

COMPETITIVE EQUILIBRIUM WITH LOCAL PUBLIC GOODS

by

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1. Introduction

In the twenty years since the appearance of his first paper on the subject, the theory of pure public goods proposed by Samuelson [27] has had a profound effect on the way economists view collective consumption. The connection between sharing a public good and summing marginal rates of substitution captures the essence of collective consumption in a way that is both simple and elegant. So compelling is this conception that it is difficult to imagine any alternative worth pursuing.

But the theory of public goods is not in a very satisfactory state, and I believe the time has come to question the foundation on which it is built. Despite twenty years of effort, there is little we know about public goods that is not in Samuelson. Much of the attention of mathematical economists during this time has been devoted to the notion of Lindahl equilibrium, and we now have rigorous proofs that Lindahl equilibria exist, that they are Pareto optimal and that they are in the core.¹

But closer examination of this equilibrium concept has only confirmed what was obvious to Samuelson all along: Lindahl equilibrium is simply a technical artifact with no behavioral significance.² Most recent work has tended to accept Samuelson's conclusion that the presence of pure public goods leads to market failure, and attention has shifted to the design of non-market incentive schemes, invulnerable to the free-rider problem, that produce Pareto optimal allocations.³

¹Foley [12], Milleron [22], Roberts [24].

²For a demonstration of the failure of equivalence theorems when pure public goods are present, see Muench [23] or Milleron [22].

³Groves and Ledyard [13].

In the case of pure public goods, an approach of this sort is probably inevitable: competitive equilibrium is irrelevant, and we might as well settle for designing an optimal process for a planned economy.⁴ However, in a certain sense the current stress on "incentive compatibility" (invulnerability to the free rider problem) seems to go too far. As Hurwicz [17] has observed, no competitive mechanism meets the test of incentive compatibility provided individual agents have non-negligible influence (i.e., provided one does not have the non-atomic measure space of agents of an Aumann [2] economy). There is no reason to single out public goods for special attention: all of competitive analysis should be supplanted by the design of incentive-compatible allocation mechanisms. It seems doubtful that many economists are willing to go that far.⁵ The attention paid in recent years to incentive compatibility and to core equivalence has had the salutary effect of sensitizing the economics profession to the need to justify the hypothesis of price-taking behavior. But it is premature to jump to the conclusion that the competitive hypothesis cannot be justified except in atomless economics. By appealing to information costs and the like, it does not seem rash to conjecture that a better case can be made for price-taking behavior than current results seem to suggest. In the interim we must rely on intuition in judging whether a particular competitive model offers a plausible portrait of reality. Lindahl equilibrium fails to meet

⁴In passing, it is worth noting that the other leading attempt to formulate a notion of competitive equilibrium for pure public goods, via an analogy to joint supply (Buchanan [5], Demsetz [7]), suffers from all the deficiencies of Lindahl equilibrium (Samuelson [28]); in fact, the proposed equilibrium is just Lindahl equilibrium in disguise (Ellickson [10]).

⁵In fact Groves and Ledyard [13] assume competitive behavior with regard to private goods in their analysis.

such rough tests of plausibility no matter how one looks at it. But the negative experience with competitive equilibrium analysis in a world of pure public goods should not blind us to the possibility that it could produce more interesting results in other contexts involving collective consumption.

In the search for a plausible notion of competitive equilibrium in an economy with public goods, the primary impetus has come not from the ranks of mathematical economists but elsewhere. Almost from the beginning Samuelson's theory of public goods was attacked not for its logic⁶ but rather on grounds that the notion of a pure public good is not realistic. Few public goods are purely public. Most forms of collective consumption involve some crowding (i.e., the cost of producing a given level of service increases with the number of consumers served) and most admit at least the technical possibility of exclusion. At an early stage, a number of economists sensed that local public goods, crowded public goods for which exclusion is possible, could provide a more hospitable stage for competitive analysis than pure public goods. Thus, Tiebout [3] described the outlines of a competitive market for local public goods, and Buchanan [4] provided a theory of optimal jurisdictional size reminiscent of the textbook theory of firm size when average cost curves are u-shaped.

The work on local public goods is to my mind the most interesting new development in public goods theory since Samuelson. But unlike the theory of pure public goods, local public goods have not proved amenable to more rigorous analysis. The reason for this failure I suggest is that we have sought to adapt the constructs introduced by Samuelson to a problem that should be treated on its own terms.

⁶As Samuelson [29] has noted, few specialists on the public sector were equipped even to understand his formulation until it was presented in graphical form.

A striking feature of existing studies of local public goods, a feature that appears to have gone essentially unnoticed, is the general irrelevance of Samuelson's approach to the results that have been obtained. It is true that most of this analysis pays lip-service to the condition $\sum MRS = MRT$, but it never seems to play an essential role. It is unlikely that many who have read Tiebout's paper have taken the details of his analysis seriously. The message that has stuck, and that has made the paper justifiably famous, is the idea that a market for local public goods is not all that far-fetched. Attempts to formulate the model more rigorously have recognized that, if the Tiebout model is to make sense, jurisdictions must be populated by consumers with identical marginal rates of substitution (McGuire [2], Hamilton [14], Ellickson [9]), a feature that renders the summation of MRS's superfluous. In such models the marginal utility of a local public good bears the same relationship to price and to marginal cost that it would if the public good were private. At least insofar as the analysis of competitive markets for local public goods is concerned, the relevance of Samuelson's model of a public good seems dubious.

In a previous paper [8] I examined the question of just how far one could go in extending the constructs of pure public theory to the case of local public goods. To some extent the conventional approach does generalize. If the assignment of consumers to jurisdictions is regarded as fixed, then the Samuelson conditions for Pareto optimality generalize in the obvious way with marginal rates of substitution summed over jurisdictions rather than the economy as a whole. Lindahl equilibrium, suitably generalized, can be shown to exist, be Pareto optimal and in the core under conditions essentially equivalent to those employed in the analysis of pure public goods. But the key point is that

all of these statements are conditional on the given assignment of consumers to jurisdictions. When consumers are allowed to move, the conventional approach to public goods breaks down. The core may be empty. Even when the core is not empty, Lindahl equilibrium defined relative to one assignment need not be in the core when the assignment is treated as endogenous. A coalition may be able to improve upon the allocation by forming a new jurisdiction. And Samuelson's condition that $\sum \text{MRS} = \text{MRT}$ within each jurisdiction coupled with the usual convexity assumptions on preferences and production sets emerge as only necessary and not sufficient for Pareto optimality: these conditions are necessary and sufficient for optimality conditional on the assignment of consumers to jurisdictions, but given the wrong assignment they may be satisfied by allocations that are Pareto dominated. These difficulties lie just below the surface even in the theory of pure public goods. The convexity assumptions imposed on production sets rule out the realistic case of lumpy public investment projects (e.g., the exploration of outer space), begging the issue of whether a public good should be produced at all. But it is in the context of local public goods theory that the flaws become manifest.

How then are we to come to terms with local public goods? The idea that competitive markets could exist for some public goods seems reasonable, and the notion that local public goods could provide a construct bridging the gap between the purely public and the purely private good has intuitive appeal. But the analytical construct of a public good as conventionally defined has not provided the means for putting this theory on a rigorous foundation. What I advocate in this paper is that we dispense with Samuelson's construct, focusing instead on local public goods as a problem worth analyzing in its own right. The basic claim is that, once the blinders of conventional public goods theory have been removed, local public goods can be regarded as a special type of indivisible private good. Section 2 develops the basic conceptual framework

and demonstrates the existence, at least in an approximate sense, of competitive equilibrium with local public goods. In Section 3 we reconcile these results with the traditional approach to the subject.

The major focus of this paper is on local public goods. However, the approach we have taken offers a better way to view pure public goods as well. In the concluding section I will justify that claim.

2. Competitive Equilibrium with Local Public Goods

The central idea in this paper is that public goods can be viewed as a type of indivisible private good. We consider, therefore, an economy with two types of commodities, divisible and indivisible. Let $A = \{1, \dots, n\}$, $L = \{1, \dots, \ell\}$ and $M = \{1, \dots, m\}$ index the set of consumers, divisible commodities and indivisible commodities respectively. We will interpret M as the set of all types of public good that could be produced: for concreteness, imagine that M indexes alternative types of elementary school. It is important to note that this set, while perhaps quite large, is assumed to be finite.

Consumption bundles take the form $x(a) = (x_1(a), x_2(a))$ where $x_1(a)$ is the ℓ - dimensional vector of divisible commodities and $x_2(a)$ the m -dimensional vector of public goods allocated to consumer a . To clarify the relationship between our results and the existing literature on local public goods, we adopt a more restrictive assumption regarding consumption sets than that employed by Broome [3] or Mas-Colell [20] in their analysis of competitive equilibrium with indivisible commodities. We assume that an allocation can assign at most one unit of at most one type of indivisible commodity to any given consumer. As a concrete example not involving public goods, imagine that the set of indivisible commodities corresponds to houses of different types; what we are assuming is that an allocation assigns at most one house to any consumer.⁷

⁷This assumption has in fact been employed in models of markets with indivisible commodities: see Rosen [26], Lancaster [19] or Ellickson [11], for example. It can easily be relaxed but that would only complicate the presentation without adding

To describe in formal terms the restrictions we impose on the consumption set of consumer a , let $G = \{g_0, \dots, g_m\}$ where g_j ($j \in M$) is an m -dimensional vector of zeros except for a one in the j th place and g_0 is an m -dimensional vector of zeros. Then $x_2(a) = g_j$, $j \in M$, if consumer a is assigned one unit of the j th type of public good, $x_2(a) = g_0$ if he is assigned no public good. We then assume:

(A.1) The consumption set of consumer a conditional on the choice of

$$x_2(a) = g \in G \text{ is } X(a, g) = R_+^l \times \{g\}. \text{ The consumption set of}$$

$$\text{consumer } a \text{ is } X(a) = \bigcup_{g \in G} X(a, g).$$

Since all consumers are assumed to have the same consumption set, we will suppress the index a , writing $X(a, g) = X(g)$ and $X(a) = X$. Note that while $X(g)$ is convex for each $g \in G$, X is not. It is the non-convexity of consumption sets that distinguishes the theory of competitive equilibrium with indivisible commodities from standard competitive analysis.

Regarding consumer preferences we will assume:

(A.2) For all $a \in A$, the preference relation \succsim_a is a complete continuous preorder on X such that:

- i) For every $g \in G$ and for all $x(a) \in X(g)$ the set $\{\tilde{x}(a) \mid \tilde{x}(a) \succsim_a x(a), \tilde{x}(a) \in X(g)\}$ is convex;
- ii) For any l -dimensional vector $\xi > 0$ and for any $x(a) \in X(g)$, $(x_1(a) + \xi, g) \succsim_a (x_1(a), g)$ for all $g \in G$ (strict monotonicity in the divisible goods).⁸

Part (i) of this assumption states that, given the choice of indivisible commodity, preferences over the divisible commodities satisfy the usual convexity properties

⁸For two vectors having the same dimension $x \gg y$ means $x_i > y_i$ for all i , $x > y$ means $x_i \geq y_i$ for all i and $x \neq y$, and $x \succeq y$ means $x > y$ or $x = y$.

of the standard competitive model. Part (ii) could undoubtedly be weakened, but only at the expense of greatly complicating the analysis.

Existing models of competitive equilibrium with indivisible commodities are restricted to the case of pure exchange. For purposes of this paper, however, it is clearly essential to introduce production into the economy. Regarding initial endowments, we will assume that no public goods are initially owned:

$$(A.3) \quad \text{For all } a \in A \text{ the initial endowment is } \bar{x}(a) = (\bar{x}_1(a), 0) \in X, \\ \bar{x}_1(a) \gg 0.$$

To keep matters simple, we will assume that there is no production of the divisible (private) goods and that public good inputs are not used to produce public good outputs. Since consumers choose either to consume a public good of a given type or they do not, the amount of public good j produced will be a non-negative integer which we denote n_j , the number of consumers provided the public good of type j . The basic idea behind Tiebout models of a competitive market for local public goods is that production exhibits constant or decreasing returns to scale for outputs above some level \tilde{n}_j . To avoid having to distribute profits, we will assume constant returns to scale for $n_j \geq \tilde{n}_j$.

For simplicity we will assume that to each $j \in M$ is associated a single technology set Y_j . In keeping with the public good interpretation, we will upon occasion refer to each j as a "jurisdiction." The sets Y_j are defined such that a vector $y_j := (z, n_j g_j)$ can be produced iff $y_j \in Y_j$ where z denotes a vector of private good inputs and n_j is multiplied by the vector g_j in order to have $y_j \in R^l \times Z_+^m$. We then assume:

$$(A.4) \quad \text{For each } j \in M,$$

- i) $0 \in Y_j$ (possibility of inaction);
- ii) $(z, n_j g_j) \in Y_j$ implies $(\tilde{z}, n_j g_j) \in Y_j$ for all $\tilde{z} \leq z$ (free disposability of the private commodities);

iii) There exists a closed convex cone Y_j^* owning 0 and an integer

$\tilde{n}_j \geq 0$ such that $Y_j \subset Y_j^*$ and $Y_j \cap \tilde{N}_j = Y_j^* \cap \tilde{N}_j$ where

$$\tilde{N}_j := \mathbb{R}^\ell \times \{n_j g_j \mid n_j \geq \tilde{n}_j, n_j \in \mathbb{Z}\}.$$

Part (iii) of this assumption is intended to capture the notion that production of the public good of type j exhibits constant returns to scale for $n_j \geq \tilde{n}_j$. Figure 1 portrays a typical technology set Y_j (the heavily marked points along with the horizontal lines extending to the left) and the associated set Y_j^* (the shaded cone) for the case $\ell = 1, m = 1$; in this case, production of the public good exhibits no crowding (pure publicness) for $n_j \leq 2$, some crowding as output increases from $n_j = 2$ to $n_j = 3$ and constant returns to scale for $n_j \geq \tilde{n}_j = 3$.

Since preferences are strictly monotonic in the divisible commodities and no public goods are initially owned, all equilibrium prices must be non-negative. For a price vector $p \in \mathbb{R}_+^{\ell+m}$, we will write $p = (p_1, p_2)$ where p_1 is the ℓ -dimensional vector of prices for the divisible commodities and p_2 the m -dimensional vector of prices for the indivisible commodities (public goods). Prices are normalized to lie in the price simplex $\Delta := \{p \in \mathbb{R}_+^{\ell+m} \mid |p| = \sum_{i \in L} p_{1i} + \sum_{j \in M} p_{2j} = 1\}$.

In the usual way we define for consumer a the budget set

$$\beta(p, \bar{x}(a)) := \{x(a) \in X \mid p \cdot x(a) \leq p \cdot \bar{x}(a)\} \text{ and the } \underline{\text{demand set}}$$

$$\phi(a, p, \bar{x}(a)) := \{x(a) \in \beta(p, \bar{x}(a)) \mid x(a) \succ_a \bar{x}(a) \text{ for all } \bar{x}(a) \in \beta(p, \bar{x}(a))\}.$$
 For

each jurisdiction $j \in M$ we define the supply set $\eta_j(p) := \{y_j \in Y_j \mid p y_j = \max_{Y_j} p y_j\}$.

Def: An allocation $f: A \rightarrow X$, a set of production vectors $\{y_j\}$ and a price vector $p \in \Delta$ is a competitive equilibrium (CE(\mathcal{E})) for the economy $\mathcal{E} = (\{X, \succ_a, \bar{x}(a)\}, \{Y_j\})$ if:

- i) $f(a) \in \phi(a, p, \bar{x}(a))$ for all $a \in A$;
- ii) $y_j \in \eta_j(p)$ for all $j \in M$;
- iii) $\sum_{a \in A} f(a) - \sum_{j \in M} y_j = \sum_{a \in A} \bar{x}(a)$.

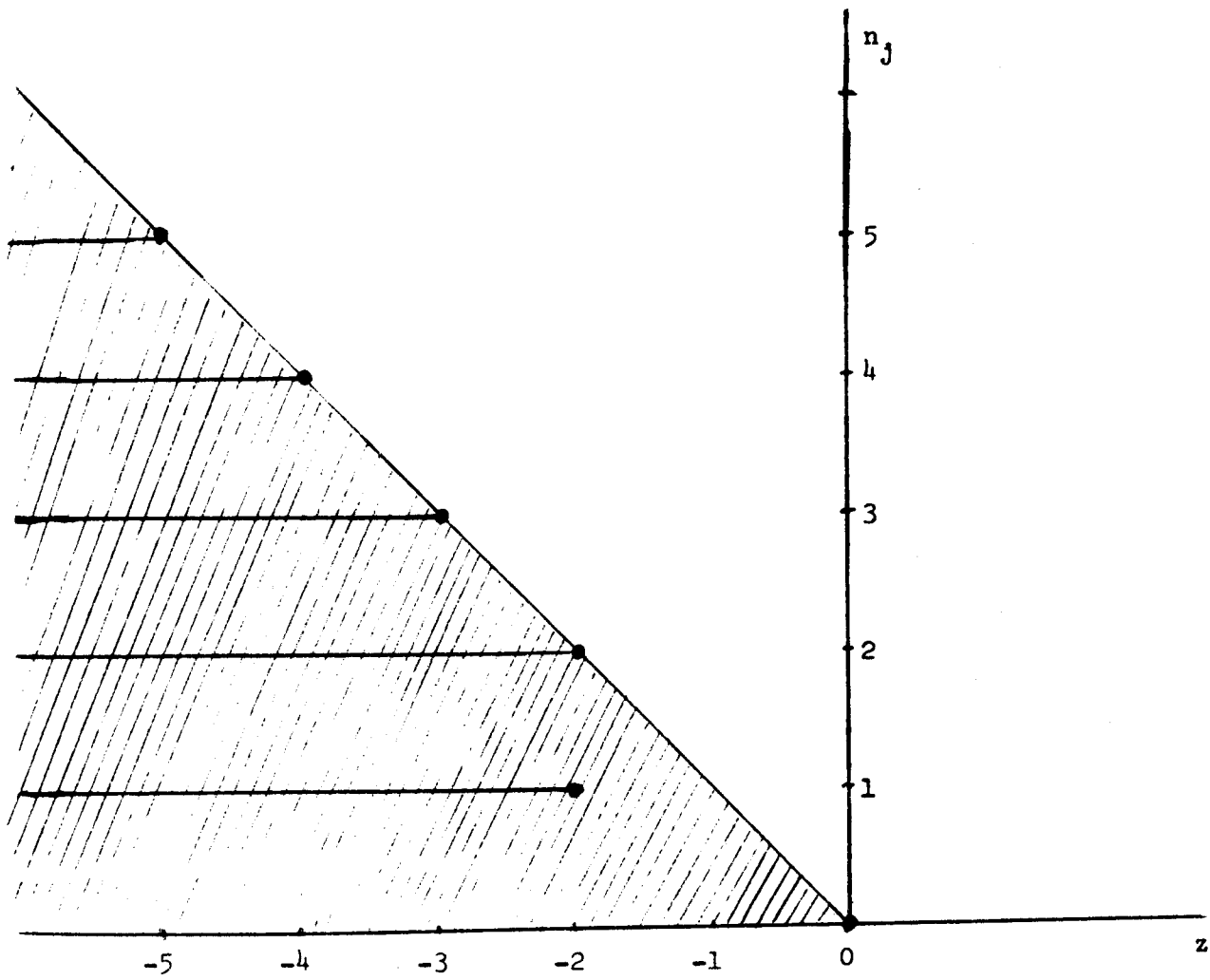


Figure 1: Production Technology Sets Y_j and Y_j^*

In attempting to prove existence of a competitive equilibrium for \mathcal{E} , we run into trouble because the consumption sets X and the production sets Y_j are non-convex. In the manner that has become standard in dealing with non-convexities, we resort to "convexification" of the economy. Beginning with the demand side of the model, suppose that we consider the convex hull $X^* := \text{co } X$ of the consumption set X and that we regard the demand correspondence as a mapping into X^* rather than X : i.e., $\phi: A \times \Delta \times X \rightarrow X^*$. The correspondence is well-defined since $X \subset X^*$. To employ the standard proofs of existence of a competitive equilibrium, we require that ϕ be non-empty, compact- and convex-valued and upper-hemi-continuous (u.h.c.) in p for every p such that $p_1 \gg 0$. However, when indivisible commodities are present, the demand correspondence ϕ can fail to be either convex-valued or u.h.c.

As a way around this problem Broome [3] adopts the approach, pioneered by Starr [30], of constructing a "convexified" version of the economy. In this construction the consumption sets X and the upper contour sets $\{\tilde{x} \in X \mid \tilde{x} \succ_a x\}$ are replaced by their convex hulls, and a synthetic preference relation \succ_a^* is defined for all $x \in X^*$. Working with synthetic preferences turns out to be rather awkward so we choose instead to follow the lead of Hildenbrand, Schmeidler and Zamir [16] in dealing directly with the demand correspondence.

As in Broome [3] we replace each consumption set X by its convex hull X^* . In order to appeal to the standard existence theorems for competitive equilibrium, we require not only that X^* be convex but that it be closed.

Lemma 1: X^* is closed.

Proof: $X^* := \text{co} X = \text{co} \bigcup_{g \in G} X(g)$. The $X(g)$ are non-empty, closed and convex sets in $\mathbb{R}^{\ell+m}$ by A.1 and all have the same recession cone \mathbb{R}_+^{ℓ} , so X^* is closed by Cor. 9.8.1 of Rockafellar [25].

Now consider the demand correspondence $\phi(a, p, \bar{x})$ as defined above but regarded

as a map into X^* , $\phi: A \times \Delta \times X \rightarrow X^*$. Define $\phi^*(a, p, \bar{x}) := \text{co}\phi(a, p, \bar{x})$. The correspondence $\phi^*: A \times \Delta \times X \rightarrow X^*$ is convex-valued by construction, and we have the following result:

Lemma 2: Assume $\phi: A \times \Delta \times X \rightarrow X^*$ is compact-valued and u.h.c. at $p \in \Delta$. Then $\phi^*: A \times \Delta \times X \rightarrow X^*$ is compact-valued and u.h.c. at p .

Proof: Immediate from Proposition AIII.4 of Hildenbrand and Kirman [15].

Provided that $p_1 \gg 0$, ϕ will be compact-valued and $\phi(a, p, \bar{x})$ non-empty since the budget set $\beta(p, \bar{x})$ will then be compact and \succ_a is assumed to be a complete continuous preorder (assumption A.1).⁹ So the main fact we need is that ϕ is u.h.c. at least for all p such that $p_1 \gg 0$. But ϕ can fail to be u.h.c. even when $p_1 \gg 0$ in the situation which Broome [3] refers to as "the problem of the edge." The difficulty that can arise is illustrated in Figure 2 for the case of one divisible and one indivisible commodity ($l = m = 1$). The consumption set is $X = X(g_0) \cup X(g_1) = \{(\xi, 0) \mid \xi \geq 0\} \cup \{(\xi, 1) \mid \xi \geq 0\}$. Let the initial endowment $\bar{x} = (1, 0)$, let $\tilde{x} = (0, 1)$ and assume $\tilde{x} \succ_a \bar{x}$. Consider a sequence of price vectors $p^n \rightarrow p = (\frac{1}{2}, \frac{1}{2})$ where $p_2^n > p_1^n$. Then $\phi(a, p^n, \bar{x}) = \{\bar{x}\}$, but $\phi(a, p, \bar{x}) = \{\tilde{x}\}$ so every subsequence $\{x_n\}$ with $x_n \in \phi(a, p^n, \bar{x})$ converges to $\{\bar{x}\} \neq \phi(a, p, \bar{x})$. Thus, ϕ is not u.h.c. at p . Even though the assumption that $p_1 \gg 0$ and $\bar{x}_1 \gg 0$ assures the existence of commodity bundles $x \in X$ with $p \cdot x < p \cdot \bar{x}$, this is not enough to guarantee that ϕ is u.h.c. at p when the consumption set is non-convex. In terms of the example, what is required is that there exist an x in $X(g_1)$ with the property $p \cdot x < p \cdot \bar{x}$; i.e., that $\tilde{x} \in \phi(a, p, \bar{x})$ not be on the edge of the consumption set. While Broome is forced simply to argue that the "problem of the edge" is unlikely to occur, in the present context

⁹Note that in the present context if the upper and lower contour sets for \succ_a are closed in X , they will also be closed in X^* .

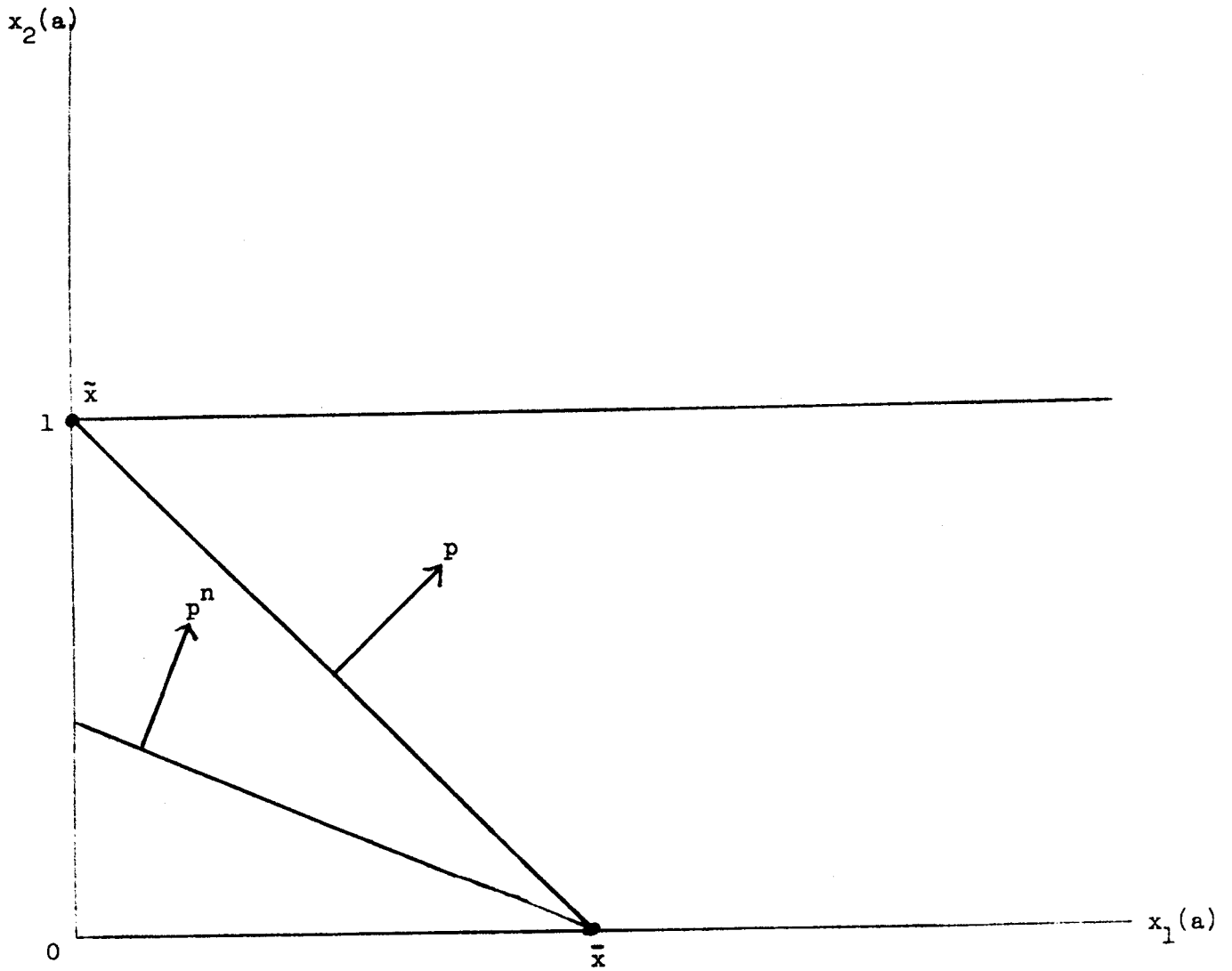


Figure 2: Failure of Demand Correspondence To Be Upperhemicontinuous

we can eliminate the problem by imposing an additional quite reasonable assumption: no consumer is willing to exchange his endowment for a commodity bundle consisting of indivisible goods alone.¹⁰

(A.5) For all $a \in A$ and for all $g \in G$, $\bar{x}(a) \succ_a (0, g)$.

It is then possible to assert the following:

Lemma 3: If \mathcal{E} satisfies A.1-A.3 and A.5, then $\phi(a, p, \bar{x})$ is non-empty, compact-valued and u.h.c. for all p with $p_1 \gg 0$.

Proof: We have already noted in passing how one proves that $\phi(a, p, \bar{x})$ is non-empty and compact-valued. The proof that ϕ is u.h.c. requires only a minor modification of the standard proof (cf. Hildenbrand and Kirman [15], pp. 151-152) which we leave to the reader. Combining Lemma 3 with Lemma 2, we then have:

Lemma 4: If \mathcal{E} satisfies A.1-A.3 and A.5, then $\phi^*(a, p, \bar{x})$ is non-empty, compact- and convex-valued and u.h.c. for all p with $p_1 \gg 0$.

To handle the supply side, we replace each Y_j by Y_j^* and define $\eta_j^*(p) := \{y_j \in Y_j^* | p y_j = \max p Y_j^*\}$. Since Y_j^* is by construction a closed convex cone owning 0, the convexified supply correspondence meets all of the requirements of the standard competitive model and the convexification $\mathcal{E}^* := (\{X^*, \succ_a, \bar{x}(a)\}, \{Y_j^*\})$ of \mathcal{E} then satisfies the assumptions of the existence theorem in Debreu [6, pp. 83-84].¹¹ Using Debreu's theorem we can immediately assert:

¹⁰ $\bar{x}(a) \succ_a (0, g_0)$ is already implied by the strong monotonicity of \succ_a in the divisible goods and the assumption that $\bar{x}_1(a) \gg 0$

¹¹Irreversibility for the total production set follows from our assumption that public goods are never inputs and private goods never outputs. Since no public goods are initially owned and public goods will be produced only if they can be sold for a non-negative profit, we need only assume free disposability for the private commodities.

Theorem 1: A CE (\mathcal{E}^*) exists for an economy \mathcal{E} satisfying assumptions A.1-A.5.

If $f^*(a) \in X$ for all $a \in A$ and $y_j^* \in Y_j$ for all $j \in M$, then a CE(\mathcal{E}^*) will be a competitive equilibrium for \mathcal{E} . But in general an exact result of this sort is too much to hope for, and the best we can achieve is some sort of approximation to equilibrium.

Def.: An allocation $\hat{f} : A \rightarrow X^*$, a set of production vectors $\{y_j\}$ and a price vector $p \in \Delta$ is an approximate competitive equilibrium for \mathcal{E} if:

- i) $\hat{p} \cdot \hat{f}(a) = \hat{p} \cdot \bar{x}(a)$ for all $a \in A$;
- ii) $\hat{p} \cdot \hat{y}_j \geq \hat{p} \cdot y_j$ for all $y_j \in Y_j$;
- iii) $\#\{a \in A \mid f(a) \notin \phi(a, \hat{p}, \bar{x}(a))\} + \#\{j \in M \mid \hat{y}_j \notin \eta_j(\hat{p})\} \leq l + m - 1$; and
- iv) $\sum_{a \in A} \hat{f}(a) - \sum_{j \in M} \hat{y}_j = \sum_{a \in A} \bar{x}(a)$.

Theorem 2: An approximate competitive equilibrium exists for an economy satisfying assumptions A.1-A.5.

Proof: Let $(\{f^*(a)\}, \{y_j^*\}, p^*)$ be a CE(\mathcal{E}^*) for \mathcal{E} and let $\hat{p} = p^*$. By definition of CE(\mathcal{E}^*), $f^*(a) \in \phi^*(a, \hat{p}, \bar{x}(a))$ for all $a \in A$, $y_j^* \in \eta_j^*(\hat{p})$ for all $j \in M$ and $\sum_{a \in A} f^*(a) - \sum_{j \in M} y_j^* = \sum_{a \in A} \bar{x}(a)$. It is easy to show that $\eta_j^*(p) = \text{con}_j(p)$ for all $j \in M$. The Shapley-Folkman theorem¹² then implies that there exist $\hat{f}(a) \in \phi^*(a, \hat{p}, \bar{x}(a))$ for all $a \in A$ and $\hat{y}_j \in \eta_j^*(\hat{p})$ for all $j \in M$ with $\sum_{a \in A} \hat{f}(a) - \sum_{j \in M} \hat{y}_j = \sum_{a \in A} \bar{x}(a)$ and $\#\{a \in A \mid f(a) \notin \phi(a, \hat{p}, \bar{x}(a))\} + \#\{j \in M \mid \hat{y}_j \notin \eta_j(\hat{p})\} \leq l + m - 1$. $\hat{f}(a) \in \phi^*(a, \hat{p}, \bar{x}(a))$ implies $\hat{p} \cdot \hat{f}(a) = \hat{p} \cdot \bar{x}(a)$ for all $a \in A$ (recall that \succ_a is strictly monotonic in the private commodities) and $\hat{y}_j \in \eta_j^*(\hat{p})$ implies $\hat{p} \cdot \hat{y}_j \geq \hat{p} \cdot y_j$ for all $y_j \in Y_j$.

¹²The original proof of the Shapley-Folkman theorem is in Starr [30]. For a short proof, see Keiding [18].

The important point to note is that the bound $l + m - 1$ does not depend on the number of consumers in the economy. If the size of the non-convexities in the consumption and technology sets is assumed uniformly bounded for all consumers and jurisdictions, then it is possible to show that an approximate equilibrium exists for \mathcal{E} in which all consumers maximize utility, all jurisdictions maximize profits and feasibility is violated by an amount that is bounded independent of the size of the economy (see Arrow and Hahn [1], Ch. 7 for a discussion of the general approach). We will not go into further detail since, at this stage, the main conclusion should be clear: economies with local public goods can be treated as economies with indivisible private goods, and the same techniques apply in either case to show that when non-convexities are small relative to the size of the market we can find an allocation that is very nearly a competitive equilibrium.

3. Interpretation of the Approximation Theorem

Theorem 2 provides a reasonable foundation for the theory of competitive equilibrium with local public goods, but it represents a substantial departure from the approach to this problem that has been adopted in the local public goods literature (Tiebout [31], McGuire [21], Hamilton [14]). In this literature:

- i) the proposed equilibrium is exact, not approximate;
- ii) there is no mention of fractional assignments; and
- iii) as a consequence of (ii), the sole reason why competitive equilibrium may fail to exist is the presence of unexhausted scale economies in the production of public goods.

To gain some insight into why approximation is essential to a rigorous formulation, suppose that $(f^* : A \rightarrow X^*, \{y_j^*\}, p^*)$ is a competitive equilibrium for the convexified economy \mathcal{E}^* . As noted earlier, if $f^*(a) \in X$ for all $a \in A$ and $y_j^* \in Y_j$ for all $j \in M$, then this equilibrium is also an exact (not approximate)

competitive equilibrium for \mathcal{E} . Clearly one reason why these conditions can fail to be met is the presence of unexhausted scale economies. However, even if scale economies are exhausted ($n_j \geq \tilde{n}_j$ for all $j \in M$ such that $y_j^* \neq 0$), equilibrium may only be approximate because the decision to consume a particular public good is an all-or-nothing choice.

Under what circumstances will a competitive equilibrium for \mathcal{E}^* or an approximate competitive equilibrium for \mathcal{E} assign to a consumer a consumption bundle $f(a) \in X^* \setminus X$? By appealing to Carathéodory's Theorem (Rockafellar [25], Theorem 17.1) we know that $f(a) \in \phi^*(a, p^*, x(a))$ implies that $f(a)$ is spanned by at most $\ell + m + 1$ vectors in $\phi(a, p^*, \bar{x}(a))$. But so far we have made no use of part (i) of assumption A.2, the convexity of upper contour sets of \succsim_a when restricted to $X(g)$ for each $g \in G$. By using this fact, we obtain the sharper result:

Theorem 3: If $f(a) \in \phi^*(a, p^*, \bar{x}(a))$, then there exist a set $\{\alpha_g \mid g \in G, \alpha_g \geq 0, \sum_{g \in G} \alpha_g = 1\}$ and for each $g \in G$ a vector¹³ $f(a, g) \in \phi(a, p^*, \bar{x}(a)) \cap X(g)$ such that $f(a) = \sum_{g \in G} \alpha_g f(a, g)$.

Proof: Immediate from Cor. 17.1.1 of Rockafellar [25].

Thus, $f(a)$ can be spanned by at most $m + 1$ vectors, each lying in the restriction of the unconvexified demand set $\phi(a, p^*, \bar{x}(a))$ to a different $X(g)$. This permits a very natural interpretation of the consumer's demand behavior in the convexified economy. If $f(a) = (x_1(a), x_2(a)) \in X^* \setminus X$, then $x_2(a) = \sum_{g \in G} \alpha_g g$ where $\alpha_g \geq 0$, $\sum_{g \in G} \alpha_g = 1$ and at least two $\alpha_g > 0$. Hence, the convexified allocation involves a fractional assignment where α_g is the fraction of consumer

¹³Of course the $f(a, g)$ are chosen only for those $g \in G$ for which $\phi(a, p^*, \bar{x}(a)) \cap X(g) \neq \emptyset$.

a assigned the public good type corresponding to $g \in G$. Though the fact that the fractions add to one is some solace, this is a pretty strange state of affairs. However, Theorem 3 implies that $f(a)$ is spanned by a set $\{f(a,g)\}$ where each $f(a,g)$ maximizes utility¹⁴ for a assuming that the consumer is constrained to choose $g \in G$. Each $f(a,g) \in \phi(a,p;\bar{x}(a))$, so each yields the same level of utility. Therefore, fractional assignments occur only when the consumer is indifferent between the commodity bundles he can obtain in two or more jurisdictions.¹⁵

This result clarifies the role that convexification plays with regard to demand correspondences: when consumers are at the point of switching from one type of public good to another, fractional assignments fill in the gaps that otherwise would occur. Theorem 2 implies that the proportion of consumers affected by this procedure will approach zero as the total number of consumers in the economy increases.¹⁶ We conclude, therefore, that provided economies to scale are exhausted, the approximate equilibrium we have described is a reasonable notion of competitive equilibrium with local public goods.

We have come a fair distance in attempting to reconcile our results with existing formulations of Tiebout equilibrium. Fractional assignments should

¹⁴We use the expression "utility" for economy of expression; the assumptions we have introduced guarantee existence of a continuous utility function representing \succsim_a .

¹⁵To be precise we also have to note the possibility that he is indifferent between choosing to consume a public good or consuming none at all.

¹⁶Since the number of consumers who receive fractional assignments is finite (bounded by $l + m - 1$, independent of the total number of consumers), the set of such consumers will have measure zero when A is a non-atomic measure space. Within a pure exchange context a competitive equilibrium will then exist that is exact, not approximate (Mas-Colell [20]).

be viewed as simply a technical device, the price that must be paid if Tiebout models are to be given a rigorous foundation. Unexhausted scale economies appear as the major barrier to the realization of a competitive equilibrium with local public goods. The main issue that remains is the following: Under what circumstances will economies to scale be exhausted? Unfortunately, Theorem 2 does not provide a fully satisfactory answer to this question. Our result does not rule out the possibility that in approximate equilibrium some producer of public goods will be operating at a scale that does not exhaust scale economies regardless of the size of the economy: some consumers may find that, no matter how many consumers are added to the economy, no one comes along who shares their particular taste in public goods.

Suppose, however, that we increase the size of the economy through replication. Changing notation, let T index the set of consumers in the original economy \mathcal{E}_1 . In the r th replication of the economy, \mathcal{E}_r , the set of consumers is indexed by $A := \bigcup_{t \in T} A_t$ where $A_t \cap A_{t'} = \emptyset$ all $t, t' \in T$; $\#A_t = r$ for all $t \in T$; and each $a \in A_t$ has the same preferences ordering \succsim_t and endowment $\bar{x}(t)$ as the consumer $t \in T$. Thus, the economy \mathcal{E}_r contains r consumers of each type $t \in T$.

Theorem 4: Assume that $\mathcal{E}_1 = (\{X, \succsim_t, \bar{x}(t)\}, \{Y_j\})$ satisfies assumptions A.1-A.5. Let $(f_r^* : A \rightarrow X^*, \{y_{jr}^*\}, p^*)$ be a competitive equilibrium for the convexified economy \mathcal{E}_r^* where $y_{jr}^* = (z^*, n_{jr}^* g_j)$ for all $j \in M$. Then:

i) There exists $r \in \mathbb{Z}$, $0 < r < \infty$, such that either $y_{jr}^* = 0$ or $n_{jr}^* \geq \bar{n}_j$ for all $j \in M$ (n_{jr}^* need not be an integer);

ii) Furthermore, there exists an approximate competitive equilibrium $(\hat{f}_r : A \rightarrow X, \{\hat{y}_{jr}\}, p^*)$ for \mathcal{E}_r where $\hat{y}_{jr} = \{\hat{z}, \hat{n}_{jr} g_j\}$ such that either $\hat{y}_{jr} = 0$ or $\hat{n}_{jr} \geq \bar{n}_j$ for all $j \in M$ (\hat{n}_{jr} need not be an integer).

Proof: By Theorem 1, a competitive equilibrium exists for \mathcal{E}_1^* . Let $(f_1^* : T \rightarrow X^*, \{y_j^*\}, p^*)$ represent this equilibrium. Define f_r^* by the equation:

$f_r^*(a) = f_1^*(t)$ for all $a \in A_t$. Then it is easy to show that $(f_r^* : A \rightarrow X^*, \{ry_j^*\}, p^*)$ is a competitive equilibrium for \mathcal{E}_r^* . Define $y_{jr}^* = ry_j^*$. Let $M' := \{j \in M \mid y_j^* \neq 0\}$ and set $r' = \min \{r \mid rn_j^* \geq \tilde{n}_j, j \in M'\}$. This r' is the r required for part (i) of the theorem (we define $n_{jr}^* = rn_j^*$). Part (ii) follows easily as a consequence of Theorem 2.

Theorem 4 states that provided the economy undergoes sufficient replication, an approximate competitive equilibrium exists in which scale economies are exhausted for all public goods that are produced. Cast in this form, our results are very close in spirit to those of McGuire [21] and Hamilton [14]. The only difference is that the equilibrium they describe is exact rather than approximate. Theorem 3 implies that fractional assignments occur only when a consumer is indifferent between the consumption bundles $(x_{1,g})$ he can obtain for two or more $g \in G$, and this may seem unlikely. Suppose we adopt the additional assumption:

(U): In a competitive equilibrium for \mathcal{E}_1^* we have for every $t \in T$ that $\phi(t, p^*, \bar{x}(t)) \cap X(g) \neq \emptyset$ for exactly one $g \in G$.

Lemma 3 guarantees that $\phi(t, p^*, \bar{x}(t)) \cap X(g) \neq \emptyset$ for at least one $g \in G$, so the effect of assumption U is to rule out the possibility that more than one type of public good is in the demand set at equilibrium.¹⁷

Theorem 5: Assume that \mathcal{E}_1 satisfies A.1-A.5 and U. Then there exists $r \in \mathbb{Z}$, $0 < r < \infty$, such that $(f_r^* : A \rightarrow X, \{ry_j^*\}, p^*)$ is a competitive equilibrium for \mathcal{E}_r .

Proof: Let $(f_1^* : T \rightarrow X^*, \{y_j^*\}, p^*)$ be a competitive equilibrium for \mathcal{E}_1^* . Assumption U implies $f_1^*(t) \in X$ for all $t \in T$. By definition, $y_j^* = (z^*, n_j^* \mathcal{E}_j)$

¹⁷We use U (for "uniqueness" of the choice of public good) to denote this assumption in order to highlight its special status relative to A.1-A.5: it cannot be verified without knowing the equilibrium price vector p^* .

and the fact that $f_1^*(t) \in X$ for all $t \in T$ implies $n_j^* \in Z_+$ for all $j \in M$. Now define $f_r^* : A \rightarrow X$ as in Theorem 4. By Theorem 4 there exists an integer r , $0 < r < \infty$, such that $(f_r^* : A \rightarrow X^*, \{ry_j^*\}, p^*)$ is a competitive equilibrium for \mathcal{E}_r^* . $rn_j \in Z_+$ so $ry_j^* \in Y_j$ for all $j \in M$; $f_r^*(a) \in X$ for all $a \in A$. Therefore, this equilibrium is a competitive equilibrium for \mathcal{E}_r .

This theorem cannot be regarded as a proper existence proof because one of its assumptions (U) can only be verified once the equilibrium price vector has been determined, but it does serve to complete the reconciliation of the approach we have adopted and that of McGuire [21] and Hamilton [14]. When assumption U is satisfied a competitive equilibrium exists that is exact. Consumers of the same type select the same type of public good, resulting in the stratification of households into homogeneous jurisdictions. The approximate competitive equilibrium appearing in Theorem 2 or Theorem 4 provides the means for translating these familiar results into rigorous terms.

4. Conclusion

In this paper I have proposed a model of local public goods that justifies the conclusions of the Tiebout model in the following sense: if the production of public goods eventually exhibits constant (or diminishing¹⁸) returns to scale, then an approximate competitive equilibrium exists. When economies of scale are exhausted, this equilibrium can be regarded as a good approximation to an exact competitive equilibrium. To obtain these results we have abandoned the conventional definition of a public good, regarding local public goods instead as a type of indivisible private good.

Although our emphasis has been on the Tiebout model, it is important to realize that the approach we have taken is no less relevant to the general theory

¹⁸Diminishing returns to scale can easily be added to the model: all that needs to be appended is a rule for distributing equilibrium profits to consumers.

of public goods. Much of the interest in the notion of a local public good stems from the belief that it could serve to bridge what Buchanan [4] has called the "awesome Samuelson gap" between purely private and purely public goods. The formal description of public goods we have described fulfills this promise. Public goods, whether local or purely public, are equivalent to indivisible private goods produced subject to (at least initial) increasing returns to scale. Pure public goods are simply the special case where all production costs are fixed.¹⁹

Of course, we have provided only a framework for the general analysis of public goods, not a complete theory. The only satisfactory solution concept comes in the extreme case where scale economies are exhausted: competitive equilibrium. Even when we focus only on the question of optimality, a rigorous statement of the conditions for an optimum is available only for the Tiebout model and the standard case of pure public good.²⁰ However, the framework we have provided does establish the formal equivalence of the problem of which public goods to provide and the problem of optimal product differentiation (see, e.g., Lancaster [19]). Solutions to one will carry over directly to the other.

¹⁹There is one important exception to the general proposition that public goods and indivisible private goods are equivalent: exclusion must be possible. If exclusion is impossible, then a public good cannot be provided by a private agent (unless it is given the power to tax or consumers are willing to contribute voluntarily). This exception presumably accounts for the emphasis which laissez faire economists place on the issue of exclusion, some going so far as to claim that only goods for which exclusion is impossible should be called public.

²⁰Observe that the quantity entering consumers' utility functions in Samuelson's model is no longer regarded as a measure of output in our model, but rather as an index of quality. Output is measured in terms of the number of consumers provided a public good of a given type (quality). Samuelson's condition, $\Sigma MRS = MRT$, is now interpreted as a necessary condition determining optimal public good quality in the case of a pure public good.

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