{1} \alpha_2^n (a_2) p(\omega | a_1, a_2) \mu_2' \left(\theta_2^k | (a_2, \omega), \theta_2^n\right) V_1^k
\end{aligned}$$

where for $n = 0, 1$

$$\begin{aligned}
\mu_2' \left(\theta_2^0 | (a_2, \omega), \theta_2^n\right) &= \sum_{m=0}^{l_2} \mu_2 \left(\theta_2^m | (a_2, \omega), \theta_2^n\right) (1 - \lambda_2^m) \\
\mu_2' \left(\theta_2^1 | (a_2, \omega), \theta_2^n\right) &= \sum_{m=0}^{l_2} \mu_2 \left(\theta_2^m | (a_2, \omega), \theta_2^n\right) \lambda_2^m
\end{aligned}$$

We can repeat the same procedure with the roles of the players being reversed to obtain the two state machine $M_i' = \left\{ \{\theta_i^n\}_{n=0}^1,\ \alpha_i,\ \mu_i' \right\} .i = 1, 2$. This pair of machines clearly satisfies (19), hence constitute a sequential equilibrium which supports the payoff profile (V_1^0, V_2^0) with the initial state (θ_1^0, θ_2^0).

If the number of the states is countable, we might not able to find the best state and the worst state. In such a case, we construct them in the following way. Suppose that M_2 has a countable number of the states. Since $\{V_1^n\}_{n=0}^{l_2}$ is bounded by assumption, there exists a least upper bound $\overline{V}_1 = \sup_{n=0,\ldots,l_2} \{V_1^n\}$ and a largest lower bound $\underline{V}_1 = \inf_{n=0,\ldots,l_2} \{V_1^n\}$. Since V_1^n (hence, $\sum_{m=0}^{l_2} \mu_2 (\theta_2^m | (a_2, \omega), \theta_2^n) V_1^m$) and α_2^n are in the compact sets ($[\underline{V}_1, \overline{V}_1]$ and $\triangle_2$ respectively), we can find a sequence $\theta_{2,}^{n(k)}, k = 1, 2, \ldots.$ such that $V_1^{n(k)} \to \overline{V}_1$, $\alpha_2^{n(k)} \to \overline{\alpha_2}$, and $\sum_{m=0}^{l_2} \mu_2 \left(\theta_2^m | (a_2, \omega), \theta_{2,}^{n(k)}\right) V_1^m \to \widetilde{V}_1 (a_2, \omega)$ as $k \to \infty$. Then, $\overline{V}_1, \overline{\alpha_2}$ and $\widetilde{V}_1 (a_2, \omega)$ satisfy

$$\begin{aligned}
\forall a_1 &\in A_1^*,\ \overline{V}_1 = (1-\delta) g_1(a_1, \overline{\alpha_2}) + \\
&\quad \delta \sum_{a_2 \in \text{supp}(\overline{\alpha_2})} \sum_{\omega \in \Omega} \sum_{m=0}^{l_2} \overline{\alpha_2} (a_2) p(\omega | a_1, a_2) \widetilde{V}_1 (a_2, \omega) \\
\forall a_1 &\in A_1^*,\ \overline{V}_1 \geq (1-\delta) g_1(a_1, \overline{\alpha_2}) +
\end{aligned}$$

$$\delta \sum_{a_2 \in \text{supp}(\overline{\alpha_2})} \sum_{\omega \in \Omega} \sum_{m=0}^{l_2} \overline{\alpha_2}(a_2)\, p(\omega|a_1, a_2)\, \widetilde{V}_1(a_2, \omega)$$

Similarly, we can obtain

$$\begin{aligned}
\forall a_1 &\in A_1^*, \underline{V}_1 = (1-\delta)\, g_1(a_1, \underline{\alpha_2}) + \\
&\delta \sum_{a_2 \in \text{supp}(\underline{\alpha_2})} \sum_{\omega \in \Omega} \sum_{m=0}^{l_2} \underline{\alpha_2}(a_2)\, p(\omega|a_1, a_2)\, \widehat{V}_1(a_2, \omega) \\
\forall a_1 &\notin A_1^*, \underline{V}_1 \geq (1-\delta)\, g_1(a_1, \underline{\alpha_2}) + \\
&\delta \sum_{a_2 \in \text{supp}(\underline{\alpha_2})} \sum_{\omega \in \Omega} \sum_{m=0}^{l_2} \underline{\alpha_2}(a_2)\, p(\omega|a_1, a_2)\, \widehat{V}_1(a_2, \omega)
\end{aligned}$$

for some $\underline{\alpha_2}$ and $\widehat{V}_1$.

Now we can replace $\widetilde{V}_1(a_2, \omega)$ and $\widehat{V}_1(a_2, \omega)$ by a randomization between $\overline{V}_1$ and $\underline{V}_1$ as before. Define $\mu_2\left(\underline{\theta}_2 | (a_2, \omega), \overline{\theta}_2\right)$ and $\mu_2\left(\underline{\theta}_2 | (a_2, \omega), \underline{\theta}_2\right)$ by equations;

$$\begin{aligned}
\widetilde{V}_1(a_2, \omega) &= \left\{1 - \mu_2\left(\underline{\theta}_2 | (a_2, \omega), \overline{\theta}_2\right)\right\} \overline{V}_1 + \mu_2\left(\underline{\theta}_2 | (a_2, \omega), \overline{\theta}_2\right) \underline{V} \\
\widehat{V}_1(a_2, \omega) &= \left\{1 - \mu_2\left(\underline{\theta}_2 | (a_2, \omega), \underline{\theta}_2\right)\right\} \overline{V}_1 + \mu_2\left(\underline{\theta}_2 | (a_2, \omega), \underline{\theta}_2\right) \underline{V}_1
\end{aligned}$$

Then, we obtain a two state machine $M_2 = \left\{\left\{\overline{\theta}_2, \underline{\theta}_2\right\}, \left\{\overline{\alpha_2}, \underline{\alpha_2}\right\}, \mu_2\right\}$ to satisfy (19) for $\overline{V}_1$ and $\underline{V}_1$. We can construct a two state machine M_1 in a similar way and (M_1, M_2) constitutes a sequential equilibrium with the payoff profile $\left(\overline{V}_1, \overline{V}_2\right)$. ■

Proof of Proposition 9

Proof. (i) Consider the following transition rule for player j in the two-state machine (or Markov) strategy: go to state P with probability $\rho_j^z(\omega, a_j)$ when the current state (for j) is $z = R, P$ and the current signal and j's action are ω and a_j (otherwise, go to state R). Consider the dynamic programming equation for the average payoff for player i when j is in state $z = R.P$,

$$V_i^z \geq (1-\delta) g_i(a_i, \alpha_j^z) + \delta E[(1 - \rho_j^z(\omega, a_j)) V_i^R + \rho_j^z(\omega, a_j) V_i^P | a_i, \alpha_j^z], \quad (42)$$

where the equality should be satisfied for $a_i \in \text{supp}\alpha_i^R \cup \text{supp}\alpha_i^P$. Consider first the case $z = R$. Subtracting δV_i^R from both sides and dividing through by $(1-\delta)$, we obtain

$$V_i^R \geq g_i(a_i, \alpha_j^R) - E[\frac{\delta}{1-\delta} \rho_j^R(\omega, a_j)(V_i^R - V_i^P) | a_i, \alpha_j^R],$$

where equality holds for $a_i \in \text{supp}\alpha_i^R \cup \text{supp}\alpha_i^P$. A similar manipulation for state $z = P$ shows

$$V_i^P \geq g_i(a_i, \alpha_j^P) + E[\frac{\delta}{1-\delta}(1 - \rho_j^P(\omega, a_j))(V_i^R - V_i^P) | a_i, \alpha_j^P],$$

where equality holds for $a_i \in \text{supp}\alpha_i^R \cup \text{supp}\alpha_i^P$. Hence, if we have an equilibrium in the two-state machine strategy, conditions (20)–(27) are satisfied with

$$x_i^R(\omega, a_j) = \frac{\delta}{1-\delta}\rho_j^R(\omega, a_j)(V_i^R - V_i^P) \text{ and} \tag{43}$$

$$x_i^P(\omega, a_j) = \frac{\delta}{1-\delta}(1 - \rho_j^P(\omega, a_j))(V_i^R - V_i^P). \tag{44}$$

Conversely, suppose that conditions (20)–(27) are satisfied. Then, (43) and (44) can be satisfied for $\rho_j^z(\omega, a_j) \in [0, 1]$, $z = R, P$, for sufficiently high δ. Hence we obtain the equilibrium condition (42) and the two-state machine equilibrium to support payoffs (V_i^R, V_i^P) for $i = 1, 2$.

(ii) Directly follows from (i). ■

Proof of Lemma 10

Proof. Let us examine the maximum number of column vectors in P that can be linearly independent. Recall that the kth row in $P(a_j)$ corresponds to the probability distribution of ω under (a_i^k, a_j). This implies that the elements of this row vector add up to one. As this is true for each row of matrix $P(a_j)$, we have

$$P(a_j)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{for each } a_j \in A_j^*.$$

This means that among the $|\Omega||A_j^*|$ column vectors in $P(A_j^*)$, $|A_j^*| - 1$ of them cannot be linearly independent. The rest of them can clearly be linearly independent by a generic choice of $p(\omega|a)$, as long as the number of columns $(|\Omega||A_j^*| - (|A_j^*| - 1) = (|\Omega| - 1)|A_j^*| + 1)$ is greater than or equal to the number of rows $(|A_i|)$ in $P(A_j^*)$. Hence, if $(|\Omega| - 1)|A_j^*| + 1 \geq |A_i|$, matrix$P(A_j^*)$ generically has full row rank and the conditions (20) and (21) can be satisfied. The same argument applies to conditions (23) and (24).

Thus we have shown that for any given A_j^*, α_i^R, α_i^R, V_i^R and V_i^P for $i, j = 1, 2$, we can generically find $\mathbf{x}_i(A_j^*)$, $\mathbf{x}_i^P(A_j^*)$ that satisfy the incentive constraints (20), (21), (23) (24) and (26). To satisfy the non-negativity conditions (22) and (25) for the punishments and rewards, which have been ignored so far, we choose a large enough numbers K_i, $L_i \geq 0$ so that

$$\widehat{\mathbf{x}}_i^R(A_j^*) \equiv \mathbf{x}_i^R(A_j^*) + \begin{pmatrix} K_i \\ \vdots \\ K_i \end{pmatrix} \geq 0 \text{ and}$$

$$\widehat{\mathbf{x}}_i^P(A_j^*) \equiv \mathbf{x}_i^P(A_j^*) + \begin{pmatrix} L_i \\ \vdots \\ L_i \end{pmatrix} \geq 0$$

for $i=1,2$. Also define

$$\widehat{V}_i^R \equiv V_i^R + K_i \text{ and}$$
$$\widehat{V}_i^P \equiv V_i^P + L_i$$

for $i=1,2$. Then, we have $\mathbf{x}_i^R(A_j^*)$, $\mathbf{x}_i^P(A_j^*)$, $\widehat{V}_i^R$and $\widehat{V}_i^P$ for $i=1,2$ that satisfy conditions (20)–(26). ■

Proof of Proposition 15

Proof. (i) When information is almost perfect with A^* ($cone(P(A_i^*))$ is close to $\Re_+^{|A_i|}$ for $i=1,2$), V_i^R is roughly a (signed) distance from $g_i(\cdot,\alpha_j^R)+\mathbf{h}_i^R$ to the face of $\Re_+^{|A_i|}$ (see Figure 4 below). Since we can make the slack positive variables $\mathbf{h}_i^R$ as large as possible for $a_i \notin A_i^*$, this distance is $\sqrt{n}\min_{a_i\in A_i^*} g_i(a_i,\alpha_j^R)$. Similarly, V_i^P is roughly a distance from $g_i(\cdot,\alpha_j^P)+\mathbf{h}_i^P$ to the face of $-\Re_+^{|A_i|}$, which is $\sqrt{n}\max g_i(a_i,\alpha_j^P)$ ($\mathbf{h}_i^P$ is 0 without loss of generality to minimize V_i^P). So, the feasibility condition (27) is satisfied if

$$\min_{a_i\in A_i^*} g_i(a_i,\alpha_j^R) > \max_{a_i\in A_i} g_i(a_i,\alpha_j^P)$$

when information is almost perfect. This is exactly the condition (31).

(ii) Straightforward from the equations (28). ■

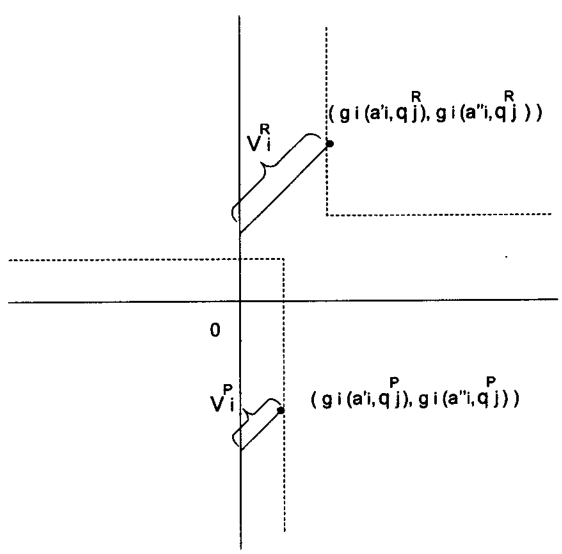


Figure 4

Appendix B: Example 2

Let q_i be the probability that player i chooses action D. Given q_j, the probability of X when player i chooses C and D are respectively $\frac{1}{3}(1-q_j)$ and $\frac{1}{3}q_j$, as X arises only when both players take the same action. Hence we have the following simple but useful observation.

Lemma 22 *When player i deviates from C to D while the opponent chooses D with probability q_j, outcome X becomes less likely iff $q_j < 1/2$.*

Let F be the sum of the expected stage payoffs under (q_1, q_2),

$$\begin{aligned} F &= (1+1)(1-q_1)(1-q_2) + (2-6)q_1(1-q_2) + (2-6)(1-q_1)q_2 \\ &= 2 - 6q_2 - 6q_1 + 10q_1q_2. \end{aligned}$$

We note that this is positive only if both players choose D with sufficiently low probability.

Lemma 23 *The sum of the stage payoffs is positive only if $q_1, q_2 < 1/3$.*

Proof. Note that $F(q_1, q_2)$ is linear in q_1 and that both $F(0, q_2) = 2 - 6q_2$ and $F(1, q_2) = 4(q_2 - 1)$ are non-positive if $q_2 \geq 1/3$. Hence $F(q_1, q_2)$, which is a convex combination of those values, is non-positive if $q_2 \geq 1/3$. Symmetric argument shows that F is non-positive if $q_1 \geq 1/3$. Hence F is positive only if $q_1, q_2 < 1/3$. ■

The following is a immediate corollary from the above two lemmata.

Corollary 24 *When the sum of the stage game payoffs is positive, outcome X becomes less likely if player i defects given player j's mixed action.*

Combining Lemma 23 and Corollary 24 with Lemma 21, we have:

Proposition 25 *For any parameter of information structure $\epsilon \in [0, 1/2)$, if the best symmetric PPE payoff v^* is not 0, then there is a (possibly asymmetric) PPE with the same total payoff $2v^*$, where in the first period (ii) no public correlation device is used,(ii) each player chooses D with probability less than $1/3$, and (iii) unilateral defection of each player makes outcome X less likely.*

Now we use this fact to show the following.

Proposition 26 *The best symmetric PPE payoff is 0 for all $\delta \in [0, 1)$ when the parameter of the information structure ϵ is equal to 0.*

Proof. Suppose $v^* > 0$ and choose the equilibrium stated in the above Proposition. When $\epsilon = 0$, we can regard Y_1 and Y_2 as a single outcome Y. Note that as D is dominant in the stage game, a player always has a short-term incentive to defect, irrespective of the opponent's mixing probability q_j.

Then the above Proposition shows that both payers must be punished when Y realizes. The associated likelihood ratio for player i given player j's mixed action is

$$L_i^{q_j} \equiv \frac{(1-q_j)\Pr(Y|D,C) + q_j \Pr(Y|D,D)}{(1-q_j)\Pr(Y|C,C) + q_j \Pr(Y|C,D)} = \frac{(1-q_j) + \frac{2}{3}q_j}{\frac{2}{3}(1-q_j) + q_j} = \frac{3-q_j}{2+q_j}$$

, and by a similar argument to the proof of Proposition 3, we have

$$2v^* \leq \left(1 - 7q_2 - \frac{1+5q_2}{L_1^{q_2} - 1}\right) + \left(1 - 7q_1 - \frac{1+5q_1}{L_2^{q_1} - 1}\right). \tag{45}$$

Note that $1 - 7q_j$ is the stage payoff when player i plays C and player j is choosing D with probability q_j, and $(1-q_j) \times 1 + q_j \times 6 = 1 + 5q_j$ is player i's current gain from defection in the same situation. As $L_i^{q_j} \leq 2/3$ for $q_j \leq 1/2$, we have

$$1 - 7q_j - \frac{1+5q_j}{L_i^{q_j} - 1} < 1 - \frac{1}{\frac{3}{2} - 1} = 1 - 2 < 0 \text{ for } i,j = 1,2 \text{ and } j \neq i.$$

which, together with (45), contradicts our presumption $v^* > 0$. Hence we conclude that best symmetric equilibrium payoff is 0 when $\epsilon = 0$. ■

Next we derive an upper bound the symmetric PPE payoffs. Let $v^*(\delta)$ be the best symmetric PPE payoff under δ. We suppress δ when no confusion ensues. If v^* is positive, the Proposition 25 shows that there is a PPE achieving the same total payoff $2v^*$, where a possibly mixed action is chosen (but no public correlation device is used) in the first period. Let q_i be the probability that player i chooses action D in the first period ($i = 1, 2$). The average payoff profile of such an equilibrium, denoted (v_1^0, v_2^0), must satisfy the following "dynamic programming" conditions.

$$v_1^0 + v_2^0 = 2v^* \tag{46}$$

$$v_i^0 = (1-\delta)(1-7q_j) + \delta \sum_\omega v_i(\omega) p(\omega|C, q_j), \text{ for } i,j = 1,2 \text{ and } j \neq i \tag{47}$$

$$v_i^0 = (1-\delta)(2-2q_j) + \delta \sum_\omega v_i(\omega) p(\omega|D, q_j), \text{ for } i,j = 1,2 \text{ and } j \neq i \tag{48}$$

In the above expression $p(\omega|a, q)$ denotes the probability of ω when a player chooses action a ($a = C, D$) and the opponent chooses D with probability q (note the symmetry of $p(\omega|\cdot,\cdot)$). The continuation payoff profile is represented by $(v_1(\omega), v_2(\omega))$. Equations (47) and (48) respectively represent player i's payoff when she plays C or D in the first period. Together they imply that player i is indifferent between C and D.

By summing up $(1-q_i) \times (47) + q_i \times (48)$ for $i = 1, 2$ and using (46), we can calculate the total payoff associated with the mixed strategy profile as

$$2v^* = (1-\delta)(2 - 6q_2 - 6q_1 + 10q_1q_2) + \delta \sum_\omega (v_1(\omega) + v_2(\omega)) p(\omega|q_1, q_2).$$

Note that the first term is $(1-\delta)$ times the sum of expected stage payoffs, which we formerly defined as F. Also note that $p(\omega|q_1, q_2)$ is the probability of ω when players mix D with probabilities q_1 and q_2. Subtract $2\delta v^* = 2\delta(v_1^0 + v_2^0)$ from both sides and divide by $(1-\delta)$ to obtain

$$2v^* = (2 - 6q_2 - 6q_1 + 10q_1q_2) + \sum_{\omega}(\Delta_1(\omega) + \Delta_2(\omega))p(\omega|q_1, q_2), \tag{49}$$

where $(\Delta_1(\omega), \Delta_2(\omega))$ represents total (as opposed to average) future payoff variations (around the "best" PPE payoff profile (v_1^0, v_2^0)):

$$\Delta_i(\omega) = \frac{\delta}{1-\delta}(v_i(\omega) - v_i^0), \text{ for } i = 1, 2. \tag{50}$$

Note that the future payoff variations $(\Delta_1(\omega), \Delta_2(\omega))$ have to satisfy some conditions. First, it must provide right incentive for each player. Subtracting (47) from (48) and dividing through by $(1-\delta)$, we have (binding) incentive constraints

$$1 + 5q_j = \sum_{\omega} \Delta_i(\omega)[p(\omega|C, q_j) - p(\omega|D, q_j)], \text{ for } i, j = 1, 2 \text{ and } j \neq i. \tag{51}$$

Note that the left hand side is the short term gain from defection, while the right hand side shows the reduction of the future payoffs. Secondly, the future payoffs $(v_1(\omega), v_2(\omega))$ should be chosen from the set of PPE payoffs, which we denote by $V^{PPE}(\delta)$. By the definition (50), this condition is represented as

$$\forall \omega \; \frac{1-\delta}{\delta}(\Delta_1(\omega), \Delta_2(\omega)) + (v_1^0, v_2^0) \in V^{PPE}(\delta) \tag{52}$$

Let us now summarize what we have found.

Lemma 27 *Let v^* be the best symmetric PPE payoff under discount factor δ. Then, there exist $q_1, q_2 \in [0, 1/2)$ and $(\Delta_1(\omega), \Delta_2(\omega))$ that satisfy the dynamic programming value equation (49), the incentive constraint (51) and the PPE condition (52) for some feasible payoff profile (v_1^0, v_2^0) such that $v_1^0 + v_2^0 = 2v^*$.*

To get an upper bound for v^*, we will relax condition (52). First, let V^F be the feasible payoff set, that is, the convex hull of stage payoffs

$$V^F = Co\{(1,1), (2,-6), (-6,2), (0,0)\}.$$

Note that $V^{PPE}(\delta) \subset V^F$. As $2v^*$ is the maximized sum of the two players' payoffs over $V^{PPE}(\delta)$, we also have $V^{PPE}(\delta) \subset \{v \mid v_1 + v_2 \leq 2v^*\}$. Hence (52) implies

$$\frac{1-\delta}{\delta}(\Delta_1(\omega), \Delta_2(\omega)) + (v_1^0, v_2^0) \subset V^F \cap \{v \mid v_1 + v_2 \leq 2v^*\}. \tag{53}$$

The part of the efficient frontier connecting two payoff profiles $(1,1)$ and $(2,-6)$ is given by

$$7v_1 + v_2 = 8,$$

and by symmetry

$$v_1 + 7v_2 = 8$$

is the other part of the efficient frontier connecting $(1, 1)$ and $(-6, -2)$. Hence any feasible payoff profile v in V^F must satisfy $7v_1 + v_2 \le 8$ and $v_1 + 7v_2 \le 8$. Therefore, (53) implies

$$\forall\omega \; \frac{1-\delta}{\delta}(\Delta_1(\omega), \Delta_2(\omega)) + (v_1^0, v_2^0) \subset \{v \mid v_1+v_2 \le 2v^*,\ 7v_1+v_2 \le 8,\ v_1+7v_2 \le 8\}. \tag{54}$$

Let us now derive an upper bound of symmetric PPE payoffs. To this end, we first find a lower bound of discount factor to support a symmetric payoff $v^* \in (0, 1)$. Fix any $v^* \in (0, 1)$. Lemma 27 shows that there is a feasible payoff profile (v_1^0, v_2^0) such that $v_1^0 + v_2^0 = 2v^*$. Then, condition (54) implies (by the first inequality on the right hand side) $\frac{1-\delta}{\delta}(\Delta_1(\omega) + \Delta_2(\omega)) + v_1^0 + v_2^0 \le 2v^*$, which is equivalent to

$$\forall\omega \; \Delta_1(\omega) + \Delta_2(\omega) \le 0. \tag{55}$$

Also the value equation (49) and Lemma 27 show

$$2v^* - (2 - 6q_2 - 6q_1 + 10q_1q_2) = \sum_{\omega}(\Delta_1(\omega) + \Delta_2(\omega))p(\omega|q_1, q_2).$$

As $(2 - 6q_2 - 6q_1 + 10q_1q_2)$ is the sum of stage payoffs, it is less than or equal to 2. This and $0 \le v^*$ imply

$$-2 \le \sum_{\omega}(\Delta_1(\omega) + \Delta_2(\omega))p(\omega|q_1, q_2). \tag{56}$$

Let r be the minimum probability of outcome X when players choose D with probabilities $q_1, q_2 \in [0, 1/2]$: $r = \min_{q_1,q_2} p(X|q_1, q_2)$ subject to $q_1, q_2 \in [0, 1/2]$. Note that $p(X|q_1, q_2) < p(Y_i|q_1, q_2)$, $i = 1, 2$ independent of $\epsilon > 0$. Clearly, $r > 0$, and (55) and the definition of r implies $\sum_\omega(\Delta_1(\omega) + \Delta_2(\omega))p(\omega|q_1, q_2) \le r \min_\omega (\Delta_1(\omega) + \Delta_2(\omega))$. Hence the condition (56) implies $-2 \le r \min_\omega (\Delta_1(\omega) + \Delta_2(\omega))$.Thus we have another condition for $(\Delta_1(\omega), \Delta_2(\omega))$;

$$\forall\omega \; -2/r \le \Delta_1(\omega) + \Delta_2(\omega). \tag{57}$$

Now we present a crucial observation that we need large payoff variations of $(\Delta_1(\omega), \Delta_2(\omega))$ in the northwest/southeast directions as $\epsilon \to 0$. That is, as we approach the information structure where the pairwise full rank condition fails, we need large payoff transfers between the players to support a positive payoff v^*.

Lemma 28 *For any (large) $K > 0$, there is (small) $\epsilon > 0$ such that for each $q_1, q_2 \in [0, 1/2]$, if $(\Delta_1(\cdot), \Delta_2(\cdot))$ satisfies conditions (51), (55) and (56), then $\forall\omega \; \Delta_1(\omega), \Delta_2(\omega) \le K$ cannot hold..*

Proof. Suppose that the assertion is not true. Then, there is a sequence $\{\epsilon^n, \Delta_1^n, \Delta_2^n, q_1^n, q_2^n\}$ such that $\epsilon^n \to 0$, as $n \to \infty$, which satisfies (51), (55), (56), and $\forall\omega\ \Delta_1(\omega), \Delta_2(\omega) \leq K$. The condition (55), (57) implied by (56) and $\forall\omega\ \Delta_1(\omega), \Delta_2(\omega) \leq K$ imply that the sequence lies in a compact set, and we can choose a converging subsequence. Let $(\Delta_1^0, \Delta_2^0 q_1^0, q_2^0)$ be its limit, where (Δ_1^0, Δ_2^0) supports C with probability more than $1/2$ for each player when $\epsilon = 0$. However, since we can regard Y_1 and Y_2 as a single outcome Y when $\epsilon = 0$, the following inequality holds as in Proposition 26.

$$
\begin{aligned}
& 2 - 6q_2 - 6q_1 + 10q_1q_2 + \sum_\omega(\Delta_1^0(\omega) + \Delta_2^0(\omega))p(\omega|q_1, q_2) \\
\leq\ & \left(1 - 7q_2 - \frac{1+5q_2}{L_1^{q_2} - 1}\right) + \left(1 - 7q_1 - \frac{1+5q_1}{L_2^{q_1} - 1}\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
& \sum_\omega(\Delta_1^0(\omega) + \Delta_2^0(\omega))p(\omega|q_1, q_2) \\
\leq\ & -6q_1 - 6q_2 - 10q_1q_2 - \frac{1+5q_2}{L_1 - 1} - \frac{1+5q_1}{L_2 - 1} \\
<\ & -4,
\end{aligned}
$$

This contradicts the fact that the limit (Δ_1^0, Δ_2^0) also satisfies (56). ■

Note that given $K > 0$, the choice of ϵ is independent of the initial choice of (v_1^0, v_2^0) and v^* in the above proof. If ϵ chosen is small enough, then $\forall\omega\ \Delta_1(\omega), \Delta_2(\omega) \leq K$ cannot hold for any (v_1^0, v_2^0) and v^*.

Now define

$$
A = \{(\Delta_1, \Delta_2)\,|\Delta_1 + \Delta_2 \leq 0 \text{ and } -2/r \leq \Delta_1 + \Delta_2\}, \text{ and}
$$

$$
B(K) = A \cap \{(\Delta_1, \Delta_2)\,|\Delta_1, \Delta_2 \leq K\}.
$$

Conditions (55), (57) and Lemma 28 implies that we can always choose (small enough) ϵ in such a way that for some ω, $(\Delta_1(\omega), \Delta_2(\omega))$ lies in the region $A \backslash B$. Let us now summarize what we have found as follows.

Proposition 29 *For any (large) $K > 0$, we can find a value of the signal distribution parameter $\epsilon > 0$ for which the following holds: Let $v^* \in (0,1)$ be the best symmetric PPE payoff under discount factor δ. Then, there exists a feasible payoff profile (v_1^0, v_2^0) such that $v_1^0 + v_2^0 = 2v^*$, where we have*

$$
\forall\omega\ \left\{\frac{1-\delta}{\delta}(A \setminus B(K)) + (v_1^0, v_2^0)\right\} \cap \{v \mid v_1 + v_2 \leq 2v^*,\ 7v_1 + v_2 \leq 8,\ v_1 + 7v_2 \leq 8\} \neq \emptyset. \tag{58}
$$

As this condition (58) becomes more stringent as $K \to \infty$, if we choose (small) ϵ that corresponds to a large K, we need a fairly large discount factor δ

to support v^*. Note that condition (58) is satisfied if δ is sufficiently large, as in Figure A. Hence, when we have the situation depicted in Figure B with small δ, condition (58) fails for any feasible payoff profile (v_1^0, v_2^0) such that $v_1^0 + v_2^0 = 2v^*$. Therefore, the value of δ given by Figure B is a lower bound of the discount factor that supports the symmetric PPE payoff v^*.

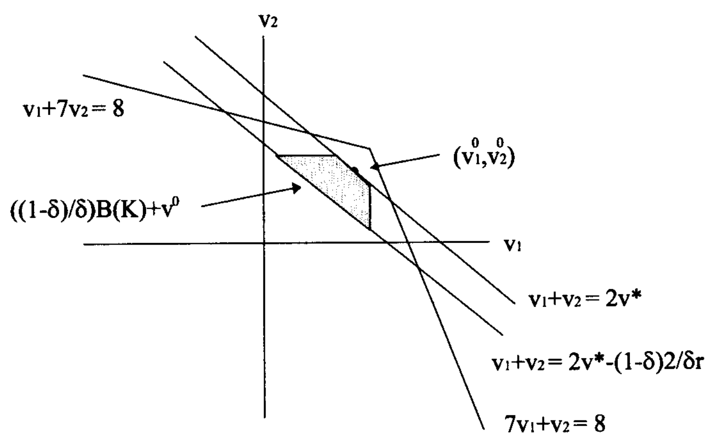


Figure A

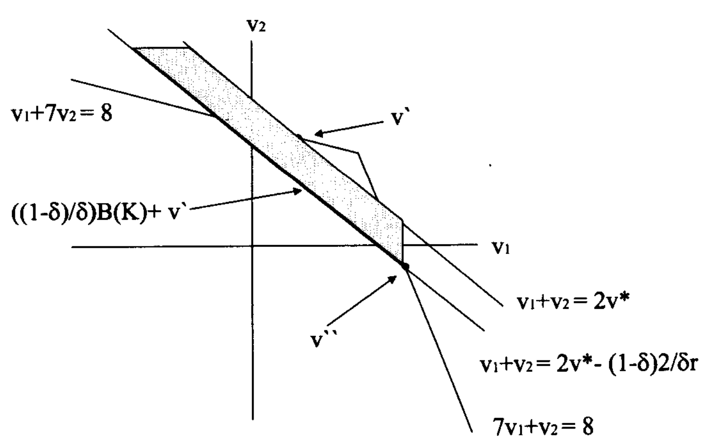


Figure B

By the definition of $B(K)$, points v' and v'' in Figure B must satisfy

$$v_1'' - v_1' = \frac{1-\delta}{\delta} K. \tag{59}$$

The value of v_1' is obtained by solving $v_1 + v_2 = 2v^*$ and $v_1 + 7v_2 = 8$, and we find $v_1' = \frac{7v^*-4}{3}$. Similarly, v_1'' is determined by $v_1 + v_2 = 2v^* - (\frac{1-\delta}{\delta})\frac{2}{r}$ and $7v_1 + v_2 = 8$, and we find $v_1'' = \frac{8-2v^*+(\frac{1-\delta}{\delta})\frac{2}{r}}{6}$. By plugging those in equation (59), we obtain a lower bound of the discount factor to support v^*;

$$\delta(v^*) = \frac{3K - \frac{1}{r}}{3K - \frac{1}{r} + 8(1 - v^*)}. \tag{60}$$

Note that this is an increasing function with $\delta(1) = 1$ and $\delta(0) \to 1$ as $K \to \infty$. This means that to support any positive value, we need a fairly large discount factor when the signal distribution parameter ϵ is small (hence K is large). The inverse function of $\delta(\cdot)$,

$$\overline{v}(\delta) = 1 - (\frac{1-\delta}{\delta})\frac{3K - \frac{1}{r}}{8} \tag{61}$$

is concave and depicted in Figure C.

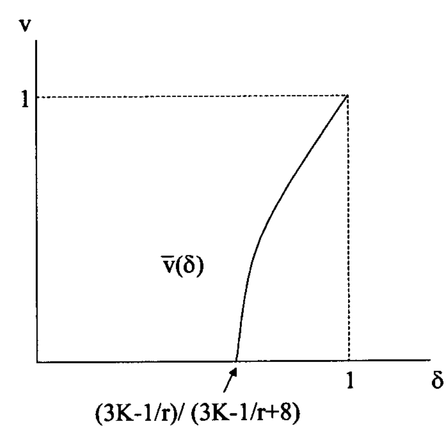


Figure C

By the definition of this function, the maximum symmetric PPE payoff under δ must be located to the right of the graph of $\overline{v}(\delta)$, and hence $\overline{v}(\delta)$ is an upper bound of the maximum symmetric PPE payoff under δ, whenever it is positive. Proposition 6 is then given by defining H by $H = \frac{3K-\frac{1}{r}}{8}$.

References

[1] Abreu, D., P. Milgrom, and D. Pearce, (1991): "Information and Timing in Repeated Partnerships," *Econometrica*, **59**, 1713-1733.

[2] Abreu, D., D. Pearce, and E. Stacchetti, (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica*, **58**, 1041-1064.

[3] Amarante, M., (2002): "Equivalence of Public Mixed-Strategies and Private Behavior Strategies in Games with Public Monitoring," unpublished.

[4] Aumann, R., (1964): "Mixed and Behavior Strategies in Infinite Extensive Games" in M. Dresher, L.S.Shapley, and A.W.Tucker (editors), *Advances in Game Theory,* Princeton: Princeton University Press, 627-650.

[5] Bhaskar, V., and I. Obara, (2002): "Belief-Based Equilibria in the Repeated Prisoners' Dilemma with Private Monitoring", *Journal of Economic Theory,* **102**, 40-69.

[6] Ely, J.C. and J. Välimäki, (2002): "A Robust Folk Theorem for the Prisoner's Dilemma", *Journal of Economic Theory,* **102**, 84-105.

[7] Fudenberg, D., D. Levine, and E. Maskin, (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, **62**, 997-1040.

[8] Fudenberg, D. and D. Levine, (1994): "Efficiency and Observability with Long-Run and Short- Run Players," *Journal of Economic Theory,* **62**, 103-135.

[9] Kandori, M., (1999): "Check Your Partner's Behavior by Randomization: New Efficiency Results on Repeated Games with Imperfect Monitoring", CIRJE-F-49, CIRJE discussion paper series.

[10] Lehrer, E., (1991): "Internal Correlation in Repeated Games", *International Journal of Game Theory,* **19**, 431-456.

[11] Mailath, G.J. and S. Morris, (2002): "Repeated Games with Almost-Public Monitoring," *Journal of Economic Theory,* **102**, 84-105.

[12] Mailath, G.J., S. A. Matthews, and T. Sekiguchi (2001): "Private Strategies in Finitely Repeated Games with Imperfect Public Monitoring", CARESS working paper #01-10.

[13] Obara, I., (1999): "Private Strategy and Efficiency: Repeated Partnership Games Revisited," unpublished.

[14] Piccione, M., (2002): "The Repeated Prisoner's Dilemma with Imperfect Private Monitoring," *Journal of Economic Theory,* **102**, 70-83.

[15] Radner, R., (1986): "Repeated Partnership Games with Imperfect Monitoring and no Discounting," *Review of Economic Studies*, **53**, 43-58.

[16] Radner, R., R. Myerson, and E. Maskin (1986): "An Example of a Repeated Partnership Game with Discounting and with Uniform Inefficient Equilibria", *Review of Economic Studies*, **53**, 59-70.

[17] T. Sekiguchi (1997): Efficiency in the Prisoners' Dilemma with Private Monitoring, *Journal of Economic Theory,* **76**, 345-361.