A MODEL OF RATIONAL NON-COMPETITIVE INTERDEPENDENCE

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ABSTRACT

The traditional approach to problems of monopoly interaction, or non-competitive interdependence, has been to make assumptions inconsistent with rationality or perfect information. An example is the Cournot oligopoly model, and its modern counterpart, the Nash model of n-person, non-cooperative games, where producers assume zero reactions of their rivals to changes in their strategies even though the actual reactions are not zero. The present paper constructs a general equilibrium model of monopoly which is consistent with rationality and perfect information. In our model, between any pair of interacting monopolists, one monopolist exhibits a prior reaction function while the other simply picks a point on the function. For both unconstrained interaction and interaction constrained by anti-monopoly laws, we derive the rational reaction functions, and characterize the resulting solutions. Results of preliminary tests of each of these models are very encouraging.

INTRODUCTION[1]

This paper deals with the problem of deriving equilibrium quantities within a general equilibrium system containing several monopolists, producers who can affect relative prices.

Since in a general equilibrium setting all relative prices are a function of all outputs, a monopolist's output decision will affect all the relative prices in the system. Hence, an individual monopolist's output change will, in general, render the remaining producers' previous outputs non-optimal with respect to the new set of prices generated by the monopolist's new output. The other producers, in recalculating their respective optimal outputs, generate a set of output reactions and again change relative prices. A rational monopolist who is aware of the relative price effects imposed on him and by the other producers will take the other producers' reactions into account when calculating his profit-maximizing output. Similarly, other monopolistic producers, realizing that their reactions are being taken into account, will determine their reactions accordingly.

The task of this paper is to describe the rational form of the reaction functions and characterize the resulting equilibria.

Our model assumes perfect information regarding reaction functions and the absence of side payments. The traditional models of economic conflict attempting to describe such a world breakdown under the assumption of rationality

1/ The authors benefited substantially from the comments on earlier drafts of this paper by Louis Makowski, Ron Heiner, Harold Demsetz, and a referee of this journal.

and perfect knowledge of reaction functions. In the standard duopoly model of Cournot, each duopolist makes output decisions based on faulty information concerning the reaction behavior of his rival. The zero output reactions assumed in this model do not describe the true reactions that occur within the model. In the duopoly model of Stackelberg we will see that the selected reaction functions are generally irrational when there is perfect information concerning output reactions.

In Section I, we specify our general model and show that a necessary condition for the existence of solutions to all monopoly interaction problems featuring rational strategy selection and perfect information regarding reactions is that between any two, interacting monopolists, one and only one of them exhibits a reaction function. In the case of m' ($m' \geq 3$) interacting firms, a solution implies a recursive set of reaction functions for $m'-1$ firms. The solutions to this interaction have a decidedly predatory, robber-barron characteristic.

In Section II, we introduce government anti-trust policy. We show that such policy changes the form of the rational reaction functions and alters the characteristics of the non-competitive equilibrium. We construct an equilibrium in a linear special case and derive its equilibrium size distribution of firms, concentration ratios, and mark-ups. Firms are each twice as large as the next smallest firm. The concentration ratio of an industry, the market share of the top t firms in an industry, asymptotically approaches $(2^t-1)/2^t$ as the number of firms grows. Further, this asymptote is approximated with an error of less than $\frac{1}{2}$ of 1% when the number of firms is at least eight. Finally, the equilibrium mark-up with at least eight firms is shown to be less than .1% of the pure monopoly mark-up.

In Section III, we note the rough empirical accuracy of our theories of monopoly for the U.S. experience before and after federal anti-monopoly laws. The central empirical result of this analysis is that U.S. anti-monopoly laws, by altering the form of rational functions, have converted a world dominated by highly inefficient monopolies into a world whose outputs are typically very close to purely competitive levels.

The analysis of the text of this paper takes as historical datum the identity of the single firm among any subset of interacting firms which is able to establish a prior reaction function over the other firms in the subset. In the appendix, we introduce a competitive process to determine the identity of this firm rather than taking it as historical datum. We then show that the results of our analyses are not substantially altered by this change in the model as long as the number of firms is not very small.

I. THE GENERAL MODEL

A. The Environment

The model we will employ is a private-property general equilibrium model containing n commodities and m firms. The economy's output allocation set is denoted

$$x = (x_1, x_2, \ldots, x_m) \ , \ x_f \geq 0, \text{ for all } f = 1, \ldots, m, \tag{1}$$

where x_f is the f^{th} firm's n-dimensional output vector. $x_f \in X_f$, the $f^{\underline{th}}$ firm's feasible output set. We will denote the set of output vectors of all firms except firm f as

$$x_{-f} = (x_1, x_2, \ldots, x_{f-1}, x_{f+1}, \ldots, x_m). \tag{2}$$

Each firm, f, can produce all commodities and has a profit function,

$$\pi_f(x) = \pi_f(x_f; x_{-f}), \quad f = 1, \ldots, m, \tag{3}$$

which summarizes firm f's technology, factor costs, and output demand conditions. Of course, $\pi_f(0; x_{-f}) = 0$. A firm may employ factors specific to that firm so that the profit functions of the various firms are not generally identical. Interfirm payments, or "side payments", are not allowed.

B. Firm Interaction Under Truly Perfect Information

A firm's strategy may be an action, which simply states what output the firm produces, or a reaction which describes the outputs a firm will produce as a function, perhaps an insensitive function, of outputs of other firms. We follow the familiar characterization of "perfect information" in game theory to the extent that our players (firms) select their strategies in succession. This is necessary for the firms to be able to know one another's strategies. But, unlike standard, "perfect information," games which would confine prior strategy selectors in our environment to choices among simple actions, we grant to each prior strategy selector the ability to form and communicate any feasible reaction strategy to all of the subsequent strategy selectors, distinguishing our information assumption by labelling it "truly perfect information".[2/] Our first strategy selector thus announces a reaction strategy to the other m-1 firms. In view of this, the second strategy selector then announces his reaction strategy to the m-2 remaining firms and so on down to the $m^{\underline{th}}$ strategy selector, who simply chooses an action. This action determines the output of the $m-1^{st}$ strategy selector which, together with the output of the $m^{\underline{th}}$ strategy selector, determines the output of the $m-2^{nd}$ strategy selector, and so on until the output of the first strategy selector is determined. Hence, firm interaction under truly perfect information can be characterized by a hierarchy of strategy selectors, with firm 1, the first strategy selector, exhibiting a reaction function,

$$x_1 = x_1(x_2,x_3,\ldots,x_m),$$

while firm 2, the second strategy selector, exhibits the reaction function,

$$x_2 = x_2(x_3,\ldots,x_m),$$

2/ While a general game theory containing unrestricted reaction strategies (as well as an absence of restrictions on more complex contingent strategies) has been produced by Howard (1), he applies conventional Nash solutions to the problem. Since such solutions do not allow players to acknowledge the effect of their strategies on the strategies of subsequent strategy selectors, the solutions are inconsistent with perfect information regarding the strategies of others or rationality. For a general game theory with perfect information regarding the strategies of others, unlimited contingent strategy formation, and rational solutions, see Thompson and Faith (8). In this latter game theory, it is shown that admitting contingent strategies other than reaction strategies is redundant. This result motivates our assumption above that the only strategies are actions or reactions.

and the third firm exhibits

$$x_3 = x_3(x_4,\ldots,x_m),$$

and so on up to the $m\text{-}1^{\underline{st}}$ firm's simple reaction function, with a solution output set easily constructed by having the $m^{\underline{th}}$ firm, viewing these reaction functions, selecting an output. This output determines the output of the $m\text{-}1^{\underline{st}}$ firm given its established reaction function, which gives the two outputs necessary for the $m\text{-}2^{\underline{nd}}$ firm to determine its output and so on until the first strategy selector's output is determined.

The $m^{\underline{th}}$ firm's output is assumed to maximize its profit in view of the various solutions which would result from his various output selections. But before we can derive the profit maximizing choice, or the reaction functions, of the other firms, we must specify the cost of becoming a prior strategy selector. We must also determine the order of priority in strategy selection. One possibility is to arbitrarily assign an order of strategy selection (corresponding, say, to the order of birth of the various firms.) A solution is then obtained by first allowing the $m\text{-}1^{\underline{st}}$ firm to select a reaction function that maximizes its profit given the m-2 prior reaction functions and in view of the various rational output choices of the $m^{\underline{th}}$ firm for the various possible reaction functions of the $m\text{-}1^{\underline{st}}$ firm. Then the $m\text{-}2^{\underline{nd}}$ firm selects a reaction function that maximizes its profit given the prior m-3 reaction functions and the various rational outputs of firms m-1 and m which result from its various reaction functions. The process continues until the m-1 rational reaction functions have been formed. Another possibility is to have the firms engage in costly competition to determine the order of strategy selection. A model of such competition is specified in the Appendix, where we demonstrate the existence of solutions to this new kind of competitive process, and compare the solutions resulting from these two possible methods of assigning priority of strategy selection. The chief result is that there is no substantial difference between the solutions resulting from the alternative assignment processes when the number of interacting monopolists is not very small. This result will hold both in the case of unconstrained interaction and when an anti-trust law restricts the set of feasible reaction functions.

So, for the text below, we assume that there is a costless assignment of hierarchial position determined, say, by the historical sequence of entry into the economy. To characterize the resulting solutions, it is instructive to consider first the case of duopoly. Here, the 2nd (now the mth) firm chooses a set of outputs, x_2^*, so as to maximize $\pi_2(x_2; x_1(x_2))$. The resulting solution determines a dependency of x_2^* on $x_1(x_2)$, a dependency which we write as

$$x_2^* = x_2^*\{x_1(x_2)\}. \tag{4}$$

This dependency is not a reaction function; it merely shows how a subsequently selected strategy depends upon a prior strategy. In view of firm 2's rational response given by (4), firm 1 chooses a reaction function $x_1^*(x_2)$ which maximizes $\pi_1(x_1(x_2); x_2^*\{x_1(x_2)\})$.

To guarantee the existence of such solutions, we can add that X_f is finite and $\Pi_f(\cdot)$ is real-valued. Sufficient conditions for the existence of a solution when X_f is infinite are given in the Appendix.

The above duopoly solution concept is equivalent in in game theoretic structure to a perfect bargaining solution concept of Schelling (Ch. 5). It is a generalization of the Stackelberg solution concept (Intriligator) in that the 1st firm presents the 2nd with a rationally selected reaction function rather than a necessarily insensitive function, one which establishes a fixed output for firm 1.

An extremely simple characterization of our solution is achieved by adding the following condition: For any positive $x_2 \in X_2$, there exists an

$x^p(x_2) \in X_1$ such that $\pi_2(x_2;\ x_1^p(x_2)) < 0$. We call such an $x_1^p(x_2)$ a set of "punishment outputs". Then, letting x^1 be the economy's output allocation which maximizes firm 1's profit over all feasible x subject to $\pi_2(x_2;\ x_1) \geq 0$, a rational reaction function of firm 1 is given by

$$x_1^*(x_2^1) = x_1^1$$

$$x_1^*(x_2) = x_1^p \quad \text{for} \quad x_2 \neq x_2^1. \tag{5}$$

Facing this, firm 2 chooses x_2^1, thus yielding x^1 as a solution. It is useful to consider the case of a Cournot terminology, wherein $\pi_f = x_f[a-b(x_1+x_2)-c]$, $(a,b,c) > 0$, $a > c$, and $x \in R_+^2$. The condition on the existence of a set of punishment outputs obviously holds in such an environment as firm 1 may, for any output of firm 2, merely set $x_1 = \frac{a-c}{b}$ in order to make $\pi_1(x_2,x_1) < 0$ for all $x_1 > 0$. As the industry's simple monopoly output, $\frac{a-c}{2b}$, for firm 1 and a zero output for firm 2 is obviously the most profitable allocation to firm 1, its reaction function is given by

$$x_1^*(0) = \frac{a-c}{2b}$$

$$x_1^*(x_2) = \frac{a-c}{b} \quad \text{if } x_2 > 0.$$

The reactions described above are extremely predatory. Such a high degree of predatory behavior does not appear to characterize the typical real world relationship between interacting firms. This may be due to the failure of the extreme assumption of truly perfect information to guide real-world relationships. The problem we have in accepting this conclusion is that we have no hint of a theory of realistically imperfect information. An alternative interpretation of the apparent empirical infrequency of extremely predatory reaction functions is that such reactions are forbidden by common law and, in several countries, by statutory law. The following sections of this paper pursue this alternative.

II. THE EFFECT OF AN ANTI-MONOPOLY POLICY

The results obtained in Section I are based on a decentralized model of monopoly decision-making without government intervention. However, in order to arrive at a more realistic calculation of monopoly behavior, we now introduce government participation in the economy in the form of federal anti-monopoly policy. Since most anti-monopoly policy ignores interactions between firms in different industries, we shall assume that firms in different industries are noninteracting and thus consider only a single m-firm industry.

A. The Revised Reaction Functions

On the basis of existing anti-monopoly laws, it is reasonable to assume that if any firm expands its output in reaction to increases in the outputs of its competitors -- either existing or entering firms -- that firm would be subject to prosecution under the law for its "predatory practices."[3/] Thus, whenever a firm increases its output for a given level of industry demand and cost we shall assume that government policy prohibits another firm from increasing its output. (We are assuming that detection of violators and enforcement of the law are carried out at zero cost.) This restriction upon the output reactions of firms precludes the use of punishment strategies.

In the absence of punishment strategies, a firm may try to induce the production of some desired industry output by "rewarding" other firms

3/ This interpretation assumes that the courts can distinguish between increases in output due to efficiency reasons, and increases for predatory or punishment reasons. Certain anti-trust cases, such as the U.S. Steel Case, 1921, lend support to this assumption. The anti-trust policy in this paper is based on past interpretations of the Sherman Anti-Trust Act. For further discussion of this law and its judicial interpretations, see Neale (4).

for their outputs; that is, decreasing its output if the other firm(s) decrease its (their) output(s) up to the desired industry output. We shall assume that such behavior will be viewed as collusion by the government policy-makers, and likewise be prohibited. Therefore, the effect of our anti-monopoly policy on a firm's choice of reaction function is to limit these choices to a class of non-increasing functions.

Thus, firm i faces only two kinds of alternatives given a change in the output choice of the firm i+k, where k=1,...,m-i. i may exhibit a zero or a negative reaction to i+k's change in output. Comparing the results of these two alternatives, firm i rationally decides upon the zero reaction function.

Our demonstration is as follows: First consider a reaction function for the first strategy selector; that is, first let i=1. Then, if i+k increases his output, say from x^o_{i+k} to x'_{i+k}, and firms i+1,...,i+k-1, i+k+1,...,m do not contract as much as i+k expands so that there is a net expansion in industry output, or $\Sigma^{m-i}_{k=1}(x'_{i+k}-x^o_{i+k}) > 0$, the result of i's contracting his output would clearly be to encourage expansion of the output of i+k and thus the aggregate output of his competitors. Inducing an increased output by his competitors could only benefit i if it ultimately permitted i a higher solution output. But x_i for any $(x_{i+1},\dots,x_m)$ may be increased independently of i's reactions to the output changes of these firms! i need only change his reaction function from $x_i(x_{i+1},\dots,x_m)$ to $x_i(x_{i+1},\dots,x_m) + \delta$. So i can alter his solution output independently of his responses to the output changes of later strategy selectors. Similarly, if i+k reduces his output and the induced output changes by the rest of the firms still results in a net decrease in industry output, i will not rationally increase his output and thereby discourage i+k from decreasing his output in the first place. Firm i could produce the desired higher output by simply committing himself to produce the desired output as a constant regardless of i+k's reaction and thereby induce i+k to produce a lower output than he would if i presented i+k with a negative reaction. Finally, if, when i+k increases his output, the rest of the firms other than i decreased their outputs so that the net output of the industry was reduced, i+k would always expand output until this was no longer the case. So there would be no equilibrium under this final possibility. Since firm 2 now faces a constant output reaction function from firm 1, we can repeat the above argument for i=2, etc. Thus, in equilibrium, each firm's rational reaction function is a constant output reaction function.

B. Equilibrium

Using this result we now illustrate equilibrium with m firms producing a homogeneous output in which each firm's hierarchical position is exogenously given. This amounts to a generalization of the Stackelberg duopoly model.[4/] We again assume a fixed hierarchy of strategy selectors, or "strategy makers," with firm 1 being the "primary maker," firm 2 the "secondary maker," and firm m the "pure taker."[5/] Also, we again adopt a Cournot technology. Thus, industry demand is assumed to be linear and of the form:

$$p = a-b\Sigma_{i=1}^{m} x_i, \tag{6}$$

where p is the price of the industry's output and, again, a and b are positive constants. And marginal costs, c, are assumed to be constant and identical for each firm so that firm f's profits can be expressed as

$$\pi_f = p(a-b\Sigma_{i=1}^{m} x_i-c), \tag{7}$$

where, to assure positive outputs, $c < a$. The condition for profit maximization for each firm is

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[4/] In the Stackelberg model of duopoly (see Intriligator (2)), one firm, called the "follower," assumes the leader will exhibit a constant output and makes his rational output choice on this assumption. The other firm, "the leader," then selects his output subject to the follower's rational response function. The result is a Stackelberg equilibrium. In our controlled monopoly mo el, the second strategy selector acts as a Stackelberg follower by choosing his output subject to constant-output reaction functions. The first strategy selector behaves as a Stackelberg leader since he chooses his reaction function (output) subject only to the profit-maximizing behavior of the other firms. Firms 2,...,m-1 introduce into the model additional relationships not described in previous models of which these authors are aware. Nevertheless, our controlled monopoly model with its (constrained) reaction functions and added relationships generates what can be interpreted as a generalized Stackelberg model. For, as derived above, adding more firms to our model merely creates a hierarchy of partial Stackelberg leaders.

[5/] As Stackelberg's "leader-follower" terminology suggests certain pricing relations that are not relevant to a single-industry model, we avoid it here.

$$p - x_f b \sum_{i=1}^{m} \frac{dx_i}{dx_f} - c = 0 \text{ , or} \tag{8}$$

$$x_f = \frac{a-c-b\sum\limits_{i\neq f} x_i}{b\left(2+\sum\limits_{i\neq f}\frac{dx_i}{dx_f}\right)} = \frac{p-c}{b\left(1+\sum\limits_{i\neq f}\frac{dx_i}{dx_f}\right)} \tag{9}$$

Since all reaction functions are constant output reaction functions, for each firm j,

$$\frac{dx_i}{dx_j} = 0 \quad \text{for all } i < j. \tag{10}$$

This yields a profit-maximizing expression for m of:

$$x_m^* = \frac{a-c-b\sum\limits_{i=1}^{m-1} x_i}{2b} = \frac{p-c}{b} \quad . \tag{11}$$

Since firm m-1 is a taker with respect to 1,..., m-2, and the latter exhibit constant output reaction functions,

$$\sum_{i=1}^{m-2} \frac{dx_i}{dx_{m-1}} = 0 \ .$$

And from (11), we know $\frac{dx_m^*}{dx_{m-1}} = -\frac{1}{2}$. Thus, using (9) and (10).

$$x_{m-1}^* = \frac{a-c-b \sum\limits_{i=1}^{m-2} x_i}{2b} = \frac{p-c}{b/2} \tag{12}$$

To obtain m-2's profit-maximizing condition, we have to calculate m-1's and m's rational responses to a change in x_{m-2}. From (1) we know that

$$\frac{dx_{m-1}^*}{dx_{m-2}} = -\ 1/2 \ . \tag{13}$$

And from (10),

$$\frac{dx^*_m}{dx_{m-2}} = \frac{\partial x^*_m}{\partial x_{m-2}} + \frac{\partial x^*_m}{\partial x_{m-1}} \cdot \frac{dx^*_{m-1}}{dx_{m-2}} = -1/4 . \tag{14}$$

Hence, again using (9),

$$x^*_{m-2} = \frac{a-c-b\sum_{i=1}^{m-3} x_i}{2b} = \frac{p-c}{b/4} . \tag{15}$$

Similarly, we find that

$$x^*_{m-3} = \frac{a-c-b\sum_{i=1}^{m-4} x_i}{2b} = \frac{p-c}{b/8} , \tag{16}$$

$$x^*_{m-4} = \frac{a-c-b\sum_{i=1}^{m-5} x_i}{2b} = \frac{p-c}{b/16} , \tag{17}$$

$$\vdots \qquad \vdots$$

The resulting size distribution of firms is obviously

$$x^*_{m-i} = x^*_m 2^i , \ i = 0,1,\ldots,m-1 . \tag{18}$$

C. Corollaries

There is a corollary concerning the "concentration ratio" of our industries. It is that the t-firm concentration ratio, the share of the top t firms in the industry, decreases as the number of firms in the industry increases. From the above theorem, the total output of the top t firms in the industry can be written:

$$K\sum_{i=m-t}^{m-1} 2^i = K2^{m-1-t}\sum_{i=1}^{t} 2^i = K2^{m-t}(2^t-1), \tag{19}$$

where $t \leq m-1$ and K is some positive number. The total output of the m firms is

$$K\sum_{i=0}^{m-1} 2^i = K(2^m-1) \tag{20}$$

Hence, the output share of the top t firms in the industry, $t \leq m - 1$, is given by:

$$S_t = \frac{2^{m-t}(2^t-1)}{2^m - 1} = \frac{2^t - 1}{2^t - \frac{1}{2^{m-t}}} \tag{21}$$

Thus we see that as the number of firms in the industry expands and thus the output becomes more competitive, the concentration ratio, S_t, for any t ($t \leq m-1$) decreases.

This decrease, however, is very slight once the number of firms in the industry becomes at all significant. For example, if $m \geq 8$, then the percentage error in using $\frac{2^t - 1}{2^t}$ as an estimate of S_t is always less than one half of one percent.[6/]

A second corollary regards the proximity of our generalized Stackelberg equilibrium to a competitive equilibrium. From (17), the equilibrium mark-up in our model is $\frac{a-c}{2^m}$ Under pure monopoly, the mark-up would be $(a-c)/2$. (This is the same as the uncontrolled monopoly mark-up since the rational maker in this industry model could not do better than he could by producing an output such that $p < c$ whenever any other firm produced a positive output.) Given the distribution of output among firms in our controlled monopoly solution, the equilibrium mark-up relative to the pure monopoly mark-up is therefore given by

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6/ Using the above analysis and the results of Part II of the Appendix, it can be shown that this same asymptote is approached, and the same approximation result holds, when competitive bidding for hierarchial position is allowed. The only difference which arises when such competition is allowed is that the concentration ratio increases, rather than decreases, to its asymptotic level as $m \to \infty$.

$$\frac{\frac{a-c}{2^m}}{\frac{a-c}{2}} = \frac{1}{2^{m-1}} \tag{22}$$

So with, say, 8 firms in the industry, the equilibrium mark-up is less than 1% of the pure monopoly mark-up. This seems safe enough to ignore for policy purposes, especially since some positive mark-up is necessary to guarantee the presence of producers in an industry. The result speaks for the powerful efficiency of the simple anti-trust policy outlined above.

IV. PRELIMINARY EVIDENCE

The rational reaction function-perfect information approach to non-competitive interdependence can be tested by attempting to verify empirically the implications of the above two models. We use the U. S. experience since we are somewhat less ignorant of it than of the experiences of other countries.

According to most accounts, no substantial monopolies other than government-granted and small, local monopolies appeared before the Civil War. After that war, the communications-transportation revolution and the emergence of the corporate form of organization apparently opened up new opportunities for large scale organizations and thus private monopolies operating in nation-wide markets. In this environment, industrial giants grew in several industries, each coming to dominate his industry by using unprofitable price-cutting as a weapon against smaller firms in order to keep them "in line." These "robber barons" were, in our terms, simply rational makers over a set of takers and their "cutthroat competition" was merely their application of punishment outputs to deviant firms.

The development of anti-trust policy in response to the obvious inefficiencies in this system took several decades and has operated, as we have suggested, to remove collusion (as well as mergers with the purpose of raising prices) and cutthroat competition. This policy implies, as we have pointed out, a hierarchy of makers in which each of the makers presents the industry with a fixed output for a given level of market demand and industry costs. Empirically, this means that the larger firms in an industry can be expected to commit themselves to an announced share of the market and retain his output regardless of the peculiar economics of individual firms. That large firms in the U. S. determine their outputs in this way rather than computing their own demand and supply curves is obvious for certain firms and has been claimed as a fairly general description by numerous institutionalist authors.

Further evidence for the controlled monopoly model was obtained from observations on relative firm sizes within selected U. S. industries. Our hypothesis, from equation (18), implies that

$$\log x_{m-2-i} = K_1 + b_1 \cdot i \ , \quad i = 0, i, \ldots, m-3 \ , \tag{23}$$

where $b_1 = \log 2$ for each industry. The hypothesis relating market share to rank which we find in the literature (Simon (6)) is

$$\log x_{m-i+1} = K_2 + b_2 \log i \ , \quad b_2 < 1 \ , \ i = 1, \ldots, m \ . \tag{24}$$

This hypothesis, which has no theoretical rationale, is clearly contrary to ours in that ours has firm size increasing more than in proportion to a firm's rank in the industry (i) while (24) has firm size increasing less than in proportion to the firm's rank.

We obtained our data from Standard and Poor's "Compustat" tape for 1971, which has data on all of the relatively large U.S. companies within industries disaggregated to the four-digit industry level.[6/] This data was used to generate least-squares fits of the two hypotheses.[7/] The regressions for (24) produced coefficients less than unity in only three industries. In each of these industries (cement, roof and wallboard, and savings and loans) it appeared that we had erred in considering the markets for their product a national rather than a local market. It is not surprising that (24) fits better than (23) for local industries as it is well-known (cf. Simon (5)) that city sizes follow a distribution such as (24).

For the remaining thirty-eight industries, the fit in (23) was better (higher R^2) than that in (24) in over 90% of the cases and the average estimated percentage excess of each firm's size over the next smallest firm's size (the average of the antilogs of the estimated coefficients in (23)) was 86% compared to the theoretical value of 100%. To us, these

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6/ We only included an industry when it (1) included 4 or more companies (for statistical reasons), (2) had a firm producing over 50 million dollars of sales (to avoid the exclusion of large producers due to their being a subsidiary of a diversified firm), (3) sold its product in a national market (to avoid local monopoly effects and interactions with firms in foreign markets), (4) sold its product to economic agents which are not substantially larger than itself (to avoid including industries in which some of the outputs are produced by vertically integrated firms, which would not be counted as part of the industry), (5) and marketed a relatively homogeneous commodity. This is a highly subjective selection of industries, but we know of no better way to provide a fair test of the hypothesis with so much of the data obviously irrelevant.

7/ We had data on both current sales and assets as measures of size, assets being perhaps better than current sales as a measure of future sales. We ran regressions for both measures of size and chose the measure for each hypothesis that yielded Durbin-Watson statistics closest to 2. The rationale here is that we wanted to be as generous as we could to each hypothesis regarding which measure of size would conform the best to the curvature assumptions of the hypothesis.

results amount to fairly strong preliminary evidence in favor of our theory. The regression results on (23) are described in Table 1.

While the fits were tight, a little more than half of the industries had b-coefficients significantly different than the theoretical value of .69 at the 5% level. Furthermore, an analysis of covariance reveals significant differences between these coefficients at the 5% level, leading us to reject the hypothesis that the true value of b_1 is the same for each industry. On the other hand we did not really expect this hypothesis to be true. Deviations from our linearity assumption were to be expected to produce deviations in the theoretical b_1 coefficients about .69. But we do not expect the deviations to be <u>systematic</u>. That is, we expect the <u>average</u> of these coefficients to be not significantly different than .69. The actual average of the b_1 coefficients, .58, is less than half of a standard deviation away from our expected value.

It is possible to test for whether deviations of b_1 about its theoretical value are due solely to non-linearities or also to the absence of legal interactions as we have characterized them. This is done by complementing the above test with a direct fit of the underlying reaction functions as specified in (10)-(14) (c.f., Thompson, Faith, and Rooney). It is of interest to note that, due to the recursive nature of our model, reaction functions are extremely easy to fit statistically while the opposite is true of received theory, which allows every firm's output to depend upon every other firm's output, thereby presenting an immense simultaneity problem.

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Table 1: Fit of Equation (54): $\log x_{m-2-i} = K_1 + b_1 i; \quad i=1,\ldots,m-3$

Industry	b_1	S_{b_1}	e^{b_1}	D.W.	R^2	m-3
gold mining	.68	.1	1.98	1.57	.85	6
coal	1.13	.21	3.10	1.66	.90	5
housing construction	.39	.04	1.48	2.37	.93	7
packaged foods	.26	.03	1.30	1.13	.88	11
dairies	.64	.13	1.90	1.78	.83	7
canned foods	.33	.02	1.39	1.55	.96	12
animal foods	1.14	.16	3.13	1.70	.95	5
biscuits	1.22	.67	3.39	3.00	.80	3
confectionary	.69	.09	2.00	2.06	.91	6
brewers	.27	.01	1.31	1.81	.97	15
distillers	.51	.03	1.67	2.53	.98	7
soft drinks	.69	.09	2.00	2.06	.91	7
tobacco	.47	.10	1.60	1.73	.83	7
forest products	.49	.04	1.63	1.67	.95	11
mobile homes	.33	.04	1.39	2.11	.90	11
home furnishings	.27	.02	1.31	1.85	.94	15
paper	.35	.04	1.42	1.28	.84	14
books	.49	.05	1.63	1.75	.95	7
drugs-ethical	.22	.02	1.25	1.21	.93	17
drugs-proprietary	.57	.07	1.77	.70	.91	9
medical & hospital supply	.38	.03	1.46	2.08	.94	10
soap	.99	.18	2.69	1.57	.91	5
cosmetics	.30	.01	1.35	2.45	.99	16
paint	.60	.11	1.83	1.74	.86	7
tires & rubber goods	.31	.02	1.36	0.89	.92	17
plastics	.33	.04	1.39	1.77	.91	8
shoes	.38	.02	1.46	1.81	.96	13
concrete gypsum and plaster	.37	.08	1.45	1.98	.83	7
aluminum	.60	.21	1.83	2.29	.80	4
motor vehicles	1.01	.22	2.75	2.14	.91	4
photographic	1.07	.07	2.92	2.38	.98	7
watches	.65	.08	1.92	1.93	.96	5
musical instruments, parts	.97	.23	2.64	2.27	.86	6
games	.41	.05	1.51	1.85	.94	6
radio-TV broadcasters	.34	.03	1.40	2.91	.95	11
wholesale foods	.52	.06	1.68	1.78	.93	9
retail lumber yards	.96	.23	2.61	2.37	.90	4
motion pictures	.55	.06	1.73	1.37	.91	9
relevant averages	.58		1.86			
relevant standard deviation	.28					

APPENDIX

THE COMPETITIVE DETERMINATION OF PRIORITIES IN STRATEGY SELECTION

The advantage that a prior strategy selector has over subsequent selectors is his ability to respond to the strategies of the other firms in the group in a way that does not maximize profits given all of the other firms' strategies. In order to have this ability to escape his own narrowly conceived rationality, a firm requires what we shall call a <u>commitment</u>. A commitment is defined as an enforceable promise to react in a specified way to the actions of another regardless of the costs incurred by the commitment-maker in carrying out the stated reaction. Although the idea of commitments is not novel to the area of conflict resolution (see, for example, Schelling (5)), its economic rationale and use in economic theory is rare.

For our model in which strategy selection priorities are determined in a competitive fashion, a firm, in order to exhibit a reaction function, must establish a commitment through its "manager", whose services, which have no alternative value, are obtained via competitive bidding by the firms. We shall give one individual, called the "top manager," the ability to enforce a firm's commitment prior to all other firms' managers. For the top manager, each firm submits m-1 bids, each bid representing the amount the firm is willing to pay to be the first strategy selector in place of a specified, alternative firm. A winning bidder is a firm whose bid against his least preferred, alternative, first strategy selector is no less than the maximum of the bids against him. The reason a winning bidder must bid as if the worst possible alternative is the actual alternative is that the manager is free to choose the bidder's alternative and will rationally choose an alternative which will maximize the bid of the winning bidder. The winning bidder, however, does not generally pay his bid to the manager; he matches the second highest bid.

Subsequent positions in the m-1 firm hierarchy of reaction functions are determined in a similar fashion. Our auction is unusual in that the bidders have different payoffs, and therefore different bids, depending on who would otherwise win the auction and on what he would do as the winning bidder. Since the reaction function chosen by the manager depends on the incentive systems he is given by his firm, if a firm presents an incentive system which leads to a relatively generous reaction to the firm which is the closest competition for the top manager, he will receive a lower competing bid and thus can be obtained at a lower salary.

We have been careful not to give to our managers to much influence on the bidding process. It would be unrealistic, for example, to allow them to encourage the bidders to exhibit reaction functions that raise the winning bid. It would be similarly unrealistic to allow them to require payments from non-winning bidders for not selecting an even worse maker from their points of view.

Overall joint profit maximization is not a general solution only because side payments are disallowed. That overall joint profit maximization (and Pareto optimality under perfect price discrimination) results when there are side payments and an absence of transaction costs in all possible transactions is shown in Thompson (7).

Part I or the following discussion contains, for the uncontrolled monopoly case, a specification of the cost of being a prior strategy selector, a derivation of the identity of a prior strategy selector, a characterization of general equilibrium solutions, and a proof of the existence of equilibrium solutions under some additional restrictions. Part II contains a similar analysis for a controlled monopoly case. In both cases, we show that the quantity solutions are close to the solutions in the text when the number of firms are not very small.

I. EQUILIBRIUM WITH UNCONTROLLED MONOPOLY

A. The Existence of Punishment Outputs

We shall assume that for each firm, there exists a punishment set of outputs. More formally,

(a.1) For each i, there exists an x_i, say x_i', such that

$\pi_j(x_j; x_1 \ldots x_{j-1}, x_{j+1}, \ldots x_i', \ldots x_m) < 0$ for all x_{-i} with $x_j > 0$ and all $j \neq i$.

If firm i is the first strategy selector, it can make a commitment which will induce each firm to produce its specified output. Faced with firm i's commitment, each of the remaining m-1 firms will rationally choose to produce their respective profit-maximizing outputs, the outputs specified by firm i.

More formally, let x_i^i and x_{-i}^i be solution values to the problem,

$$\max_x \; [\pi_i(x) - C_i(x)] \text{ subject to } \pi_f \geq 0 \text{ for all } f \neq i, \tag{25}$$

where $x_f = 0$ if $\pi_f(x_f) < 0$ for all $x_f > 0$, and where C_i is the cost to i of becoming the first strategy selector, or the strategy maker. Thus, x_f^i is the output of the f^{th} firm which maximizes the net maker profit of firm i subject to the non-negativity of profits of each of the other firms, who are strategy takers. The rational reaction function for firm i is then:

$$x_i(x_{-i}) \begin{cases} = x_i^i \text{ when } x_{-i} = x_{-i}^i \\ = x_i' \text{ otherwise.} \end{cases} \tag{26}$$

The commitment made by i guarantees that i will produce x_i' when firm f deviates from producing x_f^i even if it implies lower profits to i than some alternative values of x_i given $x_f \neq x_f^i$. Such apparently irrational behavior by firm i is rational by virtue of our assumption of profit-maximizing behavior of all firms, which implies that firm f will produce x_f^i in equilibrium rather than an alternative output.

A strategy-taker, any firm $j \neq i$, faces the problem:

$$\max \pi_j(x_j;x_{-j}) \text{ subject to}$$
$$x_{-j} \begin{cases} = x^i_{-j} \text{ if } x_j = x^i_j, \text{ and } x_k = x^i_k, \text{ all } k \neq j,i \\ = (x_1,\dots, x'_i,\dots,x_m) \text{ otherwise.} \end{cases} \tag{27}$$

This leads the j^{th} firm, knowing the rational responses of the other takers, to choose $x_j = x^i_j$. We have assumed this holds even if $\pi_j(x^i_j) = 0$ for $x^i_j > 0$. That is, the taker will choose to produce the maker's optimal output choice even though his profits there are zero and he has the equally profitable possibility of quitting business.

Given our assumption on the existence of a punishment output, the problem of the existence of an uncontrolled monopoly equilibrium when there is an arbitrary determination of the strategy maker (and thus when $C_i(x) \equiv 0$) thus reduces to a problem of the existence of an x which maximizes firm i's profit. This existence follows immediately from the minor, additional assumptions that X_f is a non-empty, compact set for each f and $\pi_i(x)$ is a continuous function. The existence of an uncontrolled monopoly equilibrium under our competitive bidding process will be established in subsection E below, after we have specified the nature of the $C_i(x)$ function under competitive bidding and examined the solution characteristics of the two models in both small and large numbers cases.

Note that disregarding the cost of becoming a maker, no firm is ever worse off by being the strategy-maker as opposed to being a strategy-taker. This is because an individual firm can always do as well by choosing its own output as having it chosen by another. Hence, each firm will have non-negative bids for the top manager's services regardless of whom he is bidding against.

B. The Two-Firm Case.

Consider two firms, i and j. The amount firm j is willing to offer to the top manager equals the difference between j's profit as a maker and j's profit as a taker. Since j's profit as a taker depends on i's choice of outputs as a maker, the cost to i of being the maker, which is the cost of just beating j's bid, is a function of the x that i would choose as maker. Hence, we can write:

$$C_i(x) = \pi_j(x^j) - \pi_j(x), \tag{28}$$

and, using (5), describe firm i's maximum maker profit as:

$$\pi_i^M = \max_x [\pi_i(x) - (\pi_j(x^j) - \pi_j(x))] \text{ subject to } \pi_j(x) \geq 0, \tag{29}$$

where $\pi_j(x^j)$ is the value of π_j implied by the solution to:

$$\max_x [\pi_j(x) - (\pi_i(x^i) - \pi_i(x))] \text{ subject to } \pi_i(x) \geq 0, \tag{30}$$

where $\pi_i(x^i)$ is the solution value of π_i implied by (9).

Solutions to (29) and (30), if they exist, yield explicit values of $\pi_i(x^i)$, $\pi_j(x^j)$, $\pi_i(x^j)$, and $\pi_j(x^i)$ from which we obtain the value of each firm's bid. These values are interpreted as i's and j's operating profit as a maker, and i's and j's operating profit as a taker, respectively.

Noting that x^i is independent of $\pi_j(x^j)$, we see from (29) that firm i is maximizing its joint-profits with firm j. Similarly, from (30), firm j is maximizing its joint-profits with firm i. If we assume that the joint-profit maximizing output is unique, then the same output vector will be chosen regardless of which firm is the strategy-maker. Hence, each firm's bid for the rights to be maker would equal zero since its profit as a maker is the same as its profit as a taker. In this case, the final determination of the strategy maker is arbitrary.

If the joint-profit maximizing output is non-unique, the two firms' bids will still be equal, but they may then be positive. For example, at x^i, let $\pi_i(x^i) = 50$, $\pi_j(x^i) = 40$, and at x^j, let $\pi_j(x^j) = 60$, $\pi_i(x^j) = 30$. Notice that both i's and j's bid will equal 20. Since the bids are equal, the selection of strategy maker is still arbitrary.

The joint-profit-maximizing solution when there are two firms contrasts sharply with the solution when there is no competition to determine a maker. In the latter case, the arbitrarily selected maker simply determines a set of outputs which maximizes his own profit and applies his punishment if the takers do not oblige him. This is generally far from a joint-profit-maximum.

Given any number of firms which produce a single output, an arbitrarily selected maker produces a simple monopoly output, flooding the market with an output which would enforce negative profits on all other active firms in the industry if any of them produced a positive output. This predatory, "robber-baron" strategy holds regardless of the nature of production costs and demand, and regardless of the number of takers, giving us a single-active-monopoly solution among any group of interacting firms selling a homogeneous product. We shall soon see, however, that this robber-baron solution is also approached in the case of competitive bidding for hierarchial position as the number of firms increases beyond two.

C. Competitive Bids to be Maker in the m-Firm Case

With m firms, $m \geq 3$, although there are m-1 competing bids with which a prospective strategy maker must contend, any prospective maker need only be concerned with the highest of his rivals' bids. This highest rival bid is the explicit cost to i of becoming the maker. Thus, (28) becomes:

$$C_i(x) = \max_{f \neq i} (\pi_f(x^f) - \pi_f(x)). \tag{31}$$

The m-1 opposing bidders are each rationally assuming that firm i will be the strategy maker if they are not. The resulting bid of each firm then measures how much a firm is willing to pay to be maker instead of being a taker of i's reaction function. We can now describe firm i's maximum maker profit, π_i^M, as:

$$\pi_i^M = \max_x \, [\pi_i(x) - \max_{f \neq i} (\pi_f(x^f) - \pi_f(x))] \text{ subject to } \pi_f \geq 0 \ , \tag{32}$$

where $\pi_f(x^f)$ is the operating profit to firm f when f is solving for its maximum maker profit.

Firm i's <u>alternative maker</u> is that firm which will be the maker if i is not. Firm i's bid when j is his alternative maker, the difference between i's profit as maker and i's profit as taker of j, is

$$\pi_i(x^i) - \pi_i(x^j) = B_{ij}(x^j) \ , \tag{33}$$

where j is i's alternative maker.

By computing maximum maker operating profit for all m firms, if these profits exist, and taker profit in a similar fashion, we can compute each firm's bids from the explicit values of maker and taker profits.

D. <u>Characterizing an m-Firm Equilibrium</u>

Distinguishing features of the m-firm case ($m \geq 3$) under competitive bidding to be maker are that at a solution there is more than one highest-bidding taker, that the solution is not a joint-profit maximum and that the solution approaches the arbitrary maker solution as m increases.

At any choice of output allocation set of the maker, i, there is either a distinct firm determining i's managerial cost - i.e., an unique f, solving (31), or there is a tie bid between some of the takers. Suppose there is an unique maximum in (31). Since the maker is responsive only to changes in the bid of the single highest-bidding taker, say, j, the maker and this taker will adopt a joint-profit maximizing relationship as in the two-firm case. Therefore, if the output choice of i is, in fact, a solution, it also corresponds to a joint-profit maximum between i and j. If the joint-profit maximizing output is unique, j's bid against i is zero. Since the remaining bids are non-negative, such an output choice is unattainable because the alternative maker's zero bid is then not higher than the other takers.

If the joint-profit maximizing output of i and j is non-unique, the same result obtains. Suppose that joint-profits between i and j are maximum and therefore equal at both x^i and x^j. Then the difference between i's maker profit at x^i and x^j,

$$\pi_i(x^i) - [\pi_j(x^j) - \pi_j(x^i)] - \pi_i(x^j) + [\pi_j(x^j) - \pi_j(x^j)]$$

$$= \pi_i(x^i) + \pi_j(x^i) - [\pi_j(x^j) + \pi_i(x^j)] = 0 \ .$$

Thus, i is indifferent between x^i and x^j which implies that i's bid is zero. Again, an inconsistency results as the other bids are non-negative. Thus, the equilibrium solution is inconsistent with the existence of an unique maximum in the alternative bids.

Therefore the optimal output choice of the strategy-maker occurs where there exists a tie in the maximum bids of some of the takers.

A solution occurring at this point does not correspond to a joint-profit maximum. Although the maker is responsive to a change in the bid of any one of his several maximum-bidding-takers, he is not concerned with the sum of their bids, which i requires for joint-profit maximization. Thus, the solution is not a joint-profit maximum.*/

The greater the number of firms, the "closer" the solution is to the arbitrary maker solution in the following sense: With a greater number of firms, there is a greater number of takers whose bids are equal to or a greater number whose bids are less than the maximum bid of the takers. If there is an expansion of those whose bids are less than the maximum, there are more firms whose non-negative profit variations are of no concern to the maker and thus more firms whose output is determined just as it is in the case of an arbitrarily selected maker. If there is an expansion of takers whose bids are maximal, then the maker internalizes less of the variation in the total profits to these takers, moving toward the extreme in which he is arbitrarily selected and therefore internalizes none of this variation.

In both the two and m-firm cases, since the solution maker need only match the highest-bidding rival firm(s), the amount going to the manager will equal the value of the second highest bid over all firms. But, whereas the manager's fee is always zero in the two-firm case when the joint profit maximizing output set is unique, it may be positive in the m-firm case.

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*/ An m-firm joint-profit maximum would mean that the maker's marginal profit (assuming differentiability) equals the sum of the other firms' marginal profits. In our case, the marginal profit of the maker is equal to each firm's marginal profit. Let there be an m-firm tie where m = 3. If marginal profit to i, the maker, equals $1 (thus, marginal profit equals $1 a piece to the takers), then marginal joint-profit equals minus $1, rather than 0.

E. A Theorem on the Existence of Equilibrium

We will be working in Euclidean space, R^y; the dimensionality y of the space equals the number of commodities (n) times the number of firms (m), or nm.

Let X, a subset of R^y, equal the feasible output set. An element, x, of this feasible output set is a y-dimensional vector of outputs of each commodity by each firm.

From the above discussion, for each firm there is a profit function, $\pi_f(x)$, f=1,...,m; defined on X. Similarly defined on X is:

(d.1) the i^{th} firm's maker profit,

$$\pi_i(x) - \max_f [\pi_f - \pi_f(x)] \quad f=1,\ldots,m;\ f \neq i\ , \tag{34}$$

where π_f is f's operating profit as a maker, a given number to i; and

(d.2) the i^{th} firm's bid function, given that f, who dictates output x, is the alternative maker,

$$B_{if}(x) = [\pi_i(x^i) - \pi_i(x)],\ f=1,\ldots,m;\ f \neq i\ . \tag{35}$$

An equilibrium is (a) a set of output allocation vectors, $\underline{x}^1, \underline{x}^2, \ldots, x^m_m$, $\underline{x}^f \in X^f$, such that each $\underline{x}^i$ maximizes the maker profit of firm i given $\pi_f = \pi_f(\underline{x}^f)$, all f≠i, and (b) a winning bidder, a firm, i, such that $\max_f B_{if}(\underline{x}^f) \geq \max_k B_{ki}(\underline{x}^i)$.

We now make the following assumptions:

(a.2) X is a non-empty, compact, convex set.

(a.3) $\pi_f(x)$ is a continuous, real-valued function, f=1,...,m.

(a.4) For any f and any given $(\pi_1,\ldots,\pi_{f-1},\pi_{f+1},\ldots,\pi_m)$, there is at most one value of $\underline{x}^f$. (This is slightly weaker than the strict convexity of the set $X_i = \{x_i\colon\ \pi(x_i) \leq \pi\}$ for all π.)

<u>Theorem</u>: Given assumptions (a.1) - (a.4), there exists an equilibrium.

The proof will consist of two parts. Part 1 will prove that there exists a set of outputs $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^m$. That is, for any i, there exists maximum maker profit, π_i^M, with consistent values of π_f, for all $f \neq i$. Part 2 will prove that there is always at least one firm which is a winning bidder, i.e., one firm whose maximum bid against his alternative makers is no less than the maximum of the bids against him.

<u>Proof</u>:*/

<u>Part 1</u>.

First we show that for given values of maker profit of other firms, firm i has a maximum maker profit. To do this we will employ the well-known theorem in analysis that a continuous, real-valued function defined over a closed and bounded set attains a maximum at some point in the set. Let

$$g_i(x) = \max_f \, [\pi_f \qquad - \pi_f(x)] \quad f=1,\ldots,m;\; f \neq i, \qquad i=1,\ldots,m. \tag{36}$$

*/ A more compact, but to us less intuitive, proof, provided by Ron Heiner, will be sent on request.

That is, $g_i(x)$ is the function describing the maximum bids against i for each point in X selected by i. Since $\pi_i(x)$ is continuous by (a.2) and the sum of two continuous functions is continuous, firm i's maker profit in (34) is continuous if $g_i(x)$ is continuous.

<u>Lemma</u>: The function $g_i(x)$ is continous.

At any point in X, and any i, there is either (a) an unique maximum bid in (36), or (b) there is an equality between the highest two, or more bids in (36).

(a) If there is an unique maximum in (36) at some point in X, then since each bid function, $B_{fi}(x)$, is continuous (the difference of two continous functions is continuous), (36) is continuous at such points in X.

(b) Let x_s be a point in X where $B_{ji}(x_s) = B_{ki}(x_s) = g_i(x_s)$, $j \neq k$. Suppose, for any $\delta > 0$, there is an $\varepsilon < \delta$, $\varepsilon > 0$, such that

$$g_i(x_s-\varepsilon) = B_{ji}(x_s-\varepsilon) > B_{ki}(x_s-\varepsilon) \text{ , and} \tag{37}$$

$$g_i(x_s+\varepsilon) = B_{ji}(x_s+\varepsilon) \leq B_{ki}(x_s+\varepsilon) \text{ , } j, k=1,\ldots,m; \quad j \neq k, \; j,k \neq i.$$

It is obvious that since each bid function is continuous and equal at x_s, $g_i(x_s)$ is continuous at x_s.

In the case where the second relation in (37) does not hold, the bid of j is a maximal bid over the entire δ-neighborhood so the continuity of $g(x_s)$ follows from the continuity of $B_{ji}(x_s)$.

In the only remaining case, where only the first relation in (37) does not hold, $B_{ji}(x) \equiv B_{ki}(x)$ about an δ-neighborhood of x_s, then either bid is maximal in that neighborhood. Since the bid functions are continuous at all points in X, then $g_i(x)$, $i=1,\ldots,m$, is continuous over all of X.

It follows from the lemma and the well-known theorem in analysis stated above that π_i^M and $\underline{x}^i$ exist for any set of values $\{\pi_f\}$ and thus for any set of vectors $\{\underline{x}^f\}$, $f \neq i$.

Now one firm's optimal maker output vector depends upon the optimal maker output vectors of other firms. This leads to the question of whether the optimal output vectors of the various firms are mutually consistent. Proving this establishes the existence of a set of output vectors, $(\underline{x}^1,\ldots,\underline{x}^m)$, such that, for each f, $\underline{x}^f$ yields maximum maker profit to firm f for the $\underline{x}^i$ of all $i \neq f$, $i,f = 1,\ldots,m$. Consider m feasible sets of nm outputs, each representing an output allocation vector arbitrarily selected by each f, or $(x^{10},\ldots,x^{m0})$. Given the values of $(x^{20},\ldots,x^{m0})$, the value of x maximizing firm 1's maker profit, x^{11}, is calculated. Using x^{11} and $x^{30},\ldots,x^{m0}$, the value of x maximizing firm 2's maximum maker profit, x^{21}, is calculated. Continuing in this manner, the output set, $(x^{11},\ldots,x^{m1})$ is attained.

The resulting transformation,

$$(x^{10},\ldots,x^{m0}) \to (x^{11},x^{21},\ldots,x^{m1}), \tag{38}$$

then is a transformation from a set of output sets, X^m, into itself. To show that there exists a consistent set of maximum maker profit over all m firms, it is sufficient to show that there exists a set of outputs, $(\underline{x}^1,\ldots,\underline{x}^m)$, which remains unchanged over the transformation (18). By the Kakutani fixed point theorem, the set, $(\underline{x}^1,\ldots,\underline{x}^m)$ exists if X^m is a compact, convex set, and the complete transformation (38) is continuous.

By assumption (a.1) - and the fact that the Cartesian product of closed, bounded, and convex sets is itself closed, bounded, and convex - we know that $\bar{X}^m$ is closed, bounded, and convex. Since each transformation in (38) is a calculation of some firm's profits, it is sufficient to show the continuity of (38) by showing the continuity of $\underline{x}^i$ as a function of

$$x^{-i} = (x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^m) \tag{39}$$

for any i, i=1,...,m. Suppose the function,

$$\underline{x}^i = \underline{x}^i(x^{-i}), \tag{40}$$

is not continuous at some $\bar{x}^{-i}$. This implies that there is an infinite sequence, $\{x^{-i}\}$, approaching $\bar{x}^{-i}$ such that

$$\bar{\underline{x}}^i = \underline{x}^i(\bar{x}^{-i}) \neq \lim_{x^{-i}\to\bar{x}^{-i}} \{\underline{x}^i(x^{-i})\}.$$

(The existence of this limit is implied by the boundedness assumption in (a.2) and the Weierstrass Theorem.) Since each firm's profit is a continuous function of $(x^1,\ldots,x^m)$, there is also an infinite sequence,

$$\{\pi_{-i}\} = \{\pi_1,\ldots,\pi_{i-1},\pi_{i+1},\ldots,\pi_m\}, \tag{41}$$

which approaches $\bar{\pi}_{-i}$ such that

$$\bar{\underline{x}}^i = \underline{x}^i(\bar{\pi}_{-i}) \neq \lim_{\pi_{-i}\to\bar{\pi}_{-i}} \{\underline{x}^i(\pi_{-i})\}.$$

Now the uniqueness of $\underline{x}^i(\overline{\pi}_{-i})$ expressed in (a.4) implies that there is a $\delta(\varepsilon), \delta > 0$, such that for any $x^i \in X$ not in an ε-neighborhood of $\underline{\overline{x}}^i$,

$$\pi^i(\underline{\overline{x}}^i, \overline{\pi}_{-i}) - \pi^i(x^i, \overline{\pi}_{-i}) > \delta(\varepsilon), \tag{42}$$

where π^i is firm i's maker profit. Then, from the linear manner in which $\overline{\pi}_{-i}$ enters i's maker profit function (34), there is an $\omega > 0$ such that for all π_{-i} satisfying $|\pi_{-i} - \overline{\pi}_{-i}| < \omega$, and for all $x^i \in X$ not in an ε-neighborhood of $\underline{\overline{x}}^i$,

$$\pi^i(\underline{\overline{x}}^i, \pi_{-i}) - \pi^i(x^i, \pi_{-i}) > \delta(\varepsilon). \tag{43}$$

Consider the ε-neighborhood of

$$\lim_{\pi_{-i} \to \overline{\pi}_{-i}} \underline{x}^i ,$$

and select an ε sufficiently small that the intersection of this neighborhood and the ε-neighborhood of $\underline{\overline{x}}^i$ is empty. If the $\underline{x}^i$ in the former neighborhood are indeed profit maximizing, for all $\underline{x}^i$ in that neighborhood,

$$\pi^i(\underline{\overline{x}}^i, \pi^M_{-i}) - \pi^i(\underline{x}^i, \pi^M_{-i}) < 0, \tag{44}$$

for all $\pi^M_{-i} \to \overline{\pi}^M_{-i}$ generating the ε-neighborhood of

$$\lim_{\pi^M_{-i} \to \overline{\pi}^M_{-i}} x^i .$$

This is a direct contradiction of the immediately preceding inequality (43).

Hence, $\underline{x}^i$ is a continuous function of x^{-i} and likewise the transformation (38) is continuous. This is sufficient for the Kakutani fixed point theorem to apply; and therefore, the set $(\underline{x}^1, \ldots, \underline{x}^m)$ exists.

Part 2.

We shall now prove that -- given the array of maximum maker profits in (34), and therefore an array of bids against all alternative makers described in (35) -- a winning bidder exists. Consider the matrix B, representing the bids of each firm against the others, with zeroes along the main diagonal:

$$B = \begin{bmatrix} 0 & B_{12} & B_{13} & \cdots & B_{1m} \\ B_{21} & 0 & B_{23} & \cdots & B_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & B_{m3} & & 0 \end{bmatrix}. \qquad (45)$$

From the definition of a solution maker, i is a solution maker if $\max_j B_{ij} \geq \max_k B_{ki}$, that is, if the maximum bid by i exceeds the maximum of the bids against i. In B, i is a solution maker if the maximum of the elements in the $i^{\underline{th}}$ row exceeds the maximum of the elements in the $i^{\underline{th}}$ column. Let B_{qr} be a maximal element of B. Then, $B_{qr} \geq \max_k B_{kq}$ so that q is a maker. Hence, there is always a winning bidder.

II. EQUILIBRIUM WITH ANTI-MONOPOLY POLICY

This part of the Appendix shows that the introduction of competition in making commitments, and thus of competitive bidding for each position in the hierarchy, makes the model of equilibrium with anti-monopoly policy in the text decidedly more complex but does not substantially alter its conclusions. As long as the number of firms is not very small.

Referring to the model with anti-monopoly policy in the text, while competitive bidding for hierarchial position may increase the costs to all firms, it does not, of course, increase the costs of the pure taker. Since all firms have the same variable cost functions, the variable maker profit of each firm is, in equilibrium, equal to the simple variable profit of the pure taker. It also follows from the equality of variable costs between our firms that every firm is indifferent to any position in the hierarchy so that all bids for each position in the hierarchy are identical. One's position in the hierarchy is determined by the priority of his commitment. Hence, the first auction is for the position of primary maker, the second for the secondary maker, etc. In the first auction, there are m bidders, in the second there are m-1, and so on until, finally in the m-1st auction, there are only two bidders. Let us see how these auctions alter the rationally chosen outputs from those selected in the model with no competition for hierarchal position.

The pure taker obviously has the same output choice function as in the model with no competitive bidding for hierarchal positions, so again

$$x_m^* = \frac{a-c-b\sum_{i=1}^{m-1} x_i}{2b} = \frac{p-c}{b}, \tag{46}$$

where x_i^* is the solution output of firm i in the present model. But the output of the m-1st firm is now sensitive to a bid of the $m^{\underline{th}}$ firm for his hierarchal position. The bid of m for position m-1 equals

$$B_{m,m-1}(x_{m-1}) = \max[0, \pi_{m,m-1}-\pi_m(x_{m-1})],$$

where $\pi_{m,m-1}$ is the operating profit m would make if he were the m-1$^{\underline{st}}$ maker. Firm m-1 therefore selects an x_{m-1} which maximizes

$$\pi^M_{m-1} = \pi_{m-1}-\max[0, \pi_{m,m-1}-\pi_m(x_{m-1})] \text{ subject to (36) and the given values, } x_1,\ldots,x_{m-2} \quad . \tag{47}$$

If the solution x_{m-1} were such that $B_{m,m-1}(x_{m-1}) > 0$, then the output that satisfies (47) would be a joint profit maximum subject to (46) and $x_1,\ldots x_{m-2}$. This joint profit is $(p-c)(x_{m-1}+x_m)-d_m-d_{m-1}$, which, as above, reaches its maximum at $x_{m-1}+x_m = \frac{p-c}{b}$. But then, from (46), x^*_{m-1} would equal zero. Since variable profits exceed zero at some positive outputs, firm m-1 would do better as the pure taker. Hence, $B_{m,m-1}(x^*_{m-1})=0$. But if so, the variable profits are equal for both firms m and m-1. Therefore,

$$x^*_{m-1} = x^*_m \, , \tag{48}$$

and using (36),

$$x^*_{m-1} + x^*_m = \frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b/2} = 2\left(\frac{p-c}{b}\right) \quad . \tag{49}$$

This solution may be constructed by starting firm m-1 at its output in the previous model, an output which maximizes his operating profit and exceeds x_m, and then making it pay m's bid to be maker to a lawyer. It is then obvious from (47) that it pays m-1 to reduce his output in order to reduce m's bid against him. This occurs until $x_{m-1} = x_m$, at which point the bids become zero. It then no longer pays m-1 to reduce his output for there is no further reduction in m's bid that is possible.

To compute m-2's optimal output, we need the profit of firms m and m-1 as a function of x_{m-2}. Using (49), and computing, from now on, profits as variable profits (profits net of d_i), we have

$$\pi_{m-1} = \pi_m = \left[a-c-b\sum_{i=1}^{m-2} x_i - b\left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b/2}\right)\right]\left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b}\right) = \frac{\left(a-c-b\sum_{i=1}^{m-2} x_i\right)^2}{9b} . \quad (50)$$

Firm m-2's maker profits can now be written, using (49),

$$\pi^M_{m-2} = \left[a-c-b\sum_{i=1}^{m-2} x_i - b\left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3b/2}\right)\right] x_{m-2} - \max[0, \pi_{m,m-2} - \pi_m(x_{m-2})] . \quad (51)$$

Assume that $\pi_{m,m-2} - \pi_m(x^*_{m-2}) \geq 0$. Then, using (50) and (51),

$$\pi^M_{m-2} = \left(\frac{a-c-b\sum_{i=1}^{m-2} x_i}{3}\right) x_{m-2} + \frac{\left(a-c-b\sum_{i=1}^{m-2} x_i\right)^2}{9b} - \pi_{m,m-2} .$$

Maximizing this profit, we find

$$x^*_{m-2} = \frac{a-c-b\sum_{i=1}^{m-3} x_i}{4b} , \text{ and} \quad (52)$$

$$\pi_{m-2}(x^*_{m-2}) = \frac{\left(a-c-b\sum_{i=1}^{m-3} x_i\right)^2}{16b} \quad (53)$$

Substituting x^*_{m-2} into (49),

$$x^*_{m-2} = x^*_{m-1} = x^*_m \quad (54)$$

The assumption that $\pi_{m,m-2} - \pi_m(x^*_{m-2}) \geq 0$ is satisfied, for at x^*_{m-2} the bid of m (and of m-1) is zero. Even if we removed the constraint that bids be non-negative, the above exercise shows that m-2 would reduce his profit by contracting his output so as to make $\pi_{m,m-2} - \pi_m(x_{m-2})$ negative.

Similarly, if the output solution to the problem,,

$$\max_{x_i} \; [(p-c)x_i - \max_j(\pi_{ji}-\pi_j(x_i))], \; i < m, \tag{55}$$

exceeds x_m, then, because bids are positive in such a solution, this output solution is also the solution to the general problem:

$$\max_{x_i} \; [(p-c)x_i - \max_j \; [0, \; \pi_{ji} - \pi_j(x_i)\}] \; . \tag{56}$$

Performing the maximization in (55) for firm m-3, using (52), (53) and (54), we maximize

$$\left[\left(a-c-b\sum_{i=1}^{m-3} x_i\right)x_{m-3} - 3/4 \left(a-c-b\sum_{i=1}^{m-3} x_i\right) x_{m-3} + \frac{\left(a-c-b\sum_{i=1}^{m-3} x_i\right)^2}{16b}\right]$$

with respect to x_{m-3}. The solution output can be written:

$$x^*_{m-3} = \frac{a-c-b\sum_{i=1}^{m-4} x_i}{3b} = \frac{a-c-b\sum_{i=1}^{m-3} x_i}{2b} \quad . \tag{57}$$

This output is twice the output of firm m-2 and thus is also a solution to (56). We also find, using (52), (54) and (57) that variable profits for each firm are:

$$\pi^*_m = \pi^*_{m-1} = \pi^*_{m-2} = \pi^{M*}_{m-3} = \frac{\left(a-c-b\sum_{i=1}^{m-4} x_i\right)^2}{36b} \quad . \tag{58}$$

Thus variable maker profit to the m-4$\underline{\text{th}}$ firm is

$$\pi^M_{m-4}(x_{m-4}) = \left[a-c-b\sum_{i=1}^{m-4} x_i - \frac{5}{6}\left(a-c-b\sum_{i=1}^{m-4} x_i\right)\right] x_{m-4} + \frac{\left(a-c-b\sum_{i=1}^{m-4} x_i\right)^2}{36b} \quad .$$

Maximizing this with respect to x_{m-4}, we find

$$x^*_{m-4} = \frac{a-c-b\sum_{i=1}^{m-5} x_i}{5b/2} = \frac{a-c-b\sum_{i=1}^{m-4} x_i}{3b/2} \quad . \tag{59}$$

This is twice the output of firm m-3 and four times the outputs of firms m, m-1, and m-2. We also find that

$$\pi^*_m = \pi^*_{m-1} = \pi^*_{m-2} = \pi^{M*}_{m-3} = \pi^{M*}_{m-4} = \frac{\left(a-c-b\sum_{i=1}^{m-5} x_i\right)^2}{100b} .$$

These profits form the bids for the m-5$^{\underline{th}}$ position in the hierarchy, and the procedure continues until we reach the top position. The distribution of outputs, moving on to m-5, m-6 and m-7 and again indexing the output of the m$^{\underline{th}}$ firm to unity, is easily seen to be

1, 1, 1, 2, 4, 8, 16, 32.

The obvious generalization is that

$$x^*_m = x^*_{m-1} \text{ and } x^*_{m-2-i} = x^*_m \cdot 2^i,\ i = 0,1,\ldots, m-3 . \tag{60}$$

To prove that this generalization is, in fact, the solution distribution of firms, we provide an inductive proof. In particular, we shall prove that if the hypothesized distribution holds for i = r, i.e., if $x_{m-2-i}/x_{m-1-i} = 2$ for any i such that $0 \leq i \leq r$, then it holds for $i = r + 1$, i.e., $x_{m-3-r}/x_{m-2-r} = 2$. To do this, we first note that variable profit to the m-3-rth firm is

$$\pi_{m-3-r} = \left(a-c-b\sum_{i=1}^{m} x_i\right) x_{m-3-r} + \frac{\left(a-c-b\sum_{i=1}^{m} x_i\right)^2}{b} .$$

By hypothesis, for the firms from m to m-2-r we have:

$$x_{m-2-r} = \frac{a-c-b\sum_{i=1}^{m-3-r} x_i}{b(2^r+1)/2^{r-1}} \tag{61}$$

and

$$\sum_{i=m-2-r}^{m} x_i = \frac{a-c-b\sum_{i=1}^{m-3-r} x_i}{b} \left(\frac{2^{r+1}+1}{2^{r+1}+2}\right) \quad , \quad r \geq 0 \tag{62}$$

Using (52),

$$\pi_{m-3-r} = \left(a-c-b\sum_{i=1}^{m-3-r} x_i - b\sum_{i=m-2-r}^{m} x_i\right) x_{m-3-r} + \frac{\left(a-c-b\sum_{i=1}^{m-3-r} x_i - b\sum_{i=m-2-r}^{m} x_i\right)^2}{b}$$

$$= \frac{1}{2^{r+1}+2}\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right) x_{m-3-r} + \frac{\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right)^2}{(2^{r+1}+2)^2 b}$$

Maximizing this expression with respect to x_{m-3-r} ,

$$0 = \frac{1}{2^{r+1}+2}\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right) - \frac{bx_{m-3-r}}{2^{r+1}+2} - \frac{2\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right)}{(2^{r+1}+2)^2}$$

$$= \frac{2^{r+1}\left(a-c-b\sum_{i=1}^{m-3-r} x_i\right)}{(2^{r+1}+2)^2} - \frac{bx_{m-3-r}}{2^{r+1}+2}$$

$$x_{m-3-r} = \frac{a-c-b\sum_{i=1}^{m-3-r} x_i}{b\left(\frac{2^r+1}{2^r}\right)} \quad .$$

Using (51), we see that

$$\frac{x_{m-3-r}}{x_{m-2-r}} = \frac{(2^r+1)/2^{r-1}}{(2^r+1)/2^r} = 2.$$

This establishes the theorem.

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