

**A MODEL OF RATIONAL NON-COMPETITIVE INTERDEPENDENCE**

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**Discussion Paper Number 47  
March 1974  
[Revision of #42]**

**Preliminary Report on Research in Progress  
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## ABSTRACT

The traditional approach to problems of monopoly interaction, or non-competitive interdependence, has been to make assumptions inconsistent with rationality or perfect information. An example is the Cournot oligopoly model, and its modern counterpart, the Nash model of n-person, non-cooperative games, where producers assume zero reactions of their rivals to changes in their strategies even though the actual reactions are not zero. The approach adopted in this paper is to construct a general equilibrium model of monopoly which is consistent with rationality and perfect information. We show that between any pair of interacting monopolists, one and only one monopolist exhibits an output reaction function while the other simply picks a point on the function. For both the case of uncontrolled monopoly and monopoly behavior under governmental anti-monopoly policy, we derive the rational firm's reaction function, demonstrate the existence of equilibrium, and characterize the resulting solutions. Results of preliminary tests of each of these models are very encouraging.

## INTRODUCTION

This paper deals with the problem of deriving equilibrium quantities within a general equilibrium system containing several monopolists, producers who can affect relative prices.

Since in a general equilibrium setting all relative prices are a function of all outputs, a monopolist's output decision will affect all the relative prices in the system. Hence, an individual monopolist's output change will, in general, render the remaining producers' previous outputs non-optimal with respect to the new set of prices generated by the monopolist's new output. The other producers, in recalculating their respective optimal outputs, generate a set of output reactions and again change relative prices. A rational monopolist who is aware of the relative price effects imposed on him and on the other producers will take the other producers' reactions into account when calculating his profit-maximizing output. Similarly, other monopolistic producers,<sup>1/</sup> realizing that their reactions are being taken into account, will determine their reactions accordingly.

The task of this paper is to describe the rational form of the reaction functions, demonstrate the existence of a general equilibrium given these rational reaction functions, assuming they are known to all producers, and characterize the resulting equilibria.

Our model assumes perfect information regarding reaction functions, and prohibitively high transactions costs of making multi-lateral or collusive agreements. The traditional models of economic conflict attempting to describe such a world breakdown under the assumption of rationality

and perfect knowledge of reaction functions. In the standard duopoly model of Cournot, each duopolist makes output decisions based on faulty information concerning the reaction behavior of his rival. The zero output reactions assumed in this model do not describe the true reactions that occur within the model. In the duopoly model of Stackelberg we will see that the selected reaction functions are generally irrational when there is perfect information concerning output reactions.

In Section I, we specify our general model and show that a necessary condition for the existence of solutions to all monopoly interaction problems featuring rational strategy selection and perfect information regarding reactions is that between any two, interacting monopolists, one and only one of them exhibits a reaction function. In the case of  $m'$  ( $m' \geq 3$ ) interacting firms, a solution implies a recursive set of reaction functions for  $m'-1$  firms. We specify a competitive process to determine which firm among a set of interacting firms is able to establish a reaction function over all of the other firms in the set.

In Section II, we derive the rational reaction functions and prove a theorem on the existence of a general equilibrium in a special case of the general model. We call this the "uncontrolled monopoly" case.

In Section III, we introduce government anti-trust policy. We show that such policy changes the form of the rational reaction functions and alters the characteristics of the controlled monopoly solution. We construct an equilibrium in a linear special case and derive its equilibrium size distribution of firms and concentration ratios. The outputs of the three smallest firms are seen to be equal while the larger firms are

each twice as large as the next smallest firm. And the concentration ratio of an industry asymptotically increases as the number of firms grows and the industry output approaches a competitive level. The asymptote is where  $(2^t - 1)/2^t$  is the market share of the top  $t$  firms in an industry.

In Section IV, we note the rough empirical accuracy of our theories of monopoly for the U. S. experience before and after federal anti-monopoly laws. A basic result of this analysis is that U. S. anti-monopoly laws, by altering the form of rational functions, have converted a world with highly inefficient monopolies into a world in which an industry with only a few firms produces an output which is very close to a competitive level.

## I. THE GENERAL MODEL

### A. The Environment

The model we will employ is a private-property general equilibrium model containing  $n$  commodities and  $m$  firms. The economy's output allocation set is denoted

$$x = (x_1, x_2, \dots, x_m), \quad x_f \geq 0, \text{ for all } f=1, \dots, m, \quad (1)$$

where  $x_f$  is the  $f^{\text{th}}$  firm's  $n$ -dimensional output vector. We will denote the set of output vectors of all firms except firm  $f$  as

$$x_{-f} = (x_1, x_2, \dots, x_{f-1}, x_{f+1}, \dots, x_m). \quad (2)$$

Each firm,  $f$ , can produce all commodities and has a profit function,

$$\pi_f(x) = \pi_f(x_f; x_{-f}), \quad f=1, \dots, m, \quad (3)$$

which summarizes firm  $f$ 's technology, factor costs, and output demand conditions. Of course,  $\pi_f(0; x_{-f}) = 0$ . A firm may employ factors specific to that firm so that the profit functions of the various firms are not generally identical.

A group of monopolists is said to be "interacting" when the output choice of some monopolist in the group affects the output of another monopolist in the group. We assume that the group of all  $m$  monopolists is interacting.

### B. The Necessity of a Strategy Maker and a Strategy Taker

A strategy of a firm is a description of what the firm will do. It may be an action, which simply states what output the firm will produce.

It may be a reaction strategy, which states what outputs a firm will produce given the outputs of other firms. It may be a contingent reaction, which states what reaction strategies a firm will adopt given the reaction strategies of other firms. Or it may be a contingent strategy, a generalization of the strategies of reaction and contingent reaction, which states what strategy a firm will adopt given the strategies of other firms. A contingent strategy may include, for example, the infinite statement: "I will produce  $x_1'$  if you say you will react with  $x_2'$  when I produce  $x_1'$  if I say I will react with  $x_1''$  when you produce  $x_2''$  if you say you will react with  $x_2''$  when I produce  $x_1'''$  if I say I will react with  $x_1''''$  when you produce  $x_2''''$  if you say . . . ."

We assume perfect information regarding the strategies of others and "rationality," i.e., that each firm selects the strategy that maximizes his profit given the responses of the others firms to its strategy. The conventional definition of perfect information in game theory has each player knowing the solution strategies of others but does not permit a player to know how the strategies of others vary with his own strategies.<sup>2/</sup> We permit the latter and distinguish our information assumption by labelling it "truly perfect information." For truly perfect information, in any set of interacting firms, one firm must determine its strategy before the other firms in the set, where the response strategies of the interacting firms are known a priori by the prior selector because they are the rational responses to the given strategy of the prior selector.

A strategy of a prior strategy selector in a set of interacting monopolists

describes his response to each of the possible strategies of the other monopolists in the set. Therefore, each of the possible strategies of the prior strategy selector always contains a reaction strategy. It follows that no firm can increase its profits by responding to a prior strategy with a strategy contingent on the subsequent strategy of the prior strategy selector. Since the  $m^{\text{th}}$  strategy selector faces only prior strategy selectors, this proposition means that the  $m^{\text{th}}$  strategy selector can achieve maximum profits by adopting a simple action, thereby triggering a chain of simple reactions up through the first strategy selector in the economy. To see this, consider the  $m^{\text{th}}$  strategy selector. He faces the prior strategies of the other firms and sees that the final set of outputs must be consistent with the reaction strategies contained in the prior strategies of each of the other firms. Hence, if he responds with a simple action, he will have a free choice of outputs consistent with the prior reaction strategies. But if he responds with a contingent strategy, thus giving further choices to the prior strategy selectors, he can only reduce his original choice out of the same set of possible outputs without expanding the set of possible outputs.

Hence, between any pair of interacting monopolists,  $i$  and  $j$ , firm  $i$ , the prior strategy selector, called a strategy-maker (or maker) relative to  $j$ , exhibits an output reaction function,

$$x_i = h_i(x_j) , \tag{4}$$

from which firm  $j$ , called a strategy-taker (or taker) relative to  $i$ , chooses its output. The reaction function is a cross section for given outputs of  $i$ 's other takers and given reaction functions of his makers. Similarly,



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$j$ 's output choice is, in general, dependent upon the reaction functions of his other makers and the outputs of his own takers.

C. Maker-Taker relationships over the entire set of firms

It is possible that there is only one maker in the entire set of  $m$  firms. While there must be at least one maker in a group of  $m$  interacting monopolists, this maker's optimal strategy may prevent interactions between the remaining firms. In this case, firm  $i$  is the maker and the other firms are takers from  $i$ . This must be the case, as we shall see in Section II, when monopolies are unregulated and there exists a "punishment output" from firm  $i$ .

In the case that reaction functions are constrained by a certain anti-trust law, then all of the firms are mutually interacting. This is shown in Section III.

For now, assume there is a maker besides individual  $i$ . And assume that  $j$ , a taker of  $i$ , is a maker with respect to  $k$ . Then  $k$  is a taker with respect to  $i$ . This is because when  $k$  selects an output and thus determines  $j$ 's output according to  $j$ 's reaction function, he is also determining  $i$ 's output according to the reaction function  $i$  presents to  $j$ . If  $k$  were committed to a certain reaction to  $i$ , he would similarly be committed to a certain reaction to  $j$ . But  $k$  is a taker of  $j$  so an inconsistency would arise. Hence, if  $i$  is a maker to  $j$ , and  $j$  is a maker to  $k$ , then  $i$  is a maker to  $k$ . The maker relation is transitive.

Summarizing, there is a binary relation, call it  $\mathcal{M}$ , over the set of all mutually interacting firms,  $S = (1, 2, \dots, m')$ , such that:

- (a) either  $i \mathcal{M} j$  or  $j \mathcal{M} i$  for all  $i, j \in S$ ,
- (b)  $i \mathcal{M} j$  and  $j \mathcal{M} k$  implies  $i \mathcal{M} k$  for all  $i, j, k \in S$ .

Since non-reflexive, complete, transitive, binary relation over a finite set determines a unique maximal element in the set, there is one member of  $S$ , say 1, such that  $1 \mathcal{M} j$  for all  $j \in S, j \neq 1$ . We call this firm the "primary maker." Since the same exercise can be repeated for the firms in  $S' = (2, \dots, m')$ , there exists a firm, say 2, such that  $2 \mathcal{M} j$  for all  $j \in S', j \neq 2$ . We call this firm the "secondary" maker. This exercise continues onto the last firm  $m'$ , who is not a maker with respect to anyone. We call this firm the "pure taker."

We thus obtain a hierarchy of makers, with the primary maker exhibiting a reaction function,

$$x_1 = x_1(x_2, x_3, \dots, x_{m'}) ,$$

while the secondary maker exhibits the reaction function,

$$x_2 = x_2(x_3, \dots, x_{m'}) ,$$

and the third firm exhibits

$$x_3 = x_3(x_4, \dots, x_{m'}) ,$$

and so on up to the  $m'-1^{\text{st}}$  firm's simple reaction function.

Solution output sets for these  $m'$  firms are easily constructed for a given set of reaction functions. The pure taker selects an output that maximizes his profit (assuming for now the existence of such an output) given all of these reaction functions. The  $m'-1^{\text{st}}$  firm then follows his established reaction function, which gives the two outputs, necessary for the  $m'-2^{\text{nd}}$  firm to determine his output. This process continues until the primary maker's output is determined. We shall employ this process

in Section III below, where all  $m$  monopolists will be mutually interacting.

Before we can derive the rational forms of the reaction functions, we must specify the cost of being a maker and determine who will be makers with respect to whom.

D. Determination of the Strategy-Maker

The advantage that a strategy maker has over his takers is not his ability to "act later." Both firms respond to one another, and no time sequence of actions need be involved. His advantage is his ability to respond to any given strategy of the taker in a way that does not maximize his profits given the taker's strategy. In order to have this ability to escape his own narrowly conceived rationality, a firm requires what we shall call a commitment.

A commitment is defined as an enforceable promise of an individual to react in a specified way to the actions of another individual regardless of the costs incurred by the commitment-maker in carrying out the stated reaction. We give one, and only one, individual, called the "lawyer," the ability to provide individuals with commitments. Thus, in order to be the strategy-maker and exhibit a reaction function, the firm must make a commitment through the lawyer.<sup>3/</sup>

The services of the lawyer are obtained via competitive bidding by the firms. Each firm submits  $m-1$  bids, each bid representing the amount the firm is willing to pay to be maker instead of a specified, alternative maker. A winning bidder, if one exists, is a firm whose bid against his least preferred alternative maker is no less than the maximum of the bids against him. The reason a winning bidder must bid as if the worst possible

alternative is the actual alternative is that the lawyer is free to choose the bidder's alternative maker and will rationally choose an alternative maker which will maximize the bid of his most-preferred winning bidder. The winning bidder, however, does not generally pay his bid to the lawyer; he need only match the second highest bid. Our auction is unusual in that the bidders have different payoffs; and therefore, different bids depending on who would otherwise win the auction and on what he would do as the winning bidder.

While our lawyer is a "deus ex machina" imposed upon the model to attain a solution, in the real world, we do sometimes observe such individuals. Moreover, we do observe firms in the real world devoting real resources to making commitments and communicating their validity.<sup>4/</sup> In our model we have summarized these activities into bids for the services of a lawyer. This serves the purpose of abstracting from the socially unnecessary resource drains which firms in the real world create by competing with other firms for prior commitments.

The feature of the model that rules out overall joint profit maximization as a general solution is simply that firms do not collude in forming their bids to be maker. The bids that one firm presents to the lawyer are not contingent on the bids of third parties. That overall joint profit maximization (and Pareto optimality under perfect price discrimination) results when there is an absence of transaction costs in all possible transactions containing conditions regarding the behavior of third parties is shown in Thompson (8).

Section II contains, for the uncontrolled monopoly case, a specification of the cost of being a maker, a derivation of the identity of the maker, a characterization of general equilibrium solutions, and a proof of the

existence of equilibrium solutions under some additional restrictions. Section III contains similar analysis for a controlled monopoly case.

## II. GENERAL EQUILIBRIUM WITH UNCONTROLLED MONOPOLY

### A. The Existence of Punishment Outputs

We shall assume until Section III that for each firm, there exists an output vector which implies negative profits from any positive output vector to each of the other firms regardless of the outputs of the remaining firms. Such a set of outputs will be called a punishment set of outputs.

More formally,

(a.1) For each  $i$ , there exists an  $x_i$ , say  $x_i'$ , such that

$$\pi_j(x_j; x_1 \dots x_{j-1}, x_{j+1}, \dots, x_i', \dots, x_m) < 0 \text{ for all } x_{-i} \text{ with } x_j > 0 \text{ and all } j \neq i.$$

Consider a pair of firms,  $i$  and  $j$ , where  $i$  is the strategy-maker and  $j$  is the strategy-taker. Firm  $i$  can induce firm  $j$  to produce any specified output that does not generate negative profits to  $j$  by making a commitment to produce a punishment set of outputs when  $j$  produces any output not equal to the specified output. Firm  $j$ , as a profit-maximizing firm facing  $i$ 's output reaction function, will produce the specified output as it yields  $j$  its highest profits.

Since firm  $i$  has a punishment set of outputs, it can make a commitment which will induce each firm to produce its specified output. Faced with firm  $i$ 's commitment, each of the remaining  $m-1$  firms will rationally choose to produce their respective profit-maximizing outputs, the outputs specified by firm  $i$ .

More formally, let  $x_i^i$  and  $x_{-i}^i$  be solution values to the problem,

$$\max_x [\pi_i(x) - C_i(x)] \text{ subject to } \pi_f \geq 0 \text{ for all } f \neq i, \quad (5)$$

where  $x_f = 0$  if  $\pi_f(x_f) < 0$  for all  $x_f > 0$ , and where  $C_i$  is the cost to  $i$  of becoming maker. Thus,  $x_f^i$  is the output of the  $f^{\text{th}}$  firm which maximizes the net maker profit of firm  $i$  subject to the non-negativity of profits to any taker. The rational reaction function for firm  $i$  is then:

$$x_i = h_i(x_{-i}), \text{ where} \tag{6}$$

$$x_i^i = h_i(x_{-i}^i) \text{ and}$$

$$x_i^i = h_i(x_{-i}), \text{ all } x_{-i} \neq x_{-i}^i.$$

The commitment made by  $i$  guarantees that  $i$  will produce  $x_i^i$  when firm  $f$  deviates from producing  $x_f^i$  even if it implies lower profits to  $i$  than some alternative values of  $x_i$  given  $x_f \neq x_f^i$ . Such apparently irrational behavior by firm  $i$  is rational by virtue of our assumption of profit-maximizing behavior of all firms. This implies that firm  $f$  will produce  $x_f^i$  in equilibrium rather than an alternative output.

A strategy-taker, any firm  $j \neq i$ , faces the problem:

$$\max \pi_j(x_j; x_{-j}) \text{ subject to}$$

$$x_{-j} = x_{-j}^i \text{ if } x_j = x_j^i, \text{ and } x_k = x_k^i, \text{ all } k \neq j, i; \tag{7}$$

$$x_{-j} = (x_1, \dots, x_i^i, \dots, x_m) \text{ otherwise.}$$

This leads the  $j^{\text{th}}$  firm, knowing the rational responses of the other takers, to choose  $x_j = x_j^i$ . We have assumed this holds even if  $\pi_j(x_j^i) = 0$  for  $x_j^i > 0$ . That is, the taker will choose to produce the maker's optimal output choice even though his profits there are zero and he has an equally profitable possibility (e.g., quitting business).

The  $m-1$  takers are obviously non-interacting; the only response that counts when a taker changes his output is the response of the maker when punishment outputs exist and monopolists are uncontrolled.

Note that disregarding the cost of becoming a maker, no firm is ever worse off by being the strategy-maker as opposed to being a strategy-taker. This is because an individual firm can always do as well by choosing its own output as having it chosen by another. Hence, each firm will have a non-negative bid for the lawyer's services regardless of whom he is bidding against.

B. The Two-Firm Case.

Consider two firms,  $i$  and  $j$ . The amount firm  $j$  is willing to offer to the lawyer equals the difference between  $j$ 's profit as a maker and  $j$ 's profit as a taker. Since  $j$ 's profit as a taker depends on  $i$ 's choice of outputs as a maker, the cost to  $i$  of being the maker, which is the cost of just beating  $j$ 's bid, is a function of the  $x$  that  $i$  would choose as maker. Hence, we can write:

$$C_i(x) = \pi_j(x^j) - \pi_j(x), \quad (8)$$

and, using (5), describe firm  $i$ 's maximum maker profit as:

$$\pi_i^M = \max_x [\pi_i(x) - (\pi_j(x^j) - \pi_j(x))] \text{ subject to } \pi_j(x) \geq 0, \quad (9)$$

where  $\pi_j(x^j)$  is the value of  $\pi_j$  implied by the solution to:

$$\max_x [\pi_j(x) - (\pi_i(x^i) - \pi_i(x))] \text{ subject to } \pi_i(x) \geq 0, \quad (10)$$

where  $\pi_i(x^i)$  is the solution value of  $\pi_i$  implied by (9).

Solutions to (9) and (10), if they exist, yield explicit values of  $\pi_i(x^i)$ ,  $\pi_j(x^j)$ ,  $\pi_i(x^j)$ , and  $\pi_j(x^i)$  from which we obtain the value of each firm's bid. These values are interpreted as  $i$ 's and  $j$ 's operating profit as a maker, and  $i$ 's and  $j$ 's operating profit as a taker, respectively.

Noting that  $x^i$  is independent of  $\pi_j(x^j)$ , we see from (9) that firm



i is maximizing its joint-profits with firm j. Similarly, from (10), firm j is maximizing its joint-profits with firm i. If we assume that the joint-profit maximizing output is unique, then the same output vector will be chosen regardless of which firm is the strategy-maker. Hence, each firm's bid for the rights to be maker would equal zero since its profit as a maker is the same as its profit as a taker. In this case, the final determination of the strategy maker is arbitrary.

If the joint-profit maximizing output is non-unique, the two firms' bids will still be equal, but they may then be positive. For example, at  $x^i$ , let  $\pi_i(x^i) = 50$ ,  $\pi_j(x^i) = 40$ , and at  $x^j$ , let  $\pi_j(x^j) = 60$ ,  $\pi_i(x^j) = 30$ . Notice that both i's and j's bid will equal 20. Since the bids are equal, the selection of strategy maker is still arbitrary.

It is now easy to show that the solution to a Cournot duopoly problem with truly perfect information and rational reactions implies a joint profit maximizing total output and an arbitrary distribution of outputs between the duopolists with a correspondingly arbitrary bid.

### C. The m-Firm Case

But these results do not extend beyond two monopolists. For with m firms,  $m \geq 3$ , although there are m-1 competing bids with which a prospective strategy maker must contend, any prospective maker need only be concerned with the highest of his rivals' bids. This highest rival bid is the explicit cost to i of becoming the maker. Thus, (8) becomes:

$$C_i(x) = \max_{f \neq i} (\pi_f(x^f) - \pi_f(x)) . \quad (11)$$

The  $m-1$  opposing bidders are each rationally assuming that firm  $i$  will be the strategy maker if they are not. The resulting bid of each firm then measures how much a firm is willing to pay to be maker instead of being a taker of  $i$ 's reaction function. We can now describe firm  $i$ 's maximum maker profit,  $\pi_i^M$ , as:

$$\pi_i^M = \max_x [\pi_i(x) - \max_{f \neq i} (\pi_f(x^f) - \pi_f(x))] \text{ subject to } \pi_f \geq 0, \quad (12)$$

where  $\pi_f(x^f)$  is the implied profit to firm  $f$  when  $f$  is solving for its maximum maker profit.

Firm  $i$ 's alternative maker is that firm which will be the maker if  $i$  is not. Firm  $i$ 's bid when  $j$  is his alternative maker, the difference between  $i$ 's profit as maker and  $i$ 's profit as taker of  $j$ , is

$$\pi_i(x^i) - \pi_i(x^j) = B_{ij}(x^j), \quad (13)$$

where  $j$  is  $i$ 's alternative maker.

By computing maximum maker operating profit for all  $m$  firms, if these profits exist, and taker profit in a similar fashion, we can compute each firm's bids from the explicit values of maker and taker profits.

#### D. Characterizing an $m$ -Firm Equilibrium

Distinguishing features of the  $m$ -firm case ( $m \geq 3$ ) are that at a solution there is an equality between the bids of some of the takers and that the solution is not a joint-profit maximum.

At any choice of output allocation set of the maker,  $i$ , there is either a distinct individual determining  $i$ 's legal fee - i.e., an unique  $f$ , solving (11), or there is a tie bid between some of the takers. Suppose there is an unique maximum in (11). Since the maker is responsive only

to changes in the bid of the single highest-bidding taker, say, j, the maker and this taker will adopt a joint-profit maximizing relationship as in the two-firm case. Therefore, if the output choice of i is, in fact, a solution, it also corresponds to a joint-profit maximum between i and j. If the joint-profit maximizing output is unique, j's bid against i is zero. Since the remaining bids are non-negative, such an output choice is unattainable because the alternative maker's zero bid is then not higher than the other takers.

If the joint-profit maximizing output of i and j is non-unique, the same result obtains. Suppose that joint-profits between i and j are maximum and therefore equal at both  $x^i$  and  $x^j$ . Then the difference between i's maker profit at  $x^i$  and  $x^j$ ,

$$\begin{aligned} & \pi_i(x^i) - [\pi_j(x^j) - \pi_j(x^i)] - \pi_i(x^j) + [\pi_j(x^j) - \pi_j(x^i)] \\ &= \pi_i(x^i) + \pi_j(x^i) - [\pi_j(x^j) + \pi_i(x^j)] = 0 . \end{aligned}$$

Thus, i is indifferent between  $x^i$  and  $x^j$  which implies that i's bid is zero. Again, an inconsistency results as the other bids are non-negative. Thus, the equilibrium solution is inconsistent with the existence of an unique maximum in the alternative bids.

Therefore the optimal output choice of the strategy-maker occurs where there exists a tie in the maximum bids of some of the takers.

A solution occurring at this point does not correspond to a joint-profit maximum. Although the maker is responsive to a change in the bid of any one of his several maximum-bidding-takers, he is not concerned with the sum of their bids, which is required for joint-profit maximization. Thus, the solution is not a joint-profit maximum. In case there is an m-way tie with all firm's bids equal to zero at the solution, we have a

separate joint-profit maximum between the maker and each taker; however, this is still not an m-firm joint-profit maximum.<sup>5/</sup>

In both the two and m-firm cases, since the solution maker need only match the highest-bidding rival firm(s), the amount going to the lawyer will equal the value of the second highest bid over all firms. But, whereas the lawyer's fee is always zero in the two-firm case when the joint profit maximizing output set is unique, it may be positive in the m-firm case.

E. A Theorem on the Existence of Equilibrium

We will be working in Euclidean space,  $R^y$ ; the dimensionality  $y$  of the space equals the number of commodities ( $n$ ) times the number of firms ( $m$ ), or  $nm$ .

Let  $X$ , a subset of  $R^y$ , equal the feasible output set. An element,  $x$ , of this feasible output set is a  $y$ -dimensional vector of outputs of each commodity by each firm.

From the above discussion, for each firm there is a profit function,  $\pi_f(x)$ ,  $f=1, \dots, m$ ; defined on  $X$ . Similarly defined on  $X$  is:

(d.1) the  $i^{\text{th}}$  firm's maker profit,

$$\pi_i(x) = \max_f [\pi_f - \pi_f(x)] \quad f=1, \dots, m; f \neq i, \quad (14)$$

where  $\pi_f$  is  $f$ 's operating profit as a maker, a given number to  $i$ ; and

(d.2) the  $i^{\text{th}}$  firm's bid function, given that  $f$ , who dictates output  $x$ , is the alternative maker,

$$B_{if}(x) = [\pi_i(x^i) - \pi_i(x)], \quad f=1, \dots, m; f \neq i. \quad (15)$$

An equilibrium is (a) a set of output allocation vectors,  $\underline{x}^1, \underline{x}^2, \dots, \underline{x}^m \in X^m$  such that each  $\underline{x}^i$  maximizes the maker profit of firm  $i$  given  $\pi_f = \pi_f(\underline{x}^f)$ , all  $f \neq i$ , and (b) a winning bidder, a firm,  $i$ , such that

$$\max_f B_{if}(\underline{x}^f) \geq \max_k B_{ki}(\underline{x}^i) .$$

We now make the following assumptions:

(a.2)  $X$  is a non-empty, compact, convex set.

(a.3)  $\pi_f(x)$  is a continuous, real-valued function,  $f=1, \dots, m$ .

(a.4) For any  $f$  and any given  $(\pi_1, \dots, \pi_{f-1}, \pi_{f+1}, \dots, \pi_m)$ , there is at most one value of  $\underline{x}^f$ . (This is slightly weaker than the strict convexity of the set  $X_i = \{x_i : \pi(x_i) \leq \pi\}$  for all  $\pi$ .)

Theorem; Given assumptions (a.1) - (a.4), there exists an equilibrium.

The proof will consist of two parts. Part 1 will prove that there exists a set of outputs  $\underline{x}^1, \underline{x}^2, \dots, \underline{x}^m$ . That is, for any  $i$ , there exists maximum maker profit,  $\pi_i^M$ , with consistent values of  $\pi_f$ , for all  $f \neq i$ .

Part 2 will prove that there is always at least one firm which is a winning bidder, i.e., one firm whose maximum bid against his alternative makers is no less than the maximum of the bids against him.

Proof: 6/

Part 1.

First we show that for given values of maker profit of other firms, firm  $i$  has a maximum maker profit. To do this we will employ the well-known theorem in analysis that a continuous, real-valued function defined over a closed and bounded set attains a maximum at some point in the set.

Let

$$g_i(x) = \max_f [\pi_f - \pi_f(x)] \quad f=1, \dots, m; f \neq i, \quad (16)$$

$$i=1, \dots, m.$$

That is,  $g_i(x)$  is the function describing the maximum bids against  $i$  for each point in  $X$  selected by  $i$ . Since  $\pi_i(x)$  is continuous by (a.2) and the sum of two continuous functions is continuous, firm  $i$ 's maker profit in (14) is continuous if  $g_i(x)$  is continuous.

Lemma: The function  $g_i(x)$  is continuous.

At any point in  $X$ , and any  $i$ , there is either (a) an unique maximum bid in (16), or (b) there is an equality between the highest two, or more bids in (16).

(a) If there is an unique maximum in (16) at some point in  $X$ , then since each bid function,  $B_{fi}(x)$ , is continuous (the difference of two continuous functions is continuous), (16) is continuous at such points in  $X$ .

(b) Let  $x_s$  be a point in  $X$  where  $B_{ji}(x_s) = B_{ki}(x_s) = g_i(x_s)$ ,  $j \neq k$ . Suppose, for any  $\delta > 0$ , there is an  $\epsilon < \delta$ ,  $\epsilon > 0$ , such that

$$g_i(x_s - \epsilon) = B_{ji}(x_s - \epsilon) > B_{ki}(x_s - \epsilon), \text{ and} \quad (17)$$

$$g_i(x_s + \epsilon) = B_{ji}(x_s + \epsilon) < B_{ki}(x_s + \epsilon), \quad j, k=1, \dots, m;$$
$$j \neq k, \quad j, k \neq i.$$

It is obvious that since each bid function is continuous and equal at  $x_s$ ,  $g_i(x_s)$  is continuous at  $x_s$ .

In the case where the second relation in (17) does not hold, the bid of  $j$  is a maximal bid over the entire  $\delta$ -neighborhood so the continuity of  $g(x_s)$  follows from the continuity of  $B_{ji}(x_s)$ .

In the only remaining case, where only the first relation in (17) does not hold,  $B_{ji}(x) \equiv B_{ki}(x)$  about an  $\delta$ -neighborhood of  $x_s$ , then either

bid is maximal in that neighborhood. Since the bid functions are continuous at all points in  $X$ , then  $g_i(x)$ ,  $i=1, \dots, m$ , is continuous over all of  $X$ .

It follows from the lemma and the well-known theorem in analysis stated above that  $\pi_i^M$  and  $\underline{x}^i$  exist for any set of values  $\{\pi_f\}$  and thus for any set of vectors  $\{x^f\}$ ,  $f \neq i$ .

Now one firm's optimal maker output vector depends upon the optimal maker output vectors of other firms. This leads to the question of whether the optimal output vectors of the various firms are mutually consistent. Proving this establishes the existence of a set of output vectors,  $(\underline{x}^1, \dots, \underline{x}^m)$ , such that, for each  $f$ ,  $\underline{x}^f$  yields maximum maker profit to firm  $f$  for the  $\underline{x}^i$  of all  $i \neq f$ ,  $i, f = 1, \dots, m$ . Consider  $m$  feasible sets of  $nm$  outputs, each representing an output allocation vector arbitrarily selected by each  $f$ , or  $(x^{10}, \dots, x^{m0})$ . Given the values of  $(x^{20}, \dots, x^{m0})$ , the value of  $x$  maximizing firm 1's maker profit,  $x^{11}$ , is calculated. Using  $x^{11}$  and  $x^{30}, \dots, x^{m0}$ , the value of  $x$  maximizing firm 2's maximum maker profit,  $x^{21}$ , is calculated. Continuing in this manner, the output set,  $(x^{11}, \dots, x^{m1})$  is attained.

The resulting transformation,

$$(x^{10}, \dots, x^{m0}) \rightarrow (x^{11}, x^{21}, \dots, x^{m1}), \quad (18)$$

then is a transformation from a set of output sets,  $X^m$ , into itself.

To show that there exists a consistent set of maximum maker profit over all  $m$  firms, it is sufficient to show that there exists a set of outputs,  $(\underline{x}^1, \dots, \underline{x}^m)$ , which remains unchanged over the

transformation (18). By the Kakutani fixed point theorem, the set,  $(\underline{x}^1, \dots, \underline{x}^m)$  exists if  $X^m$  is a compact, convex set, and the complete transformation (18) is continuous.

By assumption (a.1) - and the fact that the Cartesian product of closed, bounded, and convex sets is itself closed, bounded, and convex - we know that  $X^m$  is closed, bounded, and convex. Since each transformation in (18) is a calculation of some firm's profits, it is sufficient to show the continuity of (18) by showing the continuity of  $\underline{x}^i$  as a function of

$$\underline{x}^{-i} = (\underline{x}^1, \dots, \underline{x}^{i-1}, \underline{x}^{i+1}, \dots, \underline{x}^m) \quad (19)$$

for any  $i, i=1, \dots, m$ . Suppose the function,

$$\underline{x}^i = \underline{x}^i(\underline{x}^{-i}), \quad (20)$$

is not continuous at some  $\bar{x}^{-i}$ . This implies that there is an infinite sequence,  $\{x^{-i}\}$ , approaching  $\bar{x}^{-i}$  such that

$$\bar{x}^i = \underline{x}^i(\bar{x}^{-i}) \neq \lim_{x^{-i} \rightarrow \bar{x}^{-i}} \{\underline{x}^i(x^{-i})\}.$$

(The existence of this limit is implied by the boundedness assumption in (a.2) and the Weierstrass Theorem.) Since each firm's profit is a continuous function of  $(x^1, \dots, x^m)$ , there is also an infinite sequence,

$$\{\pi_{-i}\} = \{\pi_1, \dots, \pi_{i-1}, \pi_{i+1}, \dots, \pi_m\}, \quad (21)$$

which approaches  $\bar{\pi}_{-i}$  such that

$$\bar{x}^i = \underline{x}^i(\bar{\pi}_{-i}) \neq \lim_{\pi_{-i} \rightarrow \bar{\pi}_{-i}} \{\underline{x}^i(\pi_{-i})\}.$$



Now the uniqueness of  $\underline{x}^i(\bar{\pi}_{-i})$  expressed in (a.4) implies that there is a  $\delta(\epsilon), \delta > 0$ , such that for any  $x^i \in X$  not in an  $\epsilon$ -neighborhood of  $\underline{x}^i$ ,

$$\pi^i(\underline{x}^i, \bar{\pi}_{-i}) - \pi^i(x^i, \bar{\pi}_{-i}) > \delta(\epsilon), \quad (22)$$

where  $\pi^i$  is firm  $i$ 's maker profit. Then, from the linear manner in which  $\pi_{-i}$  enters  $i$ 's maker profit function (14), there is an  $\omega > 0$  such that for all  $\pi_{-i}$  satisfying  $|\pi_{-i} - \bar{\pi}_{-i}| < \omega$ , and for all  $x^i \in X$  not in an  $\epsilon$ -neighborhood of  $\underline{x}^i$ ,

$$\pi^i(\underline{x}^i, \pi_{-i}) - \pi^i(x^i, \pi_{-i}) > \delta(\epsilon). \quad (23)$$

Consider the  $\epsilon$ -neighborhood of

$$\lim_{\pi_{-i} \rightarrow \bar{\pi}_{-i}} \underline{x}^i,$$

and select an  $\epsilon$  sufficiently small that the intersection of this neighborhood and the  $\epsilon$ -neighborhood of  $\underline{x}^i$  is empty. If the  $\underline{x}^i$  in the former neighborhood are indeed profit maximizing, for all  $\underline{x}^i$  in that neighborhood,

$$\pi^i(\underline{x}^i, \pi_{-i}^M) - \pi^i(\underline{x}^i, \pi_{-i}^M) < 0, \quad (24)$$

for all  $\pi_{-i}^M \rightarrow \bar{\pi}_{-i}^M$  generating the  $\epsilon$ -neighborhood of

$$\lim_{\pi_{-i}^M \rightarrow \bar{\pi}_{-i}^M} \underline{x}^i.$$

This is a direct contradiction of the immediately preceding inequality (23).

Hence,  $\underline{x}^i$  is a continuous function of  $x^{-i}$  and likewise the transformation (18) is continuous. This is sufficient for the Kakutani fixed point theorem to apply; and therefore, the set  $(\underline{x}^1, \dots, \underline{x}^m)$  exists.

Part 2.

We shall now prove that -- given the array of maximum maker profits in (14), and therefore an array of bids against all alternative makers described in (15) -- a winning bidder exists. Consider the matrix B, representing the bids of each firm against the others, with zeroes along the main diagonal:

$$B = \begin{bmatrix} 0 & B_{12} & B_{13} & \dots & B_{1m} \\ B_{21} & 0 & B_{23} & \dots & B_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{m1} & B_{m2} & B_{m3} & \dots & 0 \end{bmatrix}. \quad (25)$$

From the definition of a solution maker, i is a solution maker if  $\max_j B_{ij} \geq \max_k B_{ki}$ , that is, if the maximum bid by i exceeds the maximum of the bids against i. In B, i is a solution maker if the maximum of the elements in the i<sup>th</sup> row exceeds the maximum of the elements in the i<sup>th</sup> column. Let  $B_{qr}$  be a maximal element of B. Then,  $B_{qr} \geq \max_k B_{kq}$  so that q is a maker. Hence, there is always a winning bidder.

III. ANTI-MONOPOLY POLICY AND MULTIPLE-MONOPOLY

The results obtained in Section I are based on a decentralized model of monopoly decision-making without government intervention. However, in order to arrive at a more realistic calculation of monopoly behavior, we now introduce government participation in the economy in the form of federal anti-monopoly policy. Since most anti-monopoly policy ignores interactions between firms in different industries, we shall assume that firms in different industries are noninteracting and thus consider only a single m-firm industry.

A. The Revised Reaction Functions

On the basis of existing anti-monopoly laws, it is reasonable to assume that if any firm expands its output in reaction to increases in the outputs of its competitors -- either existing or entering firms -- that firm would be subject to prosecution under the law for its "predatory practices."<sup>7/</sup> Thus, whenever a firm increases its output for a given level of industry demand and cost we shall assume that government policy prohibits another firm from increasing its output. (We are assuming that detection of violators and enforcement of the law are carried out at zero cost.) This restriction upon the output reactions of firms precludes the use of punishment strategies.

In the absence of punishment strategies, a firm may try to induce the production of some desired industry output by "rewarding" other firms for their outputs; that is, decreasing its output if the other firm(s) decrease its (their) output(s) up to the desired industry output. We shall assume that such behavior will be viewed as collusion by the government policy-makers, and likewise be prohibited. Therefore, the effect of our anti-monopoly policy on a firm's choice of reaction function is to limit these choices to a class of non-increasing functions.

With the elimination of punishment and reward outputs, no firm is able to force the production of any output allocation set on the other firms. Since there is no single firm determining the outputs of all other firms, with  $m \geq 3$ , some of the firms which were previously takers, and did not exhibit a reaction function, are now free to react to the outputs of some other firms. We are thus back to the model of IC, in which a hierarchy of makers appears. Firm  $i$ ,  $1 < i < m$ , selects its reaction function in light of the reactions of firms' 1 through  $i-1$  to the outputs of firms  $i$  through  $m$ . That is, the reaction functions of preceding makers are parameters

in any given firm's choice of a reaction function. Firm  $i$  is a maker with respect to  $i+k$  but is a taker with respect to  $i-j$  ( $k=1, \dots, m-i$ ;  $j=1, \dots, i-1$ ). Firm  $m$  exhibits no output reaction function but chooses its profit-maximizing output subject to the reaction functions of the other firms and is a pure taker.

Since the restricted reaction functions are such that all derivatives of the functions are non-positive, firm  $i$  faces two alternatives given a change in the output choice of the firm  $i+k$ .  $i$  may exhibit a zero or a negative reaction to  $i+k$ 's change in output. Comparing the results of these two alternatives, firm  $i$  rationally decides upon the zero reaction function. The reasoning is as follows: If  $i+k$  increases his output, and the rest of the firms other than  $i$  do not contract as much as  $i+k$  expands so that there is a net expansion in industry output, the result of  $i$ 's contracting his output would be to encourage the expansion of the aggregate output of his competitors. This would only be desirable if it permitted  $i$  a higher solution output, but  $i$ 's output level may be chosen independently of the derivatives of  $i$ 's reaction functions. Similarly if  $i+k$  reduces his output and the induced expansion by the rest of the firms still permits a net decrease in industry output,  $i$  will not rationally increase his output and thereby discourage  $i+k$  from decreasing his output in the first place. Firm  $i$  could produce the desired higher output by simply committing himself to produce the desired output as a constant regardless of  $i+k$ 's reaction and thereby induce  $i+k$  to produce a lower output than he would if  $i$  presented  $i+k$  with a negative reaction. If, when  $i+k$  increases his output, the rest of the firms other than  $i$  decreased their outputs so that the net output of the industry was reduced,  $i+k$  would always expand (since  $i$  cannot have a positive reaction) so that there is no equilibrium under this supposition. Thus, in equilibrium, each firm's rational reaction function

is a constant output reaction function.

B. Equilibrium with no competition for hierarchal position.

Using this result we now illustrate equilibrium with  $m$  firms producing a homogeneous output in which each firm's hierarchal position is exogenously given. This amounts to a generalization of the Stackelberg duopoly model.<sup>8/</sup> Consider  $m$  firms, each producing a homogeneous output. We assume a hierarchy of makers, with firm 1 being the primary maker, firm 2 the secondary maker, and firm  $m$  the pure taker. These positions are, in the analysis of this section, imposed upon the model and not the result of any bidding process. Industry demand is assumed to be linear and of the form:

$$p = a - b \sum_{i=1}^m x_i \quad (26)$$

where  $p$  is the price of the industry's output and  $a$  and  $b$  are positive constants. Marginal costs are assumed to be constant and identical for each firm so that

$$C_f = cx_f + d_f, \quad (27)$$

where  $c$  and  $d_f$  are positive constants and, to assure positive outputs,  $c < a$ . The condition for profit maximization for each firm is

$$p - x_f b \sum_{i=1}^m \frac{dx_i}{dx_f} - c = 0, \text{ or} \quad (28)$$

$$x_f = \frac{a - c - b \sum_{i \neq f} x_i}{b \left( 2 + \sum_{i \neq f} \frac{dx_i}{dx_f} \right)} = \frac{p - c}{b \left( 1 + \sum_{i \neq f} \frac{dx_i}{dx_f} \right)} \quad (29)$$

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$$x_f = \frac{a - c - b \sum_{i \neq f} x_i}{b \left( 2 + \sum_{i \neq f} \frac{dx_i}{dx_f} \right)} = \frac{p - c}{b \left( 1 + \sum_{i \neq f} \frac{dx_i}{dx_f} \right)} \quad (29)$$

Since all reaction functions are constant output reaction functions, for each firm  $j$ ,

$$\frac{dx_i}{dx_j} = 0 \quad \text{for all } i < j.$$

This yields a profit-maximizing expression for  $m$  of:

$$x_m^* = \frac{a-c-b \sum_{i=1}^{m-1} x_i}{2b} = \frac{p-c}{b} \quad (30)$$

Since firm  $m-1$  is a taker with respect to  $1, \dots, m-2$ , and the latter exhibit constant output reaction functions,

$$\sum_{i=1}^{m-2} \frac{dx_i}{dx_{m-1}} = 0.$$

And from (30), we know  $\frac{dx_m^*}{dx_{m-1}} = -\frac{1}{2}$ . Thus, using (29) and (30),

$$x_{m-1}^* = \frac{a-c-b \sum_{i=1}^{m-2} x_i}{2b} = \frac{p-c}{b/2} \quad (31)$$

To obtain  $m-2$ 's profit-maximizing condition, we have to calculate  $m-1$ 's and  $m$ 's rational responses to a change in  $x_{m-2}$ . From (31) we know that

$$\frac{dx_{m-1}^*}{dx_{m-2}} = -1/2.$$

And from (30),

$$\frac{dx_m^*}{dx_{m-2}} = \frac{\partial x_m^*}{\partial x_{m-2}} + \frac{\partial x_m^*}{\partial x_{m-1}} \cdot \frac{dx_{m-1}^*}{dx_{m-2}} = -1/4.$$

Hence, again using (29),

$$x_{m-2}^* = \frac{a-c-b \sum_{i=1}^{m-3} x_i}{2b} = \frac{p-c}{b/4} \quad (32)$$

Similarly, we find that

$$x_{m-3}^* = \frac{a-c-b \sum_{i=1}^{m-4} x_i}{2b} = \frac{p-c}{b/8} \quad (33)$$

$$x_{m-4}^* = \frac{a-c-b \sum_{i=1}^{m-5} x_i}{2b} = \frac{p-c}{b/16} \quad (34)$$

⋮  
⋮  
⋮

The resulting size distribution of firms is obviously

$$x_{m-i}^* = x_m^* 2^i, \quad i = 0, 1, \dots, m-1 \quad (35)$$

C. Equilibrium with competition for positions in the hierarchy

This subsection shows that the introduction of competition in making commitments, and thus of competitive bidding for each position in the hierarchy, makes the above analysis decidedly more complex but does not substantially alter its conclusions.

Competitive bidding for hierarchial position increases the costs of all firms except, of course, the costs of the pure taker. Since all firms have the same variable cost functions, the variable maker profit of each firm is, in equilibrium, equal to the simple variable profit of the pure taker. It also follows from the equality of variable costs between our firms that every firm is indifferent to any position in the hierarchy so



that all bids for each position in the hierarchy are identical. One's position in the hierarchy is determined by the priority of his commitment. Hence, the first auction is for the position of primary maker, the second for the secondary maker, etc. In the first auction, there are  $m$  bidders, in the second there are  $m-1$ , and so on until, finally in the  $m$ -1st auction, there are only two bidders. Let us see how these auctions alter the rationally chosen outputs from those selected in the model with no competition for hierarchal position,

The pure taker obviously has the same output choice function as in the model with no competitive bidding for hierarchal positions, so again

$$x_m^* = \frac{a-c-b \sum_{i=1}^{m-1} x_i}{2b} = \frac{p-c}{b}, \quad (36)$$

where  $x_i^*$  is the solution output of firm  $i$  in the present model. But the output of the  $m$ -1st firm is now sensitive to a bid of the  $m$ <sup>th</sup> firm for his hierarchal position. The bid of  $m$  for position  $m-1$  equals

$$B_{m,m-1}(x_{m-1}) = \max[0, \pi_{m,m-1} - \pi_m(x_{m-1})],$$

where  $\pi_{m,m-1}$  is the operating profit  $m$  would make if he were the  $m-1$ <sup>st</sup> maker. Firm  $m-1$  therefore selects an  $x_{m-1}$  which maximizes

$$\pi_{m-1}^M = \pi_{m-1} - \max[0, \pi_{m,m-1} - \pi_m(x_{m-1})] \text{ subject to (36) and the given values, } x_1, \dots, x_{m-2}. \quad (37)$$

If the solution  $x_{m-1}$  were such that  $B_{m,m-1}(x_{m-1}) > 0$ , then the output that satisfies (37) would be a joint profit maximum subject to (36) and  $x_1, \dots, x_{m-2}$ . This joint profit is  $(p-c)(x_{m-1}+x_m)^{-d_m-d_{m-1}}$ , which, as above, reaches its maximum at  $x_{m-1}+x_m = \frac{p-c}{b}$ . But then, from (36),  $x_{m-1}^*$  would equal zero. Since variable profits exceed zero at some positive outputs, firm m-1 would do better as the pure taker. Hence,  $B_{m,m-1}(x_{m-1}^*)=0$ . But if so, the variable profits are equal for both firms m and m-1. Therefore,

$$x_{m-1}^* = x_m^* \quad (38)$$

and using (36),

$$x_{m-1}^* + x_m^* = \frac{a-c-b \sum_{i=1}^{m-2} x_i}{3b/2} = 2 \left( \frac{p-c}{b} \right) \quad (39)$$

This solution may be constructed by starting firm m-1 at its output in the previous model, an output which maximizes his operating profit and exceeds  $x_m$ , and then making it pay m's bid to be maker to a lawyer. It is then obvious from (37) that it pays m-1 to reduce his output in order to reduce m-1's bid against him. This occurs until  $x_{m-1} = x_m$ , at which point the bids become zero. It then no longer pays m-1 to reduce his output for there is no further reduction in m's bid that is possible. To compute m-2's optimal output, we need the profit of firms m and m-1 as a function of  $x_{m-2}$ . Using (39), and computing, from now on, profits as variable profits (profits net of  $d_i$ ), we have

$$\pi_{m-1} = \pi_m = \left[ a-c-b \sum_{i=1}^{m-2} x_i - b \left( \frac{a-c-b \sum_{i=1}^{m-2} x_i}{3b/2} \right) \right] \left( \frac{a-c-b \sum_{i=1}^{m-2} x_i}{3b} \right) = \frac{\left( a-c-b \sum_{i=1}^{m-2} x_i \right)^2}{9b} \quad (40)$$

Firm m-2's maker profits can now be written, using (39),

$$\pi_{m-2}^M = \left[ a-c-b \sum_{i=1}^{m-2} x_i - b \left( \frac{a-c-b \sum_{i=1}^{m-2} x_i}{3b/4} \right) \right] x_{m-2} - \max[0, \pi_{m,m-2} - \pi_m(x_{m-2})] . \quad (41)$$

Assume that  $\pi_{m,m-2} - \pi_m(x_{m-2}^*) \geq 0$ . Then, using (40) and (41),

$$\pi_{m-2}^M = \left( \frac{a-c-b \sum_{i=1}^{m-2} x_i}{3} \right) x_{m-2} + \frac{\left( a-c-b \sum_{i=1}^{m-2} x_i \right)^2}{9b} - \pi_{m,m-2} .$$

Maximizing this profit, we find

$$x_{m-2}^* = \frac{a-c-b \sum_{i=1}^{m-3} x_i}{4b} , \text{ and} \quad (42)$$

$$\pi_{m-2}(x_{m-2}^*) = \frac{\left( a-c-b \sum_{i=1}^{m-3} x_i \right)^2}{16b} \quad (43)$$

Substituting  $x_{m-2}^*$  into (39),

$$x_{m-2}^* = x_{m-1}^* = x_m^* \quad (44)$$

The assumption that  $\pi_{m,m-2} - \pi_m(x_{m-2}^*) \geq 0$  is satisfied, for at  $x_{m-2}^*$  the bid of m (and of m-1) is zero. And m-2 obviously would reduce his profit by contracting his output so as to make  $\pi_{m,m-2} - \pi_m(x_{m-2})$  negative because (42) and (43) show that even if m-2 could profit from negative bids, he would still maintain a zero-bid solution. Since this holds for any firm, if the output solution to the problem,

$$\max_{x_i} [(p-c)x_i - \max_j (\pi_{ji} - \pi_j(x_i))] , i < m, \quad (45)$$

exceeds  $x_m$ , then, because bids are positive in such a solution, this

output solution is also the solution to the general problem:

$$\max_{x_i} [(p-c)x_i - \max_j [0, \pi_{ji} - \pi_j(x_i)]] \quad (46)$$

Performing the maximization in (45) for firm m-3, using (42), (43) and (44), we maximize

$$\left[ \left( a-c-b \sum_{i=1}^{m-3} x_i \right) x_{m-3} - 3/4 \left( a-c-b \sum_{i=1}^{m-3} x_i \right) x_{m-3} + \frac{\left( a-c-b \sum_{i=1}^{m-3} x_i \right)^2}{16b} \right]$$

with respect to  $x_{m-3}$ . The solution output can be written:

$$x_{m-3}^* = \frac{a-c-b \sum_{i=1}^{m-4} x_i}{3b} = \frac{a-c-b \sum_{i=1}^{m-3} x_i}{2b} \quad (47)$$

This output is twice the output of firm m-2 and thus is also a solution to (46). We also find, using (42), (44) and (47) that variable profits for each firm are:

$$\pi_m^* = \pi_{m-1}^* = \pi_{m-2}^* = \pi_{m-3}^{M*} = \frac{\left( a-c-b \sum_{i=1}^{m-4} x_i \right)^2}{36b} \quad (48)$$

Thus variable maker profit to the  $m-4^{\text{th}}$  firm is

$$\pi_{m-4}^M(x_{m-4}) = a-c-b \sum_{i=1}^{m-4} x_i - \frac{5}{6} \left( a-c-b \sum_{i=1}^{m-4} x_i \right) x_{m-4} + \frac{a-c-b \sum_{i=1}^{m-4} x_i}{36b}$$

Maximizing this with respect to  $x_{m-4}$ , we find

$$x_{m-4}^* = \frac{a-c-b \sum_{i=1}^{m-5} x_i}{5b/2} = \frac{a-c-b \sum_{i=1}^{m-4} x_i}{3b/2} \quad (49)$$

This is twice the output of firm m-3 and four times the outputs of firms m, m-1, and m-2. We also find that

$$\pi_m^* = \pi_{m-1}^* = \pi_{m-2}^* = \pi_{m-3}^{M^*} = \pi_{m-4}^{M^*} = \frac{a-c-b \sum_{i=1}^{m-5} x_i}{100b} .$$

These profits form the bids for the m-5<sup>th</sup> position in the hierarchy, and the procedure continues until we reach the top position. The distribution of outputs, moving on to m-5, m-6 and m-7 and again indexing the output of the m<sup>th</sup> firm to unity, is easily seen to be

$$1, 1, 1, 2, 4, 8, 16, 32.$$

The obvious generalization is that

$$x_m^* = x_{m-1}^* \text{ and } x_{m-2-i}^* = x_m^* \cdot 2^i, \quad i = 0, 1, \dots, m-3 . \quad (50)$$

To prove that this generalization is, in fact, the solution distribution of firms, we provide an inductive proof. In particular, we shall prove that if the hypothesized distribution holds for  $i = r$ , i.e., if  $x_{m-2-i}^*/x_{m-1-i}^* = 2$  for any  $i$  such that  $0 \leq i \leq r$ , then it holds for  $i = r + 1$ , i.e.,

$x_{m-3-r}^*/x_{m-2-r}^* = 2$ . To do this, we first note that variable profit to the m-3<sup>rd</sup> firm is

$$\pi_{m-3-r} = \left( a-c-b \sum_{i=1}^m x_i \right) x_{m-3-r} + \frac{\left( a-c-b \sum_{i=1}^m x_i \right)^2}{b} .$$

By hypothesis, for the firms from  $m$  to  $m-2-r$  we have:

$$x_{m-2-r} = \frac{a-c-b \sum_{i=1}^{m-3-r} x_i}{b(2^{r+1})/2^{r-1}} \quad (51)$$

and

$$\sum_{i=m-2-r}^m x_i = \frac{a-c-b \sum_{i=1}^{m-3-r} x_i}{b} \left( \frac{2^{r+1}+1}{2^{r+1}+2} \right), \quad r \geq 0 \quad (52)$$

Using (52),

$$\begin{aligned} \pi_{m-3-r} &= \left( a-c-b \sum_{i=1}^{m-3-r} x_i - b \sum_{i=m-2-r}^m x_i \right) x_{m-3-r} + \frac{\left( a-c-b \sum_{i=1}^{m-3-r} x_i - b \sum_{i=m-2-r}^m x_i \right)^2}{b} \\ &= \frac{1}{2^{r+1}+2} \left( a-c-b \sum_{i=1}^{m-3-r} x_i \right) x_{m-3-r} + \frac{\left( a-c-b \sum_{i=1}^{m-3-r} x_i \right)^2}{(2^{r+1}+2)^2 b} \end{aligned}$$

Maximizing this expression with respect to  $x_{m-3-r}$ ,

$$\begin{aligned} 0 &= \frac{1}{2^{r+1}+2} \left( a-c-b \sum_{i=1}^{m-3-r} x_i \right) - \frac{bx_{m-3-r}}{2^{r+1}+2} - \frac{2 \left( a-c-b \sum_{i=1}^{m-3-r} x_i \right)}{(2^{r+1}+2)^2} \\ &= \frac{2^{r+1} \left( a-c-b \sum_{i=1}^{m-3-r} x_i \right)}{(2^{r+1}+2)^2} - \frac{bx_{m-3-r}}{2^{r+1}+2} \\ x_{m-3-r} &= \frac{a-c-b \sum_{i=1}^{m-3-r} x_i}{b \left( \frac{2^r+1}{2^r} \right)} \end{aligned}$$

Using (51), we see that

$$\frac{x_{m-3-r}}{x_{m-2-r}} = \frac{(2^r+1)/2^{r-1}}{(2^r+1)/2^r} = 2 .$$

This establishes the theorem.

D. Corollaries

There is a rather remarkable corollary concerning the "concentration ratio" of our industries. It is that the  $t$ -firm concentration ratio, the share of the top  $t$  firms in the industry ( $t < m-2$ ), increases with the number of firms in the industry. From the above theorem, the total output of the top  $t$  firms in the industry can be written:

$$K \sum_{i=m-2-t}^{m-3} 2^i = K 2^{m-3-t} \sum_{i=1}^t 2^i = K 2^{m-2-t} (2^t - 1) ,$$

where  $t \leq m-3$  and  $K$  is some positive number. The total output of the  $m-2-t$  firms from firm  $m-2$  to firm  $t-1$  is then given by

$$K \sum_{i=0}^{m-3-t} 2^i = (2^{m-2-t} - 1)K ,$$

while the total output of firms  $m$  and  $m-1$  is, of course,  $2K$ . Hence, the output share of the top  $t$  firms in the industry,  $t \leq m-3$ , is given by:

$$S_t = \frac{2^{m-2-t} (2^t - 1)}{2^{m-2-t} (2^t - 1) + 2^{m-2-t} - 1 + 2} = \frac{2^t - 1}{2^t + \frac{1}{2^{m-2-t}}} . \quad (53)$$

Thus we see that as the number of firms in the industry expands and thus the output becomes more competitive, the concentration ratio,  $S_t$ , for any  $t$  ( $t \leq m-3$ ) increases.

This increase, however, is very slight once the number of firms in the industry becomes at all significant. For example, if  $m \geq 10$ , then the percentage error in using  $\frac{2^{t-1}}{2^t}$  as an estimate of  $S_t$  is always less than one quarter of one percent.

From (36), the equilibrium mark-up in our model is  $bx_m$ . Under pure monopoly, the mark-up would be  $b\sum_{i=1}^m x_i$ . This is the same as the uncontrolled monopoly mark-up since the rational maker in this industry model could not do better than he could by producing an output such that  $p < c$  whenever any other firm produced a positive output. Given the distribution of output among firms in our controlled monopoly solution, the equilibrium mark-up relative to the pure monopoly mark-up is therefore given by

$$\frac{x_m}{\sum_{i=1}^m x_i} = \frac{1}{2^{m-2} + 1} .$$

So with, say, 10 firms in the industry, the equilibrium mark-up is less than  $\frac{1}{4}$  of 1% of the pure monopoly mark-up. This seems safe enough to ignore for policy purposes, especially since some positive mark-up is necessary to guarantee the presence of producers in an industry. The result speaks for the powerful efficiency of the simple anti-trust policy outlined above.

#### IV. PRELIMINARY EVIDENCE

The rational reaction function-perfect information approach to non-competitive interdependence can be tested by attempting to verify empirically the implications of the above two models. We use the U. S. experience



since we are somewhat less ignorant of it than of the experiences of other countries.

According to most accounts, no substantial monopolies other than government-granted and natural, local monopolies appeared before the Civil War. After that war, the communications-transportation revolution and the emergence of the corporate form of organization apparently opened up new opportunities for large scale organizations and thus private monopolies operating in nation-wide markets. In this environment, industrial giants grew in several industries, each coming to dominate his industry by using unprofitable price-cutting as a weapon against smaller firms in order to keep them "in line." These "robber barons" were, in our terms, simply rational makers over a set of takers and their "cutthroat competition" was merely their application of punishment outputs to deviant firms.

The development of anti-trust policy in response to the obvious inefficiencies in this system took several decades and has operated, as we have suggested, to remove collusion (as well as mergers with the purpose of raising prices) and cutthroat competition. This policy implies, as we have pointed out, a hierarchy of makers in which each of the makers presents the industry with a fixed output for a given level of market demand and industry costs. Empirically, this means that the larger firms in an industry can be expected to commit themselves to an announced share of the market and retain this share regardless of the peculiar economics of individual firms. That large firms in the U. S. determine their outputs in this way rather than computing their own demand and supply curves is obvious for certain firms and has been claimed as a fairly general

description by numerous institutionalist authors.

Further evidence for the controlled monopoly model was obtained from observations on relative firm sizes within selected U. S. industries.

Our hypothesis, from equation (50), implies that

$$\log x_{m-2-i} = K_1 + b_1 \cdot i, \quad i = 0, 1, \dots, m-3, \quad (54)$$

where  $b_1 = \log 2$  for each industry. The hypothesis relating market share to rank which we find in the literature (Simon (5)) is

$$\log x_{m-i+1} = K_2 + b_2 \log i, \quad b_2 < 1, \quad i = 1, \dots, m. \quad (55)$$

This hypothesis, which has no theoretical rationale, is clearly contrary to ours in that ours has firm size increasing more than in proportion to a firm's rank in the industry ( $i$ ) while (55) has firm size increasing less than in proportion to the firm's rank.

We obtained our data from Standard and Poor's "Compustat" tape for 1971, which has data on all of the relatively large U. S. companies within industries disaggregated to the four-digit industry level.<sup>9/</sup> This data was used to generate least-squares fits of the two hypotheses.<sup>10/</sup> The regressions for (55) produced coefficients less than unity in only three industries. In each of these industries (cement, roof and wallboard, and savings and loans) it appeared that we had erred in considering the markets for their product a national rather than a local market. It is not surprising that (55) fits better than (54) for local industries as it is well-known (cf. Simon (5)) that city sizes follow a distribution such as (55).

For the remaining thirty-eight industries, the fit in (54) was better (higher  $R^2$ ) than that in (55) in over 90% of the cases and the average estimated percentage excess of each firm's size over the next smallest firm's size (the average of the antilogs of the estimated coefficients in (54)) was 86% compared to the theoretical value of 100%. To us, these results amount to fairly strong preliminary evidence in favor of our theory. The regression results on (54) are described in Table 1.

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Table 1: Fit of Equation (54):  $\log x_{m-2-i} = K_1 + b_1 i; i=1, \dots, m-3$

Industry	$b_1$	$S_{b_1}$	$e^{b_1}$	D.W.	$R^2$	m-3
gold mining	.68	.1	1.98	1.57	.85	6
coal	1.13	.21	3.10	1.66	.90	5
housing construction	.39	.04	1.48	2.37	.93	7
packaged foods	.26	.03	1.30	1.13	.88	11
dairies	.64	.13	1.90	1.78	.83	7
canned foods	.33	.02	1.39	1.55	.96	12
animal foods	1.14	.16	3.13	1.70	.95	5
biscuits	1.22	.67	3.39	3.00	.80	3
confectionary	.69	.09	2.00	2.06	.91	6
brewers	.27	.01	1.31	1.81	.97	15
distillers	.51	.03	1.67	2.53	.98	7
soft drinks	.69	.09	2.00	2.06	.91	7
tobacco	.47	.10	1.60	1.73	.83	7
forest products	.49	.04	1.63	1.67	.95	11
mobile homes	.33	.04	1.39	2.11	.90	11
home furnishings	.27	.02	1.31	1.85	.94	15
paper	.35	.04	1.42	1.28	.84	14
books	.49	.05	1.63	1.75	.95	7
drugs-ethical	.22	.02	1.25	1.21	.93	17
drugs-proprietary	.57	.07	1.77	.70	.91	9
medical & hospital supply	.38	.03	1.46	2.08	.94	10
soap	.99	.18	2.69	1.57	.91	5
cosmetics	.30	.01	1.35	2.45	.99	16
paint	.60	.11	1.83	1.74	.86	7
tires & rubber goods	.31	.02	1.36	0.89	.92	17
plastics	.33	.04	1.39	1.77	.91	8
shoes	.38	.02	1.46	1.81	.96	13
concrete gypsum and plaster	.37	.08	1.45	1.98	.83	7
aluminum	.60	.21	1.83	2.29	.80	4
motor vehicles	1.01	.22	2.75	2.14	.91	4
photographic	1.07	.07	2.92	2.38	.98	7
watches	.65	.08	1.92	1.93	.96	5
musical instruments, parts	.97	.23	2.64	2.27	.86	6
games	.41	.05	1.51	1.85	.94	6
radio-TV broadcasters	.34	.03	1.40	2.91	.95	11
wholesale foods	.52	.06	1.68	1.78	.93	9
retail lumber yards	.96	.23	2.61	2.37	.90	4
motion pictures	.55	.06	1.73	1.37	.91	9
average			1.86			

FOOTNOTES

\* The authors benefited substantially from the comments of Louis Makowski, Ron Heiner and a referee of this journal on earlier drafts of this paper.

1. These relative price effects are not theoretically interesting with respect to competitive firms. Since individual competitive firms have no effect on relative prices, they do not consider relative price effects when choosing their outputs.

2. While a general game theory containing contingent strategies has been produced by Howard (1), he makes the conventional perfect information assumption and therefore winds up with a conventional set of Nash solutions.

3. Although the idea of commitments is not novel to the area of conflict resolution (see, for example Schelling (5)), its economic rationale and use in economic theory is rare. A recent exception is contained in Thompson (7). Here, commitments characterize an equilibrium distribution of property between countries through each country's commitment to protect, with all of its resources, any part of its capital stock from foreign aggression.

4. Examples of communicating commitments are: (1) when a producer of diamonds or gold stockpiles his commodity beyond the quantity rationalized by unpredicted demand fluctuations. Thus, the producer is exhibiting his ability and desire (by incurring higher production costs) to flood the market in the event of entry or increased production by his competitors: (2) strikes and lockouts when there exist only small differences in contract negotiations. This establishes the union's, or the firm's, willingness to engage in irrational (non-profit-maximizing) behavior when its desires are not met: and (3) when nations establish commitments

by placing the power to make certain decisions beyond their control; for example, a "doomsday machine" which can destroy the whole world, but once turned on, cannot be turned off.

5. An  $m$ -firm joint-profit maximum would mean that the maker's marginal profit (assuming differentiability) equals the sum of the other firms' marginal profits. In our case, the marginal profit of the maker is equal to each firm's marginal profit. Let there be an  $m$ -firm tie where  $m = 3$ . If marginal profit to  $i$ , the maker, equals \$1 (thus, marginal profit equals \$1 a piece to the takers), then marginal joint-profit equals minus \$1, rather than 0.

6. A more compact proof, provided by Ron Heiner, will be sent on request.

7. This interpretation assumes that the courts can distinguish between increases in output due to efficiency reasons, and increases for predatory or punishment reasons. Certain anti-trust cases, such as the U. S. Steel Case, 1921, lend support to this assumption. The anti-trust policy in this paper is based on past interpretations of the Sherman Anti-Trust Act. For further discussion of this law and its judicial interpretations, see Neale (4).

8. In the Stackelberg model of duopoly (see Intriligator (2)), one firm, the "leader," believes that his rival will behave as a Cournot duopolist. That is, the rival firm, called the "follower," assumes the leader will exhibit a constant output; and thus the follower makes his rational output choice based on this assumption about his rival's output. The leader then selects his output subject to the follower's rational output choice-function. If the two duopolists behave as each believes, the result is a Stackelberg equilibrium. In our controlled monopoly model, the pure taker acts as a

Stackelberg follower by choosing his output subject to constant-output reaction functions. The pure maker behaves as a Stackelberg leader since he chooses his reaction function (output) subject only to the profit-maximizing behavior of the other firms. Firms 2,...,m-1 introduce into the model additional maker/taker relationships not described in previous models of which these authors are aware. Nevertheless, our controlled monopoly model with its (constrained) reaction functions and added maker/taker relationships generates what can be interpreted as a generalized Stackelberg model. For, as derived above, adding more firms to our model merely creates a hierarchy of makers, or partial Stackelberg leaders.

9. We only included an industry when it (1) included 4 or more companies (for statistical reasons), (2) had a firm producing over 50 million dollars of sales (to avoid the exclusion of large producers due to their being a subsidiary of a diversified firm), (3) sold its product in a national market (to avoid local monopoly effects and interactions with firms in foreign markets), (4) sold its product to economic agents which are not substantially larger than itself (to avoid including industries in which some of the outputs are produced by vertically integrated firms, which would not be counted as part of the industry), (5) and marketed a relatively homogeneous commodity. This is a highly subjective selection of industries, but we know of no better way to provide a fair test of the hypothesis with so much of the data obviously irrelevant.

10. We had data on both current sales and assets as measures of size, assets being perhaps better than current sales as a measure of future sales. We ran regressions for both measures of size and chose the measure for each hypothesis that yielded Durbin-Watson statistics closest to 2. The rationale here is that we wanted to be as generous as we could to each hypothesis regarding which measure of size would conform the best to the curvature assumptions of the hypothesis.

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