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ON THE EXISTENCE OF WALRASIAN EQUILIBRIUM IN LARGE-SQUARE ECONOMIES

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An existence theorem for Walrasian equilibrium is demonstrated for an economy with a continuum of consumers and an infinite-dimensional commodity space, such as ℓ_1 or c_o, having an "order-compatible" basis.

Two parameters in the description of a general equilibrium model are the number of commodities and the number of participants, or agents. If both parameters are finite the model will be said to be a member of the class of small-squares of economies. If neither is finite it belongs to the class of large-square economies. For small squares, the mathematical theory of the existence of Walrasian equilibrium has been rather definitively established (see, for example, McKenzie (1981)). Recent contributions which obviate the need for individual preferences to be complete or transitive, for example Mas-Colell (1974), Gale and Mas-Colell (1975), and Shafer and Sonnenschein (1975), provide the means in this paper to extend existence theory from small-square economies to certain kinds of large-squares.

The principal reason for introducing an infinite number of agents has been to justify the competitiveness of Walrasian equilibrium (Aumann (1964)). The rationale for an infinite-dimensional commodity space has been to introduce an unending time horizon (Bewley (1973)) or to incorporate product differentiation (Mas-Colell (1975) and Jones (1983)) into general equilibrium theory. Brown and Lewis (1981) and Blad and Keiding (1983) have also established results for models with a continuum of agents and an infinite-dimensional commodity space.

There is a tension between these two infinities. It is known that the competitive properties of Walrasian equilibrium in models with a continuum of agents requires some restrictions on the commodity space so as to preclude "truly large-square" models where "the number of commodities is as large as the number of agents". In these truly large-square models, the potential competition from large numbers of agents is offset by the variety of commodities in the sense that infinitesimal agents may trade commodities for which there are no good substitutes.

Encouraged by the findings in Ostroy (1984) for a special class of economies in which the existence of Walrasian equilibrium required only the existence of a supporting hyperplane, the original aim of this paper was to demonstrate a similar result for a more general version of a truly large-square model in order to highlight the differences between it and models with more competitive properties where, with a continuum of agents, the commodity space guarantees greater substitutability. Unfortunately, the methods employed here do not lead to this desired result. The assumptions imposed on the infinite-dimensional commodity space make it too small to be truly large-square. For example, we shall also demonstrate that, as in the nonatomic models of Aumann, Bewley and Mas-Colell, the core and WE allocations coincide. Further, I doubt that an existence theorem for a truly large-square model, e.g., one without core equivalence, can be obtained in anything like the generality typically associated with the study of Walrasian equilibrium.

The Samuelson overlapping generations model involves an infinite number of agents and commodities and existence theorems for it have been obtained by Balasko, Cass and Shell (1980a,b) and by Wilson (1981). However, because it employs a σ-finite measure space of atomic agents rather than a finite, nonatomic measure space and because it admits as prices linear functions that are not necessarily continuous, it lies outside the scope of the large-square models I am considering.

In the model presented here, the key assumption on the commodity space is that it has an "order-compatible" Schauder basis. The fact that the basis is order-compatible will mean that our results apply to spaces such as ℓ_p, $1 < p < \infty$ and c_0 but not to $L_p[0,1]$, $1 < p < \infty$, $C[0,1]$, or ℓ_∞.

For models with a finite number of agents, existence theorems have been obtained by Peleg and Yaari (1970) and Stigum (1973) for the space of sequences with the product topology, by Bewley (1972) for L_∞ with the Mackey topology, and by Jones (1984) for a ca(K) with the weak-star topology. On Bewley's model, see also Bojan (1974), Barkuki (1977), Brown and Lewis (1981), Toussaint (1982), Florenzano (1982), and Magill (1983). Brown (1983) has demonstrated an existence theorem for Banach lattices including L_p, $1 < p < \infty$ and Mas-Colell (1983) for spaces including L_∞ and ca(K) when they are given their weak-star topologies.

Besides the restriction on the commodity space, there are three other features of this model to which I should like to call attention. The first concerns the problem of pricing when the relevant portion of the commodity space has an empty interior. The presence of a nonempty interior to a convex set permits the conclusion that its boundary has a kind of finite steepness from which the existence of a closed hyperplane supporting a boundary point of the convex set may be derived (see Gale (1967)). Without the interiority assumption, the steepness condition is not assured and the existence of closed hyperplanes, i.e., prices, becomes problematic. In the model below, the relevant part of the commodity space is a positive cone and in the infinite-dimensional cases considered here this will always have an empty interior. To overcome the resulting difficulty, I shall invoke an assumption explicitly bounding the steepness of certain convex sets, similar to bounding marginal rates of substitution (cf., Jones (1983), Mas-Colell (1983)).

The second feature of the model concerns the relation between finite and nonatomic representations of an economy. Kannai (1970), Hildenbrand (1974) and others have shown how a nonatomic model can be obtained as the limit of a sequence of models with a finite but increasing number of atoms of diminishing size. The building block for the nonatomic model is the finite model. I shall turn this around, starting with the nonatomic model as the basic object out of which a finite model is constructed. This is accomplished by partitioning the set of agents into a finite number of groups, each of which has "coagulated" into an indivisible unit. Each such unit has an endowment that is indistinguishable in scale from that of an atom, but, because the preferences of an individual are typically assumed to be complete whereas the preferences of the group are not, there is a gap between the preferences of a single large-scale individual and a nonatomic group of agents. However, because completeness of individual preferences has been shown to be unnecessary to demonstrate the existence of Walrasian equilibrium in small-square economies, this gap becomes inconsequential. Thus, Mas-Colell's formulation of a small-square model (1974) may be regarded as deriving from certain restrictions imposed on Vind's formulation (1964) (see also Cornwall (1969) and Richter (1971)) of a nonatomic model. One advantage of building the finite model from the nonatomic one is that it minimizes the difficulties in passing from an existence theorem for the small-square model to an existence theorem for the large-square model from which it was derived. A disadvantage is that it precludes a demonstration of the convexifying effects of large numbers, i.e., it cannot be demonstrated that individually non-convex preferences lead to group preferences that are convex.

The remaining feature of the model concerns an extension of the measure space of agents into a linear space. There is a one-to-one correspondence between the elements of a σ-algebra of a measure space and its characteristic functions. The linear space derived from the measure space consists of finite linear combinations of characteristic functions. The extension of the measure space to a linear space brings with it an extension of the set of possible agents. For example, for any group E there is now, for each $\alpha > 0$, a group $\alpha\chi_E$, where χ_E is the characteristic function of E. By construction this group will have endowments that are an α-multiple of E's endowments and preferences that are similarly scaled to E.

The extension from a measure space to a linear space of agents leads to an extension of the definition of allocations from (positive) vector measures to (positive) linear operators. The advantage of the operator description of allocations is that it leads to a characterization of Walrasian equilibrium that does not directly involve prices! This characterization (see the Auxiliary Lemma) says that, subject to certain qualifications, a Walrasian equilibrium is the same as an allocation in what I shall call the <u>core of the linear space of agents</u>. Such an allocation, involving as it does only restrictions on quantities, is easier to verify than the usual statement of Walrasian equilibrium, involving as it does restrictions on prices and quantities. This result is a generalization of the Debreu-Scarf Theorem (1963) suitable for nonatomic economies with an infinite number of types of agents.

This alternative definition of Walrasian equilibrium yields a simple characterization of the core equivalence property: the core of an economy coincides with its Walrasian equilibria whenever the core of the measure space of agents (the core) coincides with the core of the linear space of agents (Walrasian equilibria). It follows as a corollary of our method of demonstrating existence of Walrasian equilibrium that the large-square model considered here exhibits core equivalence.

2. Preliminaries

2.1 Commodities and Prices

The commodity space is a Banach lattice with positive cone Y_+. The space of prices is the continuous linear functionals on Y, denoted by Y^*. If $q \in Y^*$ and $y \in Y$, the value of q at y is denoted by qy.

The zero elements of Y and Y^* are denoted by $\underset{\sim}{0}$.

The set of positive prices in Y^* is

$$Y_+^* = \{q \in Y^*: qy \geq 0, \text{ all } y \in Y_+\} .$$

The set of strictly positive quantities in Y_+ is

$$Y_{++} = \{y \in Y_+: qy > 0, \text{ all } q \in Y_+^* \setminus \{\underset{\sim}{0}\}\}.$$

The norm interior of Y_+, int Y_+, is contained in Y_{++} and if Y were finite-dimensional $Y_{++} = \text{int } Y_+$. However, Y_{++} may be nonempty when int $Y_+ = \phi$. For example, if $Y = \ell_1$, then $y = (1, 1/2, 1/4, \ldots) \in Y_{++}$ but int $Y_+ = \phi$.

The space Y is said to have a (Schauder) basis if there exists a sequence $\{e_k\} \subset Y$ such that for each $y \in Y$ there is a sequence of scalars $\{\beta_k\}$ having

$$\lim_{\ell} \left|\left| y - \sum_{k=1}^{k=\ell} \beta_k e_k \right|\right| = 0,$$

where $||\cdot||$ is the norm on Y. For example, if $Y = \ell_p$ or $L_p[0,1]$, $1 \leq p < \infty$, it has a basis. Since a space exhibiting a basis is necessarily separable, the non-separable spaces ℓ_∞ and $L_\infty[0,1]$ cannot exhibit a basis.

The key restriction imposed on Y is that it has an "order-compatible" basis, i.e., (Y.1) $y \in Y_+$ implies $y = \Sigma\beta_k e_k$ and for each k, $\beta_k \geq 0$

The fact that $\{e_k = (0,\ldots,\underset{k}{1},0,\ldots)\}$ is a basis for ℓ_p, $1 < p < \infty$, and c_o makes it clear that their positive cones satisfy (Y.1). There are spaces such as $L_p[0,1]$ or $C[0,1]$ with a basis but not one that satisfies (Y.1) (see Lindenstrauss and Tzafriri (1977)) and therefore our results do not apply to these spaces.

REMARK 1: The set Y_{++} is called the quasi-interior of the cone Y_+. It follows from a result in Peressini (1967, Proposition 4.6, p. 187-88) that if Y satisfies (Y.1) then Y_{++} is nonempty.

1.2 Agents

The measure space of agents is $(A,\mathscr{A},\lambda)$ where A is the unit interval, $\mathscr{A}$ its Borel subsets, and λ is Lebesgue measure. The linear space of agents is $X = X(A,\mathscr{A},\lambda)$, the set of finite linear combinations of characteristic functions defined by elements of $\mathscr{A}$. Thus, $x = \Sigma\alpha_i\chi_{E_i} \in X$, where α_i is scalar and χ_{E_i} is the characteristic function of $E_i \in \mathscr{A}$. X_+ is the positive cone of X, i.e., those x for which all $\alpha_i \geq 0$. The space X will be given the L^1-norm so that its closure is $L^1(A,\mathscr{A},\lambda)$. The norm of x is denoted by $|x|$.

1.3 Allocations

Denote by $\hat{z}$ a countably additive Y_+-valued measure on $\mathscr{A}$. $\hat{z}$ describes an allocation of commodities in which $\hat{z}(E)$ is the total amount of commodities assigned to E. The measure $\hat{z}$ is bounded if the per capita assignment of commodities is bounded, i.e.,

$$\sup \{||\hat{Z}(E)||/\lambda(E): \lambda(E) > 0\} < \infty.$$

Clearly, if $\hat{Z}$ is bounded then it is nonatomic, i.e.,

$$||\hat{Z}(E)|| \neq 0 \text{ implies there exists } F \in \mathscr{A} \text{ with } 0 \neq ||\hat{Z}(F)|| \neq ||Z(E)||.$$

Let $M_+[\mathscr{A},Y]$ be the set of bounded Y_+-valued measures on $\mathscr{A}$ and let $B_+[X,Y]$ be the set of bounded positive linear operators on X. To every $\hat{Z} \in M_+$ there corresponds a unique $Z \in B_+$ defined by

$$Zx = \Sigma\alpha_i\hat{Z}(E_i), \quad \text{when} \quad x = \Sigma\alpha_i\chi_{E_i}.$$

Conversely, any $Z \in B_+$ defines a unique $\hat{Z} \in M_+$ defined by

$$Z\chi_E = \hat{Z}(E).$$

This 1-1 correspondence between M_+ and B_+ means that it is merely a matter of convenience as to whether we regard allocations as (bounded) vector measures or positive linear operators. Although vector measures are defined on the measure space describing the actual sets of agents in the economy while linear operators are defined on its linear algebraic extension, we shall see that the latter has a useful role to play in the definition of Walrasian equilibrium.

Denote by $B_{++}[X,Y]$ the set of strictly positive linear operators, i.e., those $Z \in B_+$ such that

$$Z\chi_E \neq \underset{\sim}{0} \quad \text{whenever} \quad \lambda(E) \neq 0.$$

REMARK 2: In our description of allocations as vector measures we are following the Vind-Cornwall-Richter approach in which individuals disappear and attention is focused on λ-nonnull groups. There is a more individualistic approach to nonatomic economies pioneered by Aumann (1964) in which allocations are defined by (Bochner) integrable functions $z: A \to Y_+$, where $z(a)$ is the allocation assigned to individual $a \in A$. Every such integrable function yields a Y_+-valued measure $\hat{Z}$ according to $\hat{Z}(E) = \int_E z d\lambda$. If z is essentially bounded (ess sup $\{||z(a)||: a \in A\} < \infty\}$), then $\hat{Z} \in M_+$. Thus, the individualistic description of allocations is included within the description of allocations via groups. For spaces satisfying the Radon-Nikodym Property (Diestel and Uhl (1977, Chapter III)), the converse holds. These spaces have the property that for any $\hat{Z} \in M_+$ there exists an essentially bounded integrable function z such that $\hat{Z}(E) = \int_E z d\lambda$. Thus, for spaces with the Radon-Nikodym Property -- such as $\mathbb{R}^n$ or ℓ_1 -- the representation of allocations as vector measures or as integrable functions is a matter of convenience. However, in a space such as c_o, which does not exhibit this property, the vector measure description may provide added generality.

3. The Model And The Theorem

Following the Vind-Cornwall-Richter formulation of a nonatomic model, I shall first describe an exchange economy by the pair $(\succ, \hat{T})$, where $\succ$ describes preferences and $\hat{T} \in M_+$ defines the initial allocation of commodities to groups of agents. Then I shall extend these definitions from groups to the linear space of agents.

An allocation $\hat{Z}$, a member of M_+, is feasible if $\hat{Z}(A) = \hat{T}(A)$. The allocation $\hat{Z}$ agrees with $\hat{Z}'$ on E if for all $F \in \mathcal{A}$, $F \subset E$, $\hat{Z}(F) = \hat{Z}'(F)$.

Preferences are defined by the mapping $\succ: \mathcal{A} \to 2^{M_+ \times M_+}$. To say that $(\hat{Z}', \hat{Z}) \in \succ(E)$ is to say that all the members of E unanimously prefer $\hat{Z}'$ to $\hat{Z}$, i.e., $(\hat{Z}', \hat{Z}) \in \succ(F)$ for all $F \in \mathcal{A}$, $F \subset E$.

Information about preferences can be summarized by the mapping $\hat{S}: M_+ \times \mathscr{A} \to 2^{Y_+}$ derived from $\succ$. The set of elements $y \in \hat{S}(\hat{Z},E)$ is obtained from $\succ(E)$ by taking all those $\hat{Z}'$ such that

(i) $\hat{Z}'$ agrees with $\hat{Z}$ on $A\setminus E$

(ii) $(\hat{Z}',\hat{Z}) \in \succ(E)$, and

(iii) $\hat{Z}'(E) = y$.

The following assumptions on $\succ$ are given as they are reflected in $\hat{S}$.

(S.1): $\hat{S}(\hat{Z},E) = \{\underset{\sim}{0}\}$ whenever $\lambda(E) = 0$.

(S.2): $\hat{S}(\hat{Z},\cup E_m) = \Sigma\hat{S}(\hat{Z},E_m)$ whenever $\{E_m\}$, $m = 1,\ldots,n$ are pairwise disjoint.

Since $\hat{Z} \in M_+$ implies that $\hat{Z}(E) = \underset{\sim}{0}$ whenever $\lambda(E) = 0$, (S.1) says that null sets of agents, receiving null aggregate quantities of commodities, have null preference sets. (S.2) would follow from an assumption on $\succ$ that preferences do not exhibit external effects, i.e., if $\hat{Z}'$ agrees with $\hat{Z}$ on E, then $\hat{S}(\hat{Z}',E) = \hat{S}(\hat{Z},E)$.

The model $(\succ,\hat{T})$ will be referred to in its more convenient summary form $(\hat{S},\hat{T})$.

A <u>Walrasian equilibrium</u> (WE) for $(\hat{S},\hat{T})$ is a pair $(\hat{Z},q) \in M_+ \times Y^*$ such that

(1) $\hat{Z}(A) = \hat{T}(A)$

(2) $q\hat{Z}(E) = q\hat{T}(E)$, all $E \in \mathscr{A}$

(3) $y \in \hat{S}(\hat{Z},E)$ implies $qy > q\hat{T}(E)$, all E with $\lambda(E) \neq 0$.

REMARK 3: Assume Y has the Radon-Nikodym Property described in Remark 2. The model $(\hat{S},\hat{T})$ might have originated from the individualistic representation (u,t). Then $\hat{T}(E) = \int_E t\, d\lambda$ so that initial allocations for groups are built up out of the initial allocations for individuals. Similarly, let $u: A \times Y_+ \to \mathbb{R}$ be a measurable function from which $\hat{S}$ is derived: when $\hat{Z}(E) = \int_E z d\lambda$, then $\hat{S}(\hat{Z},E) = \{y = \int_E z' d\lambda: u(a,z'(a)) > u(a,z(a)),$ a.e. on $E\}$. If $(\hat{S},\hat{T})$ were derived from (u,t) in this way, the definition of $(\hat{Z},q)$ as a WE for $(\hat{S},\hat{T})$ would be equivalent to the definition of (z,q) as a WE for (u,t) where (1) $\int_A z d\lambda = \int_A t d\lambda$, and for a.e. $a \in A$, (2) $q(z(a) - t(a)) = 0$, and (3) $u(a,y) > u(a,z(a))$ implies $q(y-z(a)) > 0$.

The relation between the aggregate or group and the individualistic formulations of nonatomic models is the subject of Debreu (1967) and Armstrong (1982).

Just as there is a unique linear extension of $\hat{T} \in M_+$ to $T \in B_+$, let us also define $S: B_+ \times X_+ \to 2^{Y_+}$ as the linear extension of $\hat{S}$, where

$$S(Z,x) = \Sigma\, \alpha_i \hat{S}(\hat{Z},E_i), \quad \text{when} \quad x = \Sigma\, \alpha_i \chi_{E_i} \quad \text{and} \quad \alpha_i > 0.$$

To interpret, we may regard Z as a replicated version of $\hat{Z}$ in which the group $\alpha\chi_E$ receives the total quantities $\alpha\hat{Z}(E)$. S is the corresponding notion of replicated group preferences in which $S(Z,\alpha\chi_E) = \alpha\hat{S}(\hat{Z},E)$. Note that unlike the usual definition of a replica economy with a finite number of types, here there may be a continuum of types.

The pair (S,T), a linear extension of $(\hat{S},\hat{T})$, is simply a replicated version of $(\hat{S},\hat{T})$. It is readily verified that $(\hat{Z},q)$ is a WE for $(\hat{S},\hat{T})$ if and only if (Z,q) is a WE for (S,T), i.e., (1) $Z\chi_A = T\chi_A$, (2) $q(Z-T)x = 0$, all $x \in X_+$, and (3) $y \in S(Z,x)$ implies $qy > qZx$, all $x \in X_+ \setminus \{\underset{\sim}{0}\}$. The advantage of the latter formulation is that it leads to a characterization of WE that will be used below, in the proof of existence of WE and its coincidence with the core. For the remainder of the paper I shall use the linear extension B_+ of the space of allocations and the linear extension (S,T) of the model $(\hat{S},\hat{T})$.

The remaining assumptions on preferences and initial allocations are now given. Rather than stating them in terms of $(\hat{S},\hat{T})$ and then extending them, I shall give them directly on (S,T).

Besides the requirement that $T \in B_+$, assume

(T.1): $T \in B_{++}$,

(T.2): $T\chi_A \in Y_{++}$.

(T.2) says that aggregate initial endowments satisfy a quasi-interiority condition. (T.1) says that nonnull subsets of X_+ have nonnull initial endowments. Note that the stipulation $T \in B_+$ implies the converse of (T.1): if $|x| = 0$, then $Tx = \underset{\sim}{0}$.

For $(Z,x) \in B_+ \times X_+$, assumptions on S are:

(S.3): $S(Z,x)$ is monotonic — i.e., if $|x| \neq 0$, then
(a) $Zx \in \text{cl } S(Z,x)$ and
(b) $y \in \text{cl } S(Z,x)$ implies $y + Y_+ \setminus \{\underset{\sim}{0}\} \subset S(Z,x)$, where $\text{cl} \equiv$ closure,

(S.4): $S(Z,x)$ is convex,

(S.5): S has open graph in $Y_+ \times B_+ \times X_+$ when Y_+ and X_+ are given their respective norm topologies and B_+ is given the strong operator topology.

(S.3) is a strong monotonicity condition while (S.4) is standard and essential.

(S.5) is a key restriction upon which a limiting argument from small-square to large-square economies is based. The sequence $\{Z_n\}$ converges to Z is the strong operator topology if for each $x \in X$, $\lim Z_n x = Zx$. (S.5) says that if $\{y_n\} \subset Y_+$, $\{x_n\} \subset X_+$, $\{Z_n\} \subset B_+$ are such that $y_n \to y$, $x_n \to x \neq 0$, $Z_n \to Z$ and $y \in S(Z,x)$, then for some $\bar{n}$, and all $n > \bar{n}$, $y_n \in S(Z_n,x_n)$. This implies that if $x \neq 0$, $S(Z,x)$ is open in Y_+.

Let us examine (S.5) in more familiar settings. When the large-square model is specialized to form a small-square model, in Section 5.2, it will be seen that (S.5) is exactly the continuity condition used in the demonstrations of WE. Next, consider an individualistic representation (u,t), described in Remark 3, from which (S,T) might be derived. Each $Z \in B_+$ is representable by a $z\colon Z \to Y_+$ such that $Z\chi_E = \int_E z d\lambda$. If $y \in S(Z,\chi_E)$, there is a z' such that $y = \int_E z' d\lambda$ and $u(a,z'(a)) > u(a,z(a))$ a.e. on E.

Assume that $u(a, \ )$ is continuous. Then, for any $\epsilon > 0$ there is a $\delta > 0$ such that $u(a,N_\delta[z'(a)]) > u(a,N_\delta[z(a)])$ a.e. on $E \setminus E^o$, where $N_\delta[y]$ is a δ-neighborhood of Y_+ and $\lambda(E^o) < \epsilon$.

When $Z_n \to Z$, $y \in S(Z,\chi_E)$ and $\lambda(E) \neq 0$, (S.5) implies that for sufficiently large n, $y \in S(Z_n,\chi_E)$. To see what this entails, let Z_n be represented by z_n. It can be shown that $Z_n \to Z$ in measure (i.e., for any $\delta > 0$, $\lim \lambda\{a\colon ||z_n(a) - z(a)|| > \delta\} = 0$) and for any $\{E_n\} \subset E$ with $\lim \lambda(E_n) = 0$, $\lim ||\int_{E_n} z_n \, d\lambda|| = 0$. Therefore, the continuity of u implies that for any $\epsilon > 0$ there is an n such that $u(a,z'(a)) > u(a,z_n(a))$ a.e. on $E \setminus E_n$ and $\lambda(E_n) < \epsilon/2$. Again, by continuity we can find z_n' such that a.e. on $E \ \ E_n \cup F_n$ $u(a,z_n'(a)) > u(a,z_n(a))$, $z'(a) - z_n'(a) \in Y_+$ and $\lambda(E_n \cup F_n) < \epsilon$. Note that $\lim \int_{E \setminus E_n \cup F_n} (z'-z_n') \, d\lambda \in Y_+ \setminus \{0\}$

while $\lim \int_{E_n \cup F_n} z_n d\lambda = \underset{\sim}{0}$. (S.5) says that we may choose z_n' in such a way that the excess $\int_{E \setminus E_n \cup F_n} (z'-z_n')\, d\lambda$, which is arbitrarily large in comparison to $\int_{E_n \cup F_n} z_n d\lambda$, can be redistributed to $E_n \cup F_n$ so that $y = \int_E z d\lambda = \int_E z_n' \, d\lambda$ and $u(a,z_n'(a)) > u(a,z_n(a))$ a.e. on $E_n \cup F_n$.

The above restrictions do not capture the irreflexivity of the underlying preferences of the members of a group. For example, we cannot say that for $\lambda(E) \neq 0$, $Z\chi_E \notin S(Z,\chi_E)$, as we could if E were a single individual. For a given Z, there are many Z' such that $Z'\chi_E = Z\chi_E$ and the possibility cannot be ruled out that some of them are unanimously preferred by E to Z. To preserve this possibility while also relying upon the theorems for small-square models, another assumption must be made for which some additional constructions are required.

Let $\pi = \{E\}$ be a partition of A into a finite number of elements from $\mathscr{A}$. Use π to define the finite-dimensional subspace of X,

$$X(\pi) = \{x \in X: x = \sum_{E \in \pi} \alpha_E \chi_E\}.$$

If Z' is a linear operator on $X(\pi)$, it can be extended to all of X by taking

$$Zx = \sum_i \alpha_i (\sum_{E \in \pi} \alpha_{E(i)} Z'\chi_E),$$

where $x = \Sigma\, \alpha_i \chi_{E_i}$ and $\alpha_{E(i)} = \lambda(E \cap E_i)$. This construction illustrates the method by which an allocation for a model with a finite number of agents is made into a replica allocation for a nonatomic model. The scalar $\alpha_{E(i)}$ is the fraction of the members of E_i who are of type E.

Let $B_\pi \subset B(X,Y)$ be the set of operators on X that can be built up as extensions of operators on $X(\pi)$. In contrast to allocations in the space B, for allocations in B_π there is the conclusion,

if Z, $Z' \in B_\pi$ and $Z\chi_E = Z'\chi_E$, $E \in \pi$, then Z agrees with Z' on E.

Let S_π be the restriction of S to the domain $(B_\pi)_+ \times X_+(\pi)$. Note that if $E' \notin \pi$ and therefore $\chi_E \notin X_+(\pi)$, its preferences are ignored by S_π. Irreflexivity of preferences appears as

(S.6): $Z\chi_E \notin S_\pi(Z,\chi_E)$ whenever $Z \in (B_\pi)_+$ and $E \in \pi$.

The final assumption acknowledges the fact that the commodity space may not be finite-dimensional and therefore Y_+ may have an empty interior. Thus, without further qualification, even if the feasible allocation Z were such that $Z\chi_A$ belonged to the Y_+-boundary of the convex set $S(Z,\chi_A)$ — a necessary condition for WE, given (S.3) — there need not be a $q \in Y^*\setminus\{\underset{\sim}{0}\}$ defining a supporting hyperplane to $S(Z,\chi_A)$ passing through $Z\chi_A$. Before we can prove the existence of a WE pair (Z,q), the prior condition that such supporting hyperplanes exist must be ensured.

To this end, let $Z \in B_{++}$ and define the function $d_Z: Y \times X_+\setminus\{\underset{\sim}{0}\} \to$ by

$$d_Z(y;x) = \begin{cases} -\infty, & \text{if } y \notin \alpha S(Z,x), \text{ all } \alpha > 0. \\ \sup\{\alpha > 0: y \in \alpha S(Z,s)\}, & \text{otherwise.} \end{cases}$$

Note that when $Z \in B_{++}$, and (S.2) is satisfied then $\underset{\sim}{0} \notin \text{cl } S(Z,x)$. Therefore, $d_Z(y;x)$ lies in $[-\infty,\infty)$; and if $d_Z(y;x) > -\infty$, then $d_Z(y;x) > 0$. $d_Z(y;x)$ is a measure of the distance between point y and the set $S(Z,x)$. The point y does or does not belong to $\text{cl } S(Z,x)$ as $d_Z(y;x) \geq 1$ or < 1.

Holding Z and x fixed, $d_Z(\ ;x)$ is known to be homogeneous ($\beta > 0$ implies $\beta d_Z(y;x) = d_Z(\beta y;x)$) and, because $S(Z,x)$ is convex, it is super-additive ($d_Z(y+y';x) \geq d_Z(y;x) + d_Z(y';x)$). Therefore, $d_Z(\ ;x)$ is concave.

Properties of this function are demonstrated in Phelps (1963, Proposition 2, pp. 398-99]).

The subdifferential of the homogeneous concave function $d_Z(.;x)$ is a mapping defined by

$$\partial d_Z(y;x) = \{q \in Y^*: qy = d_Z(y;x), \quad qy' \geq d_Z(y';x), \quad \text{all} \quad y' \in Y\}.$$

Note that because $d_Z(\ ;x)$ is homogeneous, if $\beta > 0$ then $\partial d_Z(\beta y;x) = \partial d_Z(y;x)$.

The reason for introducing d_Z is the following: <u>if</u> $q \in \partial d_Z(y;x)$ <u>and</u> $d_Z(y;x) = \alpha$, <u>the hyperplane defined by</u> q <u>passing through</u> y <u>supports</u> cl α $S(Z,x)$ <u>at</u> y. Thus, y is on the boundary of $S(Z,x)$ iff $\alpha = 1$.

If $d_Z(\ ;x)$ were continuous at y, that would suffice to conclude $\partial d_Z(y;x) \neq \phi$ and, if y belonged to the interior of Y_+, the fact that $d_Z(\ ;x)$ is concave would imply its continuity. However, when Y_+ has no interior, its continuity cannot be presumed, $\partial d_Z(y;x)$ may be empty, and a supporting hyperplane cannot be ensured.

To demonstrate the existence of WE these possibilities must be precluded. This is accomplished by the following assumption.

(S.7) For any $\gamma > 0$ and $Z \in B_{++}$, there exists a weak-star compact subset $C \subset Y_+^* \setminus \{\underset{\sim}{0}\}$ such that if $(y,x) \in Y_+ \times X_+ \setminus \{\underset{\sim}{0}\}$ and $|x|^{-1}||y|| < \gamma$, then

$$\partial d_Z(|x|^{-1}\ y;x) \cap C \neq \phi.$$

There are two parts to (S.7). First, there is the assumption that for any $(y,x) \in Y_+ \times X_+ \setminus \{\underset{\sim}{0}\}$, $\partial d_Z(y;x) \neq \phi$. This would itself be a consequence of the continuity of d_Z at y. A $q \in \partial d_Z(|\chi_E|^{-1}\ y;\chi_E)$ measures the <u>per capita</u> marginal rate of substitution among commodities for the group E. Second, (S.7) says that not only are <u>per capita</u> marginal rates of substitution well-defined for any group, but they also do not vary "too much" among groups of agents.

REMARK 4: It may be helpful to give an analog of (S.7) when the model (S,T) is derived from an individualistic representation (u,t) as in Remark 3. Assuming $Z \in B_{++}$, assume there exists a $z: A \to Y_{++}$ such that $Z\chi_E = \int_E z d\lambda$. Let $S(z(a),a) = \{y: u(a,y) > u(a,z(a))\}$. Therefore, if there exists $q \in Y^*$ such that $\inf q[S(z(a),a)] = qz(a) \neq 0$, then q represents the marginal rate of substitution of commodities for individual a at $z(a)$.

Suppose $u(a,\cdot)$ is a concave function on Y_+ and let $\partial u(a,y) = \{q \in Y^*: u(a,y) - q(y-y') \geq u(a,y'),\ \text{all } y' \in Y_+\}$. It is readily verified that $q \in \partial u(a,z(a))$ implies $\inf q[S(z(a),a)] = qz(a)$. Let $\partial u_\gamma(a) = \{\partial u(a,y): ||y|| < \gamma\}$ and say $\partial u_\gamma(a) \neq \phi$ if $\partial u(a,y) \neq \phi$, for all $||y|| < \gamma$. It would suffice as an alternative to (S.7) to have

$$\lambda\{a: \partial u_\gamma(a) \cap C \neq \phi\} = \lambda(A).$$

Call the above restrictions (Y.1), (T.1-2) and (S.1-7) on the commodity space, initial allocations and preferences, respectively, the "stated conditions". The main result of this paper is the following:

EXISTENCE THEOREM: Under the stated conditions on (S,T) there exists a Walrasian equilibrium.

4. The Auxiliary Lemma and the Core Equivalence Property

Say that a feasible allocation Z for (S,T) is in Walrasian position if

(WP): $Tx \notin S(Z,x)$, all $x \in X_+ \setminus \{\underset{\sim}{0}\}$.

It is readily verified that if (Z,q) is a WE for (S,T), then Z is in Walrasian position. More interesting is the converse.

Compare the definition of Z as satisfying Walrasian position with $\hat{Z}$ as an allocation in the core of $(\hat{S},\hat{T})$;

(Core): $\hat{T}(E) \not\in \hat{S}(\hat{Z},E)$, all E with $\lambda(E) \neq 0$.

Thus, Z is in Walrasian position if it is in the core, not simply of the actual space of agents $\mathscr{A}$, but the linear space X_+.

If the economy $(\hat{S},\hat{T})$ were to be replicated so that for every "individual type" E there were groups of agents having preferences and endowments that were positive scalar multiples of $\hat{T}(E)$ and $\hat{S}(Z,E)$ and if the allocation Z were also in the core of the replicated economy, the result of Debreu and Scarf (1963) suggests that Z would be a WE. Because T and $S(Z,\cdot)$ are each linear on X_+, <u>such replication is a built-in feature of</u> (S,T). Thus, the following result is an extension of the Debreu-Scarf Theorem to economies with an infinite number of types of agents and an infinite-dimensional commodity space. Aubin (1979) gives a version of this result for small-square economies with the additional restriction that individual preferences can be represented by concave functions.

AUXILIARY LEMMA: <u>Under the stated conditions on</u> (S,T) <u>if</u> Z <u>is a feasible allocation in Walrasian position, there exists a</u> $q \in Y^*$ <u>such that</u> (Z,q) <u>is a Walrasian equilibrium.</u>

<u>PROOF:</u> Let $x \in X_+ \setminus \{\underset{\sim}{0}\}$. From (T.1), $Tx \neq \underset{\sim}{0}$. Since $Tx \not\in S(Z,x)$, it follows from (S.3) that $Zx \neq \underset{\sim}{0}$. Therefore, $Z \in B_{++}$.

Define

$$K = \{S(Z,x) - Tx\colon x \in X_+\}$$

1. K is a convex cone with vertex $\underset{\sim}{0}$.

Let $|x| = 0$. $T \in B_+$ implies $Tx = \underset{\sim}{0}$; and by (S.1), $S(Z,x) = \{\underset{\sim}{0}\}$. Therefore, $S(Z,x) - Tx = \underset{\sim}{0} \in K$.

If $y \in \{S(Z,x) - Tx\}$ and $y' \in \{S(Z,x') - Tx'\}$, then for all $\alpha > 0$, $(\alpha y + y') \in \{S(Z,x'') - Tx''\}$, where $x'' = \alpha x + x'$. Thus, K is a convex cone with vertex $\underset{\sim}{0}$.

2. $K \cap (-Y_+ \setminus \{\underset{\sim}{0}\}) = \phi$.

Let $|x| = 0$. From Step 1, $\{S(Z,x) - Tx\} \cap (-Y_+ \setminus \{\underset{\sim}{0}\}) = \phi$.

Let $x \in X_+ \setminus \{\underset{\sim}{0}\}$. Suppose there is a $y \in S(Z,x)$ such that $(y - Tx) \in (-Y_+ \setminus \{\underset{\sim}{0}\})$. But by (S.3) this would imply $Tx \in S(Z,x)$, contradicting the hypothesis of the Lemma. Again $\{S(Z,x) - Tx\} \cap (-Y_+ \setminus \{\underset{\sim}{0}\}) = \phi$.

3. $\mathrm{cl}\ K \cap (-Y_{++}) = \phi$.

Suppose $v_n \in K$ where $v_n = y_n - Tx_n$ and $y_n \in S(Z,x_n)$. Let $v = \lim v_n$. If $v = \underset{\sim}{0}$ there is nothing more to prove so assume $v \neq \underset{\sim}{0}$.

Let $q_n \in \partial d_Z(Tx_n;x_n) \subset C$. Therefore, by (T.1) and the definition of ∂d_Z,

$$q_n Tx_n = d_Z(Tx_n;x_n) > 0 \tag{1a}$$

$$q_n y_n > d_Z(y_n;x_n) \tag{1b}$$

But $d_Z(y_n;x_n) > d_Z(Zx_n;x_n) > d_Z(Tx_n;x_n)$. Therefore (1a-b) imply

$$q_n(y_n - Tx_n) = q_n v_n > 0 \tag{2}$$

Since T is bounded, there is a $\gamma > 0$ such that $|x_n|^{-1}\ ||Tx_n|| < \gamma$. Further, $\partial d_Z(Tx_n;x_n) = \partial d_Z(|x_n|^{-1}\ Tx_n;x_n)$. Therefore, by (S.7) if $q_n \in \partial d_Z(Tx_n;x_n)$ there is a $q \in Y^*_{++}$ such that a subsequence of $\{q_n\}$, taken to be the sequence itself, converges in the weak-star topology to q. Therefore

$$q_n v \to qv \tag{3}$$

Since C is norm bounded and $\{q_n\} \subset C$, the hypotheses that $v_n \to v$ in norm and $q_n \to q$ in the weak-star topology imply that

$$q_n v_n - qv = q_n(v_n - v) + (q_n - q)v \to 0. \tag{4}$$

Conditions (2-4) imply $qv > 0$. Since $q \in Y_+^* \setminus \{\underset{\sim}{0}\}$ and $v \neq \underset{\sim}{0}$, $v \notin (-Y_{++})$; otherwise, $qv < 0$.

4. $\inf qK > 0$ for some $q \neq \underset{\sim}{0}$.

A result in Klee (1948, Corollary 1, p. 769) is that if K is a convex cone with vertex $\underset{\sim}{0}$, then there exists a $q \neq \underset{\sim}{0}$ such that $\inf qK > 0$ if and only if $\mathrm{cl}\, K \neq Y$. Therefore, Step 4 follows from Step 3.

5. (Z,q) is a WE.

From (S.3) and condition (1), $Y_+ \setminus \{\underset{\sim}{0}\} \subset \{S(Z,\chi_A) - T\chi_A\}$. Therefore, $\inf qK = 0$ implies $q \in Y_+^* \setminus \{\underset{\sim}{0}\}$ and by (T.2) this implies $qT\chi_A > 0$. By a well-known argument it follows that $qT\chi_E > 0$ whenever $T\chi_E \neq \underset{\sim}{0}$ and therefore if $y \in S(Z,\chi_E)$, then $q(y - T\chi_E) > 0$. By (T.1), $T\chi_E \neq \underset{\sim}{0}$ whenever $\lambda(E) \neq 0$ and therefore condition (3) defining WE is satisfied. Q.E.D.

It is clear from their definitions that an allocation in Walrasian position for (S,T) is in the core of (S,T), i.e., $Tx \notin S(Z,x)$, all $x \in X_+ \setminus \{\underset{\sim}{0}\}$ implies $Tx \notin S(Z,x)$, all $x \in \chi \setminus \{\underset{\sim}{0}\}$ where $\chi = \{\chi_E : E \in \mathscr{A}\}$. By the Auxiliary Lemma, this is nothing other than the well-known result that a WE is in the core. In general, the condition of being in Walrasian position is stronger than merely being in the core, even in nonatomic economies.

However, in certain models where the commodity space is not "too large" the core and Walrasian position are equivalent conditions. This is the case for the model of this paper, as it is for the large-square models examined by Bewley (1973) and Mas-Colell (1975).

CORE EQUIVALENCE THEOREM: Under the stated conditions on (S,T), the conditions of being an allocation in the core, in Walrasian position, or Walrasian equilibrium are equivalent.

PROOF: Since WE => WP => core, the three are equivalent if core => WE.

We repeat the definition of K and introduce the definitions of K_1 and J:

$$K = \{S(Z,x) - Tx\colon x \in X_+\}$$

$$K_1 = \{S(Z,x) - Tx\colon x \in \text{co}\ \chi\}$$

$$J = \{S(Z,x) - Tx\colon x \in \chi\}$$

where co χ is the convex hull of $\chi = \{\chi_E\colon E \in \mathcal{A}\}$.

To say that Z is in the core is to say that

$$J \cap -Y_+ = \{\underset{\sim}{0}\}.$$

If it can be shown that this implies cl $K \cap -Y_{++} = \{\underset{\sim}{0}\}$, then steps 4 and 5 in the proof of the Auxiliary Lemma can be used to show that there is a q such that (Z,q) is a WE.

1. cl $J \cap -Y_{++} = \phi$. Since $J \subset K$, the same argument used in step 3 of the Auxiliary Lemma to show $K \cap -Y_+ = \{\underset{\sim}{0}\}$ implies cl $K \cap -Y_{++} = \phi$ may be applied here to yield the desired conclusion.

Let us say that $Z[\chi]$ is dense in $Z[\text{co } \chi]$ if for any $x = \Sigma\beta_j \chi_{E_j} \in$ co χ, where $\{E_j\}$ are pairwise disjoint, and any $\epsilon > 0$, there exists $F_j \subset E_j$, necessarily pairwise disjoint, such that

$$||Z\chi_{\cup F_j} - Zx|| < \epsilon .$$

2. $Z \in B_+$ implies $Z[\chi]$ is dense in $Z[\text{co } \chi]$. Let

$$Zx = \sum_{k=1}^{k=\infty} \beta_k(x)e_k \quad \text{and} \quad Z^\ell x = \sum_{k=1}^{k=\ell} \beta_k(x)e_k.$$

Therefore,

$$(Z-Z^\ell)x = \sum_{k>\ell} \beta_k(x)e_k.$$

If $x \in$ co χ, then $Zx \leq Z\chi_A = T\chi_A$. Writing

$$T\chi_A = \sum_{k=1}^{k=\infty} \tau_k(\chi_A)e_k \quad \text{and} \quad T^\ell \chi_A = \sum_{k=1}^{k=\ell} \tau_k(\chi_A)e_k,$$

we know that because $\{e_k\}$ is order-compatible $\tau_k(\chi_A) \geq \beta_k(x)$. Therefore

$$\sup \{||(Z-Z^\ell)x||: x \in \text{co } \chi\} \leq ||(T-T^\ell)\chi_A|| = ||\sum_{k>\ell} \tau_k(\chi_A)e_k||, \tag{5}$$

which implies that the LHS $\to 0$ as $\ell \to \infty$.

Similarly,

$$\sup \{||(Z-Z^\ell)x||: x \in \chi\} \to 0 \quad \text{as} \quad \ell \to \infty. \tag{6}$$

Now, by Lyapunov's Theorem on the convexity of the range of a vector measure,

$$Z^\ell[\chi] = Z^\ell[\text{co } \chi] \quad \text{for all } \ell, \tag{7}$$

where the equality is understood in the same sense of $Z[\chi]$ as dense in $Z[\text{co } \chi]$ except that $\epsilon = 0$.

The conjunction of (5), (6) and (7) implies the desired conclusion.

3. cl J = cl K_1: It suffices to show that J is dense in K_1. Let $y = S(Z,x) - Tx$, where $x = \Sigma\alpha_i \chi_{E_i} \varepsilon$ co χ. Now, x can be rewritten (if necessary) as a linear combination of pairwise disjoint $\{E_j\}$ such that $x = \Sigma\, \beta_j \chi_{E_j}$, $\beta_j \varepsilon [0,1]$. Therefore, there exists y_j such that $\Sigma y_j = y$ and $y_j \varepsilon \beta_j [S(Z,\chi_{E_j}) - T\chi_{E_j}]$. Thus, there exists Z' such that $Z'\chi_{E_j} \varepsilon S(Z,\chi_{E_j})$ and

$$y_j = \beta_j [Z'-T]\chi_{E_j}$$

By 2, $Z'[\chi]$ is dense in Z'[co χ], $T[\chi]$ is dense in T[co χ] and it readily follows that $(Z'-T)[\chi]$ is dense in $(Z'-T)$[co χ]. Therefore, there exists $F_j \subset E_j$ such that

$$y_j' = (Z'-T)\chi_{F_j}$$

and y_j' is close to y_j, or $y = \Sigma y_j$ is close to $y' = \Sigma\, y_j'$, as was to be demonstrated.

4. cl $K \cap -Y_{++} = \phi$. Steps 1 and 2 imply that cl $K_1 \cap -Y_{++} = \phi$. But $K = \{\alpha\, K_1\colon \alpha > 1\}$ and cl $K = \{\alpha$ cl $K_1\colon \alpha > 1\}$. Therefore, cl $K \cap -Y_{++} = \phi$; otherwise if $y\ \varepsilon$ cl $K \cap -Y_{++}$, then there would be an $\alpha > 0$ such that $\alpha^{-1} y\ \varepsilon$ cl $K_1 \cap -Y_{++}$, a contradiction. Q.E.D.

5. Proof of the Existence Theorem

The strategy of the proof is first to show that a large-square model can be suitably restricted to mimic a small-square one. Once this is done theorems for small-square models can be applied to demonstrate the existence of WE and therefore of an allocation in Walrasian position (WP). This is accomplished in Sections 5.1 and 5.2, below. In Section 5.3, I shall show how the existence of an allocation in WP for the small-square model can be extended to imply the existence of an allocation in WP with the original large-square space of agents but the small-square commodity space. In step 5.4, the existence of an allocation in WP will be extended to the original model with both a continuum of agents and an infinite-dimensional commodity space.

5.1 The economy (S^{ℓ},T^{ℓ}).

By (Y.1), each Tx can be written as

$$Tx = \Sigma\ \tau_k(x)e_k,$$

where $\{\tau_k(x)\}$ is the unique sequence of nonnegative scalars defining Tx.

Let $\bar{y} = T\chi_A$ and define

$$\bar{y}(\ell) = \sum_{k\geq\ell} \tau_k(\chi_A)e_k.$$

Construct the finite-dimensional subspace

$$Y^{\ell} = \{y \in Y\colon y = \sum_{k<\ell} \beta_k e_k + \beta_{\ell}\ \bar{y}(\ell)\}.$$

(Of course, Y^{ℓ} is equivalent to $\mathbb{R}^{\ell}$, assuming $\bar{y}(\ell)$ is not spanned by $\{e_1,\ldots,e_{\ell-1}\}$.) Use Y^{ℓ} to define B^{ℓ} as the set of bounded linear operators from X to Y^{ℓ} and B^{ℓ}_{+} as those for which $Z[X_+] \subset Y^{\ell}_{+}$.

Let $T^{\ell} \in B^{\ell}_{+}$ be defined by

$$T^{\ell}x = \sum_{k<\ell} \tau_k(x)e_k + \Sigma\ \alpha_i\ |\chi_{E_i}|\ \bar{y}(\ell), \quad \text{when} \quad x = \Sigma\ \alpha_i\chi_{E_i}.$$

T^{ℓ} is a <u>revision</u> of the initial allocation. Regarding each e_k as one unit of a distinct commodity, T^{ℓ} gives each group E exactly the amounts of the first $(\ell-1)$ commodities it had in T. However, its allocation of the remaining commodities is revised to ensure an equal per capita distribution of the "composite commodity," $\bar{y}(\ell)$. Thus, E receives the amount $\lambda(E)\bar{y}(\ell)$.

As a member of B_+^ℓ, T^ℓ satisfies conditions (T.1-2).

$S^\ell\colon B_+^\ell \times X_+ \to 2_+^{Y^\ell}$ is the restriction of the original preference mapping S to the set of allocations defined by B_+^ℓ. On its domain, S^ℓ satisfies conditions (S.1-7).

5.2 The small-square economy (S_π^ℓ, T_π^ℓ) and the existence of WE.

Let π be a partition of A having n nonnull elements. In Section 3, the preparations for assumption (S.6) included the definition of B_π. B_π^ℓ will indicate the subset of B_π in which the range of the operator lies in Y^ℓ. Note that elements of B_π^ℓ can be put in 1-1 correspondence with $\ell \times n$ matrices.

$T_\pi^\ell \in (B_\pi^\ell)_+$ is a further revision of T in which $T_\pi^\ell \chi_{E_i} = T^\ell \chi_{E_i}$, $E_i \in \pi$. For $E \notin \pi$, $T_\pi^\ell \chi_E$ need not agree with $T^\ell \chi_E$. T_π^ℓ is the description of initial endowments in a small-square economy.

The description of preferences in a small-square is given by $S_\pi^\ell\colon (B_\pi^\ell)_+ \times X_+(\pi) \to 2_+^{Y^\ell}$, which is the further restriction of the preference mapping to the domain of allocations described by $\ell \times n$ matrices. Note that for $x \notin X_+(\pi)$, its preferences are ignored in S_π^ℓ.

It is now useful to eliminate some of the notational burden from (S_π^ℓ, T_π^ℓ) so as to regard it as a small-square model on its own terms.

As an element of $(B_\pi^\ell)_+$, T_π^ℓ can be described by $(t_1, \ldots, t_n)$, where $t_i = T^\ell \chi_{E_i} \in \mathbb{R}_+^\ell$ when $E_i \in \pi$. By (T.1-2),

(a) $\quad t_i \in \mathbb{R}_+^\ell \setminus \{\underset{\sim}{0}\}$ and $\Sigma\, t_i \in \operatorname{int} \mathbb{R}_+^\ell$.

Similarly, $Z \in (B_\pi^\ell)_+$ can be described by $(z_1,\ldots,z_n)$, where $z_i = Z\chi_{E_i}$, when $E_i \in \pi$. Define $S_i(Z) = S_\pi^\ell(Z,\chi_{E_i})$. Thus $S_i: (\mathbb{R}_+^\ell)^n \to 2^{\mathbb{R}_+^\ell}$. Further,

(b) by (S.3), $S_i(Z)$ is nonempty and its closure contains $\{\mathbb{R}_+^\ell \setminus \{\underset{\sim}{0}\} + z_i\}$;

(c) by (S.4), $S_i(Z)$ is convex

(d) by (S.5), $S_i(Z)$ has open graph in $\mathbb{R}_+^\ell \times (\mathbb{R}_+^\ell)^n$

(e) by (S.6), $z_i \notin S_i(Z)$

(S_π^ℓ,T_π^ℓ) is now described by $\{(S_i,t_i): i = 1,\ldots,n\}$, where S_i defines the preferences and t_i the initial endowments of individual i. A WE for this model is a $(Z,q) \in (\mathbb{R}_+^\ell)^n \times \mathbb{R}^\ell$ such that,

(i) $\Sigma(z_i - t_i) = \underset{\sim}{0}$,

(ii) $q(z_i - t_i) = 0$, $i = 1,\ldots,n$

(iii) $y \in S_i(Z)$ implies $q(y - t_i) > 0$, $i = 1,\ldots,n$.

Through the contributions of Mas-Colell (1974), Gale and Mas-Colell (1975), and Shafer and Sonnenschein (1975), it is know that conditions (a)-(e) imply the existence of WE. Therefore, if (Z,q) is a WE for the small-square model, Z is in Walrasian position, i.e.,

$$\Sigma\alpha_i t_i \notin \Sigma\alpha_i S_i(Z), \quad \text{all } \alpha_i \geq 0, \quad i = 1,\ldots,n, \quad \Sigma\alpha_i > 0.$$

A formal restatement of this conclusion suited to our purposes is

PRINCIPAL LEMMA: Under the stated conditions, in the small-square model (S_π^ℓ, T_π^ℓ) derived from the large-square model (S,T), there exists $Z_\pi^\ell \in (B_\pi^\ell)_+$ in Walrasian position, i.e., for all $x \in X_+(\pi) \setminus \{\underset{\sim}{0}\}$, $T_\pi^\ell x \not\in S_\pi^\ell(Z_\pi^\ell, x)$.

5.3 From WE in (S_π^ℓ, T_π^ℓ) to WE in (S^ℓ, T^ℓ).

Let $\{\pi\}$ be a sequence of successively refining partitions for A for which

$$\lim_{\pi} (\max_{E \in \pi} \{\lambda(E)\}) = 0.$$

It is known that for any $\varepsilon > 0$ there is a π_o in the sequence such that for any π refining π_o and any $E \in \mathscr{A}$, there exists an E' formed by the union of elements of π such that the λ-measure of the symmetric difference between E and E' is less than ε (see Halmos (1950, p. 170)).

By construction, for each $E \in \pi$, $T_\pi^\ell \chi_E = T^\ell \chi_E$. Further, $\{T_\pi^\ell\}$, regarded as a sequence in π, is a bounded sequence of operators. Thus, for each E, $\lim_\pi T_\pi^\ell \chi_E = T^\ell \chi_E$. This readily extends to the conclusion that for all $x \in X$, $\lim_\pi T_\pi^\ell x = T^\ell x$, or

$$T_\pi^\ell \to T^\ell.$$

To show that a limit point of $\{Z_\pi^\ell\}$ exists in B_+^ℓ, a more general argument applicable to convergence in B will be given and drawn upon again, below.

Let $B_1 = \{Z_n\}$ and B_2 be the closure of B_1 with respect to the strong operator topology. Since B is known to be closed in this topology, $B_2 \subset B$. Further, if $B_1 \subset B_+$, it is readily verified that $B_2 \subset B_+$. To ensure that any sequence in B_1 contains a convergent subsequence, B_2 must be compact.

FACT 1 (Dunford and Schwartz (1957, p. 511, exercise 2)): B_2 <u>is compact in the strong operator topology iff for each</u> x, $B_2x = \{y: y = Zx, \; Z \in B_2\}$ <u>is contained in a compact set.</u>

For all π, the range of Z^{ℓ}_{π} is contained in the same finite-dimensional Y^{ℓ}. Thus, the closure of $\{Z^{\ell}_{\pi}\}$, call it B_2, is contained in B^{ℓ}_{+}. Since the range of each B_2x is in Y^{ℓ} and is bounded, the conditions of Fact 1 are met. Therefore, there exists a $Z^{\ell} \in B^{\ell}_{+}$ and a convergent subsequence of π, assumed to be the sequence itself, such that

$$Z^{\ell}_{\pi} \to Z^{\ell}.$$

Further, Z^{ℓ} is a feasible allocation for (S^{ℓ},T^{ℓ}) since $Z^{\ell}_{\pi}\chi_A = T^{\ell}_{\pi}\chi_A = T^{\ell}\chi_A$ for all π and therefore $Z^{\ell}\chi_A = T^{\ell}\chi_A$.

LEMMA 1: <u>Let</u> Z^{ℓ}_{π} <u>satisfy the conclusions of the Principal Lemma for</u> $(S^{\ell}_{\pi},T^{\ell}_{\pi})$. <u>Then</u> $\{(Z^{\ell}_{\pi},T^{\ell}_{\pi})\}$ <u>has a limit point</u> (Z^{ℓ},T^{ℓ}) <u>which is in Walrasian position for</u> (S^{ℓ},T^{ℓ}), i.e., <u>for all</u> $x \in X_{+}\setminus\{\underset{\sim}{0}\}$, $T^{\ell}x \notin S^{\ell}(Z^{\ell},x)$.

PROOF: Suppose $T^{\ell}x' \in S^{\ell}(Z^{\ell},x')$. Then for some π there is an $x \in X_{+}(\pi)\setminus\{\underset{\sim}{0}\}$ such that x is arbitrarily close to x', $T^{\ell}_{\pi}x$ close to $T^{\ell}x'$, and $Z^{\ell}_{\pi}x$ close to $Z^{\ell}x$. By the continuity condition, (S.4) this implies $T^{\ell}_{\pi}x \in S^{\ell}_{\pi}(Z^{\ell}_{\pi},x)$. But this contradicts the hypothesis that Z^{ℓ}_{π} is a WE for $(S^{\ell}_{\pi},T^{\ell}_{\pi})$.

5.4 From WE in (S^{ℓ},T^{ℓ}) to WE in (S,T).

Let $\{\ell\}$ be the sequence of positive integers. By construction,

$$T^{\ell}\chi_E = \sum_{k<\ell} \tau_k(\chi_E)e_k + |\chi_E|\ \bar{y}(\ell)$$

$$= \sum_{k<\ell} \tau_k(\chi_E)e_k + |\chi_E|\ (\sum_{k\geq\ell} \tau_k(\chi_A)e_k),$$

while

$$T\chi_E = \sum_{k=1}^{k=\infty} \tau_k(\chi_E)e_k.$$

Thus,

$$\lim_{\ell} ||T^{\ell}\chi_E - T\chi_E|| = \lim_{\ell} ||\sum_{k>\ell} [|\chi_E|\ \tau_k(\chi_A) - \tau_k(\chi_E)]e_k|| = 0,$$

i.e.,

$$T^{\ell} \to T.$$

To show that a limit point of $\{Z^{\ell}\}$ exists in B_+, it suffices by Fact 1 to establish that for each x,

$$Kx = \{y = Z^{\ell}x:\ \ \ell = 1,2,\ldots\}$$

is contained in a compact set of Y. Since Y has a basis, compactness is characterized by

FACT 2 (Dunford and Schwartz (1957, p. 260, Corollary 5)): <u>If</u> $\{e_k\}$ <u>is a basis for</u> Y, <u>then</u> K <u>is contained in a compact set</u> (i.e., <u>is conditionally compact</u>) <u>iff it is bounded and for every</u> $\epsilon > 0$, <u>there exists</u> ℓ_o <u>such that for any</u> $y = \Sigma\beta_k e_k \in K$, <u>if</u> $\ell > \ell_o$, <u>then</u> $||\sum_{k>\ell} \beta_k e_k|| < \epsilon$.

The "tails" of all y in K must be uniformly small.

Let $y = \Sigma\beta_k^{\ell}e_k = Z^{\ell}\chi_E$ be an element of $K\chi_E$. Clearly, $y < \bar{y} = T\chi_A$ and therefore $K\chi_E$ is bounded. Further, for any ℓ,

$$y(\ell) = \sum_{k>\ell} \beta_k^\ell e_k < \bar{y}(\ell) = \sum_{k>\ell} \tau_k(\chi_A)e_k.$$

Since for any $\epsilon > 0$, there exists ℓ_o such that $||\bar{y}(\ell)|| < \epsilon$ whenever $\ell > \ell_o$, the same condition follows for $y(\ell)$. Therefore, $K\chi_E$ is conditionally compact.

If $x = \Sigma\alpha_i \chi_{E_i}$, then $Kx = \Sigma\alpha_i K\chi_{E_i}$ exhibits the same property. Therefore, the closure of $\{Z^\ell\}$ is compact in the strong operator topology. Thus, there exists $Z \in B_+$ such that $\{Z^\ell\}$, or a subsequence taken to be the sequence itself, satisfies $Z^\ell \to Z$.

Further, Z is a feasible allocation for the economy (S,T) since $Z^\ell\chi_A = T^\ell\chi_A = T\chi_A$ for all ℓ and therefore $Z\chi_A = T\chi_A$.

LEMMA 2: Let Z^ℓ satisfy the conclusion of Lemma 1 for (S^ℓ,T^ℓ). Then $\{(Z^\ell,T^\ell)\}$ has a limit point (Z,T) which is in Walrasian position for (S,T), i.e., for all $x \in X_+ \setminus \{\underset{\sim}{0}\}$, $Tx \notin S(Z,x)$.

PROOF: Suppose $Tx' \in S(Z,x')$. Because $T^\ell \to T$ and $Z^\ell \to Z$, (S.4) implies that for sufficiently large ℓ, $T^\ell x' \in S(Z^\ell,x')$. Since $Z^\ell x'$, $T^\ell x' \in B_+^\ell$ and S^ℓ is simply the restriction of S to B_+^ℓ, it follows that $T^\ell x' \in S^\ell(Z^\ell,x')$ which contradicts the hypothesis that Z^ℓ is a WE for (S^ℓ,T^ℓ).

Applying the results of the Principal Lemma, Lemmas 1 and 2, and the Auxiliary Lemma completes the proof of the Existence Theorem.

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