Control Games & Organizations

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Introduction

In decision-making bodies like legislatures, parliamentary systems, and organizations in general, coalitions turn out to be either all-powerful or totally ineffectual. This singular circumstance permits us the use of a simple mathematical construct – simple games -- to study the internal authority structure of these entities. Simple games designate a particular class of multi-person games which can be thought of as *games of control*, and are characterized by having only two types of coalitions, winning or losing, that is, the only two possible outcomes for any viable coalition are either winning or losing. This property makes them particularly well suited for the analysis of structures where the primary concern is power and authority rather than strategic or monetary considerations. The mathematical structure and properties of simple games were originally developed by von Neumann and Morgerstern [5] and later extended by Shapley [2], [3].

The present paper follows closely Shapley's analysis and modeling of the structure of simple games and its extension into organization theory, and is organized in five sections. The first three give the basic necessary results about sets, coalitions, simple games, and the local topology of command. In the fourth section we study in detail the key notion of the present paper, the control function. The last section deals with the global structure of control and its applications to organization theory.

1 Sets

The basic mathematical framework of our analysis is set theory and in this section we shall briefly review some of the elementary Boolean properties of sets. The relevant issue to note is that although we use nothing more sophisticated than simple set theory, we shall be working at three different levels of analysis, and it is necessary to distinguish very clearly between sets whose elements are individuals and sets whose elements are other sets. We shall use the following notational conventions: regular lower-case letters or numerals for individuals, italic capital letters for sets of individuals, and script letters for sets of sets of individuals. The connective $\in (\notin)$ states that an element belongs (does not belong) to a set and it will help to clarify the different levels of abstraction under consideration, i.e., $a \in A \in \mathcal{A}$. Further, we shall even differentiate between the empty set of individuals $\varnothing$ and the empty set of coalitions φ. When naming the elements of a set we shall employ both the traditional braces and the overhead bar, i.e., $S = \{a,b,c,d\}$ or $S = \overline{abcd}$. The set of all individuals in a decision-making body – i.e., a game, a voting system, or an organization – will be denoted by N, and its power set by $\mathcal{N}$. Thus, $\mathcal{N}$ is the set of all subsets of N and its elements shall usually be called *coalitions*. Sets whose elements are sets shall be normally denoted as *collections*. Set subtraction will be indicated either by "\" or by "–". Therefore, if i is any element of N, then N_i will denote $N \setminus \bar{i}$ -- i.e., $\left(N - \bar{i}\right)$ -- and $\mathcal{N}_i$ will denote the power set of N_i. A *minimal element* of a nonempty collection (of sets) $\mathcal{S}$, is a set (coalition) who belongs to $\mathcal{S}$ and has no strict subsets in $\mathcal{S}$. Complements and all other set operators are defined in the usual manner.

1.1 Definitions

Let $\mathcal{S} \subseteq \mathcal{N}$ be any nonempty collection of coalitions. Then we shall define the following special collections:

$\mathcal{S}^+$	is the set of *all supersets of elements* of $\mathcal{S}$
$\mathcal{S}^-$	is the set of *all subsets of elements* of $\mathcal{S}$
$\mathcal{S}^*$	is the set of *all complements of elements* of $\mathcal{S}$
$\mathcal{S}^m$	is the set of *all minimal elements* of $\mathcal{S}$
$\mathcal{S}^\cup$	is the *union of all elements* of $\mathcal{S}$
$\mathcal{S}^\cap$	is the *intersection of all elements* of $\mathcal{S}$

Note that $\mathcal{S}^{\cup}$ and $\mathcal{S}^{\cap}$ are not collections of coalitions (sets) but just sets of individuals. By convention $\varphi^{\cup} = \varnothing$ and $\varphi^{\cap} = N$. All the elementary Boolean properties of sets carries over to collections of sets, plus some other elementary identities on these special collections, of which the most interesting follow.

For any nonempty $\mathcal{S}, \mathcal{T} \subseteq \mathcal{N}$ we have:

$$(\mathcal{S} \cap \mathcal{T})^{*} = \mathcal{S}^{*} \cap \mathcal{T}^{*}$$
$$(\mathcal{S} \cup \mathcal{T})^{*} = \mathcal{S}^{*} \cup \mathcal{T}^{*}$$
$$(\mathcal{S} \setminus \mathcal{T})^{*} = \mathcal{S}^{*} \setminus \mathcal{T}^{*}$$
$$\mathcal{S}^{**} = \mathcal{S}$$

For the purpose of illustrating the previous definitions, consider $N = \overline{abcd}$ so that $\mathcal{N}$ has sixteen coalitions and take one of them, such as $\mathcal{S} = \left\{\overline{ab}, \overline{acd}, \overline{cd}\right\}$.

Then we have

$$\mathcal{S}^{+} = \left\{\overline{ab}, \overline{cd}, \overline{abc}, \overline{abd}, \overline{acd}, \overline{bcd}, \overline{abcd}\right\}$$
$$\mathcal{S}^{-} = \left\{\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{ab}, \overline{ac}, \overline{ad}, \overline{cd}, \overline{acd}\right\}$$
$$\mathcal{S}^{*} = \left\{\overline{cd}, \overline{b}, \overline{ab}\right\}$$
$$\mathcal{S}^{m} = \left\{\overline{ab}, \overline{cd}\right\}$$
$$\mathcal{S}^{\cup} = \overline{abcd}$$
$$\mathcal{S}^{\cap} = \varnothing$$

Later on, we shall make good use of another Boolean operation on collections of sets. The *product* of two nonempty collections of sets is defined as the collection of all nonempty intersections of elements of these collections.

Formally, let $\mathcal{S}, \mathcal{T} \subset \mathcal{N}$ be a pair of nonempty collections, then its product $\mathcal{S} \bullet \mathcal{T}$ is defined as

$$\mathcal{S} \bullet \mathcal{T} = \left\{S \cap T : S \in \mathcal{S}, T \in \mathcal{T}\right\}$$

2 Simple Games

The essential notion of a simple game is the basic concept of a winning coalition which doesn't require more elaboration and one that we take as a primitive. We represent a simple game by simply stating its players and its winning coalitions. Further, we characterize this intuitive idea of winning by the following conditions:

i) the grand coalition always wins,
ii) the empty set never wins, and
iii) any superset of a winning coalition also wins.

Probably the most elemental of all simple games is the simple majority game denoted by M_n where n, the number of players, is odd. In this game M_n, there are 2^n possible coalitions of which all of those with more than $\frac{n}{2}$ members win, and the rest, those with less than $\frac{n}{2}$ members, lose. The general simple game is denoted by $M_{n,k}$ where there are n players and it takes k out of them to win. An interesting example is the unanimity game $M_{n,n}$ where the unanimous decision of all players is needed to reach an agreement or decision. This particular type of simple game is also equivalent to an instance of what is known as a *pure bargaining game*, denoted by B_n. In B_n there is only one winning coalition – the whole body or grand coalition – and thus there are $2^n - 1$ losing coalitions.

Formally, we define a simple game G on a set of players N as an ordered pair, denoted by $\Gamma(N, \mathcal{W})$, where

$$\varphi \subset \mathcal{W} = \mathcal{W}^{+} \subset \mathcal{N} \tag{2.1}$$

The first strict inclusion tells us that the grand coalition N is always in $\mathcal{W}$ and the second tells us that the empty set $\varnothing$ is never in $\mathcal{W}$, as desired. The elements of $\mathcal{W}$ are called *winning coalitions* and since every superset of a winning coalition is winning, when defining a specific simple game it is necessary only to state $\mathcal{W}^m$, the set of *minimal winning coalitions.*

A player $i \in N$ is said to be

an *essential player*	if	$i \in \mathcal{W}^{m\cup}$
a *dummy*	if	$i \in N \setminus \mathcal{W}^{m\cup}$
a *veto player*	if	$i \in \mathcal{W}^{m\cap}$
a *master*	if	$\bar{i} \in \mathcal{W}^{m}$
a *dictator*	if	$\{\bar{i}\} = \mathcal{W}^{m}$

In what follows we present several basic results on simple games without stating the formal proofs for reason of brevity.

Theorem 2.1 *A simple game has at least one essential player and at most one dictator.*

Corollary 2.2 *If a simple game is a dictatorship, then all other players are dummies.*

A coalition $\mathcal{S} \in \mathcal{N}$ is said to be

Winning	if	$\mathcal{S} \in \mathcal{W}$
Losing	if	$\mathcal{S} \in (\mathcal{N} - \mathcal{W}) \equiv \mathcal{L}$
Blocking	if	$\mathcal{S} \in (\mathcal{N} - \mathcal{W})^{*} \equiv \mathcal{BK}$

Thus, a blocking coalition is one that can prevent the formation of any winning coalition whatsoever. In terms of coalitions the notation is as follows

$$\mathcal{W} = \{S \in \mathcal{N} : S \text{ is a } \textit{winning} \text{ coalition}\}$$
$$\mathcal{L} = \{S \in \mathcal{N} : S \text{ is a } \textit{losing} \text{ coalition}\}$$
$$\mathcal{BK} = \{S \in \mathcal{N} : S \text{ is a } \textit{blocking} \text{ coalition}\}$$

A game $\Gamma(N,\mathcal{W})$ is said to be

proper	if	$\mathcal{W}\subseteq\mathcal{N}\setminus\mathcal{W}^*$	or if $\mathcal{W}\cap\mathcal{W}^*=\varphi$	or if $\mathcal{W}\subseteq\mathcal{BK}$
strong	if	$\mathcal{W}\supseteq\mathcal{N}\setminus\mathcal{W}^*$	or if $\mathcal{W}\cup\mathcal{W}^*=\mathcal{N}$	or if $\mathcal{W}\supseteq\mathcal{BK}$
decisive	if	$\mathcal{W}=\mathcal{N}\setminus\mathcal{W}^*$		
improper	if	$\mathcal{W}\cap\mathcal{W}^*\neq\varphi$		
weak	if	$\mathcal{W}^{m\cap}\neq\varnothing$		

It is interesting to observe that in a proper game winning implies blocking while in a strong game blocking implies winning. Note also that the concepts of weak and strong are not direct opposites. Since in a dictatorship all the players other than the dictator are dummies -- i.e., they are not essential players -- we may denote a simple game as *essential* if it is not a dictatorship. Then the following results follow.

Theorem 2.3 *No essential game is both weak and strong.*

Corollary 2.4 *Only dictatorships are both weak and strong.*

Theorem 2.5 *A simple game is strong only if it has no pair of complementary blocking coalitions.*

Corollary 2.6 *A simple game is strong only if it has no pair of complementary losing coalitions.*

Theorem 2.7 *A simple game is weak only if it has a veto player.*

Theorem 2.8 *A simple game is improper if it has a non-intersecting pair of minimal winning coalitions.*

If it is possible to enlarge $\mathcal{W}$, the set of winning coalitions in $G=\Gamma(N,\mathcal{W})$, without violating conditions (2.1) defining a simple game, we shall say that G has been *strengthened.* Similarly, if we can diminish $\mathcal{W}$ preserving definition (2.1) we say that G has been *weakened.*

Theorem 2.9 *Repeated strengthening (weakening) of a game* $G=\Gamma(N,\mathcal{W})$ *will eventually make it strong (weak). Once G becomes strong (weak) it remains so.*

Corollary 2.10 *Improper games may be either strong or not, but never weak.*

The following examples will serve to better illustrate the previous concepts.

Example 2.1 Consider the game $G=\Gamma(N,\mathcal{W})$ where $N=\overline{abcd}$ and $\mathcal{W}^m=\{\overline{a},\overline{bc}\}$. Then there are ten winning coalitions $\mathcal{W}=\{\overline{a},\overline{ab},\overline{ac},\overline{ad},\overline{bc},\overline{abc},\overline{abd},\overline{acd},\overline{bcd},\overline{abcd}\}$ and six losing coalitions $\mathcal{L}=\{\varnothing,\overline{b},\overline{c},\overline{d},\overline{bd},\overline{cd}\}$. The set of complements of the winning coalitions is $\mathcal{W}^*=\{\varnothing,\overline{a},\overline{b},\overline{c},\overline{d},\overline{ad},\overline{bc},\overline{cd},\overline{bd},\overline{bcd}\}$ and the set of complements of the losing coalitions is $\mathcal{L}^*=\{\overline{ab},\overline{ac},\overline{abc},\overline{abd},\overline{acd},\overline{abcd}\}=\mathcal{BK}$, therefore the set of minimal blocking coalitions is $\mathcal{BK}^m=\{\overline{ab},\overline{ac}\}$. There is one master *a*, one dummy *d*, and there are no dictators. The game is *strong*; and because it has a pair of disjoint minimal winning coalitions, it is *improper*.

Example 2.2 Consider an assembly of 100 members that requires a 2/3 majority to pass a resolution. This is a proper, nondecisive weak game. The winning coalitions are those who have at least 67 members, the minimal winning coalitions are those with exactly 67 members, and the losing coalitions are those with less than 67 members. Coalitions with more than 33 members block, and there are no dictators, masters, dummies, or veto players.

3 Local Topology of Authority

In any organization or decision-making body each of its members is directly concerned with only a small fraction of all the official orders, requisitions, authorizations, etc., that flow through the organization. Some members may have a certain degree of discretionary power; some may even be free agents, accountable to no one. Others may be merely cogs in the machinery. In what follows we develop a formal game-theoretic model for these ideas.

3.1 Boss Sets

Let N denote the set of all members of an organization, and let i denote a generic individual member of N. In general there will be certain other individuals, or more generally, *sets* of other individuals, that i must obey regardless of his/her own wishes. We call them *boss sets* and denote them collectively by $\mathcal{B}_i$. Note the script letter. Thus, if there is an individual b who can boss i, this is indicated by $\bar{b} \in \mathcal{B}_i$, *not* by $b \in \mathcal{B}_i$.

Formally, for all $i \in N$, we assume:

$$\varphi \subseteq \mathcal{B}_i = \mathcal{B}_i^+ \cap \mathcal{N}_i \subset \mathcal{N}_i \tag{3.1}$$

which may be compared with equation (2.1). The strict inclusion ensures that $\varnothing$ is never a boss set. But all other subsets of $\mathcal{N}_i$ are eligible, subject to the condition that every superset in $\mathcal{N}_i$, of a boss set, is also a boss set.

Recalling that a *rooted tree* is a finite, acyclic graph having a distinguished node called the *root*, we find that there may be several different ways to set up a hierarchical structure of an organization, where the members of N are represented as the nodes of a rooted tree.

Example 3.1 Consider the basic chain-of-command hierarchy, where there is a leader at the top who is unbossed, and where each lower-ranked member is bossed by just those coalitions that include his immediate superior, who is in effect his personal dictator. We shall refer to this type of organizational structure as a Type I Hierarchy.

Thus, if a denotes the leader, we have $\mathcal{B}_a = \varphi$, and $\mathcal{B}_i = \{S \in \mathcal{N}_i \mid j_i \in S\}$ for all $i \in N_a$, where j_i denotes i's unique immediate superior.

Hierarchy

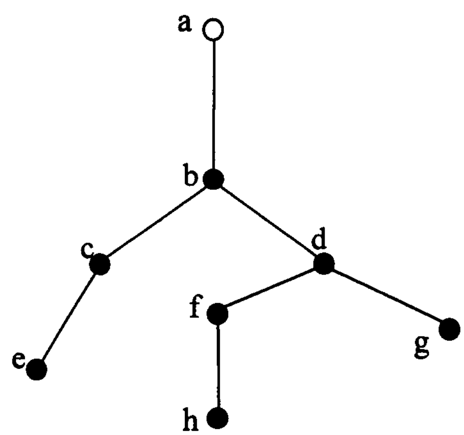

Figure 3.1

A close variation of the previous authority structure is the Type II Hierarchy, which is one where commands are not necessarily forwarded through channels. A private has to obey direct orders from his captain and his colonel as well as from his sergeant. In this case we again have a tree structure, but the boss sets now take the form $\mathcal{B}_i = \{S \in \mathcal{N}_i : S \cap J_i \neq \varnothing\}$ for all $i \in N$ where J_i is the set of all superiors of member i. For the moment, we merely observe that although these two hierarchies are in some sense equivalent, their local command structures are not the same.

3.2 Approval Sets

Just as there were some particular agents, or set of agents, that could boss any single individual, there are other set of agents, that can approve of him/her. Thus, we shall associate with every $i \in N$ another, wider collection $\mathcal{A}_i$ of coalitions in $\mathcal{N}_i$, called *approval sets*, that can approve *i*'s actions. The consent of any of these sets of *approvers* is sufficient to allow i to act, if he wishes to do so. However, it may not be able to force him to act. So approval sets are not necessarily boss sets. On the other hand, any boss set is *a fortiori* an approval set.

Formally, we have

$$\varphi \subset \mathcal{A}_i = \mathcal{A}_i^+ \cap \mathcal{N}_i \subseteq \mathcal{N}_i \tag{3.2}$$

where $\mathcal{B}_i \subseteq \mathcal{A}_i$ for all $i \in N$.

A study of equation (3.2) reveals that always $N_i \subset \mathcal{A}_i$, and that possibly $\varnothing \subset \mathcal{A}_i$. The set difference $(\mathcal{A}_i - \mathcal{B}_i)$ gives us a guide to the amount of personal discretion that individual i enjoys, if any. At one extreme, if $(\mathcal{A}_i - \mathcal{B}_i) = \mathcal{N}_i$ then i is called a *free agent.* He needs no approval since $\varnothing \in \mathcal{A}_i$ and no one can boss him, since $\mathcal{B}_i = \varphi$. At the other extreme, if $(\mathcal{A}_i - \mathcal{B}_i) = \varphi$, then i has no discretionary power, and we shall call him a *cog.* For an intermediate example of partial-discretion, consider a corporation president who is bossable by a 2/3 majority of the board of directors but is allowed to follow his own judgment so long as he has the support of a simple majority.

3.3 Command Games

Although we presented the boss and approval notions separately, they are really halves of a single *command* concept that can be expressed very naturally as a simple game.

Define for each $i \in N$, the set (collection) of *commanding coalitions* for i by

$$\mathcal{W}_i = \mathcal{B}_i \cup \{S \cup \bar{i} : S \in \mathcal{A}_i\} \tag{3.3}$$

Then $G_i = \Gamma(N, \mathcal{W}_i)$ is a well defined simple game, since (2.1) follows directly from equations (3.3), (3.2), and (3.1). We shall call G_i the *command game* for i. The ensemble of command games $G = \{G_i : i \in N\}$ then completely specifies the authority structure or constitution of the organization.

From the definition of a command game's winning coalitions we see that the free agents are dictators in their own command games while the cogs are dummies in their own command games. Note also that here we have $\mathcal{B}_i \neq \mathcal{B}_i^+$ and $\mathcal{A}_i \neq \mathcal{A}_i^+$.

Example 3.2 Returning to our two types of hierarchies in Example 3.1 (Figure 3.1), to complete their descriptions we need to define their approval sets. It seems natural in both cases to make the leader a free agent and the rest of the members cogs. The following Table gives all the respective command games:

Type I Hierarchy				Type II Hierarchy			
i	$\mathcal{B}_i^m$	$\mathcal{A}_i^m$	$\mathcal{W}_i^m$	i	$\mathcal{B}_i^m$	$\mathcal{A}_i^m$	$\mathcal{W}_i^m$
a	φ	$\mathcal{N}_i$	$\{\bar{a}\}$	a	φ	$\mathcal{N}_i$	$\{\bar{a}\}$
b	$\{\bar{a}\}$	$\{\bar{a}\}$	$\{\bar{a}\}$	b	$\{\bar{a}\}$	$\{\bar{a}\}$	$\{\bar{a}\}$
c -- d	$\{\bar{b}\}$	$\{\bar{b}\}$	$\{\bar{b}\}$	c -- d	$\{\bar{b},\bar{a}\}$	$\{\bar{b},\bar{a}\}$	$\{\bar{b},\bar{a}\}$
e	$\{\bar{c}\}$	$\{\bar{c}\}$	$\{\bar{c}\}$	e	$\{\bar{c},\bar{b},\bar{a}\}$	$\{\bar{c},\bar{b},\bar{a}\}$	$\{\bar{c},\bar{b},\bar{a}\}$
f -- g	$\{\bar{d}\}$	$\{\bar{d}\}$	$\{\bar{d}\}$	f -- g	$\{\bar{d},\bar{b},\bar{a}\}$	$\{\bar{d},\bar{b},\bar{a}\}$	$\{\bar{d},\bar{b},\bar{a}\}$
h	$\{\bar{f}\}$	$\{\bar{f}\}$	$\{\bar{f}\}$	h	$\{\bar{f},\bar{d},\bar{b},\bar{a}\}$	$\{\bar{f},\bar{d},\bar{b},\bar{a}\}$	$\{\bar{f},\bar{d},\bar{b},\bar{a}\}$

Note that in a Type I hierarchy all the command games are dictatorships, while in a Type II hierarchy many command games have more than one master, and hence no dictator.

It will be useful to have an alternative way of speaking of $\mathcal{B}_i$, $\mathcal{A}_i$, and $\mathcal{W}_i$, by introducing certain set-to-set functions that map $\mathcal{N}$ into itself.

The *boss function*	$\beta(S)=\{i\in N:S\in\mathcal{B}_i^+\}$
The *approval function*	$\alpha(S)=\{i\in N:S\in\mathcal{A}_i^+\}$
The *command function*	$\omega(S)=\{i\in N:S\in\mathcal{W}_i\}$

That is, the boss function $\beta(S)$ is the set of all individuals that must obey an order issued by coalition S; the approval function $\alpha(S)$ is the set of all individuals that can act (if they wish to do so) with the approval of S; and the command function $\omega(S)$ is the set of all individuals that are commanded by coalition S.

From the previous definitions it can easily be seen that $\beta(S)\subseteq\alpha(S)\subseteq\omega(S)$ for all $S\in\mathcal{N}$. Note also that these three functions are *monotonic* in the sense that if $T\subseteq S$ then $f(T)\subseteq f(S)$ for $f=\beta,\alpha,\omega$.

Most statements about $\mathcal{B},\mathcal{A},\mathcal{W}$ translate easily into synonymous statements about β,α,ω, which are in a sense their inverse functions. Thus, if we are given β we can recover $\mathcal{B}$ from the relation $\mathcal{B}_i=\{S\in\mathcal{N}:i\in\beta(S)\}$ and similarly for $\mathcal{A}_i$ and $\mathcal{W}_i$, if given α and β respectively.

4 Control Games

The command games described the local patterns of authority of a decision-making body. However, these games do not suffice to give an adequate account of the global distribution of authority throughout an organization. We will attempt that task by developing the notion of *control.* We shall start with a *control function* γ similar in form to the boss β, approval α, and command ω functions previously defined. Then, we shall reverse the inversion to obtain a set (collection) of *controlling coalitions* $\mathcal{C}_i$ and thus derive a set of *control games* $H_i = \Gamma(N, \mathcal{C}_i)$ similar in form to the command games $G_i = \Gamma(N, \mathcal{W}_i)$. It is important to emphasize the fact that the notion of control is a derived concept, i.e., no new information is contained in the function γ or in the games H_i's. Nevertheless, a substantial amount of calculation may be required to obtain the control games H_i's from the command games G_i's.

4.1 The Control Function

Similar to the α, β, and ω functions, the *control function* $\gamma(S)$ represents the set of individuals that coalition S can control, regardless of possible opposition from any or all of the other members of the organization. In defining $\gamma(S)$ we must recognize, on the one hand, the possibility of indirect control -- i.e., members outside S being co-opted to join with S in bossing other outsiders -- and on the other hand, the possibility that some members of S may not have full control over their actions -- i.e., they require consent from some set of approvers before they can participate in the bossing of others.

Formally, let F be the set of all free agents of an organization with N members. For each $S \in \mathcal{N}$ we construct $\gamma(S)$ with the aid of an increasing sequence of sets:

$$\gamma_0 \subseteq \gamma_1 \subseteq \gamma_2 \subseteq \gamma_3 \subseteq \cdots$$

which we shall call the *control sequence* for S.

It begins with the free agents:

$$\gamma_0 = F \cap S \tag{5.1}$$

and builds recursively according to the rule

$$\gamma_k = \beta(\gamma_{k-1}) \cup \{S \cap \alpha(\gamma_{k-1})\} \tag{5.2}$$

for $k = 1,2,3,\ldots$

Lemma 4.1 *There is a nonnegative integer $k^* \leq n-1$ such that the control sequence increases strictly up to the term γ_{k^*} and is constant thereafter.*

Thus, we can now define the control function in several different ways:

$$\gamma(S) = \gamma_{k^*}(S) \quad \text{or} \quad \gamma_{n-1}(S) \quad \text{or} \quad \lim_{k\to\infty} \gamma_k(S) \quad \text{or} \quad \bigcup_{k=0}^{\infty} \gamma_k(S)$$

which by Lemma 4.1 are all equal.

If $S = N$ rule (5.2) simplifies to

$$\gamma_k(N) = \alpha(\gamma_{k-1}(N))$$

Note that it is by no means inevitable that $\gamma(N) = N$. Individuals in $\gamma(N)$ will be called *controllable*, while those in $N \setminus \gamma(N)$ *uncontrollable.*

4.2 Properties

We present some results on control without stating the proofs.

Theorem 4.2 *The control function γ is monotonic.*

Corollary 4.3 *For all* $S, T \in \mathcal{N}$

$$\gamma(S \cup T) \supseteq \gamma(S) \cup \gamma(T)$$
$$\gamma(S \cap T) \subseteq \gamma(S) \cap \gamma(T)$$

Theorem 4.4 *If* S *contains no free agents, then* $\gamma(S) = \varnothing$. *In particular we have* $\gamma(\varnothing) = \varnothing$.

Therefore (taking $S = N$) we obtain the remarkable conclusion that, if no one in an organization is free, then no one is controllable.

Theorem 4.5

$$\text{If } R \subseteq \gamma(S) \quad \text{then } \gamma(S \cup R) = \gamma(S)$$
$$\text{If } R \subseteq N \setminus \gamma(S) \text{ then } \gamma(S \setminus R) = \gamma(S)$$

Broadly speaking, this result says that if outsiders controlled by a coalition are admitted to membership the coalition is not strengthened, while if insiders not under the coalition's control are expelled the coalition is not weakened.

Theorem 4.6 *Always* $\beta(\gamma(S)) \subseteq \gamma(S)$.

That is, anyone bossable by a controlled set is subject to the same control.

Theorem 4.7 *Always* $\gamma(\gamma(S)) = \gamma(S)$.

This expresses the notion of transitivity of control. It states that anyone controllable by a controlled set is subject to the same control, and conversely. In other words, a controlled set controls its own members, and no one else.

4.3 Exact Coalitions

The notion of exact coalitions will lately prove helpful in the decomposition of organizations. A fixed point of the control function will be defined as being an *exact coalition*:

$$S \in \mathcal{N}_{exact} \Leftrightarrow \gamma(S) = S$$

Thus, a coalition is an exact coalition if it controls just its own members. From Theorem 4.7 we have that the control function γ maps $\mathcal{N}$ onto $\mathcal{N}_{exact}$.

The following result specifies the actual existence of fixed points of the control function. Naturally we know that there are at least as many fixed points of the control function as there are free agents, actually, $2^{|F|} \leq |\mathcal{N}_{ex}| \leq 2^{|N|}$ where F is the set of free agents. The ensuing theorem formally establishes the nonemptiness of the collection of exact coalitions $\mathcal{N}_{ex}$.

Theorem 4.8 *Let $\mathfrak{L} = \langle \mathcal{N}, \subseteq \rangle$ be a complete lattice with a monotonic control function $\gamma : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{N}_{ex}$ is the set of all fixed points of γ (i.e., the set of all exact coalitions). Then the collection $\mathcal{N}_{ex}$ is not empty and the system $\langle \mathcal{N}_{ex}, \subseteq \rangle$ is a complete lattice.*

Corollary 4.9 *In particular (given the assumptions and results of Theorem 4.8) we have*

$$\mathcal{N}_{ex}^{\cup} = \bigcup_{S \in \mathcal{N}_{ex}} \left[\gamma(S) \supseteq S \right] \in \mathcal{N}_{ex}$$

and

$$\mathcal{N}_{ex}^{\cap} = \bigcap_{S \in \mathcal{N}_{ex}} \left[\gamma(S) \subseteq S \right] \in \mathcal{N}_{ex}$$

The following results will also be useful in decomposing organizations. We shall define an *exact partition* as a partition of N that is included in $\mathcal{N}_{ex}$. If (for nonempty collections) $\mathcal{S} = \mathcal{S} \bullet \mathcal{T}$ (recall the definition of product $\bullet$) we say that $\mathcal{S}$ is a *refinement* (or is *finer*) of $\mathcal{T}$, and/or that $\mathcal{T}$ is *coarser* than $\mathcal{S}$. For any two partitions $\mathcal{S}$, $\mathcal{T}$ such that $|\mathcal{S}| < |\mathcal{T}|$ we shall say that partition $\mathcal{S}$ is *smaller* than partition $\mathcal{T}$.

Theorem 4.10 *The product of two exact partitions of a controllable set is also an exact partition.*

Theorem 4.11 *There is no smaller common refinement of any two partitions than their product.*

Corollary 4.12 *There exists no smaller common refinement of any two exact partitions of a controllable set, than their product, which is also an exact partition.*

Theorem 4.13 *For any two nonempty disjoint coalitions S, T of a controllable set N, such that* $S \cap \alpha(T) = T \cap \alpha(S) = \varnothing$ *we then have that* $\gamma(S \cup T) = \gamma(S) \cup \gamma(T)$.

Coalitions with this property (Theorem 4.13) shall be called *additive-in-control* coalitions. An exact partition all of whose members satisfy the additive-in-control property shall be denoted an *exact additive-in-control partition.*

Theorem 4.14 *The union of any number of coalitions of an exact additive-in-control partition is also an exact partition.*

4.4 The Control Games

Now we invert the control function γ in order to obtain the corresponding class of control games. This is the reverse of the process that gave us the command function ω from the collection of commanding coalitions $\mathcal{W}$. Thus, for each $i \in N$ we define the collection of *controlling coalitions* for individual i by:

$$\mathcal{C}_i = \{S \in \mathcal{N} : i \in \gamma(S)\}$$

Unfortunately, the ordered pair $(N, \mathcal{C}_i)$ does not necessarily define a simple game according to conditions (2.1). Theorems (4.2) and (4.4) assure that $\mathcal{C}_i = \mathcal{C}_i^+$ and that $\mathcal{C}_i^+ \neq \mathcal{N}$ respectively. But nothing guarantees that $\mathcal{C}_i$ is nonempty. Indeed, $\mathcal{C}_i \neq \varphi$ if and only if $i \in \gamma(N)$. *Only controllable players can have control games!*

Therefore, we define the *control game* for each individual player $i \in \gamma(N)$ to be $H_i = \Gamma(N, \mathcal{C}_i)$. Then, the symbol *H* will denote the ensemble of control games, $H = \{H_i : i \in \gamma(N)\}$ (analogous to *G*, the ensemble of command games).

Theorem 4.15 *The set F of all free agents is a blocking coalition in every control game.*

Theorem 4.16 *Every cog is a dummy in every control game.*

Recall that a free agent (cog) is a dictator (dummy) in his/her own command game.

5 Organizations

The term organization in our model is meant to be synonym with *organizational authority structure.* We aim to develop a game theoretical model that can adequately describe and analyze the internal and external workings of a decision-making body or of any organization in general. Because the command games describe only the local patterns of authority in the past section we developed the notion of control, which then allowed us to obtain the controlling games which in turn give us the desired global structure of authority of the organization.

We pay a price, however, for defining our model in terms of the local notion of command, since the global fabric of authority may have then to contend with the logical inconsistency of improper control games. An improper command game is not necessarily a defect in the authority structure since two disjoint commanding coalitions may be subject to a common higher control. However, an improper control game is a serious defect, because it means that there are independent subsets of the organization that can send contradictory instructions to the same individual agent – instructions which, under the rules, that individual must obey.

Therefore, we have to recognize the existence of a fundamental distinction between those systems of command that are free from such logical impediments and those that are not.

5.1 Proper Organizations

In the previous section we concluded that only controllable players can have control games. Control is the key notion to keep in mind when defining an organization. Accordingly, if there exists an structured entity which does not control all of its members, we denote it as *pre-organization.* Only entities which are able to control *all of its members* are defined as *organizations.*

Any member of an organized entity who belongs to $\gamma(N)$ is called *controllable*, while those in $N \setminus \gamma(N)$ are *uncontrollable* (recall definitions in section 4.1). Thus, in a pre-organization some of its members may be uncontrollable whereas in any organization all of its members are controllable.

Formally, we denote an *organization* by the symbol $\Omega(N,G)$ where N is a finite set and G is an ensemble of simple games $\{G_i : i \in N\}$. For $\Omega(N,G)$ to be considered a *proper organization*, however, we shall require that all its control games be proper.

We can state these definitions in two ways, using the control games or using the control functions.

An entity $\Omega(N,G)$ is an *organization* if for every $i \in N$,

$$\mathcal{C}_i \neq \varphi \tag{5.3}$$

and an organization $\Omega(N,G)$ is *proper* if for every $i \in N$,

$$\mathcal{C}_i \cap \mathcal{C}_i^* = \varphi \tag{5.4}$$

Or equivalently, an entity $\Omega(N,G)$ is an *organization* if for every $Q, R \in \mathcal{N}$,

$$\gamma(N) = N \tag{5.5}$$

and an organization $\Omega(N,G)$ is *proper* if for every $Q, R \in \mathcal{N}$,

$$Q \cap R = \varnothing \Rightarrow \gamma(Q) \cap \gamma(R) = \varnothing \tag{5.6}$$

Because of the elaborate computations that may be necessary to determine the control function γ, or the ensemble of controlling collections $\{\mathcal{C}_i : i \in N\}$, it would be desirable to have some general conditions, stated in terms of the command games, that would ensure a proper organization. The following three propositions will be helpful in this regard.

Theorem 5.1 *Let $R \in \mathcal{N}$ be such that $R \setminus \bar{i}$ is a blocking coalition in G_i for every $i \in R$. Then $\gamma(N) \cap R = \varnothing$ -- i.e., the members of R are uncontrollable.*

The simplest example of this phenomenon occurs when R has two members, each one having veto power in the other's command game.

Theorem 5.2 *If there is only one free agent, then the control games are all proper.*

We shall say that an organization is *hierarchic* if its members can be represented by a rooted tree (the nodes) in such a way that in each command game G_i the essential players are all superiors or equals to player i.

We shall say that an organization is a *pyramidal-type* organization if it has a hierarchic control structure that can be represented by a rooted tree with a unique free agent – the root. Common examples of pyramidal-type classes are Type I and Type II Hierarchies, the Armed Forces, the Catholic Church, etc.

Theorem 5.3 *Every pyramidal-type organization is proper.*

Theorem 5.4 *If all the command games of an organization are proper then all the control games are also proper.*

Clearly the converse of Theorem 5.4 is not valid. For instance, in Type II Hierarchies the typical command game has several masters, and so it is improper. However, the control games are all dictatorships, and thus they are proper. It can also be easily seen that decisiveness of the command games does not suffice to ensure that the control games are decisive also. Consider the following simple example, let $N = \overline{abcde}$ and let players a, b, and c be free agents; also assume that the command games for both d and e, are of the form $M_{5,3}$. Then all the command games are decisive. Unfortunately, it can easily be calculated that $\gamma\left(\overline{ab}\right) = \overline{ab}$ and $\gamma\left(\overline{cde}\right) = \bar{c}$, from where it follows that the control games H_d and H_e each have a pair of complementary losing coalitions ($\overline{ab}$ and $\overline{cde}$ respectively) and thus are not decisive.

An organization in which each member is either a free agent or a cog will be called a *simple organization.* A *sub-organization* is simply a subset of an organization which is also an organization on its own right (i.e., it controls all of its members). Two organizations will be called *equivalent* if and only if their control functions (games, coalitions) are the same.

5.2 Applications

In the previous sections we have emphasized the procedures of command and control, of the internal contractual aspects of the organization's authority structure, or in other words, of the governance of the organization. However, most organizations or decision-making bodies have an external material purpose as well, so that at least some of the commands that flow through the authority system have a substantive content, and at least some of the power to control that the organization's members exercise, does extend beyond the external boundaries of the organization into another larger domain.

In order to set up the framework of analysis for the interactions between organizations we need to formalize an organization's external environment, i.e., to model the way how the organization members' power to control extend itself into that external dimension, i.e., into the outside world. We accomplish this by considering the notion of *simple tasks* that the organization may be called upon to perform. For reasons of simplicity we shall only contemplate tasks one at a time, and we shall treat them as black boxes because their internal structure will not interest us. On the other hand, we shall be very much concerned with the interface between the tasks and their potential *effectors* within the organization.

The problem of quantifying the distribution of *responsibility* in an organization is solved by a simple application of the Shapley-Shubik [4] power index for simple games. Terms like power or responsibility present serious difficulties of use, because besides the fact that they are multidimensional concepts, they have no established formal meaning in organization theory. Further, the Shapley-Shubik power index for simple games is not the only game-theoretic index amenable to use for our purposes. For instance, the Banzhaf power index[1] would be a serious alternative candidate to consider, however, we have not yet examined the question of which one would be the most appropriate for our use in the analysis of organizations. Nevertheless, it seems quite natural, and of great practical use, to have a reliable numerical measure of organizational responsibility, and therefore, we shall confine ourselves to the Shapley-Shubik power index specification.

The primitive basic interactions between organizations that are of interest to consider here, are interactions between two organizations both vying for the control of some external task. There are several possibilities that can be modeled: *i)* performing one external task under the vigilance of another organization (*monitoring*), or *ii)* because of control of another organization's external task, eventually a new organization emerges (*merger*). Further, this event could be the result of *iia)* full cooperation (*friendly merger*) or *iib)* open conflict (*takeover*). The situation of competition would directly aim to cover open conflict for internal tasks (for the management control of other organization) and full cooperation (the emergence of pacts and alliances among different organizations). Possibly some sort of returns to scale are at play here. Besides organization theory, more practical applications of our theory can be conducted on a wide diversity of topics such as optimal tariffs, central banks, constitutional amendments, economic analysis of conflict and coalition forming incentives, to name but a few.

[1] See Dubey P., and L. S. Shapley [1] for a discussion and comparison of both indexes.

References

[1] Dubey, P., and L. S. Shapley, 1979. Mathematical Properties of the Banzhaf Power Index. *Mathematics of Operation Research 4*: pp. 99-131.

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[5] von Neumann, J., and O. Morgerstern, 1944. *Theory of Games and Economic Behavior.* Princeton, NJ.: Princeton University Press (2nd ed., 1947).