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RAWLSIAN GROWTH
Dynamic Programming of Capital and Wealth for Intergenerational 'Maxi-min' Justice

by

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The volume of national saving in the Western countries has so far been regulated by fiscal conventions and beliefs, not by any ruling conception of intergeneration justice.[1]

Social welfare theorists have nevertheless looked ahead to the time when national saving, and the intertemporal allocation of resources generally, might be put in the service of some express conception of intergenerational justice. Among the rival standards of intergeneration optimality that have so far been considered, the utilitarian notion is undoubtedly the foremost. From Ramsey [ ] to Koopmans [ ], the sequence of generations, 1, 2, ..., are each imagined to submerge their egoistic utilities, U_1, U_2, ..., into an agreed-upon social-welfare functional, $W = W(U_1, U_2, \ldots)$. With the usual specifications of W and of tastes and technology, the result is that the optimal sequence of generation utilities rises monotonically: $U_1 < U_2 < U_3 \ldots$[2]

Utility rises in these utilitarian models because, at the efficient intertemporal allocation affording (highest) equal utilities, the sacrifice of utility by the present generation must, up to a point, permit a reallocation allowing increases in future utilities sufficient to increase the

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[1]For a time the dogma of the balanced budget worked like a charm to ward off the appetite for larger consumption. With fiscal orthodoxy now weaker, some dubious fiscal theory and widespread opposition to high interest rates and inflation, have kept a political lid on the growth of the public debt.

[2]Provided that initial capital endowments do not already permit the maintenance of the "maximum sustainable" Golden Rule level of utility, a case not considered realistic. Note that with the introduction of a finite time horizon, as in Cass [ ] and Samuelson [ ], the utilitarian model still displays rising utility under the same kind of initial conditions.

resulting W(·) -- provided that W is increasing and symmetrical and provided that the social rate of return to investment is positive. The reasoning involves little more than Fisher's theory of saving by a two-period household facing a positive rate of interest and having zero pure time preference.

Yet is it puzzling that a multi-period household, willing to exchange enjoyments in one period for larger ones at another, should constitute an allegory for the multi-generation society. The Fisherine household does not "sacrifice" enjoyment, it only defers enjoyment for the sake of larger lifetime enjoyment. Why should a generation be expected to sacrifice lifetime consumption for the sake of any other generation no less fortunate merely on the condition that the investment pay a positive dividend? The present generation might well complain that it was being made to suffer for the natural accident of its place in the merely chronological generation ordering:

> The time is out of joint; O cursed spite,
> That ever I was born to set it right!

The neo-utilitiarians' answer to this complaint is that anyone would have agreed to make some investment for future generations if he had not known <u>which</u> generation he would be born into and believed that one generation was as probable as another. But among wide numbers of people that argument has not been persuasive: Adherence to the ethic of sharing alike, where that is easy to interpret, is for many more satisfactory than the egoistic maximization of hypothetical expected-utility, however impartial the choice setting is construed to be.[3]

In his recent book on distributive justice [ ], John Rawls has proposed the substitution of the maximin criterion for the utilitarian one of maximum W. Several papers have since appeared which utilize minimum utility as the maximand in the study of optimally redistributive tax-and-transfer policies for intragenerational justice. In these studies the capital stock is subject to certain constraints, if there is capital at all, in order to abstract from intergenerational choices.[4] Offhand, it would seem equally natural to employ the Rawlsian conception of social welfare, $W = \min(U_1, U_2, \ldots)$, in the study of intergenerational justice. A society dedicated to this standard of justice would program its taxes and transfers, and resulting stocks of capital and national debt, so as to maximize the lifetime utility of the generation (or generations) having least utility -- over all foreseeable generations. This paper is an essay in the theory of maximin growth.

Optimal growth under the maximin criterion has so far been the subject of preconception more than hard analysis. In the prevailing wisdom, "maximin growth" is no growth at all. The reasoning to that conclusion proceeds satisfactorily to a point. Suppose that individuals are

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[3] Of course, there are more interior difficulties with the neo-utilitarian position. While we may be egoists, what if future generations are ascetics, or intergenerational egalitarians or somehow rendered incompetent? Several philosophers have doubted whether the neo-utilitarian thought-experiment is a meaningful method to determine the rate of national saving.

[4] Intragenerational lifetime justice with capital is the subject of papers by Hamada [ ], Ordover [ ], and Ordover and Phelps [ ].

given lump-sum grants (or other government benefits) which are differentiated according to the generation to which they belong.[5] Then, barring corner constraints (and none seems in order), any egoistic generation will have its "utility" pulled up to the next-lowest utility level simply by being awarded a sufficiently large lump-sum grant -- at the cost of a lesser utility for some other generations (unless the previous allocation was Pareto-inefficient across generations). Hence, any allocation that offered a generation a smaller utility than the utilities alloted to all other generations could not be an intergeneration maximin allocation. It follows that maximin growth equalizes the utilities of all generations, specifically, at the highest level affordable by initial conditions and foreseeable natural forces -- at least in a deterministic model where the terminal date is known, or else never known beforehand.

It does not follow from the above argument, however, that the maximin criterion at once freezes society into a steady state of equal capital and national income per head over all generations. The intertemporal character of the intergenerational utility functions needs to be very special, and quite unrealistically so, for that implication to emerge. Solow [ ] derives such a result by making the utility of the t-th "generation" a function solely of "consumption" in the t-th period. Then equal utility over generations enjoins constant consumption over time -- equal, in the

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[5]Lump-sum redistributions among persons of generally differing yet imperfectly known earning powers do not provide a reliable device for maximizing minimum (expected) utility within a generation of heterogeneous individuals, but the opportunities for redistribution between generations are presumably much more dependable.

case of an unbounded horizon, to (constant) national income. The task of the maximin programmers, in Solow's model, is simply to make those capital-investment substitutions for other other capital goods, especially for dwindling natural resources, that maximize the steady-state level of national income. A number of utilitarians, e.g. Brandt [ ] and Solow, while not at all averse to equality of utility, have regarded this alleged zero national saving feature as a serious defect of the maximin criterion. However, to establish the justice of positive national saving, at least from some initial conditions, it suffices to recognize the "generation" as itself "intergenerational", a bridge between old and young.

The following section describes a simple model of overlapping generations, and shows that the Rawlsian problem can be formulated in dynamic programming terms. A problem over the existence of a maximin solution is then discussed. The latter is formalized in section 2, where it is shown that, for initial conditions inside some domain, there is a unique optimal sequence of intergenerational "trades". A generation that adds to the capital stock, receives in exchange a moral claim to additional old age consumption. A generation that receives added capital to work with, also takes on an obligation to work harder.

Section 3 focusses on the nature of the dynamic path for the special case of a fixed supply of labor. It is shown that the initial generation increases the capital stock and 'trades' this for higher old age consumption. All other generations maintain this higher capital stock, trading with their immediate descendants exactly as their ancestors traded with them.

The variable labor case is examined in section 4. It is shown that in general the intergenerational trades differ over time. Moreover, under a certain plausible condition on households' consumer preferences, we show that, from any starting point off some stationary state locus, the capital stock changes monotonically over time, and approaches asymptotically a stationary state appropriate to the starting point.

Then in section 5, it is shown how the maximin allocation program can be "supported" by a system of private wealth-ownership and perfect decentralized markets supplemented by the institution of public grants and public debt.

Less constricted notions of the "generation" and its interests, offer other avenues of escape from the straightjacket of zero saving (if a straightjakket it is viewed). Rawls, taking the utilitarian critique perhaps too much to heart, proposes that "ties of sentiment" will insure that a poor generation will want to improve the well-being of its successor until some satisfactory level of well-being is approached.[6] We investigate Rawls's suggestion in Section 6.

There is another easy escape from zero growth. Rawls states that each generation has a high obligation to preserve the society's art and science for the next generation. It might further be maintained that a generation takes pleasure in adding to society's knowledge and indeed to society's aggregate capital and net national product. The point is considered in Section 7.

[6] In one implementation of this suggestion, Arrow [ ] makes utility of each one-period generation depend vicariously upon the consumption of the next generation as well as its own. While his assumptions led only to a regular two period cycle, Riley [ ] has shown that such an approach may also yield long run growth of capital and income.

1. Formulation of the Problem and Outline of the Solution

At the beginning of each period a new generation of identical individuals is born into the economy. Each generation can work in its first period and can consume at the end of its first and second period. All generations are alike in size, tastes and technology. Whether they will have identical endowments of capital and obligations to the old, of course, is a matter to be determined.

Consider the situation of the t-th generation born under justice, $t = 1, 2, \ldots$ It has available for use in current period production, a stock of capital, k_{t-1}, left over by the previous generations (now old). It faces a predetermined claim by the older generation for second-period consumption x_{t-1}. The two-dimensional description of the state in terms of (k_{t-1}, x_{t-1}) reflects the fact that two generations, young and old, co-exist in period t.

The optimal dynamic program must determine the fraction of the period ℓ_t, in which the young are to work, and what portion of the resulting gross output, $F(k_{t-1}, \ell_t)$, they are to consume, c_t, in each period t. The unconsumed output is the capital, k_t, of the next period:

$$k_t = F(k_{t-1}, \ell_t) - c_t - x_{t-1} \tag{1.1}$$

Gross output is related to Pigovian income, y_t, by

$$y_t = F(k_{t-1}, \ell_t) - k_{t-1} \tag{1.2}$$

where, from (1.1), y_t is the largest consumption possible if capital is kept intact, i.e. $k_t = k_{t-1}$. The production function F is linear homogeneous, concave and twice differentiable, with first derivatives $F_k(k,\ell)$ and

$F_\ell(k,\ell)$ positive everywhere. For every ℓ there is some $\bar{k}(\ell) > 0$ beyond which the net marginal product of capital is zero or negative. Also $F_k(k,\ell) \to \infty$ as $k \to 0$. Finally we suppose $F(0, \ell) = F(k,0) = 0$. Note that all variables are negative and in per capita form.

Each generation's preferences are "identical" and "egoistic". They are represented by an ordinal utility function which is (functionally) independent of t and in which only the generation's own experiences figure:

$$U_t = U(c_t, x_t, \ell_t) \tag{1.3}$$

The utility function U is strictly quasi-concave, and twice differentiable with derivatives $U_c(c,x,\ell) > 0$, $U_x(c,x,\ell) > 0$ and $U_\ell(c,x,\ell) < 0$ everywhere. Whenever it is desired to avoid corner solutions it will be assumed that $U(\cdot) \to -\infty$ as either c or x or "leisure", $1-\ell$, goes to zero.

Associated with each allocation $\{c_t, x_t, \ell_t | t=1,2,\ldots\}$ is a corresponding sequence of intergenerationally commensurate ordinal utilities $\{U_1, U_2, \ldots\}$. Such an allocation is feasible if the implied $(k_t, x_t) \geq 0$ for all t = 1, 2, ..., given the initial state (k_0, x_0). Our maximin problem is, roughly, to find from the feasible allocations one that makes the smallest of the utilities as large as possible.[7]

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[7]In the above formulation of our problem, x_o is arbitrarily given. It is nevertheless possible utlimately to select x_o in view of the past history of the old, (c_o, ℓ_o), to set the lifetime utility of the old, U_o, at whatever feasible level may be desired. In particular, one could choose x_o to maximize the minimum of $(U_o, U_1, U_2, \ldots)$, thus extending maximin justice to the old. Certainly the original expectations of the old should not be ruling.

We adopt the infinite time horizon. It is not implied that society as we know it will surely go on forever; a Rawlsian interpretation is that there is never a period t so far in the future that the probability of survival for another period is zero. The motive for the infinite horizon is mathematical, to maintain the time-independence (or stationarity) of the optimization problem from generation to generation. But these analytical gains come at the cost of some difficulties over the existence of a maximin path, at least for some subset of initial states.

To begin with, we seek the path or paths which

$$\underset{\{c_t,\, x_t,\, \ell_t,\, k_t\}}{\text{maximize}} \quad \inf\; U(c_t, x_t, \ell_t) \tag{1.4}$$

$$\text{st.}\quad k_t = F(k_t, \ell_t) - c_t - x_{t-1} \geq 0$$

with (k_o, x_o) given.

A property of the infimum function is

$$\inf_{\tau \geq t} U(c_\tau, x_\tau, \ell_\tau) = \min[U(c_t, x_t, \ell_t), \inf_{\tau \geq t+1} U(c_\tau, x_\tau, \ell_\tau)]$$

Thus, whatever the state (k_t, x_t) that generation t leaves to its immediate descendants, the latter will maximize the infimum thereby made feasible, from period t+1. Hence the max-inf problem in any period t = 1, 2, ..., can be described by the typical functional equation of dynamic programming.[8]

[8] Certainly the supremum exists. Moreover $m(k,x)$ is continuous (see section 2), hence the supremum is attained.

$$\text{i.e.,}\quad m(k_{t-1}, x_{t-1}) = \max_{(c_t,x_t,\ell_t)} \{\min[U(c_t,x_t,\ell_t), m(k_t,x_t)]\}$$

$$\text{s.t.}\quad k_t = F(k_{t-1},\ell_t) - c_t - x_{t-1} \geq 0 \qquad (1.5)$$

The 'return' at the first stage $m(k_o,x_o)$ is of course the solution to (1.4).

Before discussing the solution to (1.5) for arbitrary initial conditions, it is convenient to consider the "Golden Rule" state. Since the production set is bounded from above, and $U \to -\infty$ as $x \to 0$, there exists a finite state (k^G, x^G) which is maximal (and hence golden) over all stationary states $\{(k_t, x_t) = (k, x)\}$. That is, after choosing the optimal levels of c and ℓ, it affords the highest (stationary) level of utility, denoted U^G.

Suppose the initial state is (k^G, x^G). Because the Golden Rule path is efficient, any path with $U_t > U^G$ for some t requires $U_{t'} < U^G$ for some other period t', and hence $\min[U_1, U_2, \ldots] < U^G$. Then all such paths are inferior to the stationary sequence $\{(k_t,x_t) = (k^G,x^G)\}$ implying that the latter is a Rawlsian allocation.

Now if x_o were larger (smaller) than x^G, the utility possibility of any generation — generation 1, for example — would be decreased (increased), given that other generations continued to enjoy $U_t = U^G$. But up to a point each increase in x_o could be exactly compensated by an increase in k_o since $F_k(k,\ell) > 0$.

Hence we may define a locus of 'as-good-as-golden' initial states $(k^G(x_o), x_o)$ having the property that, for the given x_o, $k^G(x_o)$ is the smallest k^o from which there exists some feasible path affording $U_t = U^G$ for all t = 1, 2, ... From such an initial state, the latter path is again a Rawlsian allocation. For if some generation could enjoy $U_t > U^G$, then it could have enjoyed U^G with less k_{t-1}; but then k_{t-2} could have been smaller, and so on, leading to the implication that $k_o = k^G(x_o)$ was unnecessarily large for $U_t = U^G$, a contradiction.

The as-good-as-golden initial state locus is the upward-sloping curve labelled $m(k,x) = U^G$ in Figure 1. From our derivation of this locus, it follows that all initial states lying to the left must have a lower return. Moreover, in section 2, it is demonstrated that for all $k < k^G(x)$, the return $m(k,x)$ is strictly increasing in k, therefore all "iso-m contours" in this region are "thin".

To the right of the as-good-as-golden locus the situation is quite different. In section 4 it is shown that at least one point on any iso-m contour must be a stationary solution. But the value of the return for stationary states is bounded from above by U^G, therefore for any (k,x), $m(k,x) \leq U^G$. That is, U^G is the maximum return from any initial state.

If in addition we assume free disposal, it is always possible to move immediately to the as-good-as-golden locus, implying

$$m(k,x) = U^G \quad \text{for all } k > k^G(x) \tag{1.6}$$

Suppose initial conditions are such that the latter holds. Clearly generation 1 can use $k_o - k^G(x_o)$ units of capital to increase its own consumption, and hence utility, while maintaining the infimum, U^G, for all future generations. Alternatively, half of the 'surplus' capital can

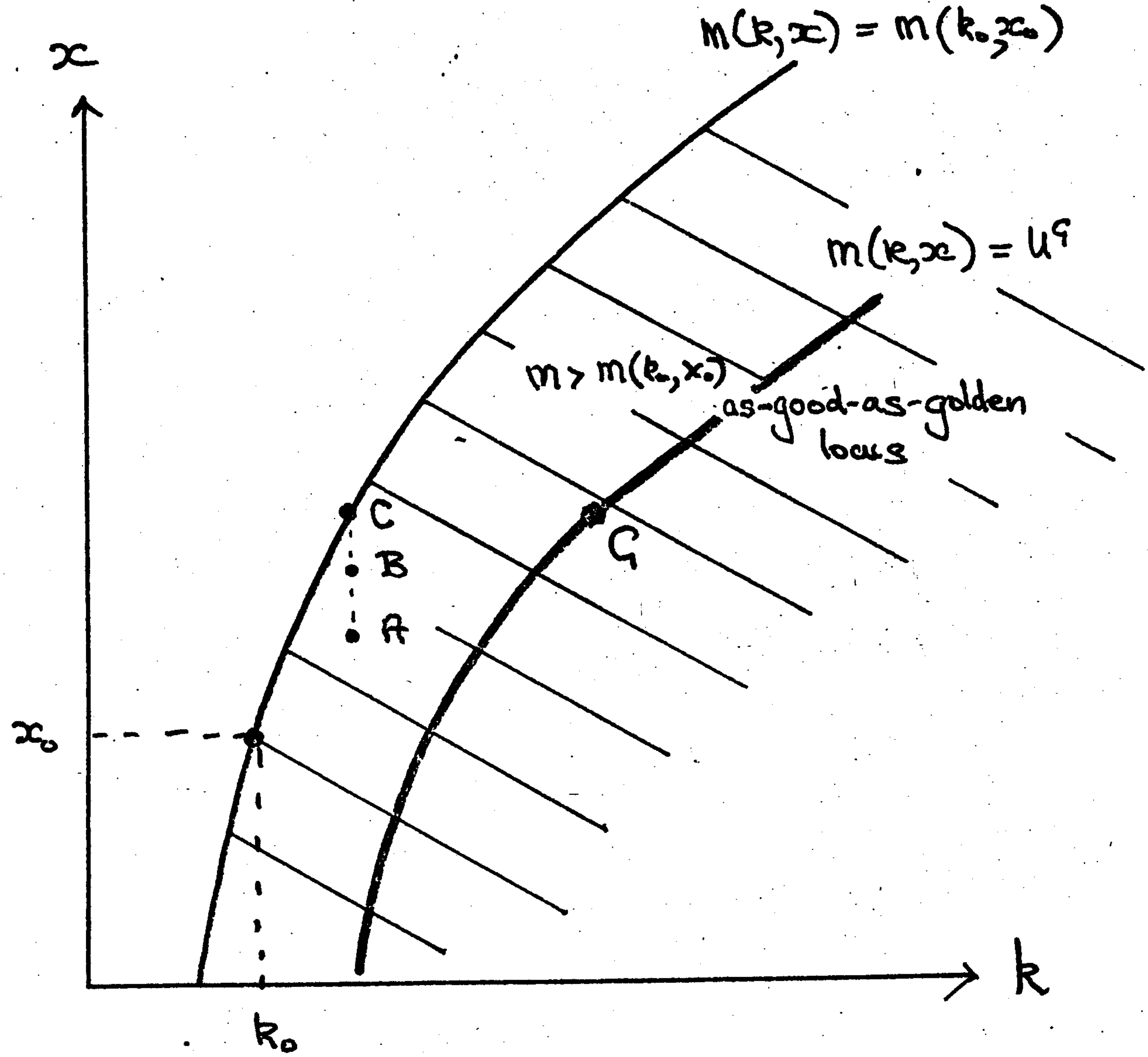

Figure 1.

be invested yielding a higher total output in the following period. Continuing this process indefinitely yields (since $F_k > 0$) a sequence of utility levels all strictly greater than U^G. But from the above discussion the infimum of this sequence is U^G, therefore we have a sequence with no minimum utility which is strictly preferred by all generations to the sequence $\{U_t\} = \{U^G\}$. Thus, there is no Rawlsian 'maxi-min' solution.

For the remainder of the paper we focus on initial states strictly inferior to the as-good-as-golden states, i.e. $k_o < k^G(x_o)$. Here we provide an outline of the solution, with a more vigorous discussion following in section 2.

Certainly the infimum for all generations beginning with the second, cannot be less than the infimum over all generations.

$$\text{i.e.} \quad (k_1,x_1) \in [(k,x)|m(k,x) \geq m(k_o,x_o)]$$

This set is the shaded region in Figure 1. All points in the interior yield a return strictly greater than $m(k_o,x_o)$.

Suppose the solution (k_1^*,x_1^*) is a point such as A. Since $m(k_o,x_o)$ is the infimum, we must have

$$U_1^* \geq m(k_o,x_o) \tag{1.7}$$

Then from Figure 1, there is another point B yielding an infimum for all future generations in excess of $m(k_o,x_o)$ and also a higher old age consumption for the first generation. But this contradicts the assumption that $m(k_o,x_o)$ is the infimum.

Therefore, the solution (k_1^*,x_1^*) must be a point such as C, lying on the boundary of the feasible set.

$$\text{i.e.} \quad m(k_1^*, x_1^*) = m(k_o, x_o)$$

From (1.7) $U_1^C \geqslant m(k_o,x_o)$. But if the inequality were strict, there would be a point lying directly below C such that $U_1^B > m(k_o,x_o)$ and $m(k_1^B,x_1^B) > m(k_o,x_o)$; again a contradiction. Hence $U_1^C = m(k_o,x_o)$.

Finally, since (k_1^*,x_1^*) lies on the boundary, the above arguments can be repeated for period 2 and for all those following. We therefore have

$$U_t^* = m(k_t^*,x_t^*) = m(k_o,x_o) \qquad t = 1, 2, \ldots \tag{1.8}$$

Thus, whenever $k_o < k^G(x_o)$, the max-inf solution involves a sequence of intergenerational trades which exactly maintain the first period return $m(k_o,x_o)$. Moreover, the return is just attained by all generations hence a max-inf solution is also a Rawlsian maximin solution, distributing utility equally among all generations.

Intuitively one would expect a unique Rawlsian allocation, given that tastes are assumed to be strictly quasi-concave. That such is indeed the case, follows from the more formal analysis of section 2.

2. The Return Function

We begin with a demonstration that $m(k,x)$ is continuous, thereby justifying the assumption made in (1.5), that the solution to the dynamic programming problem is the maximum rather than the supremum.

Suppose that the economy is initially in a state (k_0,x_0), that is, it begins with a capital stock of k_0 and a debt to the previous generation of x_0. Since the dynamic programming solution $m(k_0,x_0)$ is an infimum, there exists, for any $\varepsilon > 0$, a feasible sequence of vectors

$$\{\underline{f}_t\} = \{c_t,x_t,k_t,\ell_t | t = 1,2,\ldots\} \tag{2.1}$$

such that

$$\min_{t>0} \{U_t\} > m(k_0,x_0) - \frac{1}{2}\varepsilon \tag{2.2}$$

From our assumptions about U and F, the sequence $\{\underline{f}_t\}$ is bounded from below by a strictly positive vector. In particular, for sufficiently small δ, $k_t - \delta > 0$ for all t. Then it is feasible to reduce capital by δ in all periods and follow a new sequence $\{\hat{c}_t,x_t,k_t-\delta,\ell_t\}$ where $\hat{c}_t$ is given by:

$$(k_t-\delta) - \hat{c}_t - x_{t-1} = F(k_{t-1}-\delta,\ell_t) \tag{2.3}$$

Since c_t satisfies (2.3) with $\delta = 0$ and since F is concave we have,

$$c_t - \hat{c}_t \leq [F_k(k_{t-1}-\delta,\ell_t) - 1]\delta \tag{2.4}$$

The right hand side of the expression approaches zero with δ. Therefore for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\hat{U}_t = U(\hat{c}_t,x_t,\ell_t) > U_t - \frac{1}{2}\varepsilon, \text{ for all t}$$

implying $$\min_{t>0} \{\hat{U}_t\} > \min_{t>0} \{U_t\} - \frac{1}{2}\varepsilon$$

Combining this result with (2.2), and noting again that $\hat{U}_t$ is feasible for the initial state $(k_0-\delta, x_0)$, we have finally

$$m(k_0-\delta, x_0) \geq \min_{t>0} \{\hat{U}_t\} > m(k_0, x_0) - \varepsilon \quad .$$

Since $m(k,x)$ is a non-decreasing function of k, the continuity of m with respect to k, is established.

Arguing almost identically we can also establish that $m(k,x)$ is a continuous function of x.

We next prove that m is a semi-strictly quasi-concave function. Since m is also continuous, an immediate implication is that m is quasi-concave, and hence that iso-m contours have the general shape depicted in Figure 1.

Theorem 2.1. The return function $m(k,x)$ is semi-strictly quasi-concave.

It will be convenient to use the notation $\underset{\sim}{z}^{\nu}$ to mean the convex combination $\nu \underset{\sim}{z}' + (1 - \nu)\underset{\sim}{z}''$ of any two vectors $\underset{\sim}{z}'$ and $\underset{\sim}{z}''$.

Then for any two initial states (k_0', x_0'), (k_0'', x_0'') such that $m(k_0', x_0') < m(k_0'', x_0'')$, we must show that

$$m(k_0^{\nu}, x_0^{\nu}) > m(k_0', x_0') \qquad 0 < \nu < 1$$

Corresponding to (k_0'', x_0'') is an optimal sequence of vectors $\{c_t'', x_t'', k_t'', \ell_t''\}$ such that

$$U(c_t'', x_t'', \ell_t'') = U_t'' \geq m'' > m' \tag{2.5}$$

Similarly for (k_0', x_0') there is an optimal sequence $\{c_t', x_t', k_t', \ell_t'\}$

$$\text{s.t.} \qquad U_t' \geq m' \tag{2.6}$$

Next consider the initial state (k_0^ν, x_0^ν). Since the production function is assumed to be concave, the sequence of vectors $\{c_t^\nu, x_t^\nu, k_t^\nu, \ell_t^\nu\}$ is certainly feasible. Moreover from (2.5), (2.6) and the assumption that U is strictly quasi-concave, there exists a $\delta = \delta(\nu) > 0$

$$\text{s.t.} \quad U_t^\nu = U(c_t^\nu, x_t^\nu, \ell_t^\nu) > U_t' + \delta(\nu)$$

$$\geq m' + \delta(\nu)$$

Then
$$m(k_0^\nu, x_0^\nu) \geq \inf_{t>0} \{U_t^\nu\} > m' + \delta(\nu)$$

$$> m'$$

Q.E.D.

Note that Theorem 2.1 implies that there can be no 'thick' iso-m contours. However, we have assumed that $F_k(k,\ell)$ exceeds unity only for $k < \bar{k}(\ell)$, and $0 \leq \ell \leq 1$. Therefore there is some maximum sustainable (golden rule) utility level U^G, and for any given x, m(k,x) is a strictly increasing continuous function of k, up to some point $k^G(x)$ where $m(k,x) = U^G$.

The previous section outlined the implications of beginning to the right of this 'good-as-golden' locus, therefore here we consider the alternative possibility,

$$\text{i.e.} \quad k_0 < k^G(x_0).$$

If the first generation is to leave its immediate descendents with a capital stock of k_1 and a debt of x_1, the best it can do for itself is achieve a utility level

$$U_1 = W(k_1,x_1|k_0,x_0) = \max_{\ell_1} U(F(k_0,\ell_1) - x_0 - k_1,x_1,\ell_1) \qquad (2.7)$$

Clearly W is strictly increasing in k_0 and x_1 and strictly decreasing in k_1 and x_0. Furthermore given the concavity of F and strict quasi-concavity of U, it is a straightforward matter to check that $W(k_1,x_1)$ is a strictly quasi concave function. Thus W-indifference curves must be as depicted in Figure 2.

Since the infimum over all generations beginning with the first, cannot exceed the infimum over all generations beginning with the second, we also have

$$m(k_1x_1) \geq m(k_0,x_0) = m_0$$

In addition, since m_0 is the infimum, we must have;

$$U_1 = W(k_1,x_1) \geq m_0.$$

The set of states

$$P_{2...} = [(k_1,x_1)|m(k_1,x_1) \geq m_0]$$

is the preferred set, under the Rawlsian criterion, for all future generations taken together. Similarly the set of states

$$P_1 = [(k_1,x_1)|W(k_1,x_1) \geq m_0]$$

is the preferred set for the first generation. We know a solution exists, therefore the intersection of these two sets $P_1 \cap P_{2...}$ is non empty. Moreover, the indifference curves are "thin", thus any point in the interior yields a first generation utility strictly greater than m_0, and at the same time raises the infimum for all future generations. But this contradicts

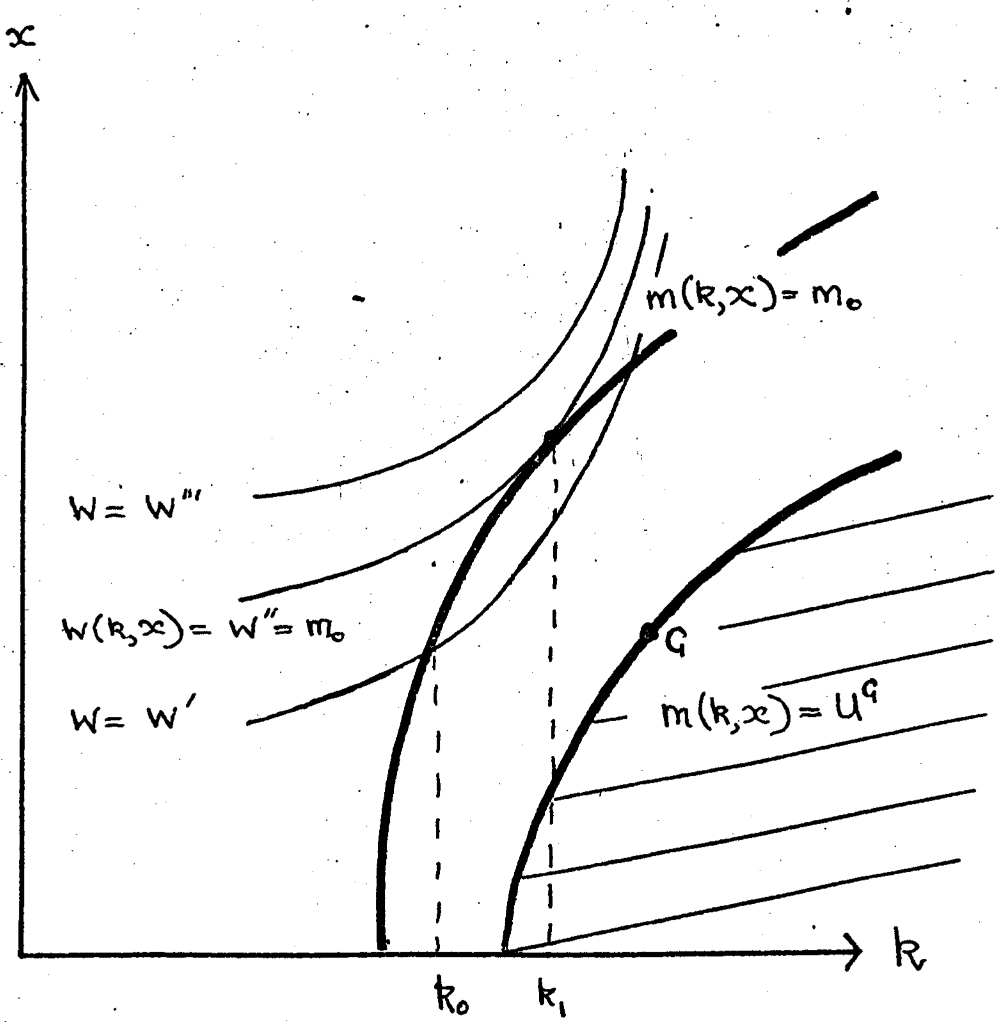

Figure 2.

the assumption that m_0 is the infimum hence int $(P_1 \cap P_{2\ldots})$ is empty.

It follows that the indifference contour $W(k_1,x_1) = m_0$ must just touch the iso-m contour $m(k_1,x_1) = m_0$ as shown in Figure 2. From the quasi-concavity of m and strict quasi-concavity of W, the intersection (k_1^*,x_1^*) is unique. But $m(k_1^*,x_1^*) = m_0$ therefore the above arguments hold with (k_1^*,x_1^*) as the new 'initial' state and we have

$$U_2^* = m(k_2^*,x_2^*) = m_0$$

Finally, applying this argument inductively for all t, it must be true that

$$U_t^* = m(k_t^*,x_t^*) = m_0$$

Summarizing all these results we have the following theorem.

Theorem 2.2. Given the assumptions of section 1, about $U(c_t,x_t,\ell_t)$ and $F(k_t,\ell_{t+1})$, a unique Rawlsian maxi-min solution exists whenever the maximized infimum is less than the golden rule utility level. Further, the optimal path $\{c_t^*,x_t^*,k_t^*,\ell_t^*\}$ distributes utility equally among all future generations.

We conclude this section by noting that, while it fits with intuition to draw the iso-m contours as smooth curves, this is not implied by the above results. Proof of differentiability follows from an examination of the left- and right-derivatives at any point (k_0,x_0) on the iso-m contour $m(k,x) = m_0$.

Suppose the optimal first period vector for the initial state (k_0,x_0) is $(c_1^*,x_1^*,k_1^*,\ell_1^*)$. If k_0 is increased by δk, no generation is hurt by an increase in the debt to the past, δx, which preserves the first period vector

$$\text{i.e.} \quad x_0 + \delta x = F(k_0 + \delta k, \ell_1^*) - k_1^* - c_1^*$$

Since F is concave and

$$x_0 = F(k_0, \ell_1^*) - k_1^* - c_1^*$$

we must have

$$\frac{\delta x}{\delta k} \geq F_k(k_0 + \delta k, \ell_1^*) \quad .$$

The efficient trade-off between initial capital and debt must be at least as large as this feasible trade-off. Therefore, letting $\delta k \rightarrow 0$, we have

$$\left.\frac{dx^+}{dk}\right|_{\substack{k=k_0 \\ m=m_0}} \geq \lim_{\delta k \rightarrow 0} \left(\frac{\delta x}{\delta k}\right) \geq F_k(k_0, \ell_1^*)$$

Similarly it can be shown that the left side derivative must satisfy

$$\left.\frac{dx^-}{dk}\right|_{\substack{k=k_0 \\ m=m_0}} \leq F_k(k_0, \ell_1^*)$$

But if any of the inequalities are strict, the iso-m contour is strictly quasi-convex in the neighborhood of (k_0, x_0), contradicting the results of Theorem 2.1. Thus there can be no inequalities and the iso-m contour is differentiable, with slope $F_k(k_0, \ell_1)$ equal to the optimal gross marginal product of capital

$$\text{i.e.} \quad \left.\frac{dx}{dk}\right|_{\substack{k=k_0 \\ m=m_0}} = F_k(k_0, \ell_1^*) \qquad (2.8)$$

We now analyse the nature of the optimal dynamic path. The discussion begins with what turns out to be a rather special case and then, in section 4, moves to the general solution.

3. The Optimal Path with Fixed Labor.

In a simple over-lapping generation model two types of inter-generational transfer are possible. Today's young can trade either consumption when young, or leisure when young in return for increased consumption when old. In both cases the return to future generations is the additional capital stock thereby made available.

It therefore appears that even without variable labor, it might be optimal to make a sequence of inter-generational trades of capital for future consumption. However it turns out that the dynamics in the fixed labor case take on a particularly simple form. Specifically, after an initial adjustment in the capital stock, the Rawlsian economy settles into a stationary state.

As outlined in section 1, the young (t+1)-th generation make a decision at the beginning of their working life based on a (k_t, x_t) offer by the old t-th generation. However we can also describe the optimal path as a sequence of decisions made after production. Subtracting the previously determined claims of the old (x_t), the young have at their disposal total assets

$$a_{t+1} = F(k_t, \ell_{t+1}) - x_t \tag{3.1}$$

These assets can be used either for immediate consumption or as inputs for next periods production.

$$\text{i.e.}\quad a_{t+1} = c_{t+1} + k_{t+1} \tag{3.2}$$

The production constraint can then be rewritten as;

$$a_{t+1} = F(a_t - c_t,\ \ell_{t+1}) - x_t \tag{3.1$'$}$$

Since in this view of the problem, the young do not make decisions until after production, their own (slave) labor must have been determined by the previous generation. Therefore the state of the world at the point of decision can be described by the pair (a_{t+1}, ℓ_{t+1}). Generation (t+1) must decide whether or not it is optimal to trade lower future leisure (higher ℓ_{t+2}) for increased future assets (higher a_{t+2}). Formally we have the following dynamic programming problem

$$r(a_{t+1}, \ell_{t+1}) = \max\ (\min\ [U(c_{t+1}, x_{t+1}, \ell_{t+1}),\ r(a_{t+2}, \ell_{t+2})])$$

subject to (3.1)'

Just as it has been shown that $m(k,x)$ is strictly increasing in k, for $m < U^G$, so it can be demonstrated that $r(a,\ell)$ is strictly increasing in a over the same range. Therefore if ℓ_t is fixed, all trades for increased future assets are eliminated, and as long as the return $r(a_{t+1}, \ell_{t+1})$ is less than the golden rule level, the solution is simply

$$a_{t+1} = a_t \qquad t = 1,2,\ldots \tag{3.3}$$

That is, total assets received and total assets passed on to the future must be identical. One can easily confirm that the optimal sequence $\{c_t, x_t, k_t \mid t = 1,2,\ldots\}$ is also stationary.

It remains to analyze the initial period. Suppose, for simplicity, that there are no generations prior to the just era.[9] Then there is no initial claim of the old ($\dot{x}_0 = 0$) and the production constraints in the first two periods must be as follows;

$$a_1 = F(k_0, \bar{\ell})$$

$$a_2 = F(k_1, \bar{\ell}) - x_1.$$

Since it is optimal to maintain constant total assets, and since the optimal consumption when old, x_t, is strictly positive, it follows immediately that $k_1 > k_0$. That is, with a fixed supply of labor and no initial debt, it is optimal to save during the first period of the just era, and from then on maintain the enlarged capital stock.

4. Optimal Growth - The General Case

Before discussing the case in which time worked is also a control variable, it may be helpful to recapitulate. We have shown that for any pre-determined state (k_{t-1}, x_{t-1}) lying to the left of the 'as-good-as-golden' locus, the optimal path lies on a smooth quasi-concave iso-m contour

$$m(k,x) = m(k_{t-1}, x_{t-1})$$

Since this contour is everywhere upward sloping in (k, x) space, it can also be expressed in the form

$$x = x_m(k) \tag{4.1}$$

---

[9] An alternative possibility - that the economy had previously been following a Ramsey-Koopmans path - is discussed below.

Furthermore, the optimal state at time $\underline{t}$ is the point of tangency of this contour and a member of the family of derived indifference curves

$$W(k_t, x_t | k_{t-1}, x_{t-1}) = \overline{W}$$

For convenience, the gradient of the W-indifference curve through (k_{t-1}, x_{t-1}) is denoted by $g(k_{t-1}, x_{t-1})$

$$\text{i.e.} \quad g(k_{t-1}, x_{t-1}) = \frac{\partial w}{\partial k_t} \Big/ \frac{\partial w}{\partial x_t} \Bigg|_{\substack{k_t = k_{t-1} \\ x_t = x_{t-1}}} \tag{4.2}$$

As depicted in figure 3, $g(k_{t-1}, x_{t-1})$ is exceeded by the slope of the iso-m contour at (k_{t-1}, x_{t-1}). Hence, the optimal state at the end of period t lies to the right.[10]

Suppose we now allow k_{t-1} to vary, at the same time varying x_{t-1} in such a way that the return is held constant

$$\text{i.e.} \quad x_{t-1} = x_m(k_{t-1}) \tag{4.3}$$

Given the differentiability of U and F, the slope of the W-contour, at $(k_{t-1}, x_m(k_{t-1}))$, is certainly a continuous function of k_{t-1}.

Then either $g(k_{t-1}, x_m(k_{t-1})) < x'_m(k_{t-1})$ for all larger k_{t-1}, or there is some point $(\hat{k}, x_m(\hat{k}))$ such that

$$g(\hat{k}, x_m(\hat{k})) = x'_m(\hat{k}) \tag{4.4}$$

---

[10] Reversing inequalities the following argument goes through almost identically if g_{t-1} is smaller.

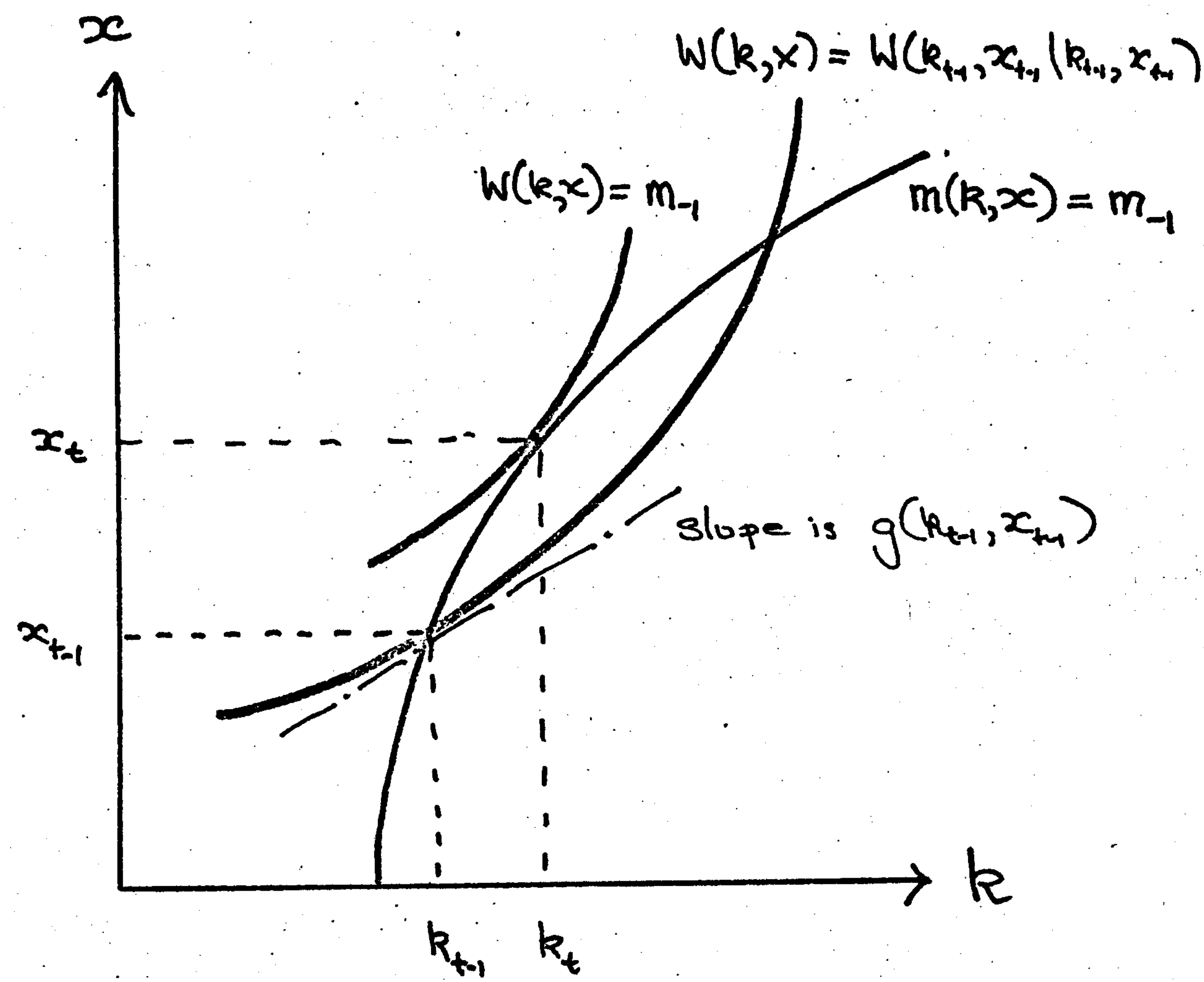
x
W(k,x) = W(k_{t-1}, x_{t-1} | k_{t-1}, x_{t-1})
W(k,x) = m_{-1}
m(k,x) = m_{-1}
x_t
slope is g(k_{t-1}, x_{t-1})
x_{t-1}
k
k_{t-1}
k_t

Figure 3

If the former, $\{(k_\tau, x_\tau)|\tau \geq t\}$ is increasing and unbounded. But this is inconsistent with production conditions. Thus, the latter must hold, that is, there is at least one stationary point on any iso-m contour. We can summarize this result as follows:

Theorem 4.1 Among the initial states with the same Rawlsian return, there is at least one for which it is optimal to remain at the initial state.

To analyze non-stationary solutions we note again that, for any generation the predetermined and final states must lie on the same iso-m contour, i.e. $x = x_m(k)$. Then substituting for x_{t-1} and x_t in the derived utility function $W(k_t, x_t | k_{t-1}, \tilde{x}_{t-1})$, the decision for generation $\underline{t}$ can be represented as,

$$\max_{k_t} W(k_t, x_m(k_t) | k_{t-1}, x_m(k_{t-1}))$$

From (2.7) this in turn can be written as:

$$W^* = \max_{k_t, \ell_t} U(F(k_{t-1}, \ell_t) - x_m(k_{t-1}) - k_t,\ x_m(k_t),\ \ell_t)$$

All the functions are differentiable and our assumptions preclude corner solutions, therefore the following first order conditions must be satisfied.

$$W^*_{k_t} = -U_{c_t} + U_{x_t} x'_m(k_t) = 0 \qquad (4.5)$$

$$W^*_{\ell_t} = U_{c_t} F_{\ell_t} + U_{\ell_t} = 0 \qquad (4.6)$$

From (2.8), the slope of the iso-m contour is the optimal gross marginal product of capital. Therefore, the necessary conditions for optimality can be rewritten as:

$$\frac{-U_{\ell_t}}{U_{c_t}} = F_\ell(k_{t-1},\ell_t); \quad \frac{U_{c_t}}{U_{x_t}} = F_k(k_t,\ell_{t-1}) \qquad (4.7)$$

Furthermore from Theorem 2.2 these conditions define a unique maximum, hence the second order necessary conditions must be satisfied.[11]

$$\text{i.e.} \quad W^*_{k_tk_t} < 0 \qquad \Delta = \begin{vmatrix} W^*_{k_tk_t} & W^*_{k_t\ell_t} \\ W^*_{\ell_tk_t} & W^*_{\ell_t\ell_t} \end{vmatrix} > 0 \qquad (4.8)$$

Now consider an increase in k_{t-1} along the iso-m contour. Differentiating the first order conditions, totally with respect to k_{t-1} yields:

$$\begin{bmatrix} W^*_{k_tk_t} & W^*_{k_t\ell_t} \\ W^*_{\ell_tk_t} & W^*_{\ell_t\ell_t} \end{bmatrix} \begin{bmatrix} \frac{dk_t}{dk_{t-1}} \\ \frac{d\ell_t}{dk_{t-1}} \end{bmatrix} = - \begin{bmatrix} W^*_{k_tk_{t-1}} \\ W^*_{\ell_tk_{t-1}} \end{bmatrix} \qquad (4.9)$$

where $W^*_{k_tk_{t-1}} = (-U_{c_tc_t} + U_{c_tx_t} x'_m(k_t)) \, [F_{k_{t-1}} - x'_m(k_{t-1})]$

and $W^*_{\ell_tk_{t-1}} = (-U_{c_tc_t} F_{\ell_t} + U_{c_t\ell_t}) \, [F_{k_{t-1}} - x'_m(k_{t-1})] + U_{c_t} F_{k_{t-1}\ell_t}$

---

[11] One can extend the arguments of section 2 to show that along any iso-m contour, $x'_m(k)$ is continuous. This leaves open the possibility of discontinuities in W^*_{kk}. However, even if this were the case, (4.9) would yield the same qualitative implications for both left- and right-side derivatives.

But (k_{t-1}, x_{t-1}) lies on the iso-m contour, therefore from (2.8) the square bracket in the last two terms is zero. Then applying Cramer's rule (4.9) can be solved as follows:

$$\frac{d\ell_t}{dk_{t-1}} = \frac{-U_{c_t} F_{k_{t-1}\ell_t}}{\Delta} \; W^*_{k_t k_t} \quad , \tag{4.10}$$

$$\frac{dk_t}{dk_{t-1}} = \frac{-U_{c_t} F_{k_{t-1}\ell_t}}{\Delta} \; W^*_{k_t \ell_t} \tag{4.11}$$

From (4.8) the right hand side of the first expression is strictly positive and we therefore have:

Theorem 4.2 It is optimal for generation t to increase the capital stock if and only if it is optimal for the (t+1)-th generation to work longer hours.

For the implications of (4.11), it is necessary to examine $W^*_{k_t \ell_t}$. Since $x'_m(k_t) = F_{k_t}$, the first order condition (4.5) can be rewritten as:

$$W^*_{k_t} = U_{x_t}\left[F_{k_t} - \frac{U_{c_t}}{U_{x_t}}\right] = 0$$

Therefore $$W^*_{k_t \ell_t} = -U_{x_t}\left[\frac{\partial}{\partial C_t}\left(\frac{U_{c_t}}{U_{x_t}}\right) F_{\ell_t} + \frac{\partial}{\partial \ell_t}\left(\frac{U_{c_t}}{U_{x_t}}\right)\right]$$

small c

and using (4.8), (4.11) implies

$$\frac{dk_t}{dk_{t-1}} \gtreqless 0 \text{ if and only if } \frac{\partial}{\partial \ell_t}\left(\frac{U_{c_t}}{U_{x_t}}\right) \gtreqless \frac{U_{\ell_t}}{U_{c_t}} \frac{\partial}{\partial c_t}\left(\frac{U_{c_t}}{U_{x_t}}\right) \tag{4.12}$$

Given strict quasi-concavity, the right hand side of this inequality is strictly positive. While theory does not sign the left hand side, it seems reasonable to argue that the amount of future consumption an individual is willing to give up for one more unit of present consumption decreases, when individuals work longer hours in the first period. Equivalently the precentage increase in the marginal utility of present consumption is greater than that for future consumption, as a result of a one per cent increase in leisure time.[12]

If this additional assumption is satisfied for all feasible (c,x,ℓ), the two terms are of opposite sign and it follows that

$$\frac{dk_t}{dk_{t-1}} > 0 \tag{4.13}$$

at every point along any iso-m contour.

But if (4.13) is true everywhere, we have immediately

$$k_{t-1} \lesseqgtr k_t \leftrightarrow k_t \lesseqgtr k_{t+1} \tag{4.14}$$

That is, the optimal path is either stationary or strictly monotonic. Suppose, as in Figure 4, that k_o lies to the left of the (lowest) stationary solution $\hat{k}$. From (4.14), $k_1 = k_o$ implies $k_t = k_o$ for all t, contradicting our assumption that $\hat{k}$ is the lowest stationary value of k.

---

12 $\frac{\partial}{\partial \ell}\left(\frac{U_c}{U_x}\right) = \frac{1}{h}\frac{U_c}{U_x}\left\{\frac{h}{U_x}\frac{\partial}{\partial h}(U_x) - \frac{h}{U_c}\frac{\partial}{\partial h}(U_c)\right\}$

where $h = 1 - \ell$ is leisure time.

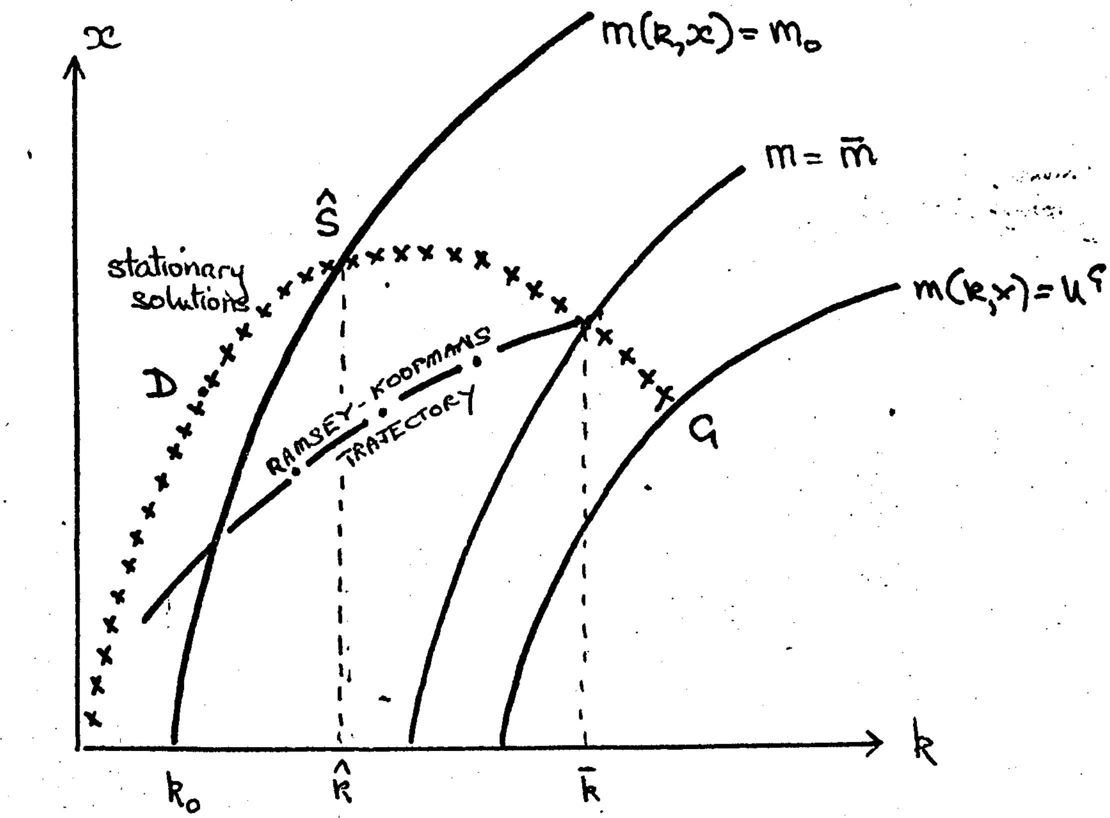

Figure 4a.

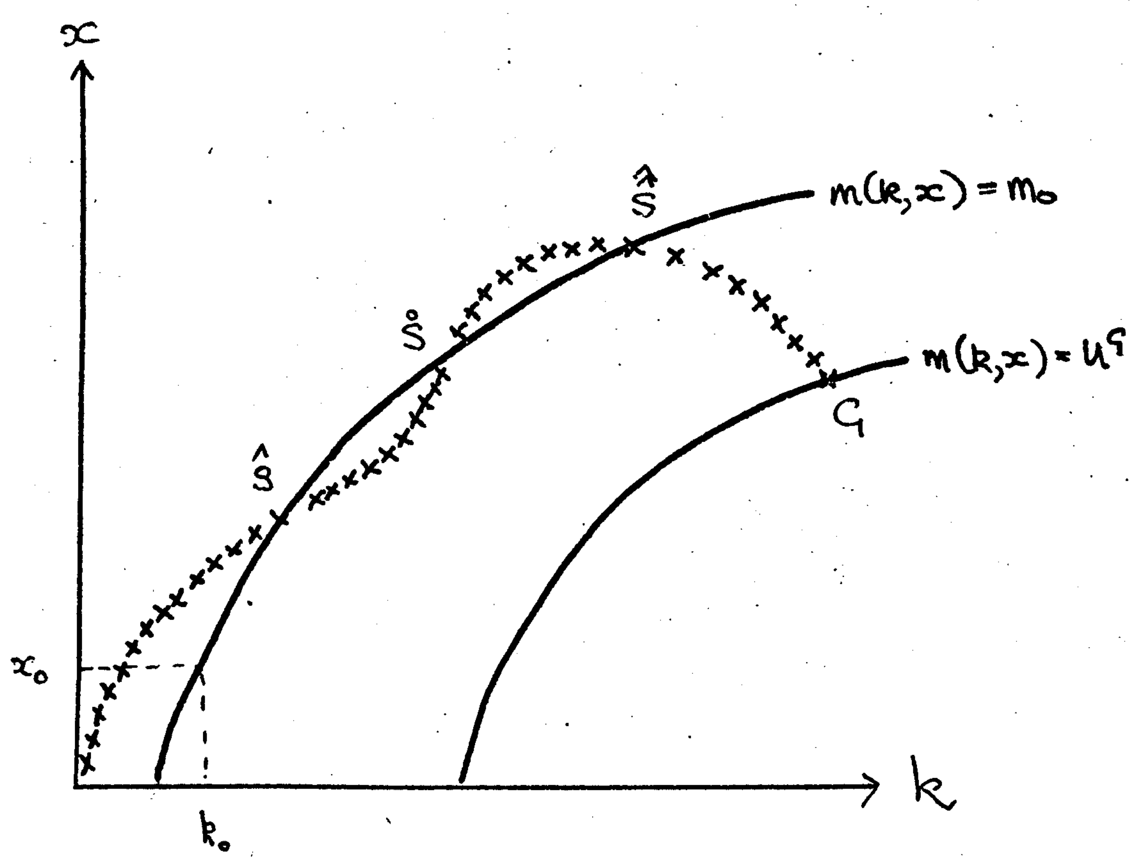

Figure 4b.

Then suppose $k_1 < k_o$. From (4.14) the optimal capital sequence $\{k_t | t=1,2...\}$ is strictly decreasing. Since $k_t \geq 0$ the sequence must approach some limit $\hat{\hat{k}}$, again contradicting our assumption that $\hat{k}$ is the lowest stationary solution.

Therefore $k_1 > k_o$ and the optimal path is a strictly increasing sequence which approaches asymptotically the stationary point $\hat{k}$. Arguing almost identically it can be established that, whatever the initial conditions, the optimal path must approach monotonically a stationary asymptote. We therefore have the following theorem:

Theorem 4.3 If over the set of feasible consumption vectors

$$\frac{\partial}{\partial \ell}\left(\frac{U_c}{U_x}\right) < \frac{U_\ell}{U_c}\frac{\partial}{\partial_c}\left(\frac{U_c}{U_x}\right)$$

the optimal sequence $\{(k_t, x_t) | t \geq 1\}$ is strictly a monotonic sequence, approaching asymptotically some stationary solution $(\hat{k}, \hat{x})$.

The 'well behaved' case in which the locus of stationary solutions can be expressed as a unique function of k is depicted in figure 4a. That such cases exist is demonstrated in the appendix, where a special Cobb-Douglas case is analyzed.

Multiple stationary solutions are depicted in figure 4b. For all initial points to the left of $\overset{o}{S}$ the optimal path follows a monotonic approach towards $\hat{S}$, and if the initial point is beyond $\overset{o}{S}$ the optimal path approaches $\hat{\hat{S}}$. In the borderline case it is a matter of indifference as to which of these points should be approached.

While we have argued that the assumption in Theorem 4.3 is plausible, it would be incomplete to ignore the implications of reversing the inequality. For concreteness we consider the special case

$$U = x(c^{\beta} + (1-\ell)^{\alpha}) \qquad 0 < \alpha < 1$$

For β in $(0,1]$, U is strictly quasi-concave. Moreover, since the feasible region is bounded, it is possible to apply a continuity argument and show that, over this region, U is strictly quasi-concave for all $\beta < \bar{\beta}$, where $\bar{\beta} > 1$.

It is easy to check that the assumption in Theorem 4.3 is satisfied if $\beta < 1$. When $\beta = 1$, the inequality becomes an equality and from (4.12) we have

$$\frac{dk_t}{dk_{t-1}} = 0, \text{ everywhere} \tag{4.14}$$

If (4.14) is satisfied, the optimal choice (k_1, x_1) is independent of the initial point on an iso-m contour. Moreover, (k_1, x_1) lies on this contour so every future choice is also (k_1, x_1) implying that the latter is a stationary solution. Therefore, in this border-line case, it is optimal to jump directly to a stationary point and remain there.

When $\beta > 1$, the inequality in Theorem 4.3 is reversed and from (4.12) we have

$$\frac{dk_t}{dk_{t-1}} < 0, \text{ everywhere}$$

implying $\qquad k_{t-1} \underset{(>)}{<} k_t \leftrightarrow k_t \underset{(<)}{>} k_{t+1}$

Therefore, unless the initial state is a stationary solution, oscillations must result. It is left as an open question whether the optimal path is necessarily a damped cycle, or whether it might approach some limit cycle.

We conclude with a brief discussion of initial conditions. If the 'just era' begins with the first generation, there are no initial claims of the old ($x_o = 0$) and, assuming Theorem 4.3 holds, the capital stock grows thereafter (see Figure 4a). Alternatively, suppose the economy has been moving along a Ramsey-Koopmans trajectory. It can be shown that this must lie below the locus of stationary solutions for all $k < \bar{k}$, the asymptotic capital stock.

Therefore, for all $k_o < \bar{k}$ the introduction of the Rawlsian criterion again results in growth of the capital stock.

5. Decentralization

Having determined the optimal path, how might a planner decentralize such an economy, assuming that individuals respond egoistically? From the previous section (equation (4.7)), marginal rates of substitution must, for every generation equal marginal productivities. Then given constant returns to scale and quasi-concavity of preferences, optimal decisions by individual agents can be achieved with the introduction of a sequence of wages $\{\omega_t\}$ and interest rates $\{\rho_t\}$. To complete the decentralization consumer budgets must be balanced by an optimal sequence $\{\beta_t^1, \beta_t^2\}$ of first and second period demogrants.

When young, consumers must choose between consumption and saving according to

$$c_t + s_t = \omega_{t-1}\ell_t + \beta_t^1 \tag{5.1}$$

where ω_{t-1} is the marginal product of labor associated with the previously determined capital stock k_{t-1}.

When old, all income is consumed according to

$$x_t = (1 + \rho_t)s_t + \beta_t^2 \tag{5.2}$$

Since consumption of x_t takes place at the end of period t+1, ρ_t is the marginal product of capital associated with k_t. Combining these two expressions yields the lifetime budget constraint,

$$c_t + \frac{x_t}{1+\rho_t} = \omega_{t-1}\ell_t + \beta_t \tag{5.3}$$

where $\beta_t = \beta_t^1 + \beta_t^2/(1+\rho_t)$ is the present value of demogrants received. Since, at least for the present discussion, only the choice of β_t is of interest we shall consider the case $\beta_t^2 \equiv 0$.

In period t+1 firms borrow capital k_t and individuals save s_t. For equilibrium in the capital market the government must float public debt. Suppose it offers bonds paying one unit of consumption at the end of the present period. Then equilibrium requires that there must be an offering of d_t such bonds where

$$d_t/1+\rho_t = s_t - k_t \tag{5.4}$$

If we balance the government budget constraint, Walras' Law will automatically ensure that individual budgets are balanced. To achieve the former the government supplies a demogrant β_t at the end of period t, equal to the difference between debt payments due and the value of the debt about to be floated.

$$\text{i.e.} \quad \beta_t = \frac{d_t}{1+\rho_t} - d_{t-1} \tag{5.5}$$

Combining (5.2) and (5.4), we have

$$x_t = (1 + \rho_t)(k_t + d_t/1+\rho_t)$$

Since $1 + \rho_t = F_k(k_t, \ell_{t+1})$ this can be rewritten as

$$x_t = d_t + k_t F_k(k_t, \ell_{t+1}) \tag{5.6}$$

Now suppose that the sufficient condition for monotonicity is satisfied and that initially the capital stock is below the corresponding stationary state level $\hat{k}$. Then k_t lies to the right of k_{t-1} as depicted in Figure 5. From (2.8) the slope of the m-contour at (k_t, x_t) is the marginal product of capital $F_k(k_t, \ell_{t+1})$. It follows immediately that the size of the bond issue d_t is given by the intercept of the tangent with the x-axis.

Then to achieve the monotonic rise in the capital stock the government must float a monotonically increasing volume of debt. As depicted the government lends money in initial periods but eventually becomes a borrower from the private sector. However, it is quite possible that even in the asymptotic stationary state lending is optimal.[13]

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[13] There is also a point on the stationary locus, with zero public debt. This is the long run equilibrium for a laissez-faire economy; see also P.A. Diamond [ ].

Optimal levels of Public Debt.

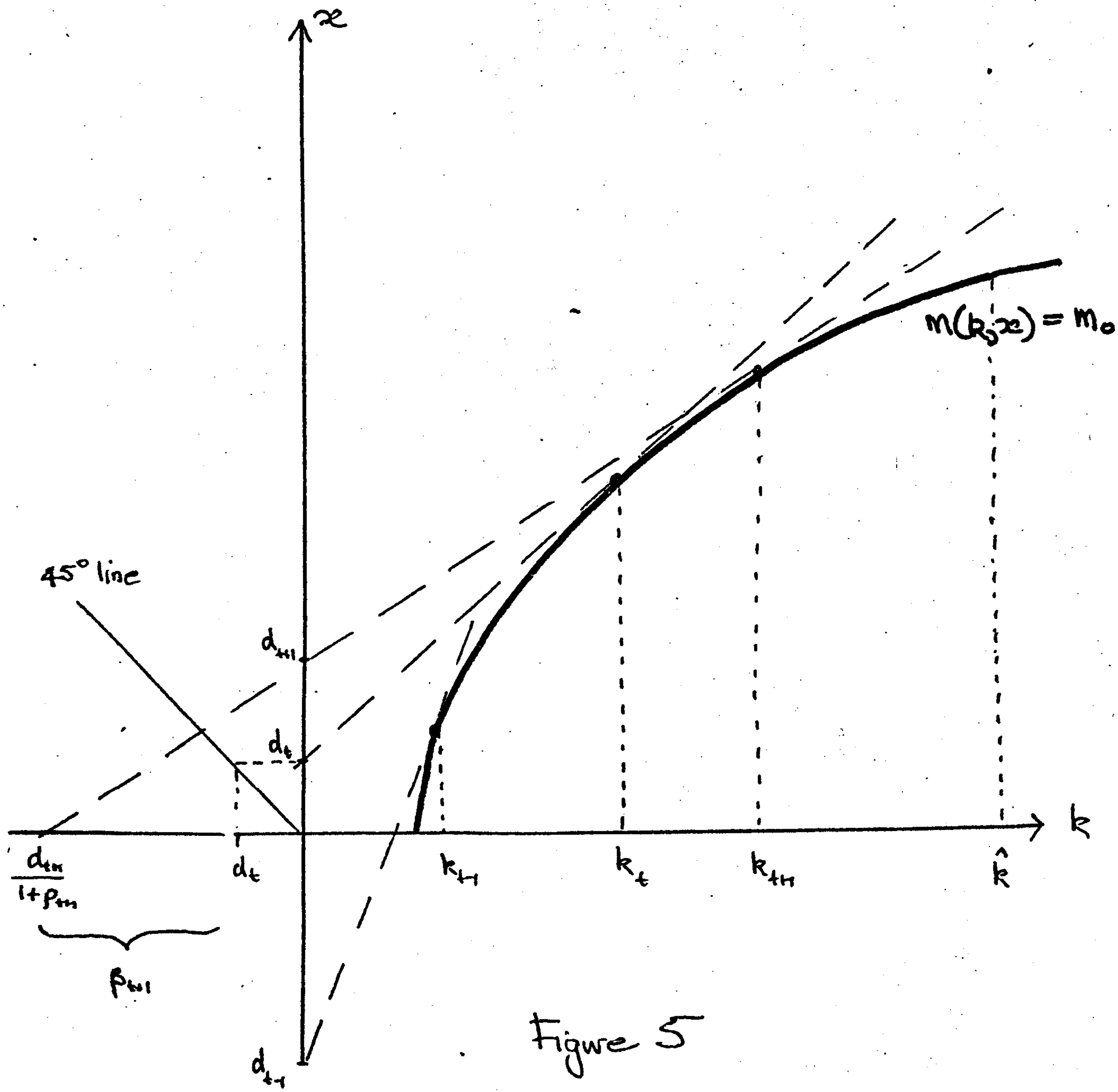

Figure 5

By extending the tangent at (k_{t+1}, x_{t+1}) to the horizontal axis we obtain the value of bonds floated at the beginning of period t+1; i.e., $d_{t+1}/(1 + \rho_{t+1})$. From the diagram it is clear that this also is directly related to the capital stock at the beginning of period t+1. Also shown on the negative half of the k-axis is the mirror image of the level of the previous periods debt. Then from (5.5) the difference between these two points is exactly the present value of the demogrant paid to the (t+1)-th generation. However, both terms in (5.5) increase with k and we cannot make general inferences about changes in the size of the demogrant over time.

6. Intergeneration Ties of Sentiment

We have been studying the intertemporal character of 'maximin' growth under conditions of generation egoism. The interests of each generation, apart from its prior interest in justice, are limited to its own consumptions (and its leisure). This maximin problem, as we have shown, has the peculiar features that generation welfare is equalized and that the asymptotic state is completely sensitive (for $k < k^G(x)$) to the initial state. Both features, especially the latter, have been regarded by some as unattractive features of the maximin criterion. But we shall argue that they are peculiar to the postulate of egoistic preferences, not to the criterion, and hence, vanish (over most if not all of the domain) as soon as the former postulate is relaxed.

Before doing so, however, we should not let the objections to these features pass without comment. One does not expect of a person who has grown up with less advantages in his formative years than someone else, that he finally match the lifetime achievements of the other person, no matter how many years he is given to do it in. Why, then, should one expect of a society of egoistic generations that it strive for some asymptotic state that is as good as the destiny of another society that is more fortunate in its initial endowments? Surely justice does not demand that the less fortunate society, by dint of its own sacrifices, eventually catch the more fortunate one. If the failure of the disparity between the two societies to vanish in the limit does not accord with the intuition of some critics, as for example Solow [ ], it is perhaps because they assume that national pride or parental pride would drive the less favored society, given enough time, to erase its initial disadvantage or even to embark on a restless quest for some more absolute state of perfection or completeness. But such national and/or parental objectives are an elective matter for each generation to decide on the basis of its instincts and cultural values, not the intrinsic dictates of justice. An egoistic generation that lacked these drives yet heeded the maximin criterion might be called unaltruistic or uninspiring or otherwise abnormal. But if it made a maximin allocation, thus to assure for future generations the possibility of economic welfare at least as great as its own realized welfare, it could not fairly be called unjust.

It should not be concluded from the foregoing that when generations choose, out of altruistic spirits or other motives, to improve opportunities for their successors that questions of intergenerational justice are thereby rendered moot. If a generation says, "We make things better for the next generation, don't we?" the question might occur why the generation does not make things still better. One expects some kind of ethical justification in reply. Like: "Further gifts to the future by us would have negative marginal utility for us, and since our utility is already smaller than what theirs can be, the consequence would be a reduction of minimum generation utility and hence a contravention of the principles of inter-generation maximin justice."

When generation utility functions are made to incorporate altruistic preferences, there is still room for the maximin criterion. Though the introduction of altruism will generally alter the allocation, the criterion functions in the same essential way. Some criterion of intergenerational justice is appropriate, even needed, in order to model the "optimum" intertemporal allocations of a society. Otherwise, there is no way, in the model, to mediate the partially conflicting interests of generations, altruistic or not -- unless, of course, their interests or preferences are in full agreement, not just similar (symmetrical) and consistent (or congruent).[14]

We can now present and discuss a rather simple example of the inter-generation maximin problem when every (homogeneous) generation possesses altruistic preferences of a certain stationary or vintage-free type. The egoistic utility function $U(c_t, x_t, \ell_t)$ is replaced and incorporated by

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[14]Rawls reaches the conclusion that there is no concept of justice between generations [, p 291]. The unhappy result follows from his ethical position that justice is a matter between parties who can gain from economic cooperation -- no one is ever obligated to accept less than what he (or a nation?) can attain operating alone -- and the economic premise that even adjacent generations cannot gain from economic cooperation. What ever the merits and problems in the first postulate, Rawls has clearly made a (rare) slip with the economic premise -- as this paper has demonstrated. They can benefit from our production of capital and we can later benefit from their working with it.

the altruistic utility function

$$V_t = V[U(c_t, x_t, \ell_t), V_{t+1}], \quad t = 1, 2, \ldots$$

where the function V has positive and continuous first derivatives everywhere. Rawls's "ties of sentiment" are here like links in a chain. Each generation gives positive weight to its own-interests and to the interests of the immediately succeeding generation, the latter expressed by the same function V. The chain creates a derived interest by any generation in the own-interests (or self-interests) of subsequent generations indefinitely into the future.

In its technocratic version, leaving aside fiscal implementability, the maximin problem becomes:

$$\text{Maximize } W(c_1, x_1, \ell_1, \ldots; k_0, x_0) = \inf [V_1, V_2, \ldots]$$

$$\text{s.t. } x_{t-1} + c_t + k_t = F(k_{t-1}, \ell_t)$$

$$\text{given } k_0 = k_o > 0, \; x_0 = x_o \geq 0.$$

By considering the analogous maximin problem in the <u>t</u>-th period under justice and upon defining

$$m(k_{t-1}, x_{t-1}) = \max W(c_t. x_t, \ell_t, \ldots; k_{t-1}, x_{t-1})$$

one obtains the dynamic- programming equation

$$m(k_{t-1}, x_{t-1}) = \max [\min \{V[U(c_t, x_t, \ell_t), V_{t+1}], m(k_t, x_t)\}]$$

$$\text{s.t. } x_{t-1} + c_t + k_t = F(k_{t-1}, \ell_t)$$

The solution to this altruistic maximin problem, when there is a solution, contrasts with the solution of the egoistic porblem in several respects. First, a range of states which are fixed points or rest points under the egoistic utility function are no longer rest points upon the introduction of the altruistic argument in the generation utility function; the locus of rest points or stable steady states shrinks, though not generally to a point. Second, with regard to at least one rest point, there will be a whole region (not just a locus which is the maximin trajectory to the rest point) within which a change of the initial state will not alter the corresponding rest point. Third, for some initial states the maximin trajectories are non-egalitarian, making V(·) increase monotonically with time, while from other initial states the maximin trajectories are egalitarian with respect to V and (it follows immediately) U as we found in the egoistic case.

These departures from the egoistic maximin solution are satisfactorily illustrated by consideration of the additive V function with a positive implicit time-discount:[15]

$$V_t = U(c_t, x_t, l_t) + \gamma V_{t+1}, \quad 0 < \gamma < 1$$

More general forms of the V function would only complicate the departures from the egoistic case if they would admit of a solution at all.

To obtain the solution to this additive altruistic maximin problem for some initial (k_0, x_0) in the region of "scarcity", we first find the intertemporal allocation that maximizes $V_1(\cdot)$. This sub-optimization problem can obviously be reduced to the familiar utilitarian problem of

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[15]We are indebted to Guillermo Calvo for discussing this problem with us.

maximizing the geometrically weighted own-utility sum, $\sum_t \gamma^{t-1} U(c_t, x_t, \ell_t)$. In the first of two cases we have to distinguish between, the "sub-optimal" allocation yields a sequence of own-utilities U_t which are monotone increasing and which approach asymptotically some Stationary Utility level $\overline{U} < U^G$. In this case, that allocation must also be the full maximin solution. For if the U_t sequence is monotone increasing then so must be the corresponding V_t sequence; hence, noting that the V_1 maximum is unique, any other allocation could only lower the minimum V, namely V_1, and thus could not be maximin. Moreover there will not arise any Strotz-Pollak problem of inconsistency causing generation 2 to select a different plan. For once (c_1, x_1, ℓ_1) is given the subsequent allocation maximizing V_2 also maximizes V_1. The quantity of steady-state capital in the Stationary state is determined by the familiar condition $F_k(\overline{k}, \overline{\ell}) = \gamma^{-1} > 1$, together with the usual marginal equivalence regarding the quantity of employment, $\overline{\ell}$.

In the other case, the allocation that solves the sub-optimization problem would make U_t decline monotonically and asymptotically down to $\overline{U}$. Then V_t would also be declining asymptotically down to $(1-\gamma)\overline{U}$. The first generation under justice would be exploiting its position as first in the sequence of generations to award itself higher V_2 at the expense of subsequent generations thereby made worse off than it, which would not be maximin. In such a case the maximin solution must, from an initial state in the region of scarcity, equalize generation utilities at the highest feasible level. Since constant V_t implies constant U_t over time, this maximin allocation is identical to the egoistic maximin allocation if the U functions in the two problems are identical. Outside the region, $k \leq k^G(x)$, some but not all generations can be assigned a V_t in excess of the maximum-sustainable Golden Rule V^G. It follows that while there are many "maxinf" solutions there exists no (true or generalized) maximin solution.

A complete discussion of the altruistic maximin problem would require analysis of the implementation of optimal maximin program by, say, taxes and transfers in a setting of perfect markets. In the non-egalitarian region of the state space, where utilities are rising from one generation to the next, it may be that voluntary private bequests will suffice without benefit of "fiscal policy" (apart from any lump-sum grant to the initial old); if so, that would be fortunate for it may be, as suggested by some results of Barro [ ], that in a world of perfect certainty and foresight variations in the volume of lump-sum grants and taxes "as long as current generations are connected to future generations by a chain of operative intergenerational transfers". But this difficult matter is perhaps not worth discussing except in a model of heterogeneous generations operating in imperfect (that is, realistic models of) markets and beset by uncertainties.

7. The Instinct to Invest

In the Freudian theory of private behavior, gratification is imputed to activities of both disposal and accumulation. Yet the hypothesis of national gratification from national capital accumulation has not so far made its mark on the theory of optimal growth.[16]

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[16] An exceptional study is M. Kurz, [ ].

Society's investments in knowledge and in artistic capital are rewarding for the generation making them because of the enjoyment and usefulness that subsequent generations as well as contemporaries are anticipated to derive from them. These two kinds of capital are intergenerational collective goods. Investments in them present a type of intergeneration externality (or expected externality) that operates differently from the altruistic type of intergeneration externality discussed in the previous section.

Does investment by a nation in every durable good produce this pride of creation? Suppose that the great bulk of capital investments do not confer such satisfactions, being just so much nondescript hardware to their absentee owners. Then the psychic rewards which are special to some types of investments would influence the investment mix selected by the current generation but it is not evident that they would influence the "aggregate stock" of capital that the generation (faithful to maximin) would leave to the next generation; that is, the utility possibilities of the next generation might not experience a net increase. "We are leaving you less nuts and bolts than were left us," the current generation might say, "in consideration of the addition we have made to the stock of technologic and artistic capital".

But it is imaginable that many a society takes satisfaction from the growth of aggregate national capital, from an increase of the next generation's production possibilities. The simplest representation of this idea introduces the quantity of net investment into each generation's (egoistic) utility function:

$$U_t = U(c_t, x_t, \ell_t, k_t - k_{t-1})$$

with the first derivative of U with respect to the fourth argument positive, perhaps everywhere. In this case, it is obvious that, under the maximin criterion, there may very well be no rest point short of the Golden Rule state -- if the instinct to invest is strong enough relative to the marginal utility of consumptions. The maximin criterion would preclude a sequence of capital deepening beyond the Golden Rule point of capital saturation. If, on the other hand, the marginal utility of adding to capital relative to the marginal utility of consumption, the results are entirely similar to those of the previous section in which future utility possibilities, rather than production possibilities, have utility for the present generation. In both cases there is a region of initial states in which the original capital endowment is sufficiently small that despite the regulation of the maximin criterion, capital <u>and</u> (egoistic) generation utility rise monotonically toward some stationary-state. Thus the satisfactions from national investment, like the altruistic interest in subsequent generations' satisfactions, are capable of dissolving the egalitarian property of the maximin allocation that would otherwise obtain.

8. Concluding Remarks

The principal messages of this paper are presumably clear. The application of the intergeneration maximin criterion is not generally a bar to the growth of capital. Unless the economy happens to be in an efficient stationary state initially, the maximin criterion will not lock the economy forever in that state. It is true that the maximin allocation (where it exists) is intergenerationally egalitarian with regard to utility if intergeneration externalities are excluded. That such intergeneration equality should result from the maximin criterion does not seem a telling objection to the use of that criterion when by hypothesis the generations, while just, are perfect egoists. In any case, the maximin criterion does not generally preclude the growth of utilities if initial capital is sufficiently scarce and if the generations possess an altruistic interest in the future utility possibilities or take pride in future production possibilities.

Ethical theory, as Rawls has himself insisted, is uncertain and provisional like knowledge in general, especially the theory of human behavior. Without being able to foresee the final verdict on the maximin criterion, we nevertheless find it significant that no anomalies or conundrums have been turned up by our study of maximin as a standard for the allocation of resources among generations -- especially when "growth" has been considered a critical stumbling block for the maximin criterion. The only difficulty that the maximin criterion has encountered in our analysis occurs where the initial capital stock is so large that some generations can be allocated a utility exceeding the Golden Rule amount while by implication not all generations can be so favored. Yet even this difficulty can be laid to the unboundedness of the time horizon rather than to the criterion itself. Moreover it is a question whether our ethical principles should be asked to meet all manner of hypothetical conditions however counterfactual in actual experience. In a significant sense, the unrestricted domain is undefinable, so no criterion could ever be certified universally robust.

It remains to be seen whether the maximin criterion will stand up under various extensions of the model. What are the consequences of placing an upper bound on the lifetime of the earth or mankind? What are the effects of making population growth an endogenous variable to be optimized along with capital and wealth?[17] What are the consequences of introducing exhaustible natural resources?[18] Of introducing investments in human beings and in the technology? Finally, to put an end to an indefinite list of questions, one wonders what can be said once we admit that future tastes, values and consumption possibilities are uncertain.

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[17] Calvo [ ] has analyzed this problem when generations overlap as here though without provision for variability of manhours in employment emphasized in the present paper.

[18] Solow [ ] has studied the fixed-population maximin problem without generation overlap. Koopmans [ ] gives a utilitarian treatment of a variable-population problem without capital.

APPENDIX: A Cobb Douglas Example

Suppose, for the sake of simplicity, that tastes are given by $U = [cx(1 - \ell)]^{\theta}$ and gross output is produced according to $F = 2(k\ell)^{1/2}$.

First, we solve for the locus of stationary solutions, that is, stationary points satisfying the dynamic first order conditions (4.7), and the production constraint (1.1).

We have,

$$\frac{-U_{\ell}}{U_x} = F_k F_{\ell} = 1 \rightarrow x + \ell = 1 \tag{A.1}$$

$$\frac{U_c}{U_x} = F_k \rightarrow \frac{x}{c} = y^{-1/2} \tag{A.2}$$

$$c + x = F(k,\ell) - k \rightarrow x = \frac{2y^{1/2} - y}{1 + y^{1/2}} \tag{A.3}$$

where $y = k/\ell$.

Combining (A.1) and (A.3) it is a straight-forward matter to show that ℓ must decline (and x increase) with y until $y^{1/2} = -1 + \sqrt{3}$ and that the opposite is true for larger y.

Also combining (A.2) and (A.3) we have

$$\frac{x}{k} = \frac{2y^{-1/2} - 1}{1 + y^{1/2}} \tag{A.4}$$

so x/k declines as y increases. Since x is increasing for lower values of y, k must also be increasing in y. For $y^{1/2} > -1 + \sqrt{3}$, x declines and ℓ

increases with y. But $y = k/\ell$ so k must continue to increase with y. Therefore, the capital-labor ratio always changes with k around the stationary state locus. Note that the marginal product of capital is simply $y^{-1/2}$ therefore F_k declines monotonically around this locus as k increases. Finally as $x/k \to 0$, $y^{1/2} \to 2$ and ℓ approaches 1, therefore $k \to 4$.

Hence, the locus of stationary solutions is a single peaked curve in (k,x) space, from the origin to the point (4,0). At the turning point

$$F_k = y^{-1/2} = \frac{1}{2}(1 + \sqrt{3}) > 1$$

thus the golden rule point G must lie on the downward sloping section of the curve. All this is depicted in Figure 4(a).

Next we examine steady state levels of the public debt. Utilizing (5.6) we obtain,

$$\frac{d}{k} = \frac{x}{k} - y^{-1/2}$$

and substituting from (6.4) yields

$$\frac{d}{k} = \frac{y^{-1/2} - 2}{1 + y^{1/2}} \tag{A.5}$$

As k increases and we move around the locus of stationary solutions, y increases and the right hand side declines, reaching zero where

$$F_k = y^{-1/2} = 2$$

This is the laissez-faire stationary state (the point D in Figure 4a). Since $F_k^D > 1$, it lies to the left of the Golden Rule point. As Diamond [ ] has noted this need not be the case. In fact, if we introduce the more general production function, $F = 2k^{\alpha}\ell^{1-\alpha}$, it is possible to show that for α less than some $\overline{\alpha}$ the laissez faire point lies to the right.

Combining (A.1), (A.3), (A.4) and (A.5) we have

$$d = \frac{(y^{1/2} - 2y)}{1 + 3y^{1/2} - y}$$

Differentiating, it can be shown that d increases with y and hence with k in the range $F_k > 5$ and decreases thereafter. Thus even for this very malleable case the long run equilibria cannot be characterized simply by the level of the public debt.[19]

Finally, it is possible to confirm that moving around the locus of stationary solutions, the demogrant β declines from zero to some minimum then increases again reaching zero at the laissez faire point. To the right of this point β is positive, increasing steadily up to some maximum and then declining again towards the Golden Rule point.

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19 The same is true for the value of the public debt, i.e., d/F_k.