INDIVISIBILITY, HOUSING MARKETS AND PUBLIC GOODS

By

Bryan Ellickson*

Department of Economics, U.C.L.A.
and
The Rand Corporation

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The indivisibility of housing commodities should be obvious to anyone who thinks about the problem for a moment. Apart from the fact that one only has to look at a house to realize that this is so, the issue is forced by the logic of formal general equilibrium theory. In the grammar of axiomatic competitive analysis, all commodities must be time dated and indexed by location. If we impose the natural requirement that a consumer cannot be in two or more places at the same time, then the introduction of a spatial dimension means that consumption sets are necessarily non-convex.

Of course, there is nothing in this logic that requires the introduction of indivisible commodities. One could assume that the restrictions of consumption sets to specific locations are convex and that commodities, indexed by location, are perfectly divisible. But to do so obscures the basic reason for the non-convexity of spatial models. The impossibility of being in two different places at the same time is a reflection of the <u>indivisibility of the consumer</u>, and this source of indivisibility carries over to many different aspects of the consumer choice problem. Choosing a job, a house, a neighborhood each has the aspects of an all-or-nothing choice, and in each case the discreteness of the choice process seems to be intimately tied up with the identity of the consumer as an individual. This paper will present the case for the importance of recognizing this source of indivisibility to the development of a coherent theory of housing markets and local public goods.

The difficulties in constructing a theory that allows for indivisibility are more psychological than real. Recent advances in mathematical economics have demonstrated that perfect divisibility is not essential to the theory of perfect competition. But the old ways die hard. Years of tradition have ensconced perfect divisibility at the core of economic analysis, the apparent sine qua non for the osculation of smoothly bending curves with separating hyperplanes that drives the engine of competition. Indivisibility, on the other hand, conjures up images of corner solutions, scale economies and market failure. Against this background it is small wonder that economists have chosen to ignore indivisibility in their efforts to model housing markets and local public goods.

Matters are not improved by the language in which the new results from mathematical economics are presented. Conclusions reached through an appeal to the weak star convergence of Borel measures on an infinite dimensional commodity space, set in the context of a non-atomic measure space of consumers, are unlikely to reach a large audience. Nevertheless, the ideas involved are really quite straightforward and very compelling. To help bridge the communications gap, in this paper I will suppress references to the technicalities of measures, sigma algebras and the like which underlie my approach. There won't be any theorems either. (None are needed because Mas-Colell[1975,1977] provides everything that is required.) Instead we will work through a series of examples illustrating how one models economies with housing commodities or local public goods as markets for indivisible commodities.

1. Pure Exchange With a Finite Number of Indivisible Commodities

In setting up this first example, we establish some notation that will be used throughout the paper. Commodities fall into one of two classes, divisible and indivisible, which we label one and two respectively. We assume a non-atomic measure space of agents, denoted A. The commodity bundle allocated to consumer $a \in A$ is written

$$x(a) = (x_1(a), x_2(a))$$

where $x_1(a)$ is a vector of divisible commodities and $x_2(a)$ is the bundle of indivisible commodities. As a varies over the set of agents A, $x(a)$ indicates the allocation received by each of the consumers. Thus, we can regard the entire allocation as a function x defined over the set A.

In all of our examples we will assume that there is only one divisible commodity so that $x_1(a)$ is a scalar. The description of $x_2(a)$ requires more elaboration. Associated with the class of indivisible commodities is a compact metric space K, called the set of characteristics. Each point $z \in K$ represents a description of a particular type of indivisible commodity. For example, if we assume that K is a subset of R^n, then the components of $z \in K$ could be square feet of floor space, lot size, number of bedrooms, quality and so forth.

In this paper we are going to assume that each consumer chooses at most one unit and at most one type of indivisible commodity.[1] If the set K is finite, say $K = \{z_1, \ldots, z_m\}$, then $x_2(a)$ can be given a

[1] Mas-Colell's theory requires no such restriction, but this assumption will simplify our presentation considerably.

simple representation. For example if $m = 5$ and the consumer chooses one unit of the fourth type of indivisible commodity, then $x_2(a) = (0,0,0,1,0)$. If K is not finite, say $K = [0,1]$, then the representation of $x_2(a)$ is more complex. We will defer discussion of this problem to a later section. For now we confine ourselves to the case where K is finite. However, in order to establish a comparable notation for the finite and infinite case, we will write $x_2(z_i)$ for the ith component of the function x_2 rather than the more usual x_{2z_i}.

The distribution of endowments in the economy is given by a function $e: A \to R^{m+1}$ where e is shorthand for the m+1-tuple of functions

$$(e_1, e_2(z_1), \ldots, e_2(z_m)),$$

one for each type of commodity. In our applications, we will be interested primarily in the aggregate endowment, obtained by integrating e over the set of consumers. Accordingly, we define $b = \int_A e_1$ and

$$c = (c_1, \ldots, c_m) = (\int_A e_2(z_1), \ldots, \int_A e_2(z_m)).$$

At this juncture we need to address a possible source of confusion regarding the integrals we have just introduced. In our examples, we will have no difficulty in evaluating the integrals (only freshman calculus is involved), but readers more used to urban economics than the continuum of agents literature may find the results slightly puzzling. When urban economists integrate over a density of consumers, the answer equals N, the number of consumers. But in the continuum of agents context, there is no natural choice for N (because the number of consumers is infinite). An alternative approach is adopted where the integral of such a density over a subset of consumers equals the fraction of those consumers in the economy as a whole, and, as a result, the integral over the entire set of consumers equals one.

As a consequence of this procedure, the integral $\int_A e$ has a somewhat different interpretation from what one might expect: it equals the average or mean endowment of the economy and not the total. Similar comments apply to the aggregation of the functions

$$x = (x_1, x_2(z_1), \ldots, x_2(z_m))$$

describing the allocation of commodities to consumers. The integral

$$\int_A x = (\int_A x_1, \int_A x_2(z_1), \ldots, \int_A x_2(z_m))$$

represents the average amount of each of the commodities allocated to consumers. When we clear markets we set $\int_A x = \int_A e$ (which, because each integral is really an m+1-tuple of integrals, equates demand and supply in all of the markets), so that equating average allocation to average endowment can be regarded as essentially equivalent to equating the "total" allocation to the "total" endowment.

Our examples will all involve perfectly competitive markets with price-taking consumers and firms. The natural notation for the vector of prices is $p = (p_1, p_2)$ where p_1 is the price of the divisible commodity and p_2 is itself an m-tuple of prices, one for each type of indivisible commodity. However, rather than using p_{2z_i} for the ith component of p_2, we will use $h(z_i)$ to denote the price of the indivisible commodity of type z_i. As with the consumption vector x_2, this establishes a comparable notation for the case in which K is finite and the case in which K is infinite. As the letter "h" is supposed to suggest, the function h defined on K can be interpreted as a "hedonic" price function.

Having completed the preliminaries, we are ready for the first example. Consider an economy with three types of indivisible commodity, and let $K = \{1,2,3\}$ be the set of characteristics. For concreteness, imagine that these commodities represent three different types of houses of size (or "quality", or whatever) equal to 1, 2 or 3. All consumers have the same utility function where $U(x(a)) = x_1(a)(1+z)$ if consumer a chooses the indivisible commodity of type z and $U(x(a)) = x_1(a)$ if none of the indivisible commodities is chosen.

Recall that the mean endowment of the divisible commodity is denoted b and the mean endowment vector for the indivisible commodities, c. To simplify the computations which follow, we will assume equal means for each of the indivisible commodities: $c = (k,k,k)$ where $k > 0$. Letting $p_1 = 1$ as numeraire, mean income ("per capita GNP") for the economy is computed as follows:

$$Y = \int_A pe$$
$$= p_1\int_A e_1 + h(1)\int_A e_2(1) + h(2)\int_A e_2(2) + h(3)\int_A e_2(3) \quad (1)$$
$$= b + k(h(1)+h(2)+h(3)) \ .$$

where we have used pe to denote the scalar product

$$pe = p_1e_1 + p_2e_2$$
$$= p_1e_1 + h(1)e_2(1) + h(2)e_2(2) + h(3)e_2(3)$$

One of the difficulties of using any general equilibrium model is that computations can quickly become very complicated. In the present instance we want to allow consumers to have different incomes, but if we start with an arbitrary distribution of endowments the model becomes very complex (there is no conceptual problem -- particularly because everything is linear -- but the calculations are unpleasant). To simplify matters we adopt a strategy of imposing a form a priori on the final income distribution. Justification for this unusual procedure will be given at the end of the example.

We will assume that the equilibrium income distribution is uniform on the interval $[(1-\sigma)Y,(1+\sigma)Y]$ where Y is the mean income given by equation (1) and σ is a fixed parameter lying between 0 and 1. Letting y represent the income of some particular consumer, we can write $y = \gamma Y$. Our assumption on the equilibrium income distribution is then equivalent to the assertion that γ is uniformly distributed on the interval $[1-\sigma,1+\sigma]$ with density $(2\sigma)^{-1}$.

With regard to the indivisible commodities, consumers have four choices open to them: either to consume one of the indivisible commodities or to consume only the divisible commodity. For a consumer with income y, let $V(z,y)$ represent the maximum utility achievable if $z \varepsilon K$ is chosen and $V(0,y)$ the maximum if only the divisible commodity is consumed, where the maximum is computed conditional on the consumer's income and market prices. (Thus, $V(0,y)$ and $V(z,y)$ are just the values of the indirect utility function.) In each case, the consumer maximizes utility by spending all of his income net of housing costs, $y-h(z)$, on the divisible commodity. We conclude immediately:

$$V(0,y) = y \tag{2}$$
$$V(z,y) = (y-h(z))(1+z)$$

Because all of the consumers in this economy agree that indivisible commodities with higher z are better and because they all have the same tastes, it is obvious that equilibrium will stratify consumers by income with the wealthiest choosing $z = 3$, the next wealthiest choosing $z = 2$, the next $z = 1$ and the poorest getting no house at all. Our next step, therefore, is to calculate the value of income which separates each of these income classes.

A consumer will be indifferent between a house with characteristic $z = 1$ and no house at all if $V(1,y) = V(0,y)$. Using equations (2), this condition becomes $(y-h(1))2 = y$ so that $y = 2h(1)$ gives the income separating those who get no house and those who get a house of type $z = 1$. Using the definition $y = \gamma Y$, we obtain the first transition parameter:

$$\gamma_1 = (2h(1))/Y \qquad (3a) \quad .$$

Similarly, by setting $V(1,y) = V(2,y)$ we obtain the value of γ which separates the consumers choosing $z = 1$ from those who choose $z = 2$:

$$\gamma_2 = (3h(2)-2h(1))/Y \qquad (3b) \quad .$$

And finally by setting $V(2,y) = V(3,y)$ we obtain the third transition parameter:

$$\gamma_3 = (4h(3)-3h(2))/Y \qquad (3c) \quad .$$

Because there are four markets in this economy, by Walras' law we only have to clear three. We will equate demand and supply in the three housing markets, leaving the market for the divisible commodity to clear automatically. Consumers whose income parameter γ lies in the interval $\Gamma(1) = [\gamma_1,\gamma_2]$ will choose houses of type $z = 1$. Each consumer demands one house and, in integrating these demands, we weight by the density $(2\sigma)^{-1}$. Therefore, the market clearing equation for houses of type $z = 1$ is the following:

$$\int_{\Gamma(1)} (2\sigma)^{-1} d\gamma = c_1 = k \qquad (4a) \quad .$$

The market clearing equation for houses of type $z = 2$ is obtained in the same way where we define $\Gamma(2) = [\gamma_2,\gamma_3]$:

$$\int_{\Gamma(2)} (2\sigma)^{-1} d\gamma = c_2 = k \qquad (4b) \quad .$$

Finally, consumers whose parameter γ lies in the interval $\Gamma(3) = [\gamma_3, 1+\sigma]$ between γ_3 and the top of the income distribution will choose houses of type $z = 3$, leading to the market clearing equation:

$$\int_{\Gamma(3)} (2\sigma)^{-1} d\gamma = c_3 = k \qquad (4c) \quad .$$

Evaluating the integrals appearing in equations (4a)-(4c), we find:

$$\begin{aligned} \gamma_2 - \gamma_1 &= 2\sigma k \\ \gamma_3 - \gamma_2 &= 2\sigma k \qquad (5) \\ 1 + \sigma - \gamma_3 &= 2\sigma k \end{aligned}$$

Using equations (3a)-(3c) to substitute for γ_1, γ_2, and γ_3 in equations (5), we obtain

$$\begin{aligned} -4h(1) + 3h(2) &= 2\sigma kY = \xi_1 \\ h(1) - 3h(2) + 2h(3) &= \sigma kY = \xi_2 \qquad (6) \\ 3h(2) - 4h(3) &= [2\sigma k - (1+\sigma)]Y = \xi_3 \end{aligned}$$

where we have defined the expressions on the right-hand side to be ξ_1, ξ_2 and ξ_3 respectively. Letting $\xi = (\xi_1, \xi_2, \xi_3)$ and recalling that

$$p_2 = (h(1), h(2), h(3))$$

we can write the system (6) in matrix form (where the definition of B should be self-evident):[2]

$$Bp_2 = \xi \qquad (7)$$

Provided that B is non-singular, equation (7) has the solution $p_2 = B^{-1}\xi$. Solving explicitly, we obtain the following solution to the market clearing equations (6):

$$h(1) = [1+\sigma-6\sigma k]Y/2$$
$$h(2) = [1+\sigma-5\sigma k]2Y/3 \qquad (8)$$
$$h(3) = [1+\sigma-4\sigma k]3Y/4$$

Equations (8) do not yet constitute a complete solution to the model because they depend on the mean income Y. But Y is itself determined by equation (1). Substituting the equations (8) for the prices h(1), h(2) and h(3) into equation (1) gives mean income as a function of the exogenous parameters alone:

$$Y = \{1 - (k/12)[23(1+\sigma)-112\sigma k]\}^{-1}b \qquad (9)$$

Thus, we can use equation (9) to determine the equilibrium value for Y and, by inserting this value into equations (8), determine the equilibrium prices h(1), h(2) and h(3).

An explicit numerical example will help give a better sense of the model. Suppose that the number of houses per capita of each type is .25 and that $\sigma = .6$ (so that income is uniformly distributed between .4Y and 1.6Y). Then equation (9) implies that Y = 12b/7 and equations

[2] Note that we follow here the mathematical tradition that does not make distinctions between row and column vectors, arguing that the appropriate definition is always obvious from the context.

(8) imply that $h(1) = 3b/5$, $h(2) = 34b/35$ and $h(3) = 9b/7$. We find, therefore, that per capita income and the prices of all three types of houses are proportional to the per capita endowment of the divisible commodity, b. To obtain integer solutions, let $b = 35$. We then obtain $Y = 60$, $h(1) = 21$, $h(2) = 34$ and $h(3) = 45$.

Using this explicit numerical example, we can perform some consistency checks on the model. The per capita income from the sale of houses is $k(h(1)+h(2)+h(3)) = .25(21+34+45) = 25$ while that from the sale of the divisible commodity is $p_1 b = b = 35$. We conclude that total per capita income is 60, as before. Equations (3a)-(3c) give the transition points separating the income classes. Evaluating these equations we obtain $\gamma_1 = .7$, $\gamma_2 = 1$ and $\gamma_3 = 1.3$ which neatly subdivide the interval $[.4,1.6]$ into four equal-sized parts. We conclude that the upper 25 per cent of the income distribution is getting the houses of type $z = 3$, the next 25 per cent get the middle quality houses, the next 25 per cent the worst houses and the bottom 25 per cent is left out in the cold. By spot-checking for a few different incomes, the reader can verify that all consumers are maximizing their utility at the equilibrium prices, so indeed we do have a competitive equilibrium.[3]

The one flaw in this presentation from the purist's point of view is that we have not specified the distribution of initial endowments, choosing instead to specify a priori the form of the equilibrium income distribution. However, once we have obtained the solution it is always

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[3] The best way to see that this is so is to plot $V(0,y)$, $V(1,y)$, $V(2,y)$ and $V(3,y)$ as functions of the income parameter γ.

possible to specify an initial distribution of endowments that would yield the given income distribution at the given equilibrium prices (otherwise we would not have found a solution). The basic idea involved is probably best illustrated by using the numerical example given above.

Suppose we divide the population of consumers into four classes: class 1 consumers own one house each of type $z = 1$, class 2 one house of type $z = 2$, class 3 one house of type $z = 3$ and class 4 no houses at all. Assume that one fourth of the consumers in the economy fall into each category. Our aim is to distribute the divisible commodity to consumers in such a way that a uniform income distribution on the interval [24,96] will emerge. This would be easy except for the fact that three of the classes earn income from selling houses: an income of 21 for class 1, 34 for class 2 and 45 for class 3. The procedure we follow is to dole out the divisible commodity (which is income since $p_1 = 1$) in such a way that we produce the desired uniform distribution.

Starting with group 4, we give them the divisible commodity uniformly distributed from 24 to 42; group 1 receives the divisible commodity uniformly distributed from 21 to 39 (which, when added to their income from the sale of houses, yields an income distribution uniform on [42,60]); group 2 receives a uniform distribution from 26 to 44 (which, when added to the income of 34 from houses, amounts to a uniform distribution of total income from 60 to 78); and finally group 4 receives a uniform distribution from 33 to 51 (yielding a final income distribution uniform on [78,96]). This procedure requires an average amount of the divisible commodity equal to

$$(.25)[(24+42)/2 + (21+39)/2 + (26+44)/2 + (33+51)/2] = 35$$

which is exactly the amount that we have available to distribute.

The point of this construction is that if we started with this initial distribution of endowments and solved the model without imposing the a priori constraint on the final income distribution, then we would obtain exactly the same solution.

2. A Peculiarity of Pure Exchange

An interesting aspect of the model presented in the last section emerges when we consider the case $k = 1/3$. This represents the pleasant state of affairs in which there is one-third of a house per capita of each of the three types, so that there is a house for everyone.

Suppose again that the income distribution parameter $\sigma = .6$. Using equation (9) we find that average income $Y = 5b/3$, where b is the per capita endowment of the divisible commodity. Equations (8) imply that the equilibrium prices are $h(1) = b/3$, $h(2) = 2b/3$ and $h(3) = b$. Equations (3a)-(3c) give transition points $\gamma_1 = .4$, $\gamma_2 = .8$ and $\gamma_3 = 1.2$. Recalling that, when $\sigma = .6$, γ is uniformly distributed on the interval $\Gamma = [.4,1.6]$, we see that γ_2 and γ_3 divide the income distribution (as represented by Γ) into three equal parts, and the population stratifies into three income classes matched to the three types of houses in just the way one would expect. The reader can verify that this solution passes all of the consistency checks imposed on the previous numerical example, so we again have a competitive equilibrium.

However, if we go back over the derivation of these equations, we see that we are now making an unwarranted assumption. Because there are enough houses for everyone, we can no longer assume that $V(0,y) = V(1,y)$ for some consumer with income y. The above solution reflects this assumption (which is why $\gamma_1 = .4$, the left end-point of Γ), but all that can be required on economic grounds is that $V(0,y) \leq V(1,y)$ for all y: i.e., the lowest quality houses must be priced low enough to attract the lowest income consumer.

It seems clear on economic grounds how one should remedy the derivation. We know that all of the houses will be allocated in equilibrium, so all that is required is to replace γ_1 by $1 - \sigma$ in equation (4a). (Note that (4a) is then symmetrical with equation (4c)). Repeating the steps of the earlier derivation, we obtain the following system of market clearing equations to replace equations (6):

$$\begin{aligned} -2h(1) + 3h(2) &= [2\sigma k+1-\sigma]Y = \xi_1 \\ h(1) -3h(2) + 2h(3) &= \sigma kY = \xi_2 \\ 3h(2) - 4h(3) &= [2\sigma k-(1+\sigma)]Y = \xi_3 \end{aligned} \tag{10}$$

or, in matrix form,

$$Bp_2 = \xi \tag{11}$$

But at this point something quite remarkable happens: the matrix B is singular so that the system is indeterminate! This result is not so surprising if we think about it for a moment. Suppose that $c_2 = c_3 = 1/3$ as above (one-third house per capita of the middle and high quality houses), but $c_1 > 1/3$. We obtain exactly the same system of market clearing equations (10), but now we know how to solve the system. Low quality houses are in excess supply for any price greater than zero and so $h(1) = 0$ in equilibrium. With this additional constraint, the equations do have a unique solution.

In fact if we impose the constraint $h(1) = 0$, the equations have a solution even in the case where $c_1 = c_2 = c_3 = 1/3$ considered above. Sparing the reader the details of the calculation, I will simply supply the answers (for the case $\sigma = .6$, $k = 1/3$): $Y = 90b/67$, $h(2) = 24b/67$ and $h(3) = 45b/67$. The transition points are $\gamma_2 = .8$ and $\gamma_3 = 1.2$ so that the income distribution is again split neatly into thirds to match the different types of houses.

We can even say something more. If we let $c_2 = c_3 = 1/3$ and leave the other parameters the same, then the solution we have just described is the limit of the solutions obtained to the model if we let c_1 approach 1/3 from above. On the other hand, the solution described at the beginning of this section is the limit of the solutions obtained from equations (10) as c_1 approaches 1/3 from below. Thus, we have a discontinuity at $c_1 = 1/3$, a result that (with the advantage of hind-sight) makes perfect economic sense: this is precisely the point at which one more house produces a glut while one less produces a shortage.

3. Production With a Finite Number of Indivisible Commodities

The fact that the price of the lowest quality houses can be zero if houses are in excess supply should not be too disquieting at the theoretical level, but still there is something strange about the notion that people would be willing to sell houses at zero price. Intuitively, we expect that such houses would not be around in the first place. Of course, when we react that way, implicitly we are assuming that houses get produced (so that we do not have the perfectly inelastic supply of pure exchange).

In this section we will present a simple model of the production of indivisible commodities. We assume that consumers have identical utility functions of the form given in Section 1 and the set of characteristics is again $K = \{1,2,3\}$. Because of a particular feature of the model, which we will discuss in a moment, it will not be necessary to impose the equilibrium income distribution a priori.

The main point of departure is that we now assume that each indivisible commodity is produced using the divisible commodity as input. Let $\beta(z)$ be the amount of the divisible commodity required to produce one unit of the indivisible commodity of type z, and assume that production is constant returns to scale. It is important to emphasize that the scale referred to is with respect to the number of units of the commodity of type z produced and not with respect to $\underline{z}$ (which would be meaningless).

The introduction of production greatly simplifies the computational aspects of the model. In the first place, the analogue of equation (1) is now simply:

$$Y = b \tag{12}$$

This reflects the fact that, with constant returns to scale, equilibrium profits are zero so that the only source of income is the divisible commodity. The second fact that simplifies the model is that with constant returns price equals average cost. Therefore, using the assumption that $p_1 = 1$, equilibrium price are immediately determined:

$$h(z) = \beta(z) \qquad z \in K \qquad (13)$$

All that remains is to determine which commodities get produced and who receives them, and for this we need to specify the initial distribution of the divisible commodity. We will assume that the endowment of this commodity is uniformly distributed on the interval $[(1-\sigma)b,(1+\sigma)b]$ where $0 < \sigma \leq 1$. In view of equation (12), this is equivalent to assuming that the final income distribution is uniform on the interval $[(1-\sigma)Y,(1+\sigma)Y]$ which is precisely the distribution imposed a priori in Section 1. It will prove useful to parameterize the distribution in the same way by defining $y = \gamma Y = \gamma b$ and describing the income distribution as a uniform distribution of γ on the interval $\Gamma = [1-\sigma,1+\sigma]$.

At this point we can use the same procedure to solve the model that was used in Section 1. Because all consumers agree that a higher level of z is better, we know that in equilibrium there will be stratification by income class. By setting $V(0,y) = V(1,y)$, $V(1,y) = V(2,y)$ and $V(2,y) = V(3,y)$ we can solve for the transition parameters γ_1, γ_2 and γ_3 . The results are, of course, identical to equations (3a)-(3c) except that now we are able to impose the additional requirements given by equations (12) and (13). As a consequence, equations (3a)-(3c) now become:

$$\gamma_1 = (2\beta(1))/b$$

$$\gamma_2 = (3\beta(2) - 2\beta(1))/b \qquad (14)$$

$$\gamma_3 = (4\beta(3) - 3\beta(2))/b$$

To illustrate this solution to the model, we will consider two cases. In both we assume that the income distribution parameter equals .6, and we assume that the production coefficients are $\beta(1) = 2$, $\beta(2) = 3$ and $\beta(3) = 4$. The choice of these coefficients is, of course, arbitrary except that we expect that higher quality houses should be more costly to produce. I chose the numbers to increase linearly with z , but I started at 2 rather than 1 for a reason: I want to suggest some sort of increasing returns to "production of z " in order to highlight the fact that any such notion that it is z that is produced is entirely irrelevant.

Once the production coefficients have been specified, equations (14) can be solved for the transition parameters. For the choice of coefficients given above, we obtain:

$$\gamma_1 = 4/b; \quad \gamma_2 = 5/b; \quad \gamma_3 = 7/b \qquad (15) .$$

<u>Case 1: b = 10</u>

From equations (15) we find $\gamma_1 = .4$, $\gamma_2 = .5$ and $\gamma_3 = .7$. Because the income distribution parameter is $\sigma = .6$, the interval representing the income distribution is $\Gamma = [.4,1.6]$. Consumers for whom $\gamma \in [.4,.5]$ choose houses of the lowest quality, those for whom $\gamma \in [.5,.7]$ choose the middle quality houses and those for whom $\gamma \in [.7,1.6]$ choose the highest quality houses. Or, in other words, 1/12 of the consumers choose the worst houses, 2/12 the next highest quality and the remaining 9/12 the best houses.

Because this all seems so simple, it is a good idea to check the consistency of the model. The inputs used to produce the houses (in per capita terms) are found by multiplying the per capita demand for each type of house by the appropriate production coefficient:

$$\beta(1)(1/12) + \beta(2)(2/12) + \beta(3)(9/12) = 11/3$$

where we have substituted in the values for the production coefficients. The demand for the divisible commodity is found by integrating the demand functions of the individual consumers. Letting $\Gamma(1) = [.4,.5]$, $\Gamma(2) = [.5,.7]$ and $\Gamma(3) = [.7,1.6]$ we find that this per capita demand is

$$\int_{\Gamma(1)}(Y-h(1))(5/6)d\gamma + \int_{\Gamma(2)}(Y-h(2))(5/6)d\gamma + \int_{\Gamma(3)}(Y-h(3))(5/6)d\gamma$$

$$= Y - (h(1) + 2h(2) + 9h(3))/12 = 10 - 11/3 = 19/3.$$

The per capita input of the divisible commodity used in production plus the amount consumed should equal the amount initially available, and since 11/3 + 19/3 = 10, the solution checks.

Case 2: $b = 5$

From equations (15), we find $\gamma_1 = .8$, $\gamma_2 = 1$ and $\gamma_3 = 1.4$. Consumers in the interval [.4,.8] are priced out of the housing market altogether, those in the interval [.8,1] choose houses of type $z = 1$, those in the interval [1,1.4] choose houses of type $z = 2$ and those in [1.4,1.6] choose houses of type $z = 3$.

As the two cases we have discussed are intended to illustrate, the "comparative statics" of this simple model clearly works just as one would like: if production costs increase, consumers demand lower quality houses; if the lower tail of the income distribution is sufficiently low, one can get "squatter settlements"(consumers priced out of the housing market); and so forth.

To illustrate how models of this sort can be used for policy analysis, suppose that in Case 2 the government decides to subsidize the consumption of housing of those priced out of the market through a tax on income in excess of the mean. Let $S = [.4,.8]$ denote the class of consumers to be subsidized and $T = [1,1.6]$ the class that will be taxed, where both S and T are subsets of Γ.

Define $g(\gamma)$ as the subsidized housing price to a consumer with income parameter $\gamma \in S$. Assume that the government sets the subsidized price such that the consumer is indifferent between consuming no house and choosing a house of the lowest quality: i.e., $V(0,y) = V(1,y)$ for a consumer with income y . Using equations (2) with the definition $y = \gamma b$ and replacing $h(1)$ by $g(\gamma)$, this condition implies that $g(\gamma) = \gamma b/2$. The required subsidy is then $s(\gamma) = \beta(1) - g(\gamma) = 2 - 2.5\gamma$ where we have used the fact that $b = 5$ and $\beta(1) = 2$.

The total budget B_s required for this subsidy program (in per capita terms) is found by integrating the function s over the set S :

$$B_s = \int_S s(\gamma)(2\sigma)^{-1}d\gamma = 1/6$$

We assume that the housing subsidy is financed through a proportional tax on income in excess of the mean. Letting τ denote the tax rate, a consumer with income parameter $\gamma \in T$ will pay a tax

$$t(\gamma) = \tau(\gamma b - b).$$

Total tax revenue (per capita) is then

$$B_t = \int_T t(\gamma)(2\sigma)^{-1}d\gamma = 3\tau/4$$

Setting $B_s = B_t$ yields the equilibrium tax rate, $\tau = 2/9$.

Before the subsidy program, 1/3 of the consumers were priced out of the housing market, 1/6 bought the lowest quality houses, 1/3 the middle quality and 1/6 the highest quality houses (see case 2 above). Assuming that the subsidy is tied to consumption of housing, after the subsidy is implemented all consumers with $\gamma \in S$ will choose a house of quality $z = 1$. Consumers with income parameter $\gamma \in [.8,1]$ are not taxed and, since we found that $\gamma_2 = 1$, they will continue to choose the lowest quality houses.

To determine the housing choices of consumers for whom $\gamma \in T$ it is necessary to consider the effect of the tax. The relationship between the gross income y^* of a consumer and income net of the tax, y , is given by $y = y^* - \tau(y^*-b)$ or

$$y^* = (y-\tau b)(1-\tau)^{-1}.$$

Defining γ^* by setting $y^* = \gamma^* b$, we obtain the post-tax transition parameters (γ_j^*) as functions of the corresponding pre-tax parameters (γ_j):

$$\gamma_j^* = (\gamma_j-\tau)(1-\tau)^{-1} = (9\gamma_j-2)/7 \qquad (j = 1,2,3)$$

Using the values for the transition parameters γ_j calculated earlier, we find that $\gamma_1^* \simeq .74$, $\gamma_2^* = 1$ and $\gamma_3^* \simeq 1.51$. Therefore, we conclude that, after the subsidy program goes into effect, half of the consumers choose the lowest quality houses, approximately 43 per cent choose middle quality houses and the remaining 7 per cent choose the highest quality houses.

4. Production With a Continuum of Indivisible Commodities

As our final example we consider an economy in which the set of characteristics is infinite. To keep the exposition relatively simple, we will assume that this set is one-dimensional, letting $K = [0,\infty)$.[4] As in the finite case, the introduction of production greatly simplifies matters and, therefore, we will ignore the corresponding pure exchange model in this paper.

We adopt the same assumptions employed in the preceding sections. All consumers have identical preferences described by a utility function

[4] Although the set $K = [0,\infty)$ is, of course, non-compact, thereby violating one of the requirements of Mas-Colell's pure exchange model, the set of indivisible commodity types actually produced in the production economies we are considering will be compact.

taking on the value $x_1(a)(1+z)$ if a house with characteristics $z \varepsilon K$ is chosen and the value $x_1(a)$ if no house is chosen, where $x_1(a)$ is the amount of the divisible commodity consumed.

We noted in section 1 that when K is infinite the representation of the bundle of indivisible commodities allocated to consumer a, $x_2(a)$, is rather subtle. In the finite case, $x_2(a)$ is a unit vector (e.g., (0,0,0,1,0)), a representation that makes no sense when K is infinite. Consequently, when K is infinite $x_2(a)$ is given a different interpretation: if the consumer chooses a house with characteristics z, then we let $x_2(a) = \delta(z)$ where $\delta(z)$ is the probability measure on K that assigns mass one to the point z. (In the physics literature, $\delta(z)$ is called a Dirac delta function.) We will continue to assume that the consumer chooses at most one house, so $x_2(a)$ is equal to one of the Dirac delta functions $\delta(z)$, $z \varepsilon K$.[5]

The easiest way to gain some understanding of this formalism is to consider the budget constraint of a typical consumer:

$$p_1 x_1(a) + \int_K h x_2(a) = p_1 e_1(a) \qquad (16)$$

where most of this equation is interpreted just as in the earlier examples. In particular, since we will be assuming constant returns to scale in the production of housing, consumer income equals the value of the initial endowment of the divisible commodity, $p_1 e_1(a)$. h is the

[5] In Mas-Colell[1975] consumers are allowed to consume more than one unit of a indivisible commodity and more than one type of indivisible commodity, provided only that the total units consumed is bounded by some finite integer. In the more general model, the commodity bundle of indivisible commodities is then a finite sum of Dirac delta functions.

hedonic price function, $h: K \to R_+$, which gives the price of a house as a function of its characteristics. The integral $\int_K hx_2(a)$ is then the analogue of the scalar product used in the finite case to represent the housing expenditure of consumer a.

If the consumer chooses a house with characteristics z , then, by definition of the measure $\delta(z)$, this integral equals $h(z)$, the price of a house with characteristics z . Thus, the budget constraint (16), conditional on the choice of a house of type z , becomes:

$$p_1x_1 + h(z) = p_1e_1 \qquad (17)$$

a form that should look familiar to readers acquainted with the literature on urban housing markets or, more generally, "hedonic theory".[6]

Consumers are assumed to maximize the utility function described above subject to the budget constraint (17). In the usual way, the solution to this constrained maximum problem conditional on the choice of z can be represented by the indirect utility function:

$$V(z,y) = (y-h(z))(1+z) \qquad (18)$$

where we let y denote the income of the consumer. By now the construction should look very familiar!

Assuming that the hedonic function is differentiable, the optimal choice of a type of house is found in the usual way by setting $\partial V/\partial z = 0$. Differentiating (18) this first-order condition implies:

$$y = h(z) + (1+z)\ dh/dz \qquad (19)$$

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[6] The standard reference on hedonic theory is Rosen[1974].

To solve for the competitive equilibrium we must first specify the initial distribution of endowments and the production technology, and here we follow essentially the same procedure as in Section 3. Consumers are endowed with the divisible commodity alone, and we again assume a uniform distribution on the interval $[(1-\sigma)b,(1+\sigma)b]$ where $0 < \sigma \leq 1$, σ a fixed parameter. Just as in Section 3, this implies that consumer income is uniformly distributed on the interval $[(1-\sigma)b,(1+\sigma)b]$ where $b = Y$, the per capita income, and we are assuming $p_1 = 1$.

Houses are produced subject to constant returns to scale using the divisible commodity as input, where the technology is described by a function $\beta\colon K \to R_+$ with $\beta(z)$ representing the amount of commodity one required to produce a house of type z. To facilitate comparison with Section 3, we will assume that the function β takes the form:

$$\beta(z) = 1+z \qquad (20)$$

which agrees with the technology of the earlier section when restricted to the subset $\{1,2,3\}$ of K.

Because production is constant returns to scale, $h(z)$ equals the average cost of producing a house with characteristics z and, since we are assuming $p_1 = 1$, this means that $h(z) = \beta(z) = 1+z$. Substituting this expression into the first-order condition (19) yields $y = 2 + 2z$ or, defining $y = \gamma Y = \gamma b$ as before,

$$z = \gamma b/2 - 1 \qquad (21)$$

which gives the choice of housing type z as a function of the consumer's income parameter γ.

In Section 3 we considered two cases, $b = 10$ and $b = 5$. When $b = 10$, equation (21) implies that the amount of housing produced in the competitive equilibrium is uniformly distributed on the interval $[1,7]$. When $b = 5$, production is uniformly distributed over the interval $[0,3]$. It seems clear that qualitatively the models with K finite and K infinite are very similar. For example, as production costs increase (i.e., as b increases), consumers shift to lower quality houses. When $b = 5$ one-third of the consumers were priced out of the market in the finite case, a phenomenon that is absent in the continuum case. However, if we restrict K to equal $[1,3]$ then one-fourth of the consumers are unable to afford houses even though a infinite variety of housing types is available.

In fact the two types of model are much more closely related than this rough comparison seems to suggest. We can view the model with a finite set of characteristics, say $K = \{1,2,3\}$, as an approximation to the continuum model where K is some infinite subset of R_+. Intuitively, one would expect that as more points are added to the finite set of characteristics (provided, of course, that they are well chosen), the approximation should improve. The demonstration that this conjecture is true is perhaps the major contribution of the pioneering paper of Mas-Colell[1975] that we have mentioned earlier.

The convergence of the finite to the continuum model can be illustrated very easily with the examples we have presented thus far. However, first we must decide how the solutions of these models are to be compared.

Let K denote the set of characteristics in the continuum model and let the subset K_a of K represent a finite approximation to the continuum of characteristics (e.g., in case 1 we have $K = [1,7]$ and could take $K_a = \{1,2.5,4,5.5,7\}$). One way to describe the solution to the continuum model is in terms of the function $F: K \to [0,1]$ giving the fraction of houses produced with characteristics less than or equal to z for each $z \in K$. We let $F_a: K \to [0,1]$ denote the analogous function for the finite model with set of characteristics K_a . The reader may find it helpful to regard F and F_a as distribution functions for probability measures on K . (Note, however, that if some consumers are priced out of the housing market, then these measures will assign mass less than one to the sets K or K_a .)

The function F for the continuum model is determined by equation (19). Letting $y = \gamma(z)Y = \gamma(z)b$, where $\gamma(z)$ is the income parameter of a consumer choosing a house with characteristics z , equation (19) implies that

$$\gamma(z) = b^{-1}\{h(z) + (1+z)\ dh/dz\} \qquad (22).$$

Defining $\Gamma(z) = \{\gamma \in \Gamma: \gamma \le \gamma(z)\}$ we conclude that

$$F(z) = \int_{\Gamma(z)} (2\sigma)^{-1} d\gamma = G(\gamma(z)) \qquad (23)$$

where G is the distribution function for the income parameter γ.

To determine the corresponding function F_a for the finite approximation to the continuum, it is necessary to recast our earlier derivation of the transition parameters in somewhat more general terms. Suppose that z and $z + \varepsilon_z$ are adjacent points in K_a where $\varepsilon_z > 0$. To find the transition parameter separating the consumers who choose houses with characteristics z from those who choose $z + \varepsilon_z$ we set

$$V(z,y) = V(z+\varepsilon_z,y) \qquad (23)$$

just as in Sections 1 and 3. Using equations (2) this implies

$$y = h(z+\varepsilon_z) + (1+z)[h(z+\varepsilon_z)-h(z)]/\varepsilon_z$$

or, letting $y = \gamma_a(z)b$,

$$\gamma_a(z) = b^{-1}\{h(z+\varepsilon_z) + (1+z)[h(z+\varepsilon_z)-h(z)]/\varepsilon_z\} \qquad (24)$$

The reader should verify that for $\varepsilon_z = 1$, equation (24) reduces to equation (3b) when $z = 1$ and to equation (3c) when $z = 2$.

Defining

$$\Gamma_a(z) = \{\gamma \in \Gamma\colon \gamma \le \gamma_a(z)\}$$

we have for $z \in K_a$ that

$$F_a(z) = \int_{\Gamma_a(z)} (2\sigma)^{-1} d\gamma = G(\gamma_a(z)) \qquad (25)$$

and we extend this function to the domain K by letting

$$F_a(z) = \max \{F_a(\zeta): \zeta \leq z \ , \ \zeta \in K_a\}$$

for all $z \in K$.

Suppose now that $z \in K$ is a point of continuity[7] of the function G and suppose as well that $z \in K_a$. Imagine now that we improve our approximation by adding points to K_a in such a way that (i) the gaps between adjacent points of K_a converge to zero (i.e., $\varepsilon_z \to 0$ for all $z \in K_a$) and (ii) every point in K is the limit of a sequence of points in K_a (i.e., "in the limit" K_a is dense in K). Letting $\varepsilon_z \to 0$ in equation (24) and comparing the result to equation (22), we find that $\gamma_a(z) \to \gamma(z)$ and, therefore, $F_a(z) \to F(z)$. Thus, the solution to the finite model approaches that of the continuum model for each $z \in K_a \cap K$ that is a point of continuity of G (and hence, by equation (23), a point of continuity of F). If $z \in K$ is such a point of continuity but $z \notin K_a$, then by considering a sequence $\{z_i\}$ in K_a which converges to z we conclude that $F_a(z) \to F(z)$ for all points of continuity z of F.

In concrete terms we could construct the approximation K_a by choosing equally spaced points of K . The function F_a will be a step function that jumps at each $z \in K_a$ and, as the spacing goes to zero, F_a converges to F at each point of continuity of F . In our

[7] In our example, where income is uniformly distributed, all points of K are points of continuity for G . However, in more general versions of the model this caveat is important.

examples, where γ is is uniformly distributed, every point of K will be a point of continuity and so we get convergence everywhere.

As an illustration, for case 2 where $K = [1,7]$ suppose that we choose K_a to divide K into M intervals of length $6/M$. Equation (24) then implies that

$$\gamma_a(1 + 6m/M) = 10^{-1}\{4 + 12m/M + 6/M\} \qquad (26)$$

for $m = 0,\ldots,M$. Fixing the ratio $m/M = m^*$ and letting $M \to \infty$ we find that in the limit equation (26) becomes

$$\gamma_a(1 + 6m^*) = .4 + 1.2\ m^* \qquad (27)$$

Recall that in case 2 with the continuum model, the production of housing was uniformly distributed on the interval $[1,7]$. Since the income parameter γ is also uniformly distributed on the interval $[.4,1.6]$, we see immediately from equation (27) that the "distribution function" F_a for the finite model will approach the uniform distribution of the continuum model at all of the rationals in K. Since the rationals are dense in K and the limit distribution is continuous, we conclude that the convergence occurs everywhere in the interval $[1,7]$.

Recalling our suggestion that F and F_a can be regarded as distribution functions for probability measures on K, what we have demonstrated is that the solution of the finite approximation model converges to the solution for the continuum model in precisely the sense that the distribution function for the binomial approaches that of a normal distribution. This is an illustration of the "weak star" convergence alluded to in the introduction. In fairness to Mas-Colell, we should remark that our demonstration, relying as it does on the many special features of our examples, does not begin to do justice to the power and generality of his result, just as the central limit theorem is a much more powerful statement than showing that the binomial distribution converges to the normal.

5. Retrospect and Prospect

The theory of housing markets proposed in this paper, based on a model of competitive equilibrium with indivisible commodities and a non-atomic measure space of consumers, represents a radical break with tradition. If such a departure from current practice is to have any chance of widespread acceptance, it should satisfy at least two criteria: (1) the models which emerge from the analysis should be easy to construct and to comprehend and (2) they should offer new insights into the operation of housing markets and cover a wider range of phenomena than the standard theory.

The examples we have discussed were selected with the first of these criteria in mind. The concept of a competitive equilibrium with indivisible commodities and the notion of a continuum of economic agents are alien to most economists. Both ideas have been developed in a highly technical and abstract setting and neither has seen much practical application. The intent of our examples is to show that, when translated into a less abstract setting, these theoretical constructs are easy to work with and lead to results that have a ready economic interpretation. (For another demonstration along these lines, see Scotchmer[1981].)

In striving for simplicity of presentation, however, we have had to pay a price. The models considered in this paper are too simple to do justice to the complexity of housing markets. The second criterion for acceptance of this type of analysis is, therefore, still at issue. In this concluding section I will attempt to address this question by placing the foregoing discussion in more general perspective and offering some hints regarding what is to come.

I will consider in turn three topics that are widely regarded as important in the analysis of housing markets: (a) the role of existing stock in the production of housing; (b) the development of the hedonic approach to housing markets; and (c) the incorporation of local public goods and neighborhood effects into housing market models.

a. Production from the Existing Stock of Housing

The crucial ingredient that gives the theoretical approach I am advocating its power is the definition of the housing commodity: the commodities are housing units and not some amorphous notion of housing services. The way in which this theory handles production from existing stock provides a nice illustration of the advantages of this definition. Consider the model with three types of houses treated in Section 3. Let

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$$

denote a typical production vector where the first three components of the vector represent outputs or inputs of houses of types 1, 2 or 3 respectively and η_4 is the input of the divisible commodity.

For example, if the conversion of one house of type 2 to one house of type 3 requires κ_c units of the divisible commodity, then the corresponding production vector is $(0,-1,1,-\kappa_c)$. Assuming that this production process exhibits constant returns to scale, we conclude immediately that if such production takes place then in equilibrium

$$-h(2) + h(3) - p_1\kappa_c = 0$$

and, therefore,

$$h(3) = h(2) + p_1\kappa_c \quad .$$

We obtain precisely the relationship we should have expected: the price of a house of type 3 equals the opportunity cost of the house of type 2 from which it is converted plus the cost of conversion.

Similarly, if a house of type 3 is produced through the demolition of a house of type 2 using κ_d units of the divisible commodity, then the production vector takes the form $(0,-1,1,-\kappa_d)$ and, again assuming constant returns to scale, we conclude that in equilibrium

$$h(3) = h(2) + p_1\kappa_d$$

if this production process takes place. In this instance the price of a house of type 3 equals the opportunity cost of the type 2 house that is demolished plus the cost of construction. (Note that $p_1\kappa_d$ also includes the cost of demolition so that, contrary to what might be supposed, demolition costs cause no problem for the functioning of the competitive process.)

Working out the implications of such a model requires much more elaboration, but enough has been said to establish our point: once housing commodities have been properly defined, the introduction of production from existing stock becomes both easy and natural.

b. The Hedonic Theory of Housing Markets

Providing a rigorous foundation for hedonic theory is perhaps the most important application of the model proposed by Mas-Colell, and in the preceding sections I have used h to denote the equilibrium price function in order to emphasize this relationship. However, the examples we have considered represent only a very specialized version of what is usually considered to be hedonic theory. The main advantage of the hedonic approach to housing markets is its ability to describe a house in terms of a large bundle of characteristics. But in all of our examples, the set of characteristics K is assumed to be a subset of the real line and, because all consumers agree that a higher level of the characteristic z is better, the various types of houses are linearly ordered.

It is important to emphasize, therefore, that the theory developed by Mas-Colell only requires that K be a compact metric space and, as a result, much more complex interrelationships among housing types can be accommodated. For example, we could take K to be a subset of R^m where the components of $z \in K$ represent various attributes of a house:

e.g., accessibility to employment, lot size, square feet of floor space, number of bathrooms, neighborhood quality and the like. (For an empirical test of a model of this type, see Ellickson[1981].)

The basic question we want to address is whether there is any reason to develop the hedonic theory of housing markets within the framework provided by Mas-Colell instead of relying on the much less formidable version presented by Rosen[1974]. One justification for turning to Mas-Colell is that he provides a proof of existence of equilibrium as well as demonstrating core equivalence and Pareto optimality of the resulting competitive allocation. In view of the technical complexity of his theory, this justification alone probably does not warrant the effort needed to learn the required techniques.

There is, however, another reason for preferring the theoretical approach of Mas-Colell that should be much more relevant to the practical theorist: it is much more powerful from the computational point of view. In practice, Rosen's version of hedonic theory has been quite useful in modeling demand behavior, less successful in treating the supply side and largely a failure in analyzing the interaction between demand and supply. The problem is that the analytical devices employed by Rosen, while quite useful in illuminating the qualitative features of an equilibrium, provide very little information concerning market clearance. Bid price functions for consumers and offer functions for firms, defined over the set K of housing characteristics, serve as the basic ingredients for Rosen's diagrammatic exposition of the determination of hedonic prices. But demand and supply clear in terms of housing units, not characteristics, and the number of housing units of each type is suppressed in the Rosen diagram.

These critical remarks regarding Rosen's model are intended to be pointed, but they are not supposed to be harsh. Rosen and Mas- Colell present different aspects of the same theory, and my point is simply that both approaches are crucial to the development of that theory. Rosen's diagram provides a splendid picture of what is going on in a market for indivisible, differentiated commodities. Mas-Colell's theory tells you how to model the equilibrium. At a few crucial junctures, Rosen notes that an application of "functional analysis" is needed to complete his theory. Mas-Colell provides that functional analysis.

What then are the computational advantages of Mas-Colell's theory? Providing a concrete answer to that question was the primary motivation for writing this paper. The examples we have constructed are, of course, far too specialized to represent what is normally meant by hedonic theory, and I do not claim that the extension to more complex models will be devoid of any complication. For example, of the three models that we considered, the pure exchange model of Section 1 was computationally the most complex. I avoided presenting the parallel pure exchange model for a continuum of characteristics because the solution is even more complicated. As in the model with production treated in Section 4, the solution of the corresponding pure exchange model is determined by equation (22) but, in the absence of production, we can no longer solve this equation simply by substituting the production coefficient $\beta(z)$ for the hedonic price $h(z)$. Instead equation (22) gives us a differential equation in dh/dz, the solution of which depends on the distribution of endowments.

It is not hard to specify endowment distributions that lead to a differential equation that I know how to solve, but it is even easier to end up with differential equations that are analytically intractable. This situation is strongly reminiscent of the difficulties faced by the "new urban economics", a movement that floundered on similar shoals.

It is clear that the difficulties arise because, in the case of a continuum of characteristics, one is dealing with an infinite dimensional commodity space. Without heroic restrictions on the distribution of preferences and endowments, tractable analytical solutions will be the exception and not the rule. The introduction of production can simplify matters greatly when, as in Section 4, it imposes a structure on the relative prices of the continuum of housing types. However, if we add another characteristic such as accessibility to employment the complexity returns (essentially because land at various locations, as an input into the production of houses, is not itself produced).

The experience we have gained from our earlier examples suggests a way out of this apparent impasse. In Section 4 we saw that, in a sense that Mas-Colell has made rigorous, a continuum of characteristics can be approximated by a finite set of characteristics. Since it is accessibility that is causing the problem, the natural approach is to divide the total land area into a finite number of "zones", treating all land within a particular zone as equivalent so far as distance to employment is concerned. Similar suggestions have been made before as a way around the corresponding difficulties faced by the new urban economics (see Mills and MacKinnon[1973]), but implementation of such suggestions was stymied both by the absence of a logical connection from the finite to the continuous models and by the awkward computational aspects of the finite approach. In the theoretical framework we have been considering, the weak star convergence result of Mas-Colell resolves the first problem (as illustrated in Section 4) and the continuum of agents setting resolves the second (as shown in the example of Section 1).

c. Neighborhoods and Local Public Goods

Finally we come to the aspect of the theory that was the primary motivation for my interest in indivisible commodities in the first place, the modeling of the market for local public goods and its interaction with the housing market. The reader will undoubtedly have noticed that, although we have discussed our examples in terms of houses, that interpretation is quite incidental. The indivisible commodities could just as well have been automobiles, washing machines or microcomputers. And, more to the point, they could be local public goods.

I have argued elsewhere (Ellickson[1979a]) that the notion of a local public good is a redundant concept in economics. Local public goods can be regarded as indivisible private goods whose production exhibits increasing returns over some initial range. When looked at in this way local public goods theory, a rather confusing body of ideas from the standard point of view, reduces to something which is easy to understand. For example, the issue of existence of a competitive equilibrium for local public goods is then identical to the question of existence of a competitive equilibrium for indivisible private goods. If scale economies are small relative to the extent of the market, then a competitive equilibrium (also known as a "Tiebout equilibrium") will exist, at least in an approximate sense. When equilibrium fails to exist, the cause is no longer a mystery: it is simply a matter of increasing returns to scale. An application of these ideas to neighborhood attributes, which can be regarded as a type of local public good, is given in Ellickson [1979b].

Two limitations of this new approach to the theory of local public goods which restrict its range of application are that: (i) the competitive equilibrium concept involves an approximate, rather than an exact, equilibrium and (ii) the number of public good types is assumed to be finite. But the theory presented in Ellickson[1979a] is based on an economy with a finite number of consumers. With a non-atomic measure space of consumers, we obtain an exact equilibrium of the form illustrated by the example of Section 3. And the extension to a continuum of public good types (where the continuous characteristic z is then interpreted as the "quantity" of the public good) leads to a model of the kind exhibited in Section 4. Because the market for houses and the market for public goods are thereby put on a comparable footing, tied together in a single choice process by the indivisibility of the consumer, the way is paved for the study of the interaction of the two types of markets.

To summarize, I have argued that the key to the development of a coherent and powerful theory of housing markets and local public goods is the concept of a competitive equilibrium for indivisible commodities. Through a series of examples I have tried to show that such an approach is computationally feasible and relatively easy to comprehend. In this concluding section I have suggested what the approach has to offer. Most economists view indivisibility as something which undermines economic theory. If my view is accepted, it should instead be regarded as a prime illustration of the power of the competitive concept and as a vehicle for extending the scope of economic analysis to a vast new domain.

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