Simple Games and Authority Structures

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Abstract

We present a formal set-theoretic descriptive model for authority structures in organizations and suggest some applications. We extend the inward-looking approach developed in earlier research[1] to include interfaces with the outside world of tasks and rival organizations. An important feature of the model is the ability to assign a numerical measure of the responsibility that each member of the organization bears (i.e., the credit or blame) for what was actually accomplished.

1 Introduction

Voting rules for political decision-making such as legislatures or committee systems are familiar but almost trivial examples of *authority structures* (AS's for short) because they can be described by just one simple game (i.e., a list of winning coalitions) à la von Neumann Morgenstern [1], and/or Shapley [3], [4], etc. More generally an AS requires a whole menagerie of interconnected simple games to capture the possible intricacies of multiple chains of command and approval, shared responsibilities for decisions, broken hierarchies, "czars", committees and subcommittees with overlapping membership, etc. that usually clutter up the organizational charts of real-life bureaucracies.

While conventional graph-theoretic organization charts can handle a good portion of this kind of detail, we think the set-theoretic approach is far more flexible and precise, and therefore we use simple games as our basic building blocks. This has the advantage (among others) of avoiding dubious measurements and comparisons of personal utilities; indeed payoffs scarcely appear until we enter the task environment. But the "Boolean" atmosphere gets rather thick. We introduce several special operations on sets of sets, and we sometimes find ourselves dealing with sets of sets of sets!

[1] L.S. Shapley, *A Boolean Model of Organizational Authority*, Seminar Notes, UCLA, 1994.

The paper is organized in six sections. Section 1 introduces the topic of research. Section 2 gives the basic set-theoretic notation and definitions of simple games. Section 3 establishes the local and global topologies of command and control, respectively, of an AS. Section 4 handles the AS interaction with its external environment. Section 5 proposes a quantitative index to assign the credit (or blame!) for the responsibility (of the members of the AS) in the actual accomplishment of tasks. Section 6 synthesizes results.

2 Simple Games

Simple games were first developed by von Neumann and Morgenstern [1], and later extended by Shapley [3]. They designate a class of multiperson games where each feasible coalition of players turn out to be either all-powerful or completely ineffectual. Simple games seem particularly well suited for the analysis of structures where the primary concern is power and authority rather than strategic or monetary considerations. Simple games are conceptually equivalent with other mathematical constructs outside game theory like clutters, coherent systems, switching functions, and block design games, among others, that have found useful applications in different fields.

2.1 Notation

Our basic mathematical framework of analysis for simple games is set theory, specifically the algebra of sets of sets. We shall use the following notational conventions: regular lower-case letters or numerals for individuals, italic capital letters for sets of individuals, and script capital letters for sets of sets of individuals (*collections*).

The connective $\in$ ($\notin$) states that an element belongs (does not belong) to a set and it will help to clarify the different levels of abstraction under consideration, i.e., $a \in A \in \mathcal{A}$. . Further, we shall even disturb some logicians by making a distinction between the empty set of individuals " $\varnothing$ " and the empty set of coalitions " $\not\mathcal{O}$ "[2]. When naming the elements of a set we shall employ both the traditional braces and the overhead bar, i.e., $S = \{a, b, c, d\}$ or $S = \overline{abcd}$. The set of all individuals in a decision-making body — i.e., a game, a voting system, or an AS — will be denoted by N, and its power set by $\mathcal{N}$. Thus, $\mathcal{N}$ is the set of all subsets of N and its elements sometimes will be called *coalitions.* Set subtraction will be indicated either by "\" or by "–". Thus, if i is any element of N, then N_i will denote $N \setminus \overline{i}$ — i.e., $(N - \overline{i})$ — and $\mathcal{N}_i$ will denote the power set of N_i. The complement of a set of sets is defined in the traditional way, i.e., $\mathcal{A}^c = \mathcal{N} - \mathcal{A}$, (equivalently $\mathcal{A}^c = \mathcal{N} \setminus \mathcal{A}$.) Also as usual the symbol $\subseteq$ denotes set inclusion and $\subset$ strict inclusion.

[2] We are dealing with two levels of Boolean algebra: "italic" and "script." Hence the two empty sets: $\varnothing \subset N$ and $\not\mathcal{O} \subset \mathcal{N}$.

2.2 Definitions

In this subsection we analyze the elementary Boolean properties of collections of coalitions.

The relevant issue to note is that although we use nothing more sophisticated than simple set theory, we shall be working at three different levels of analysis, and that is why is necessary to distinguish very clearly between sets whose elements are individuals and sets whose elements are other sets.

Let $\mathcal{S} \subseteq \mathcal{N}$ be any non-empty collection of coalitions. A *minimal element* of the collection $\mathcal{S}$ is an element of $\mathcal{S}$ (i.e., a coalition) such that it has no proper subsets in $\mathcal{S}$. Then, we define the following special collections (sets of sets):

$\mathcal{S}^+$ is the set of *all supersets* of $\mathcal{S}$

$\mathcal{S}^*$ is the set of *all complements of elements* of $\mathcal{S}$

$\mathcal{S}^m$ is the set of *all minimal elements* of $\mathcal{S}$ — equivalently — the *smallest subset* $\mathcal{T}$ *of* $\mathcal{S}$ *such that* $\mathcal{T}^+ = \mathcal{S}^+$ — or equivalently — *the set of all* $R \in \mathcal{S}$ *such that no* $T \in \mathcal{S}$ *satisfies* $T \subset R$.

$\mathcal{S}^\cup$ is the *union of all elements* of $\mathcal{S}$

$\mathcal{S}^\cap$ is the *intersection of all elements* of $\mathcal{S}$.

Note that $\mathcal{S}^\cup$ and $\mathcal{S}^\cap$ should be considered as honorary italic capital letters since they are not collections. Also, by convention $\emptyset^\cup = \varnothing$ and $\emptyset^\cap = N$. Note that in general $(\mathcal{S}^+)^m = \mathcal{S}^m$ and $(\mathcal{S}^m)^+ = \mathcal{S}^+$. Also conventionally $\emptyset^+ = \emptyset$ and $\emptyset^m = \mathcal{N}$.

All the elementary Boolean properties associated with the usual set operations for sets of individuals, translate immediately to sets of sets. However, for the special collections of sets defined above, there are some additional properties that do not correspond to the traditional ones.

In particular

$$
\begin{aligned}
(\mathcal{S} \cap \mathcal{T})^* &= \mathcal{S}^* \cap \mathcal{T}^* \\
(\mathcal{S} \cup \mathcal{T})^* &= \mathcal{S}^* \cup \mathcal{T}^* \\
(\mathcal{S} \setminus \mathcal{T})^* &= \mathcal{S}^* \setminus \mathcal{T}^* \\
\mathcal{S}^{**} &= \mathcal{S}
\end{aligned}
$$

Note that inserting $\mathcal{N}$ for $\mathcal{S}$ in the next to last identity gives $(\mathcal{N} \setminus \mathcal{T})^* = \mathcal{N} \setminus \mathcal{T}^*$, that is, the set of complements of coalitions in the complement of $\mathcal{T}$ is the complement of the set of complements of coalitions in $\mathcal{T}$.

To illustrate the previous special set definitions, consider $N = \overline{abcd}$ so that $\mathcal{N}$ is the set of all subsets of N — i.e., it has 16 coalitions — and take a coalition such as $\mathcal{S} = \{\overline{ab}, \overline{acd}, \overline{cd}\}$.

Then we have,

$$
\begin{aligned}
\mathcal{S}^+ &= \{\overline{ab}, \overline{cd}, \overline{abc}, \overline{abd}, \overline{acd}, \overline{bcd}, \overline{abcd}\} \\
\mathcal{S}^* &= \{\overline{cd}, \overline{b}, \overline{ab}\}
\end{aligned}
$$

$$\mathcal{S}^m = \{\overline{ab}, \overline{cd}\}$$
$$\mathcal{S}^\cup = \overline{abcd}$$
$$\mathcal{S}^\cap = \varnothing$$

The essential notion of a simple game is the basic concept of a winning coalition which does not require more elaboration and one that we take as a primitive. We represent a simple game by simply stating its *players* and its *winning coalitions.* Further, we characterize this intuitive idea of winning by the following conditions:

1. the grand coalition always wins,
2. the empty set of individuals never wins, and
3. any superset of a winning coalition also wins.[3]

Probably the most elemental of all simple games is the simple majority game denoted by M_n where n, the number of players, is odd. In this game M_n, there are 2^n possible coalitions of which all of those with more than $\frac{n}{2}$ members win, and the rest, those with less than $\frac{n}{2}$ members, lose.

Formally, we define a simple game on a finite set of players N as an ordered pair, denoted by $\Gamma(N, \mathcal{W})$, where

$$\varnothing \subset \mathcal{W} = \mathcal{W}^+ \subset \mathcal{N} \quad (1)$$

The first strict inclusion tells us that N is always in $\mathcal{W}$ and the second tells us that $\varnothing$ is never in $\mathcal{W}$, as desired. In the absence of any more specific interpretation, the elements of $\mathcal{W}$ will be called *winning coalitions* and since every superset of a winning coalition is winning, when defining a particular simple game it is necessary only to state $\mathcal{W}^m$, the set of minimal winning coalitions. As we shall see, $\mathcal{W}$ the set of winning coalitions, will have a variety of different interpretations in our organization theory. Players who belong to at least one minimal wining coalition are called *essential,* those who do not are called *dummies.* A *veto* player is one who belongs to every minimal winning coalition, while a *master* is one who by himself forms a minimal winning coalition. If one player alone by himself makes up the unique minimal winning coalition, we designate him a *dictator.*

Coalitions that do not win are denoted losing coalitions. A *blocking* coalition is one that can prevent the formation of any winning coalition whatsoever. A game is *proper* if the intersection between its winning coalitions and its complements is empty. If the union of these two collections equals $\mathcal{N}$ the game is *strong.* Games that are strong and proper are called *decisive.* It is interesting to observe that in a proper game winning implies blocking while in a strong

[3]Stated differently we could have used 3a. "any subset of a losing coalition also loses." But recall that our primitive concept is winning, not losing.

game blocking implies winning. Note also that the concepts of weak and strong are not direct opposites. Since in a dictatorship all the players other than the dictator are dummies we may denote a simple game as *essential* if it is not a dictatorship.

Example 1 *Consider the following game*

$G = \Gamma(N, \mathcal{W})$ where $N = \overline{abcd}$ and $\mathcal{W}^m = \{\overline{a}, \overline{bc}\}$.
Then there are ten winning coalitions,

$$\mathcal{W} = \{\overline{a}, \overline{ab}, \overline{ac}, \overline{ad}, \overline{bc}, \overline{abc}, \overline{abd}, \overline{acd}, \overline{bcd}, \overline{abcd}\}$$

and six losing coalitions,

$$\mathcal{L} = \{\varnothing, \overline{b}, \overline{c}, \overline{d}, \overline{bd}, \overline{cd}\}.$$

Thus,

$$\mathcal{W}^* = \{\varnothing, \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{ad}, \overline{bc}, \overline{cd}, \overline{bd}, \overline{bcd}\}$$

and the blocking coalitions are,

$$\mathcal{BK} = \mathcal{L}^* = \{\overline{ab}, \overline{ac}, \overline{abc}, \overline{abd}, \overline{acd}, \overline{abcd}\}.$$

So that the minimal blocking coalitions are,

$$\mathcal{BK}^m = \{\overline{ab}, \overline{ac}\}.$$

There is one master a, one dummy d, and there are no dictators. Because it has a pair of disjoint minimal winning coalitions, the game is improper; and strong.

Example 2 *Consider an assembly of 100 members that requires a 2/3 majority to pass a resolution. This is a proper, nondecisive weak game. The winning coalitions are those who have at least 67 members, the minimal winning coalitions are those with exactly 67 members, and the losing coalitions are those with less than 67 members. Coalitions with more than 33 members block, and there are no dictators, masters, dummies, or veto players.*

3 Command and Control Games

Our formal treatment of an AS begins with an account of its local structure — what might be called the "worm's-eye" view. In any AS or decision-making body — i.e., *an organizational authority structure* — each of its members is directly concerned with only a small fraction of all the official orders, requisitions, authorizations, etc., that flow through the organization. Some members may have a certain degree of discretionary power; some may even be free agents, accountable to no one. Others may be merely cogs in the machinery. The following sections develop a formal game-theoretic model for these ideas.

3.1 Bosses and Approvers

Let N denote the set of all members of an organization, and let i denote a generic individual member of N. In general there will be certain other individuals, or more generally, sets of other individuals, that i must obey. We call them *boss sets* and denote them collectively by $\mathcal{B}_i$. Note the script letter. Thus, if there is an individual b who can boss i, this is indicated by $\bar{b} \in \mathcal{B}_i$, not by $b \in \mathcal{B}_i$.

For all $i \in N$, we assume:

$$\not{\emptyset} \subseteq \mathcal{B}_i = \mathcal{B}_i^{+} \cap \mathcal{N}_i \subset \mathcal{N}_i \tag{2}$$

which may be compared with equation (1). The strict inclusion ensures that $\varnothing$ is never a boss set. But all other subsets of $\mathcal{N}_i$ are eligible, subject to the condition that every superset, in $\mathcal{N}_i$, of a boss set is also a boss set.

Recalling that a rooted tree is a finite, acyclic graph having a distinguished node called the root, we find that there may be several different ways to set up a hierarchical structure, where the members of N are represented as the nodes of a rooted tree.

Example 3 *Consider the basic chain-of-command hierarchy, where there is a leader at the top who is unbossed, and where each lower-ranked member is bossed by just those coalitions that include his immediate superior, who is in effect his personal dictator. We shall refer to this structure as a Type I hierarchy.*

Thus if a denotes the leader, we have:

$$\mathcal{B}_a = \not{\emptyset}$$

and, for all $i \in N_a$

$$\mathcal{B}_i = \{S \in \mathcal{N}_i \mid j_i \in S\}$$

where j_i denotes i's unique immediate superior (see Figure 1.)

A close variation of the previous authority structure is the Type II hierarchy, which is one where commands are not necessarily forwarded through channels. A private has to obey direct orders from his captain and his colonel as well as from his sergeant. In this case we again have a tree structure, but the boss sets now take the form

$$\mathcal{B}_i = \{S \in \mathcal{N}_i : S \cap J_i \neq \varnothing\}$$

for all $i \in N$, where J_i is the set of all superiors of member i. For the moment, we merely observe that although these two hierarchies are in some sense equivalent, their local command structures are not the same.

Just as there were some particular agents, or set of agents, that could boss any single individual, there are other set of agents, that can approve of her. Thus, we shall associate with every $i \in N$ another, wider collection $\mathcal{A}_i$ of coalitions in $\mathcal{N}_i$, called *approval sets*, that can approve i's actions. The consent of

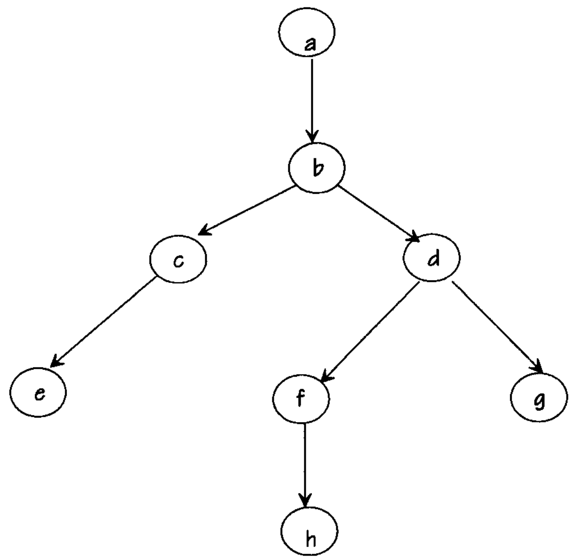

Figure 1: Type I hierarchy

any of these sets of approvers is sufficient to allow i to act, if he wishes to do so. However, it may not be able to force him to act. So approval sets are not necessarily boss sets. On the other hand, any boss set is *a fortiori* an approval set.

Formally, we have

$$\emptyset \subset \mathcal{A}_i = \mathcal{A}_i^+ \cap \mathcal{N}_i \subseteq \mathcal{N}_i \tag{3}$$

for all $i \in N$

A study of equation (3) reveals that always $N_i \in \mathcal{A}_i$, and that possibly $\varnothing \in \mathcal{A}_i$. The set difference $(\mathcal{A}_i - \mathcal{B}_i)$ gives us a guide to the amount of personal discretion that i enjoys, if any. At one extreme, if $\mathcal{A}_f - \mathcal{B}_f = \mathcal{N}_f$ then f is called a *free agent.* He needs no approval since $\varnothing \in \mathcal{A}_f$ and no one can boss him, since $\mathcal{B}_f = \emptyset$. At the other extreme, if $\mathcal{A}_c - \mathcal{B}_c = \emptyset$, then c has no discretionary power, and we shall call him a *cog.*

For an intermediate example of partial-discretion, consider a corporation president who is bossable by a $\frac{2}{3}$ majority of the board of directors but is allowed to follow his own judgment so long as he has the support of a simple majority.

3.2 Command Games

Although we presented the boss and approval notions separately, they are really halves of a single *command* concept that can be expressed very naturally as a simple game. Define for each $i \in N$, the set of *commanding coalitions* for i by:

$$\mathcal{W}_i = \mathcal{B}_i \cup \left\{ S \cup \overline{i} : S \in \mathcal{A}_i \right\} \tag{4}$$

Then $G_i = \Gamma(N, \mathcal{W}_i)$ is a well defined simple game, since (1) follows directly from equations (2), (3), and (4). We shall call G_i the *command game* for i. The command games tell every member i (of the AS) precisely who can order them around. Specifically, on receipt of an order signed by all members of coalition S:

(i) i must follow the order if $S \in \mathcal{W}_i$ and $i \notin S$ — such a set S constitutes a boss set for i ;

(ii) i will also follow the order if $S \in \mathcal{W}_i$ and $i \in S$, but only because they have signed the order — such a set S with i removed constitutes an approval set for i ;

(iii) i must ignore the order if $S \notin \mathcal{W}_i$

The ensemble of command games: $G = \{G_i : i \in N\}$ completely specifies the authority structure or constitution of the organization.

From the definition of a command game's winning coalitions we can readily see that the free agents are dictators in their own command games while the cogs are dummies in their own command games.

Example 4 *Returning to our two types of hierarchy in our previous Examples, to complete their descriptions we need to define the approval sets. It seems natural in both cases to make the leader a free agent and the rest of the members cogs. The following Table gives all the respective command games:*

	Type I			**Type II**		
i	$\mathcal{B}_i^m$	$\mathcal{A}_i^m$	$\mathcal{W}_i^m$	$\mathcal{B}_i^m$	$\mathcal{A}_i^m$	$\mathcal{W}_i^m$
a	$\emptyset$	$\mathcal{N}_i$	$\{\overline{a}\}$	$\emptyset$	$\mathcal{N}_i$	$\{\overline{a}\}$
b	$\{\overline{a}\}$	$\{\overline{a}\}$	$\{\overline{a}\}$	$\{\overline{a}\}$	$\{\overline{a}\}$	$\{\overline{a}\}$
$c-d$	$\{\overline{b}\}$	$\{\overline{b}\}$	$\{\overline{b}\}$	$\{\overline{b},\overline{a}\}$	$\{\overline{b},\overline{a}\}$	$\{\overline{b},\overline{a}\}$
e	$\{\overline{c}\}$	$\{\overline{c}\}$	$\{\overline{c}\}$	$\{\overline{c},\overline{b},\overline{a}\}$	$\{\overline{c},\overline{b},\overline{a}\}$	$\{\overline{c},\overline{b},\overline{a}\}$
$f-g$	$\{\overline{d}\}$	$\{\overline{d}\}$	$\{\overline{d}\}$	$\{\overline{d},\overline{b},\overline{a}\}$	$\{\overline{d},\overline{b},\overline{a}\}$	$\{\overline{d},\overline{b},\overline{a}\}$
h	$\{\overline{f}\}$	$\{\overline{f}\}$	$\{\overline{f}\}$	$\{\overline{f},\overline{d},\overline{b},\overline{a}\}$	$\{\overline{f},\overline{d},\overline{b},\overline{a}\}$	$\{\overline{f},\overline{d},\overline{b},\overline{a}\}$

Note that in a Type I hierarchy all the command games are dictatorships, while in a Type II hierarchy many command games have more than one master, and hence no dictator.

It will be useful to have an alternative way of speaking of boss sets $\mathcal{B}_i$, ($\mathcal{A}_i$ and $\mathcal{W}_i$;) we accomplish this by introducing certain set-to-set functions that map $\mathcal{N}$ into itself.

The *boss function* is the set of all individuals that must obey an order issued by S,

$$\beta(S) = \left\{ i \in N : S \in \mathcal{B}_i^{+} \right\} \tag{5}$$

similarly, we construct the *approval* $\alpha(S)$ and *command function* $\omega(S)$.

From the previous definitions it can easily be seen that:

$$\beta(S) \subseteq \alpha(S) \subseteq \omega(S) \text{ for all } S \in \mathcal{N}$$

Note also that these three functions — boss, approval, and command — are *monotonic* in the sense that:

$T \subseteq S \Longrightarrow f(T) \subseteq f(S)$, for $f = \beta, \alpha, \omega$.

Most statements about $\mathcal{B}$, $\mathcal{A}$, $\mathcal{W}$ translate easily into synonymous statements about β, α, and ω, which are in a sense their inverse functions. Thus, if we are given β we can recover $\mathcal{B}$ from the relation

$$\mathcal{B}_i = \{S \in \mathcal{N} : i \in \beta(S)\}$$

Similarly, if given α,(ω) we can recover $\mathcal{A}$,($\mathcal{W}$) respectively.

3.3 Control Games

The command games described the local patterns of authority. However, these games do not suffice to give an adequate account of the global distribution of authority throughout the organization. We will attempt that task by developing the notion of *control.* Since commands can themselves be commanded, and approvals approved, the "local" concepts of command and approval can be extended by an iterative procedure to a "global" concept of control. The resulting *control function* γ, is similar in form to the boss (β), approval (α), and command (ω) functions previously defined. Then, we shall reverse the inversion to obtain a set of *control games* $H_i = \Gamma(N, \mathcal{C}_i)$ similar in form to the command games $G_i = \Gamma(N, \mathcal{W}_i)$.

It is important to emphasize the fact that the notion of control is a derived concept, i.e., no new information is contained in γ or the H_i's. Nevertheless, a substantial amount of calculation may be required to obtain the control games H_i's from the command games G_i's.

Similar to the β,(α, ω) functions, the control function $\gamma(S)$ represents the set of members that coalition S can control, regardless of possible opposition from any or all of the other members of the organization. In defining $\gamma(S)$ we must recognize, on the one hand, the possibility of indirect control — i.e., members outside S being co-opted to join with S in bossing other outsiders — and on the other hand, the possibility that some members of S may not have full control over their actions — i.e., they require consent from some set of approvers before they can participate in the bossing of others.

Formally, let F be the set of all free agents in an AS with N members. For each $S \in \mathcal{N}$ we construct $\gamma(S)$ with the aid of an increasing nest of subsets of N:

$$\gamma_0 \subseteq \gamma_1 \subseteq \gamma_2 \subseteq \gamma_3 \subseteq \cdots \tag{6}$$

which we shall call the *control sequence* for S. It begins with the free agents:

$$\gamma_0 = F \cap S$$

and builds recursively according to the rule

$$\gamma_k = \beta(\gamma_{k-1}) \cup \{S \cap \alpha(\gamma_{k-1})\} \tag{7}$$

for $k = 1, 2, 3 \ldots$

Note that a coalition does not automatically control itself; indeed, without a free agent to start the ball rolling it controls nobody. Coalitions that control exactly themselves are called exact; they play a special role in the structural analysis of control [2]. We state without proof several general results and some properties of the control function.

Lemma 5 *Let $n = |N|$. There is a nonnegative integer $k^* \leq n-1$ such that the control sequence (6) increases strictly up to the term γ_{k^*} and is constant thereafter.*

Thus, we can now define the control function in several different ways:

$$\gamma(S) = \gamma_{k^*}(S) \text{ or } \gamma_{n-1}(S) \text{ or } \lim_{k \longrightarrow \infty} \gamma_k(S) \text{ or } \bigcup_{k=0}^{\infty} \gamma_k(S)$$

which by Lemma (5) are all equal.

If $S = N$ rule (7) simplifies to

$$\gamma_k(N) = \alpha(\gamma_{k-1}(N))$$

Note that it is by no means inevitable that the grand coalition is exact (i.e., $\gamma(N) = N$.) Individuals in $\gamma(N)$ will be called *controllable*, while those in $N \setminus \gamma(N)$ are denoted *uncontrollable*, (or not fully controllable) and for many purposes can be ignored. Free agents are self-controlled, and it can be shown that every AS has at least $2^{|F|-1}$ non-empty exact coalitions, where F is the set of free agents.

Theorem 6 *The function γ is monotonic.*

Corollary 7 *For all $S, T \in \mathcal{N}$*

$$\gamma(S \cup T) \supseteq \gamma(S) \cup \gamma(T)$$

$$\gamma(S \cap T) \subseteq \gamma(S) \cap \gamma(T)$$

Theorem 8 *If S contains no free agents, then $\gamma(S) = \varnothing$. In particular we have that $\gamma(\varnothing) = \varnothing$.*

Thus (taking $S = N$), if no one in an organization is free, then no one is controllable.[4]

Theorem 9

$$\textit{If } R \subseteq \gamma(S) \textit{ then } \gamma(S \cup R) = \gamma(S)$$

$$\textit{If } R \subseteq N \setminus \gamma(S) \textit{ then } \gamma(S \setminus R) = \gamma(S)$$

Broadly speaking, this says that if outsiders controlled by a coalition are admitted to membership the coalition is not strenghtened, while if insiders not under the coalition's control are expelled the coalition is not weakened.

Theorem 10 *Always* $\beta(\gamma(S)) \subseteq \gamma(S)$.

That is, anyone bossable by a controlled set is subject to the same control.

Theorem 11 *Always* $\gamma(\gamma(S)) = \gamma(S)$.

This expresses the notion of transitivity of control. It states that anyone controllable by a controlled set is subject to the same control, and conversely. In other words, a controlled set controls its own members, and no one else.

Now the stage is set to advance forward: we only need to invert the control function γ so as to obtain the corresponding class of control games. This is the reverse of the similar process that gave us the command function ω, from the ensemble of command games $G = \{G_i : i \in N\}$.

Thus, for each $i \in N$ we define the set of *controlling coalitions* for i by:

$$\mathcal{C}_i = \{S \in \mathcal{N} : i \in \gamma(S)\} \quad (8)$$

Unfortunately, the pair $(N, \mathcal{C}_i)$ does not necessarily define a simple game according to definition (1). Theorems (6) and (8) assure us that $\mathcal{C}_i = \mathcal{C}_i^+$ and that $\mathcal{C}_i^+ \neq \mathcal{N}$, respectively. But nothing guarantees that $\mathcal{C}_i$ the set of controlling coalitions for player i is not empty, and this violates (1). Actually, $\mathcal{C}_i \neq \emptyset$ if and only if $i \in \gamma(N)$. That is, we have that only controllable players can have control games.

Therefore, we define the *control game* for each $i \in \gamma(N)$ to be:

$$H_i = \Gamma(N, \mathcal{C}_i)$$

where $\mathcal{C}_i$ is given by (8).

We shall use H to denote the ensemble of control games

$$H = \{H_i : i \in \gamma(N)\}$$

Recall that a free agent (cog) is a dictator (dummy) in his own command game. Hence,

[4] In future research, we shall use this property to characterize the state of anarchy, which applies to conflict-cooperation issues.

Theorem 12 *The set F of all free agents is a blocking coalition in every control game.*

Theorem 13 *Every cog is a dummy in every control game.*

3.4 Organizations

The problem with defining our model in terms of the local notion of command is that the AS may have to contend with the logical inconsistency of improper control games. An improper command game is not necessarily a defect in the authority pattern, since two disjoint commanding coalitions may be subject to a common higher control. However, an improper control game is a serious defect, because it means that there are independent subsets of the organization that can send contradictory instructions to the same individual agent — instructions which, under the rules, that agent must obey.

Therefore, we shall have to recognize the existence of a fundamental distinction between those systems of command that are free from such logical inconsistencies, and those that are not.

3.4.1 Proper Organizations

Formally, we define an *organization* — i.e., an organizational authority structure — as a system $O(N,G)$ where N is a finite set of players and G is an ensemble of command games:

$$G = \{G_i : i \in N\}$$

where $G_i = \Gamma(N, \mathcal{W}_i)$.

However, to consider the system $O(N,G)$ a proper organization, we shall require that *all its members be controllable* and that *all its control games be proper*. Accordingly, this definition can then be stated in two different ways, either using the control games or using the control function.

Formally, an organization $O(N,G)$ is *proper* if for every $i \in N$

$$\mathcal{C}_i \neq \emptyset \tag{9}$$

and

$$\mathcal{C}_i \cap \mathcal{C}_i^* = \emptyset \tag{10}$$

Equivalently, an organization $O(N,G)$ is *proper* if for every $Q, R \in \mathcal{N}$

$$\gamma(N) = N \tag{11}$$

and

$$Q \cap R = \varnothing \Longrightarrow \gamma(Q) \cap \gamma(R) = \varnothing \tag{12}$$

We shall say that an organization is *hierarchic* if its members can be represented by a rooted tree in such a way that in each command game G_i the essential players are all superior or equal to i.

We shall say that an organization is a *pyramidal-type* organization, if it has a hierarchic control structure that can be represented by a rooted tree with a unique free agent — where the free agent is the root. Common examples of pyramidal type organizations are Type I and Type II hierarchies, the Armed Forces, the Catholic Church, etc.

Corollary 14 *Every pyramidal-type organization is proper.*

Theorem 15 *If all the command games are proper then all the control games are proper.*

Clearly the converse of Theorem (15) is not valid.

4 Tasks

In previous sections we have emphasized the procedures of command and control, of the internal contractual aspects of the organization's AS, or in other words, of the governance of the organization. However, most organizations have an external material purpose as well, so that at least some of the commands that flow through the authority system have a substantive content, and at least some of the power to control that the organization's members exercise, does extend beyond the external boundaries of the organization into another larger domain. In order to set up the framework of analysis for the next topic — that of *responsibility* — we need to formalize the organization's external environment, i.e., to model the way how the organization members' power to control extend itself into that external space or dimension, i.e., into the outside world. We shall accomplish this by considering a collection of *tasks* that the AS may be called upon to perform. For reasons of simplicity we shall only contemplate tasks one at a time, and we shall treat them as "black boxes" because their internal structure will not interest us. On the other hand, we shall be very much concerned with the interface between the tasks and their potential effectors within the organization.

We shall begin by defining the notion of *simple tasks.* In a simple task, just as in a simple game before, only two possibilities are recognized: either the job gets done or it doesn't. All other issues, no matter how important — such as cost or quality, for example — are completely ignored. Thus, when viewed abstractly, simple tasks are nothing more than simple games. As such, they fit neatly into the organization's authority structure by means of the compounding mechanism of simple games. This conceptualization leads us directly to the construction of other related games, the *task control games*, fact which in turn will allow us to make the quantification of responsibility we are searching.

4.1 Task Games

Our starting assumption will be that, for any given task, only a small fraction of the members of an organization $O(N, G)$ are likely to be occupied with it, and further, that of those members who are involved, few perhaps none, are likely to be indispensable. This is due to the existence of substitutes, i.e., other members who have equivalent skills within O. For example, a particular job might require the following team: two carpenters, four computer programmers, and one administrator — or perhaps it can be done with an alternative team of only three programmers, if one of them has had just the right training or is familiar with the lumber industry. Thus, there is a substitution principle at work when assembling task effectors, and in general there will be many different ways for an AS to assemble an effective task force.

We thus define the *capable sets* as all those working parties of *effectors* drawn from the organization's membership who, by virtue of their number, their mix of skills, the resources they can supply, their general efficiency, or their specialized experience, are adequate for the particular task at hand.

Formally, let $O(N, G)$ be a proper organization, let τ be a task for O, and let $\mathcal{E}_\tau \subseteq \mathcal{N}$ denote the set of all *τ-capable sets*. Since we, as usual, are interested in simple games, we impose the familiar conditions

$$\emptyset \subset \mathcal{E}_\tau = \mathcal{E}_\tau^+ \subset \mathcal{N} \tag{13}$$

and we then define the *task game* for τ by

$$K_\tau = \Gamma(N, \mathcal{E}_\tau) \tag{14}$$

Every thing we need to know about τ (for the present) is embodied in K_τ. In the same way that we formalized our intuitions in the definition of a simple game, the three connectives used in (13) above correspond to the following similar modelling assumptions:

1. there is at least one τ-capable set
2. every superset of a τ-capable set is also a τ-capable set
3. the empty set is not a τ-capable set

4.2 Task Control Games

Let us now change our emphasis on the effectors and let us look at the controllers instead. The approach we use is based in the compounding mechanism for simple games. The individual control games H_i will be the components (or committees, in general overlapping), while the task game K_τ will play the role of quotient.

Formally, if τ is any simple task in the reach of the organization $O(N, G)$, then the *task control game* for τ will be the compound

$$TC_\tau = K_\tau[H_i : i \in N] \tag{15}$$

In other words, a coalition controls a task *if and only if* it controls a capable set of effectors for that task.

We shall say that a task is *internal* to the organization when it is directly related to management activities. On the other hand, when a task is unrelated to administering the AS we shall say it is an *external* task.

4.3 Example

Consider the following situation: a university research team has been assembled in order to conduct a special research project. The team has six members and is led by a principal investigator (member L) and consists of two additional senior investigators (members J and M), a group of research assistants (member P), who are assisted in their work by computational and laboratory resources (member C) and further, they also have under their jurisdiction the additional help of library and reference resources (member Q).

The team allocates their research activities in accordance with Figures 4.2 and 4.3, where the expected output of the project is modeled as the successful completion of tasks T_1 and T_2.

We may easily observe that the traditional organizational chart (Figure 4.1) fails to account adequately for the implicit degrees of control and responsibility of the different research team members involved in the project.

Flow Diagram of the Organization

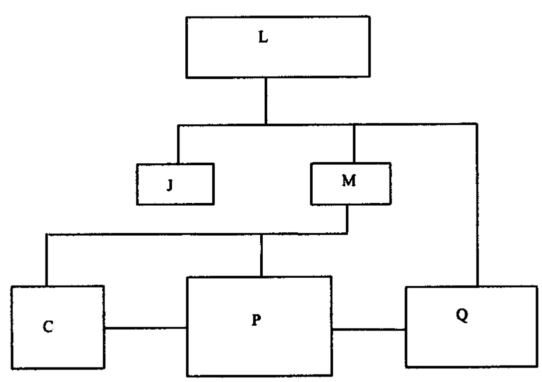

Figure 4.1

Further, an initial attempt at quantifying the authority of the investigators, for example a simple game representation of the "board" of investigators (i.e., an $M_{4,3}$ simple majority game) does not convey more specific information of the actual powers of control that the individual investigators actually have over the terminal outcomes of the project, and thus it is unsatisfactory.

One of the particular strengths of our AS model is the ability to discriminate clearly, and to elucidate, the real control structures at work. This is possible because of our use of a set-theoretical framework of analysis.

Example of Authority Structure

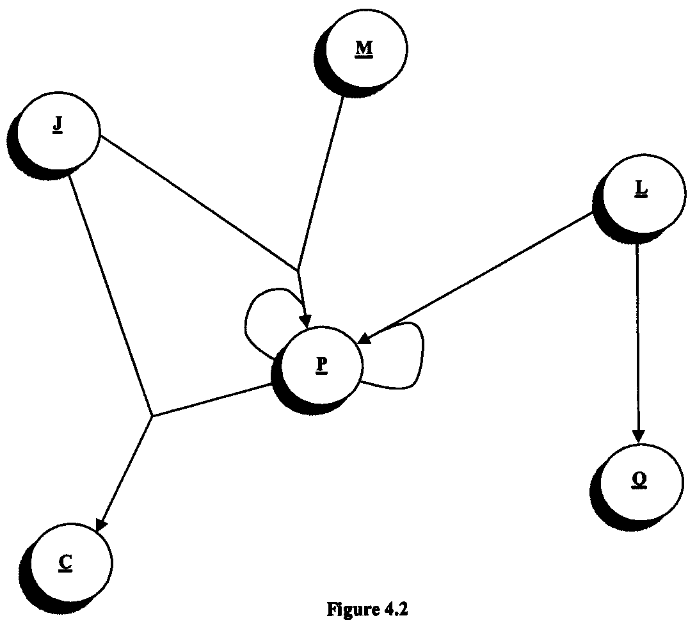

Figure 4.2

Figures 4.2 and 4.3 show the different specifications of the example; the principal and senior investigators are free agents who allocate their duties according to their particular interests and capabilities. Both senior researchers work together with the team of assistant researchers in commanding the computational resources whereas the latter two teams are essential to the successful completion of task T_1.

On the other hand the leader also works in team with the assistants researchers but he also has a direct chain of command with the reference resources. These teams are essential for the successful completion of task T_2.

Commanding coalitions

$$\mathcal{W}_L^m = \{\overline{L}\}, \ \mathcal{W}_J^m = \{\overline{J}\}, \ \mathcal{W}_M^m = \{\overline{M}\},$$
$$\mathcal{W}_P^m = \{\overline{LP}, \overline{JMP}\}, \ \mathcal{W}_C^m = \{\overline{JP}\}, \ \mathcal{W}_Q^m = \{\overline{L}\}$$

Command Games:

$$G_i = \Gamma(N, \mathcal{W}_i^m) \ \text{for } i = L, J, M, P, C, Q$$

Controlling Coalitions

$$\mathcal{C}_L^m = \{\overline{L}\}, \ \mathcal{C}_J^m = \{\overline{J}\}, \ \mathcal{C}_M^m = \{\overline{M}\},$$
$$\mathcal{C}_P^m = \{\overline{LP}, \overline{JMP}\}, \ \mathcal{C}_C^m = \{\overline{JP}\}, \ \mathcal{C}_Q^m = \{\overline{L}\}$$

Control Games

$$H_i = \Gamma(N, \mathcal{C}_i^m) \ \text{for } i = L, J, M, P, C, Q$$

Task Game for T_1

$$K_{T_1} = \Gamma\left(N, \mathcal{E}_{T_1}^m\right) \ \text{where } \mathcal{E}_{T_1}^m = \{\overline{PC}\}$$

Task Control Game for T_1

$$TC_{T_1} = K_{T_1}[H_i : i \in N] \ \text{where } H_i = \Gamma(N, \mathcal{C}_i^m) \ \text{for } i \in N$$

$$C(T_1) = \mathcal{C}_P \cap \mathcal{C}_C = \{\overline{LP}, \overline{JMP}\}^+ \cap \{\overline{JP}\}^+$$

$$C^m(T_1) = \{\overline{JP}\}$$

Task Game for T_2

$$K_{T_2} = \Gamma\left(N, \mathcal{E}_{T_2}^m\right) \ \text{where } \mathcal{E}_{T_2}^m = \{\overline{PQ}\}$$

Task Control Game for T_2

$$TC_{T_2} = K_{T_2}[H_i : i \in N] \ \text{where } H_i = \Gamma(N, \mathcal{C}_i^m) \ \text{for } i \in N$$

$$C(T_2) = \mathcal{C}_P \cap \mathcal{C}_Q = \{\overline{LP}, \overline{JMP}\}^+ \cap \{\overline{L}\}^+$$

$$C^m(T_2) = \{\overline{LP}\}$$

5 Responsibility

We propose solving the problem of quantifying the distribution of responsibility in an AS by a simple application of the Shapley-Shubik power index [7] for simple games.

Terms like "power" or "responsibility" present serious difficulties of use, because besides the fact that they are multidimensional concepts, they have no established formal meaning in organization theory. Nevertheless, it seems quite natural — and of great practical use — to have a reliable numerical measure of organizational responsibility.

5.1 The Responsibility Index

Formally, let

$$O = O(N, G) \qquad \text{where } G = \{G_i = \Gamma(N, \mathcal{W}_i) : i \in N\} \tag{16}$$

be a proper organization, and let τ be a task for O. Then, the ***responsibility index*** of member i for task τ, denoted $\rho_i(O, \tau)$, shall be defined to be the power index of player i in the task control game TC_τ for the task τ.

We can readily see that the responsibility index $\rho_i(O, \tau)$ has the following properties, which we state here without further proof

A member of the organization has full responsibility if and only if he is a dictator in the task control game

$$\rho_i = 1 \Longleftrightarrow TC_\tau = \{\bar{i}\}$$

A member of the organization has zero responsibility if and only if he is a dummy in the task control game

$$\rho_i = 0 \Longleftrightarrow i \notin TC_\tau^{m\cup}$$

Members who are symmetric in the task control games have equal responsibility for the task

$$i, j \text{ symmetric in } TC_\tau \Longrightarrow \rho_i = \rho_j$$

Every member who is indispensable for the task has as much responsibility as every other indispensable member, and more responsibility than anyone else

$$\rho_i = \max_{k \in N} \rho_k \Longleftrightarrow i \in TC_\tau^{m\cap}$$

Hence, we immediately can deduct the following results

Corollary 16 *Only a free agent can ever have total responsibility*

$$\rho_i = 1 \Longrightarrow i \in F \Longleftrightarrow \mathcal{W}_i = \{\bar{i}\}$$

Corollary 17 *Cogs are never responsible*

$$i \notin \mathcal{W}_i^{m\cup} \Longrightarrow \rho_i = 0$$

5.2 Examples

In Section 3.1 we studied two different hierarchic organizations — Type I and Type II. These two hierarchic structures are uninteresting to us now, since from the control viewpoint they are both dictatorships. Thus, if a denotes the leader, then the responsibility index for the leader is $\rho_a = 1$ for all τ, where naturally $\rho_i = 0$ for all other $i \neq a$ players.

Hence, we shall consider as more interesting examples, two similar but somewhat more permissive hierarchies, defined below:

$$\text{Type III} \qquad G_i = \Gamma\left(N, \{\overline{ij_i}\}^+\right)$$

$$\text{Type IV} \qquad G_i = \Gamma\left(N, \{\ \overline{ij} : j \in J_i\}^+\right)$$

where j_i and J_i are as before.

Thus, in a Type III hierarchy only an immediate superior can grant permission whereas in a Type IV any superior can do so.

The control games for $i \neq a$ are

$$\text{Type III}: \qquad H_i = \Gamma\left(N, \{J_i \cup \bar{i}\}^+\right)$$

$$\text{Type IV}: \qquad H_i = \Gamma\left(N, \{\overline{ai}\}^+\right)$$

For a numerical case take the following simple tree

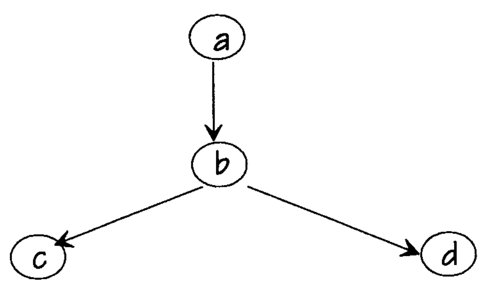

Figure 2: Example

and consider two task games: $K_\sigma = \Gamma\left(\overline{abcd}, \{\overline{cd}\}^+\right)$ and $K_\tau = \Gamma\left(\overline{abcd}, \{\bar{c}, \bar{d}\}^+\right)$.

That is, to accomplish task σ, both members c and d are indispensable effectors where anyone of them can carry out task τ, so the task control games for task σ are:

$$\text{Type III}: \qquad TC_\sigma = \Gamma\left(\overline{abcd}, \{\overline{abcd}\}^+\right)$$

$$\text{Type IV}: \qquad TC_\sigma = \Gamma\left(\overline{abcd}, \{\overline{acd}\}^+\right)$$

By properties (i) and (iii) above, we see that the responsibility indices must be:

$$\text{Type III}: \qquad \rho_a = \frac{1}{4}, \quad \rho_b = \frac{1}{4}, \quad \rho_c = \frac{1}{4}, \quad \rho_d = \frac{1}{4}$$

$$\text{Type IV}: \qquad \rho_a = \frac{1}{3}, \quad \rho_b = 0, \quad \rho_c = \frac{1}{3}, \quad \rho_d = \frac{1}{3}$$

Note that b is a dummy in the Type IV organization because he is superfluous in the "chain of approval"(i.e., he has zero responsibility.)

For task τ we have the following task control games

$$\text{Type III}: \qquad TC_\tau = \Gamma\left(\overline{abcd}, \{\overline{abc}, \overline{abd}\}^+\right)$$

$$\text{Type IV}: \qquad TC_\tau = \Gamma\left(\overline{abcd}, \{\overline{ac}, \overline{ad}\}^+\right)$$

By properties (i) and (iii) above, the responsibility indices are:

$$\text{Type III}: \qquad \rho_a = \frac{5}{12}, \quad \rho_b = \frac{5}{12}, \quad \rho_c = \frac{1}{12}, \quad \rho_d = \frac{1}{12}$$

$$\text{Type IV}: \qquad \rho_a = \frac{2}{3}, \quad \rho_b = 0, \quad \rho_c = \frac{1}{6}, \quad \rho_d = \frac{1}{6}$$

Note that with c and d no longer indispensable, the balance of responsibility shifts toward the "higher echelons". Again, b is a dummy in a Type IV organization.

Returning to our AS example in Section 4.3 we obtained the following task control games:

Task Control Game for T_1

$$TC_{T_1} = K_{T_1}[H_i : i \in N] \text{ where } H_i = \Gamma(N, \mathcal{C}_i^m) \text{ for } i \in N$$

$$C(T_1) = \mathcal{C}_P \cap \mathcal{C}_C = \{\overline{LP}, \overline{JMP}\}^+ \cap \{\overline{JP}\}^+$$

$$C^m(T_1) = \{\overline{JP}\}$$

Task Control Game for T_2

$$TC_{T_2} = K_{T_2}[H_i : i \in N] \text{ where } H_i = \Gamma(N, \mathcal{C}_i^m) \text{ for } i \in N$$

$$C(T_2) = \mathcal{C}_P \cap \mathcal{C}_Q = \{\overline{LP}, \overline{JMP}\}^+ \cap \{\overline{L}\}^+$$

$$C^m(T_2) = \{\overline{LP}\}$$

Therefore, the responsibility indexes (in each task) for all the members are:

$$\begin{array}{ll}
\rho_L(T_1) = 0 & \rho_L(T_2) = 1/2 \\
\rho_J(T_1) = 1/2 & \rho_J(T_2) = 0 \\
\rho_M(T_1) = 0 & \rho_M(T_2) = 0 \\
\rho_P(T_1) = 1/2 & \rho_P(T_2) = 1/2 \\
\rho_C(T_1) = 0 & \rho_C(T_2) = 0 \\
\rho_Q(T_1) = 0 & \rho_Q(T_2) = 0
\end{array}$$

Computation of the responsibility indexes (i.e., the power index in the task control game for the respective tasks) shows that the burden of the credit (or blame) for T_1 clearly lies between (equally shared actually) one of the senior investigators (J) and the team of assistant researchers (P) while the responsibility of task T_2 is shared (halved also) between the leader (L) and the assistant investigators (P). It seems somewhat improbable, that we could have detected the actual distribution of responsibility of the different individuals involved in the project, from the standard hierarchical organizational chart (Figure 4.1.)

6 Conclusions

Starting from basic notions of command (local) and control (global) and using simple games as our building blocks we have developed a set theoretic descriptive model of an AS. The consequence of this set-theoretic emphasis is a Boolean mathematical structure which in turn explains the essential characteristics of our model and of our approach to organization theory. One of the relevant features of our model is its ability to assign a quantitative index of responsibility for every member of the AS in the actual accomplishment of tasks. We have seen that the traditional graph theoretic approache cannot easily do this. Recall that the interaction with the external environment of the organization was modeled by defining simple task games.

It is important here to emphasize the distinction between skilled teams and responsible parties for the task at hand. For any given task there may be many teams of skilled workers in the organization that could handle it. These teams define the task game for the particular task. Nevertheless, the relevant game (for distributing responsibility among members of the AS) is the task control game which is defined by the coalitions that control the task game. All responsibility for a given task lies with the non-dummy players of the task control game; the skilled workers of the task game are "just following orders".

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