

Efficient Sorting in a Dynamic Adverse-Selection Model¹

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Abstract

We discuss a class of markets for durable goods where efficiency (or approximate efficiency) is obtained despite the presence of information asymmetries. In the model, the number of times a good has changed hands (the vintage of the good) is an accurate signal of its quality, each consumer self-selects into obtaining the vintage that the social planner would have assigned to her, and consumers' equilibrium trading behavior in secondary markets is not subject to adverse selection. We show that producers have the incentive to choose contracts that lead to the efficient allocation, and to supply the efficient output. We also provide a contrast between leasing contracts, resale contracts, and different kinds of rental contracts. Resale contracts do not lead to the efficient allocation. A specific kind of rental contract provides the appropriate incentives to consumers.

1 Introduction

Since Akerlof's (1970) seminal paper, adverse selection has been recognized to be a potential source of inefficiency in durable-goods markets. The present paper suggests that asymmetric information leads to inefficiency in standard adverse-selection models only under specific restrictions about trading opportunities. If those restrictions are removed, full information payoffs and allocations can be achieved in a competitive equilibrium even under asymmetric information.

We depart from the literature in three key ways. First, we do not study secondary markets in isolation. Secondary markets interact in important ways with the market for new goods; we take these interactions into account by integrating the markets for new units (including production) into the analysis. Second, we remove some restrictions on secondary markets implicitly present in the previous literature by allowing for unrestricted opportunities for retrading in frictionless secondary markets (except for the presence of information asymmetries). Third, we do not restrict goods to be offered on the basis of selling (or leasing) contracts. We show that a menu of rental contracts is efficient and privately optimal for competitive producers.

To understand the contrast between our findings and standard inefficiency results, consider the classic adverse selection environment (as, e.g., in Akerlof, 1970 and Wilson, 1980). Efficiency requires allocating high quality cars to high valuation consumers. In Akerlof's model, the used cars are owned by low-valuation consumers, implying that trading is essential to achieve efficiency. However, if car quality is privately known, then it is typically impossible to effect such welfare-improving trades in an incentive-compatible way. Inefficiency is thus a consequence of private information. Some recent literature has pointed out two ways in which Akerlof's model may be incomplete as a model of durable goods. But, while both modifications of Akerlof's model lead to a reduction in the distortions caused by asymmetric information, inefficiencies still remain.

The first departure from Akerlof, which has been pursued by Hendel and Lizzeri (1999, 2002) and Johnson and Waldman (2003), explores the idea that, in the case of durable goods such as cars, the identity of the owners of used goods is endogenous because consumers self-select into either the new-goods or the used-goods market. Thus, in contrast with Akerlof, in these models, the owners of used units are high-valuation consumers who chose to purchase new goods instead of purchasing used units. These papers show that the interaction between the markets for new and used units has important consequences, and that the distortions caused by adverse selection in durable-goods markets may be less drastic than was suggested by Akerlof. However, in these papers, inefficiencies due to adverse selection generally persist due to an implicit restriction on the set of markets that can be open. This arises because goods are assumed to last two periods and there is only one secondary market. Thus, while new goods tend to be allocated to those who value them most, distortions persist in the allocation of used goods. With only one secondary market, it is impossible to achieve sorting in the used-good market by allocating higher quality used goods to higher valuation used-good consumers.¹

¹An exception that is explored by Hendel and Lizzeri and Johnson and Waldman is the case in which there are only two types of consumers. In this case, only one type consumes used goods, so sorting in secondary markets is not an issue. We will see below that another exception is the case of two qualities.

The second departure from Akerlof, which has been explored by Janssen and Roy (2001, 2002), retains the basic assumption of Akerlof’s model that used goods are exogenously allocated to low-valuation consumers. They point out that, in the case of durables, it may be natural to allow markets to be open repeatedly. Janssen and Roy show that, when used-goods markets are open at every date, more welfare-enhancing trades are possible. Prices and traded qualities increase over time, and the time to trade acts as a sorting device because owners of high quality cars are more willing to wait to obtain the high prices. As in our paper, sorting takes place. However, the equilibrium allocation is not efficient in their model: the inefficiency takes the form of a delay of trade. Furthermore, such inefficiency is inescapable: it cannot be eliminated by clever contract design. This is because, in an environment in which ownership of goods is exogenous, there exists no mechanism (static or dynamic) that achieve the ex-post efficient allocation.

We show that removing both limitations of Akerlof’s model at the same time can lead to the first-best outcome, given the appropriate choice of contracts. Thus, the combination of multiple secondary markets and endogenous assignment of new goods can completely eliminate the inefficiencies caused by asymmetric information. Endogeneity of new car consumers allocates the highest-value cars to the right consumers, and the existence of multiple secondary markets allocates used units to the right used-car consumers.

The basic intuition for this result emerges most starkly in an asymmetric-information environment we call the *simple depreciation model*. We assume that all *new* units of the durable good have the same, known quality. However, goods depreciate stochastically: if the good is of quality q_n at date t , there is a positive probability that the good depreciates to quality q_{n+1} by date $t + 1$. This implies that a good produced some time in the past may be of several distinct qualities, numbered 0 (highest) through N (lowest). In a steady state, there is a distribution of qualities, with newly produced goods replacing units that have depreciated. Smoothly functioning secondary markets play two key roles in this environment: (1) they must allow high-valuation consumers whose units have depreciated to replace them with new goods, transferring the used good to lower-valuation consumers; (2) they must allocate used units efficiently among low-valuation consumers. If the quality of the good is publicly observable, trade in secondary markets achieves these two goals and leads to the efficient allocation. If, on the other hand, a prospective buyer cannot observe the quality of the good prior to purchase, adverse selection could in principle preclude efficiency, because stochastic depreciation can potentially generate quality uncertainty in secondary markets.

However, we show that, even in the presence of these information asymmetries, there is a competitive equilibrium in which a menu of rental contracts induces precisely the same allocation that would prevail if quality were observable; furthermore, per-period rental rates are exactly the same as under observable quality. In this equilibrium, all that consumers need to observe is the *vintage* of a unit, i.e. the number of distinct consumers who have used it in the past. Thus, in this model, as long as this limited amount of information about the trading history of a good is available, asymmetric information about quality is completely harmless.^{2,3}

²Note that this information is commonly available to used car consumers in the US through Carfax.

³Consider the following alternative assumptions: cars follow the depreciation process specified in the text, but a used unit can only be traded once, t periods after it is produced. This yields a model that is very similar to the

The equilibrium may be briefly described as follows. Every vintage (including vintage 0, which corresponds to new goods) is traded at a different price. In each period, a high-valuation consumer rents a new (vintage-0) unit, and stops renting that unit when it depreciates for the first time. At that point the vintage of the unit increases from zero to one. Consumers with somewhat lower valuation rent a vintage-1 unit and keep renting it until the unit depreciates again, at which point its vintage increases to 2, and is passed on to consumers with yet lower valuation. The process continues until the good “falls apart and dies”. Consumers who stop renting a unit of vintage n obtain another unit of the same vintage. Therefore, in equilibrium, the vintage of a unit is a perfect signal of its quality: a good that has had n previous consumers is of quality q_n . Note the contrast with the idea discussed by Janssen and Roy. In their model, the time that the good is kept by its original owner serves as a signal of quality. However, this is a costly signal; the first-best allocation involves immediate trade but, in the case of asymmetric information, the owner must keep the good sufficiently long in order to prove that the good is high quality. In our model, the signal of quality is the number of previous consumers of the good, and this signal is not distortionary: regardless of whether quality is observable or unobservable, the good changes hands precisely when it depreciates. The time the good is held by a consumer is random in our model; it depends on how quickly the good depreciates.

Rental contracts provide the right incentives to consumers; in particular, consumers have no reason to keep renting a unit once that unit has depreciated: better units are available at the same rental price. In contrast, a system of resale markets generates the wrong incentives: some consumers find it profitable to keep a unit after it depreciates, so that its vintage is no longer a perfect signal of its quality. The rough intuition is that, in a system of resale markets, consumers suffer a capital loss when the good changes hands (and hence its vintage increases). This loss is instead borne by the producer of the good when the good is rented. Thus, with rental contracts, the consumer has no incentive to retain depreciated units.

Given that the desirable efficiency properties of rental contracts depend on transferring the capital loss generated by a change in vintage from consumers to producers, it is important to determine which contracts would be chosen by the producers of these goods. We show that there is a competitive equilibrium in which the efficient amount of output is produced and the efficient menu of rental contracts is chosen by each firm. Thus, firms are indeed willing to bear the capital losses associated with changing vintages. It should however be emphasized that this result does require that consumers be able to observe the contracts offered by individual firms. This is a strong, albeit standard, requirement which is consistent with the spirit of competitive equilibrium analysis.

Vintage is a particularly effective signal of quality in the model just described because there is no uncertainty about the quality of the new good (it is known to be q_0), and depreciation occurs only one step at a time (e.g. from q_n to q_{n+1} , but not to q_{n+k} , for $k > 1$). In order to examine the robustness of the intuition just described, we analyze a more general model that allows both for

ones studied by Hendel and Lizzeri and Johnson and Waldman: from the point of view of prospective consumers, the quality of a unit offered on the used-car market is a random variable with a distribution determined by the depreciation rates. Under this interpretation, the only difference between our simple depreciation model and the environments studied in the papers cited above is the number of active secondary markets for a given unit.

initial uncertainty about the quality of newly produced goods, and for more general depreciation processes. Again, efficiency involves assortative matching of qualities to consumers. The quality of a unit is only observed by a consumer who has tried it in the past; thus, there is asymmetric information in secondary markets. An important difference relative to the simple depreciation model is that, in this more general model, in order to find a good of the right quality, consumers must *experiment* with different units.

We show that it is still possible to employ the vintage of the good to signal its quality and to allocate it efficiently; however, in contrast with the simple depreciation model, sorting now occurs with some delay. Each unit changes hands until it finds its right match, and the unit increases in vintage each time it changes hands. In turn, a consumer continues experimenting with a particular vintage until she gets the top quality of that vintage, and then keeps the unit until it depreciates. If the car depreciated only one step, the next consumer to obtain that unit will hold on to it until the next time it depreciates. If, on the other hand, the car depreciated $k > 1$ steps, the next consumer to obtain that unit will immediately return it to try another car of the same vintage, while the car is traded at least k times increasing in vintage with every trade (exactly k times if the car does not depreciate while this process continues). Thus, as in the previous model, units “trickle down” from consumers with high valuation to consumers with lower valuations.

We also show that, as under simple depreciation, a menu of rental contracts induces consumers to follow an experimentation policy leading to the efficient matching of goods to consumers.⁴ While full efficiency is achieved in the simple depreciation case, sorting occurs with delay in the more general model, because experimentation is required. However, the utility cost of this delay becomes negligible if retrading can happen quickly. Furthermore, the rental prices that induce consumers to follow this experimentation policy converge to the observable-quality rental prices when the time between transaction converges to zero. We show that, as a consequence, producers’ incentives are approximately in line with efficiency.

Our analysis of the general depreciation model highlights a new role for secondary markets as vehicles for facilitating experimentation. Since experimentation is only necessary when quality is not publicly observable, secondary markets are more active when quality is unobservable. To elaborate, when quality is observable, trading in secondary markets takes place only when units depreciate, and one transaction per depreciation event is enough to achieve sorting regardless of whether depreciation occurs in one step or in multiple steps. In contrast, when quality is not observable, multiple transactions per depreciation event are necessary to land the unit in the right hands. This implies that the absence of impediments to frequent retrading such as transaction costs can be more important in a world with private information about quality.

An implication of our model is that, if transactions involve rental contracts, observability of the vintage of the good is welfare-improving. The role of the observability of trading histories in the case of resale markets is less obvious. Indeed, House and Leahy (2001) provide a model in which observability of trade histories can create additional distortions. In their setup, there are

⁴As in the simple depreciation model, resale markets generate the wrong incentives. Indeed, there is an additional drawback of resale markets, because consumers incur a capital loss every time they experiment.

two car qualities; consumers are homogeneous in their valuations for quality, but the match value of a consumer/car pair deteriorates stochastically over time. Efficiency requires that the good change hands every period; however, welfare is unaffected by the identity of used-good buyers, because these all have the same valuation for quality, and the match value is idiosyncratic. Thus, conditional on the owner selling the good to some other agent, information about the quality of the good cannot improve welfare; however, adverse selection does create a distortion, since owners of good-quality units may refrain from trading. House and Leahy show that observability of trade histories introduces an additional distortion, because owners of good cars may delay selling them to signal that their car is high quality (as in Janssen and Roy).

In contrast, extensive numerical analysis of a three-quality version of our simple depreciation model indicates that the allocation when the vintage of the good is observable is more efficient than in the case in which it is unobservable. This contrast may be explained by noting that gains from trade stem from different sources in the two models. In particular, in our model, consumers are heterogeneous in their valuation for quality; as a consequence, efficiency does depend upon the identity of the agents buying a used unit: it is efficient to match high-quality used units to quality-sensitive consumers. Since trade history is a signal of the quality of traded units, observability enhances the buyer-car match.⁵

We should note that the goal of the paper is not to construct a realistic model of a durable-goods market. Rather, our model is designed to enable us to evaluate the distortions caused by adverse selection in the absence of any other friction in secondary markets. A more realistic model would acknowledge the importance of frictions such as transaction costs. However, by assuming away additional complications, we are able to isolate the role of informational asymmetries as a barrier to the efficient operation of secondary markets.

2 Preliminaries

2.1 Model

We consider a discrete-time, infinite-horizon economy. Time is measured in some specified *unit* (e.g. days, months, years), and every time *period* lasts for $\Delta \in (0, 1]$ units. There is a unit mass of infinitely lived consumers who differ in their valuation for quality, characterized by a “type” $\theta \in [\underline{\theta}, \bar{\theta}] \subset \mathbf{R}_+$, distributed according to the c.d.f. F ; the latter is assumed to have a strictly positive density. The total mass of cars equals $Y < 1$; at any time, the quality of a car may take up one of finitely many values, denoted $q_0 > q_1 > \dots > q_N \geq 0$.

Consumers discount utility streams at the instantaneous rate ρ ; thus, u utils at time $t > 0$ are worth $e^{-\rho t}u$ utils at time 0. Car qualities and consumer valuations determine instantaneous

⁵Stolyarov (2002) develops a model of trade in secondary markets with transaction costs. He shows that the probability of trade is non-monotonic in the age of the good. Tadelis (1999) develops an adverse-selection model where the name of a firm summarizes its reputation. He shows that there is active trade in names, but there is no equilibrium in which only good types buy good names. In his model, in contrast with ours, shifts in ownership are not observable. Tadelis (2002) studies a related model where moral hazard is also considered.

flow utility from consumption, as follows: if a type- θ consumer drives a quality- q car for τ units of calendar time, she receives utility equal to

$$\int_0^\tau e^{-\rho t} q \theta dt = \frac{1 - e^{-\rho \tau}}{\rho} q \theta.$$

Associating quality levels with *instantaneous* (as opposed to per-period) utility from consumption simplifies the comparison of consumption streams in economies characterized by periods of different length Δ . Finally, utility is quasi-linear in “money”. Specifically, for every Lebesgue-measurable function $q : \mathbb{R}_+ \rightarrow \{q_0, \dots, q_N\}$ and every pair of sequences $\{P_k\}_{k \geq 0}, \{t_k\}_{k \geq 0}$ in \mathbb{R}_+ , the utility of a type- θ consumer who, at each time $t \in \mathbb{R}_+$, drives a car of quality $q(t)$ and effects a (lump-sum) payment of P_k at time t_k for every $k \geq 0$, is

$$\int_0^\infty e^{-\rho t} q(t) \theta dt - \sum_{k \geq 0} e^{-\rho t_k} P_k.$$

Each period consumers receive an endowment e of ‘money’. We assume that e is finite and ‘large’, namely e is large enough that consumers can potentially afford any quality they wish to consume.⁶

In any *period*, a car may depreciate (i.e. its quality may deteriorate), or it may break down, i.e. “die”, in which case it exits the economy and is replaced by a newly-produced car. We assume throughout that the “death” of a car is publicly observable, regardless of whether or not quality is. The probability of depreciation events is assumed to be linear in the length Δ of periods; thus, depreciations are less likely to occur in a short period of time. This linearity assumption is immaterial as far as the results in Section 3 are concerned, and can be substantially relaxed in the setting of Section 4. However, if depreciation probabilities are linear in Δ , the resulting discrete-time depreciation process has a well-defined and natural continuous-time limit, which we briefly describe below.

We analyze two models of the depreciation process. In the **simple depreciation** model, the quality of a newly produced car is known to be q_0 . For $n = 0, \dots, N - 1$, at the end of each period, a car currently of quality q_n depreciates to q_{n+1} with probability $\gamma_n \Delta$; a quality- q_N car does not depreciate, but may die with probability $\gamma_N \Delta$ at the end of each period. It turns out that the analysis is independent of the length Δ of time periods, and hence of the specific functional dependence of depreciation probabilities on Δ .

In the **general depreciation** model, for $n = 0, \dots, N$, a newly produced car has quality q_n with probability $\chi_n \geq 0$, where $\sum_{n=0}^N \chi_n = 1$. Moreover, for $n = 0, \dots, N$ and $m = n + 1, \dots, N + 1$, a car of quality q_n depreciates to q_m (if $n < m \leq N$) or dies (if $m = N + 1$) with probability $\gamma_{n,m} \Delta$ in every time period. The simple depreciation case corresponds to $\chi_0 = 1$ and $\gamma_{n,m} = 0$ for $m > n + 1$. Thus, there are two generalization relative to the simple depreciation model: (1) initial quality is uncertain; (2) depreciation can occur in more than one step.

⁶The assumption that the endowment is large allows us to rule out equilibria with price ‘bubbles’, where resale prices of a good escalate because consumers expect them to rise in the future. The assumption of a large endowment is made to avoid keeping track of the wealth of each consumer. Such a problem would complicate the analysis, and seems tangential to the issue studied here.

It can be shown that, as $\Delta \rightarrow 0$, the general depreciation process has a well-defined continuous-time limit, which may be (somewhat loosely) described as follows: if, at time t , car quality equals q_n , then (i) the time until the next depreciation event is exponentially distributed, with parameter $\sum_{m=n+1}^{N+1} \gamma_{n,m}$; and (ii) conditional upon a depreciation event, the quality of the car becomes q_m with probability $\left(\sum_{m=n+1}^{N+1} \gamma_{n,m}\right)^{-1} \gamma_{n,m}$.

To avoid redundancies, we assume that the general depreciation process generates a positive mass of cars of each quality level. Formally, for every $n = 0, \dots, N$, there is at least one sequence $m_0 < \dots < m_M = n$, with $M \geq 0$, such that $\chi_{m_0} > 0$ and $\gamma_{m_\ell, m_{\ell+1}} > 0$ for all $\ell = 0, \dots, M-1$. To clarify, for $n = 0$, this requires that newly produced cars attain the highest quality level q_0 with positive probability; for $n = 1$, it requires that either newly produced cars attain quality level q_1 with positive probability, or that they attain quality level q_0 with positive probability, and that quality- q_0 cars depreciate to q_1 with positive probability; and so on.

2.2 Efficiency

We now define our reference notion of efficiency. At each time $t \geq 0$, positive assortative matching of consumer types to cars must obtain; thus, we need to determine cutoff types $\theta_0^*, \dots, \theta_N^* \in [\underline{\theta}, \bar{\theta}]$ such that types $\theta \in [\theta_0^*, \bar{\theta}]$ hold a quality- q_0 car, types $\theta_1 \in [\theta_1^*, \theta_0^*)$ hold a quality- q_1 car, and so on; types $\theta < \theta_N^*$ will not hold any car (recall that the mass Y of cars is less than 1, the mass of consumers).

In order to determine these cutoff types, we must first derive the steady-state masses of cars of each quality, as determined by the depreciation process. Let v_n^* denote the steady-state mass of cars of quality q_n . We consider the general specification of the depreciation process, as it entails only a slight penalty in terms of analytical complexity. Recall that Δ denotes the length of a period in terms of the chosen units of calendar time.

It is convenient to introduce the following notation. First, for all $n = 0, \dots, N$ and $m = n+1, \dots, N+1$, let $G_n = \sum_{\ell=n+1}^{N+1} \gamma_{n,\ell}$, so $G_n \Delta$ is the probability that a quality- q_n car depreciates at all (or dies) in a period; we assume that $G_n \leq 1$ for all $n = 0, \dots, N$. Next, denote by $\gamma_{n,n}(\Delta)$ the probability that a car of quality q_n does *not* depreciate in a single period: that is, $\gamma_{n,n}(\Delta) = 1 - G_n \Delta$.

For $n = 0, \dots, N$, the steady-state mass v_n^* must satisfy the following system of equations:

$$v_0^* = \gamma_{0,0}(\Delta)v_0^* + \chi_0 y^* \quad (1)$$

$$v_n^* = \gamma_{n,n}(\Delta)v_n^* + \chi_n y^* + \sum_{k=0}^{n-1} \gamma_{k,n} \Delta v_k^* \quad \text{for } n = 1, \dots, N \quad (2)$$

$$y^* = \sum_{n=0}^N \gamma_{n,N+1} \Delta v_n^* \quad (3)$$

$$Y = \sum_{n=0}^N v_n^*. \quad (4)$$

That is: Y is the total mass of cars (equation 4) and y^* is the mass of cars that die in each time period, and are replaced by newly produced cars (equation 3). The mass of cars of quality q_0 consists of quality- q_0 cars that have not depreciated in the previous period, and of newly-produced quality- q_0 cars (equation 1). Finally, the mass of quality- q_n cars, for $n > 0$, is given by undepreciated quality- q_n cars, newly-produced quality- q_n cars, and cars previously of higher quality that just depreciated to q_n (equation 2).

Straightforward manipulations⁷ show that the above system of equations admits a unique solution, which is independent of Δ (so our benchmark is unaffected by the duration of a period). Furthermore, a simple induction argument shows that, under the assumption on the depreciation process stated at the end of the preceding section, $v_n^* > 0$ for all $n = 0, \dots, N$.

The *ex-post* efficient steady-state allocation of cars to consumers (“efficient sorting” hereafter) can then be described as follows. First, let $\theta_{-1} := \bar{\theta}$; next, proceeding iteratively for $n = 0, \dots, N$, assuming that θ_{n-1}^* has been defined, choose θ_n^* such that

$$\forall n = 0, \dots, N, \quad F(\theta_{n-1}^*) - F(\theta_n^*) = v_n^*; \quad (5)$$

observe that $\theta_{-1} > \theta_0^* > \dots > \theta_N^*$ by construction; also, $\theta_N^* > \underline{\theta}$, because $Y < 1$.

Thus, for every $n = 0, \dots, N$, the mass of consumers with types $\theta \in [\theta_n^*, \theta_{n-1}^*]$ is equal to the mass of cars of quality q_n . As noted above, we then assign all cars of quality q_n to consumer types $\theta \in [\theta_n^*, \theta_{n-1}^*]$.

We are interested in analyzing the incentives of consumers and producers. For expository reasons, it is convenient to focus on consumers’ incentives first, assuming that production is exogenous, and then extend the analysis to the supply side of the economy. Correspondingly, we distinguish between **consumer equilibrium**, which assumes exogenous production, and **market equilibrium**, which encompasses firms’ optimal choice of output and contracts.

3 Simple Depreciation Model

3.1 Observable-Quality Benchmark and Trickle-Down

We begin by briefly analyzing consumer equilibrium in the simple depreciation model, under the assumption that quality is observable. This will serve as a benchmark, and also illustrate our notation.

Regardless of whether cars are sold or rented, the following strategies constitute a consumer equilibrium. Whenever consumers of type $\theta \in [\theta_0^*, \bar{\theta}]$ do not have a car, they obtain a car of quality q_0 and keep it as long as the car remains of quality q_0 . As soon as the car depreciates to q_1 they get rid of the car and obtain a new car of quality q_0 . Consumers of type $\theta \in [\theta_n^*, \theta_{n-1}^*]$ behave in an

⁷For any $y^* \geq 0$, Eqs. (1) and (2) determine v_0^*, \dots, v_N^* ; furthermore, adding up Eqs. (1) and (2) and solving for y^* shows that Eq. (3) is automatically satisfied. If $y^* = 0$ then $v_n^* = 0$ for all n , and since $\chi_0 > 0$, there exists y^* such that Eq. (4) holds. Substituting for y^* in Eqs (1), it becomes apparent that the solution is independent of Δ .

analogous fashion with cars of quality q_n . Clearly, the resulting equilibrium allocation is ex-post efficient; moreover, this equilibrium allocation is essentially⁸ unique.

Under *rental*, a consumer who rents a car of quality q_n pays a fee $\frac{1-e^{-\rho\Delta}}{\rho}r_n$ at the beginning of the period (hence, if she keeps renting the same quality, she pays this fee at times $0, \Delta, 2\Delta, \dots$). This can be seen as the discounted value of a constant, *instantaneous rental price* r_n to be paid throughout the period. Consequently, the per-period utility for a type- θ consumer who rents a quality- q_n car can be written as $\int_0^\Delta e^{-\rho t}[q_n\theta - r_n] dt = \frac{1-e^{-\rho\Delta}}{\rho}[q_n\theta - r_n]$. The $N + 1$ rental prices that sustain this equilibrium allocation can then be defined exactly in the same way as the sorting prices in a static model:

$$\theta_n^*q_n - r_n^* = \theta_n^*q_{n+1} - r_{n+1}^*, \quad n = 0, 1, \dots, N \quad (6)$$

where, by convention, $r_{N+1}^* = q_{N+1} = 0$. These instantaneous rental prices are defined by the indifference of marginal type θ_n^* between renting a good of quality q_n and renting a good of quality q_{n+1} . Also notice that the prices defined in equation (6) are independent of Δ .

Under *selling*, the prices that sustain this consumer equilibrium are defined by the expected present value of rental prices (where the expectation is necessary since the time until the good depreciates and is sold is stochastic). It is easy to verify that these prices are defined by the conditions

$$p_n^* - \sum_{k=1}^{\infty} (1 - \gamma_n \Delta)^{k-1} \gamma_n \Delta e^{-\rho \Delta k} p_{n+1}^* = \sum_{k=0}^{\infty} (1 - \gamma_n \Delta)^k \gamma_n \Delta \frac{1 - e^{-\rho \Delta k}}{\rho} r_n^*, \quad n = 0, 1, \dots, N \quad (7)$$

where, by convention, $p_{N+1}^* = 0$, and r_n^* is defined in equation (6). The right-hand side of this expression is the expected present value of the rental payment for a unit of a good of quality q_n , given that the consumer will stop renting it once the unit depreciates. In the left-hand side, the expected present value of revenues from resale is subtracted from the price of the quality- q_n good.

We use the term *trickle-down* to denote the process by which these goods are allocated in equilibrium: the good trickles down from the high-valuation consumers to the low-valuation ones as it declines in quality.

The equilibrium just described has the following key feature: the quality of any given unit of the good offered on the market can be exactly inferred from its *vintage*, i.e. the number of times the unit has changed hands. A unit of vintage n is of quality q_n .

Therefore, the equilibrium strategies described above can equivalently be formulated as follows: for each $n = 0, \dots, N$, consumer types $\theta \in [\theta_n^*, \theta_{n-1}^*]$ rent or buy vintage- n cars, and keep the same unit until it depreciates. Notice that, in order to implement these strategies, only the vintage of a unit must be observed, not its quality. Yet, if all consumers follow these strategies, the ex-post efficient allocation will ensue, and qualities will be fully revealed. For this reason, we deem these strategies *revealing*.

⁸If the car she is currently renting does not depreciate, a consumer is indifferent between keeping the same unit and renting another unit of the same quality.

3.2 Failure of Resale Markets under Asymmetric Information

We now turn to the analysis of the simple depreciation model with unobservable quality. Specifically, assume that consumers cannot observe the quality of a specific car without using it. Moreover, a consumer who is using a specific car at time t observes the time- t realization of the depreciation process that determines the quality of the car at time $t + 1$. It is notationally convenient (but without loss of generality) to assume that realizations of the depreciation process occur at the end of each period.

We continue to assume that the trading history of each unit is observable. Thus, the revealing strategies described in the previous section are still well-defined, and it is natural to ask whether they still constitute a consumer equilibrium.

Consider resale markets first. We show that, if there are more than two qualities, under asymmetric information, revealing strategies do not constitute a consumer equilibrium with resale. Moreover, we show that there is no consumer equilibrium with resale that achieves the efficient allocation.

By analogy with the complete-information case, we begin by analyzing trading environments consisting of $N + 1$ resale markets; a vintage- n unit is sold at a price p_n , for $n = 0, \dots, N$.

Theorem 1 (i) *If there are more than two qualities, in a system of resale markets, there is no set of $N + 1$ vintage-dependent prices that supports the revealing strategy profile.*

(ii) *If there are only two qualities, then there exists an ex-post efficient consumer equilibrium.*

Observe that both statements are true for *any* value of Δ .

Proof. (i) Assume (by contradiction) that a vintage- n car is indeed of quality q_n . Denote by $V_n(\theta)$ the ex-ante value of purchasing a vintage- n good, and then behaving as prescribed by the vintage- n revealing strategy: keep the car until it depreciates to q_{n+1} and then buy another vintage- n car. Denote by $W_n(\theta, q_n)$ the value of already owning a car of quality q_n for a consumer who is a vintage- n consumer and who follows the policy just described. We have

$$V_n(\theta) = -p_n + \frac{1 - e^{-\rho\Delta}}{\rho} q_n \theta + e^{-\rho\Delta} [\gamma_n \Delta (V_n(\theta) + p_{n+1}) + (1 - \gamma_n \Delta) W_n(\theta, q_n)]$$

$$W_n(\theta, q_n) = V_n(\theta) + p_n \tag{8}$$

Thus,

$$V_n(\theta) = \frac{\frac{1 - e^{-\rho\Delta}}{\rho} q_n \theta - p_n (1 - e^{-\rho\Delta} (1 - \gamma_n \Delta)) + e^{-\rho\Delta} \gamma_n \Delta p_{n+1}}{1 - e^{-\rho\Delta}} \tag{9}$$

In order to achieve the efficient allocation, the following condition inducing efficient ex-ante sorting must be satisfied:

$$V_N(\theta_N^*) = 0, \text{ and } V_n(\theta_n^*) = V_{n+1}(\theta_n^*), \text{ for } n = 0, \dots, N - 1. \tag{10}$$

These $N + 1$ equations determine prices p_0, p_1, \dots, p_N . To obtain the efficient allocation, these prices must induce the right ex-post keeping behavior: no consumer in $[\theta_{n-1}^*, \theta_n^*]$ should want to

keep any good of quality lower than q_n . We will show that, if equations 10 are satisfied, then the consumer of type θ_n^* strictly prefers keeping quality q_{n+1} . Note that the value of selling the good is:

$$\begin{aligned} p_{n+1} + V_n(\theta_n^*) &= p_{n+1} + V_{n+1}(\theta_n^*) = W_{n+1}(\theta_n^*, q_{n+1}) = \\ &= \frac{\frac{1-e^{-\rho\Delta}}{\rho}q_{n+1}\theta + e^{-\rho\Delta}\gamma_{n+1}\Delta p_{n+2}}{1 - e^{-\rho\Delta}(1 - \gamma_{n+1}\Delta)} + \frac{e^{-\rho\Delta}\gamma_{n+1}\Delta V_{n+1}(\theta_n^*)}{1 - e^{-\rho\Delta}(1 - \gamma_{n+1}\Delta)} \end{aligned} \quad (11)$$

where the first equality uses equation (10), the second equality uses equation 8 and the last equality is just a solution for $W_{n+1}(\theta_n^*)$. Consider the alternative strategy for the consumer of holding on to the good until it depreciates to q_{n+2} and then following the policy of buying vintage $n + 1$. Denote by W^K the value of this strategy.

$$W^K(\theta_n^*) = \frac{1 - e^{-\rho\Delta}}{\rho}q_{n+1}\theta + e^{-\rho\Delta}(\gamma_{n+1}\Delta(V_{n+1}(\theta_n^*) + p_{n+1}) + (1 - \gamma_{n+1}\Delta)W^K(\theta_n^*))$$

Thus,

$$W^K(\theta_n^*) = \frac{\frac{1-e^{-\rho\Delta}}{\rho}q_{n+1}\theta + e^{-\rho\Delta}\gamma_{n+1}\Delta p_{n+1}}{1 - e^{-\rho\Delta}(1 - \gamma_{n+1}\Delta)} + \frac{e^{-\rho\Delta}\gamma_{n+1}\Delta V_{n+1}(\theta_n^*)}{1 - e^{-\rho\Delta}(1 - \gamma_{n+1}\Delta)}.$$

Subtract the right-hand side of this equation from the expression for $W_{n+1}(\theta_n^*, q_{n+1})$ in equation 11 to obtain

$$W_{n+1}(\theta_n^*, q_{n+1}) - W^K(\theta_n^*) = - \left(\frac{e^{-\rho\Delta}\gamma_{n+1}\Delta}{1 - e^{-\rho\Delta}(1 - \gamma_{n+1}\Delta)} \right) (p_{n+1} - p_{n+2}) \quad (12)$$

If we can show that $p_n > p_{n+1}$ for every n , we can conclude that keeping quality q_{n+1} is better than selling it for type θ_n^* (and, given the continuity of payoffs with respect to θ , for all types sufficiently close to θ_n^*). We now claim that equation (10) implies that $p_n > p_{n+1}$, $n = 0, 1, \dots, N-1$. We prove this by induction: First, note that for $n = N-1$, we can rewrite equation 10 as:

$$\frac{1 - e^{-\rho\Delta}}{\rho}\theta_{N-1}^*(q_{N-1} - q_N) + e^{-\rho\Delta}\gamma_N\Delta p_N = (p_{N-1} - p_N)(1 - e^{-\rho\Delta}(1 - \gamma_{N-1}\Delta))$$

since $p_{N+1} = 0$ because $q_{N+1} = 0$. Thus, $p_{N-1} > p_N$ and the claim is true for $n = N-1$. Assume the claim is true for $n+1$ (i.e., $p_{n+1} > p_{n+2}$). We can rewrite equation (10) as:

$$\frac{1 - e^{-\rho\Delta}}{\rho}\theta(q_n - q_{n+1}) + e^{-\rho\Delta}\gamma_{n+1}(p_{n+1} - p_{n+2}) = (p_n - p_{n+1})(1 - e^{-\rho\Delta}(1 - \gamma_n\Delta))$$

so that $p_n > p_{n+1}$, which proves the inductive step.

(ii) If $N = 1$ (i.e., there are only two qualities), existence of an efficient equilibrium is established if we can show that the prices defined by equations (10) induce the correct keeping behavior. For consumers of vintage 1, this is trivial. For consumers of vintage 0, the right keeping behavior

requires that all type $\theta \geq \theta_0^*$ be willing to sell the car the moment it depreciates to q_1 . This requires that for all such θ 's,

$$p_1 + V_0(\theta) \geq \frac{1 - e^{-\rho\Delta}}{\rho} \theta q_1 + e^{-\rho\Delta} (1 - \gamma_1 \Delta) p_1 + e^{-\rho\Delta} V_0(\theta). \quad (13)$$

Rewrite equation (13) as

$$V_0(\theta) \geq \frac{\frac{1 - e^{-\rho\Delta}}{\rho} \theta q_1 - (1 - e^{-\rho\Delta} (1 - \gamma_1 \Delta)) p_1}{1 - e^{-\rho\Delta}}$$

Note that the right hand side of this equation is equal to $V_1(\theta)$ (since $p_2 = 0$). Thus, the ex-post keeping condition is equivalent to the ex-ante condition that $V_0(\theta) \geq V_1(\theta)$, implying that all types who buy vintage 0 will never want to keep quality q_1 . ■

The intuition for this result is the following. For revealing strategies to be an equilibrium with resale markets, type θ_n^* must be just willing to be a vintage- n consumer ex-ante, i.e., $V_n(\theta_n^*) = V_{n+1}(\theta_n^*)$. Furthermore, he should be willing to sell the good as soon as it becomes of quality q_{n+1} . These two conditions together imply that he should be willing to sell a vintage- n good that just depreciated to quality q_{n+1} and then buy a vintage- $(n+1)$ good whose quality (in equilibrium) is q_{n+1} . However, this cannot be optimal. The reason is that by keeping the vintage n car that is of quality q_{n+1} until it depreciates again, the consumer enjoys a quality q_{n+1} good which he can then sell for p_{n+1} . In contrast, if he buys a vintage $n+1$ car, he would still enjoy a quality q_{n+1} unit, but would only be able to sell it for p_{n+2} . Thus, it is less costly to consume q_{n+1} if one happens to have a vintage- n good than it is to consume the same quality with a vintage $n+1$ good: the resale price is higher in the first case. Note that this logic fails for $n = N - 1$ because when a good of quality $N - 1$ depreciates twice it dies, and we have assumed that this event is observable. This implies that efficiency is possible when $N = 1$.

Remark 1 *Inefficiency does not vanish in the limit as $\Delta \rightarrow 0$.*

Proof. In the proof of part (i) of Theorem 1 we showed that in a system of resale markets buyers have an incentive to keep the wrong qualities. Thus, it is enough to show that these incentives do not disappear in the limit as $\Delta \rightarrow 0$. To do this, recall that the right-hand side of equation (12) expresses the (negative) payoff from keeping quality q_{n+1} for the marginal type who should instead be keeping only q_n . We now show that this expression is bounded away from zero. Observe that

$$\lim_{\Delta \rightarrow 0} \left(\frac{e^{-\rho\Delta} \gamma_{n+1} \Delta}{1 - e^{-\rho\Delta} (1 - \gamma_{n+1} \Delta)} \right) = \frac{\gamma_{n+1}}{\gamma_{n+1} + \rho} > 0.$$

Now suppose by contradiction that the limit allocation is efficient. It is easy to show that then prices must converge to the observable-quality prices implicitly described in equation (7); consequently, $(p_{n+1} - p_{n+2})$ must converge to a finite positive quantity. But this implies that the payoff to type

θ_n^* from keeping the lower quality q_{n+1} is bounded away from zero as $\Delta \rightarrow 0$. Furthermore, since payoffs are continuous in θ , the mass of types who benefit from keeping the lower quality is bounded away from zero as $\Delta \rightarrow 0$. This contradicts the assumption that the limit allocation is efficient, and proves the claim. ■

We now show that the only candidate for an efficient consumer equilibrium under resale is the revealing strategy profile. This result, jointly with Theorem 1, implies that *there is no efficient consumer equilibrium under stochastic depreciation and resale*. By Remark 1, this is true even in the limit as $\Delta \rightarrow 0$.

We consider $K + 1$ markets, numbered $0, \dots, K \leq \infty$. New cars are traded in market 0. For all $k = 1, \dots, K$, equilibrium determines the price p_k that clears market k , as well as the (average) quality of cars traded in that market. We do not distinguish between markets where the same qualities are traded and the same prices prevail.

Proposition 1 *If there exists a system of $K + 1$ resale markets and an equilibrium strategy profile that yields the efficient allocation, then $K = N$ and the consumer equilibrium consists of the revealing strategy profile.*

Proof. Note first that, if in some market k , positive masses of goods of two or more different qualities are sold, there is a positive mass of consumers who obtain cars of the wrong quality. But this is ruled out by efficiency; hence, a single quality is traded in every market. Now consider the markets where quality q_N is traded; the price in all such markets must be equal to $\frac{1-e^{-\rho\Delta}}{\rho}\theta_N q_N$.⁹ Hence, in effect, there is a unique market for quality- q_N cars. Now suppose that qualities q_m , for $m = n, \dots, N$, are traded each in a unique market. Then, all cars of quality q_{n-1} must have the same resale value, and by a similar argument to the one for quality- q_N cars, it follows that the market for quality- q_{n-1} cars is also unique. Hence, there are exactly $N + 1$ markets, one for each quality. Furthermore, efficiency also implies that consumers of type $\theta \in [\theta_n^*, \theta_{n-1}^*]$ are only active in market n . ■

3.3 Rental and Efficiency under Asymmetric Information

We now show that the set of instantaneous rental prices $\{r_n^*\}_{n=0}^N$ that prevail under observable quality (cf. equation (6)) lead to the efficient allocation even under asymmetric information. Specifically, we consider rental contracts that specify an instantaneous rental price r_n the consumer pays for renting vintage n . Moreover, the consumer can keep renting the same unit as long as she wishes, and stop paying the rental fee the moment she wishes to return the unit (without any cancellation fees to the manufacturer). Consumers who have rented the specific unit in the past are not allowed to rent the same unit. This prevents the consumers from strategically returning cars to lower their rental payments.

Consider the ex-ante value $V_n(\theta)$ of pursuing the ex-post efficient policy of renting vintage n and keeping until (and only until) the good depreciates to q_{n+1} . It is easy to see that

⁹This requires that endowments be finite; otherwise there could be sequences of increasing prices sustained by expectations of ever-increasing resale values.

$$V_n(\theta) = \sum_{k=0}^{\infty} e^{-\rho k \Delta} \frac{1 - e^{-\rho \Delta}}{\rho} [q_n \theta - r_n^*] = \frac{q_n \theta - r_n^*}{\rho}. \quad (14)$$

Notice that $V_n(\theta)$ does not depend directly¹⁰ on the depreciation rate γ_n : the consumer immediately replaces depreciated cars, and therefore effectively enjoys a sequence of quality- q_n cars.

We now verify that, given the observable-quality rental prices (equation 6), consumers optimally follow efficient policies. This requires: (i) the ex-ante sorting condition that types $\theta \in [\theta_n^*, \theta_{n-1}^*]$ be willing to rent a vintage n car; (ii) the “ex-post keeping” condition that consumer types $\theta \in [\theta_n^*, \theta_{n-1}^*]$ not be willing to keep any quality below q_n .

Condition (i) is satisfied if for every n , $V_n(\theta_n^*) = V_{n+1}(\theta_n^*)$. Given equation (14), this condition is equivalent to $\theta_n^* q_n - r_n^* = \theta_n^* q_{n+1} - r_{n+1}^*$ which is clearly satisfied given that $\{r_n^*\}_{n=0}^N$ are defined by equation (6). That is, observable-quality rental prices are such that the self-selection conditions are satisfied even if quality is not observable.

In contrast to the case of resale, ex-post keeping (condition (ii)) is automatically satisfied under rental. No consumer has an incentive to keep any quality below the highest quality of any vintage. Suppose that a consumer of type θ is renting vintage n and he is currently consuming a quality- m good. Consuming this good for one period and then resuming tomorrow the efficient policy of only keeping quality q_n yields a payoff of $\frac{1 - e^{-\rho \Delta}}{\rho} [q_m \theta - r_n^*] + e^{-\rho \Delta} V_n(\theta)$. If instead the consumer returns quality q_m immediately to start the efficient policy today, he obtains a payoff of $V_n(\theta) = \frac{1 - e^{-\rho \Delta}}{\rho} [\theta q_n - r_n^*] + e^{-\rho \Delta} V_n(\theta)$. We can therefore conclude that, under rental, incentives to keep are always guaranteed to hold. Thus, the instantaneous rental prices $\{r_n^*\}_{n=0}^N$ guarantee that both ex-ante sorting and ex-post keeping incentives are consistent with the ex-post efficient allocation that obtains under observable quality. As above, note that the conclusion holds for *any* positive value of Δ .

Theorem 2 *If the goods are rented, there is a consumer equilibrium under asymmetric information that features the same allocation, strategies, and instantaneous rental prices as under observable quality.*

3.4 Supply Side

We now consider the incentives of car producers. We will show that, if producers are competitive, there is a market equilibrium where firms maximize profits by renting the goods at the observable-quality rental prices. Thus, efficient sorting is achieved in equilibrium. Furthermore, the efficient amount of output is supplied. Thus, the equilibrium we characterize leads to the first-best allocation in spite of the presence of asymmetric information in secondary markets.

Assume that there is a unit measure of producers, each of whom has an opportunity to produce a single unit of the good at a cost c in every period.¹¹ Firms have the same instantaneous discount factor ρ as consumers.

¹⁰Recall that rental prices are determined by cutoff types, which in turn depend upon the depreciation rates.

¹¹The assumption that a producer can produce only one unit every period ensures that each producer is ‘small’ and simplifies the analysis considerably.

Denote by $R(y)$ the per-unit expected present value of revenue as a function of the total industry output y . If this output is offered according to the efficient rental contracts, the value of this revenue can be obtained recursively as follows. Define $R_n(y)$ as the expected present value of revenue conditional on quality being q_n in the current period:

$$R_n(y) = \frac{1 - e^{-\rho\Delta}}{\rho} r_n^*(y) + e^{-\rho\Delta}((1 - \gamma_n\Delta)R_n(y) + \gamma_n\Delta R_{n+1}(y)), \quad n = 0, \dots, N$$

where $R_{N+1} = 0$ by convention. Thus,

$$R_N(y) = \frac{\frac{1 - e^{-\rho\Delta}}{\rho} r_N^*(y)}{1 - e^{-\rho\Delta}(1 - \gamma_N\Delta)}$$

and it can be verified that

$$R_n(y) = \sum_{k=0}^{N-n} \frac{\frac{1 - e^{-\rho\Delta}}{\rho} r_{n+k}(y) e^{-\rho k\Delta} \prod_{j=0}^{k-1} \gamma_{n+j}\Delta}{\prod_{j=0}^k (1 - e^{-\rho\Delta}(1 - \gamma_{n+j}\Delta))}$$

Because every new unit is born with quality q_0 , it must be the case that $R(y) = R_0(y)$. It is easy to see that $R(y)$ is decreasing and continuous in y , because so is $r_n(y)$.¹² We assume that $R(1) < c < R(0)$.¹³

Let y^* be the output defined by the solution of $R(y^*) = c$. This is the output that leads to zero profits for all firms in the industry. Since $R(y)$ is decreasing and continuous, and $R(1) < c < R(0)$, y^* exists and is unique.

We now construct a market equilibrium with the following features: a fraction y^* of firms produce each period. Active firms offer rental contracts at instantaneous prices $\{r_n^*(y^*)\}_{n=0}^N$. The remaining $1 - y^*$ firms are inactive. Thus, the equilibrium under asymmetric information is identical to the equilibrium that would obtain under observable quality.

To formalize the market equilibrium concept, we need to describe the class of contracts that firms can offer. Each firm can offer a sequence of mechanisms, one for every consumer who enters into a relationship with the firm during the lifetime of the car (recall that each firm produces a single car). For instance, the rental contracts described in the previous subsection can be viewed as a sequence of $N + 1$ mechanisms, each corresponding to a vintage; the consumer is induced to return the car as soon as it depreciates, and a returned car previously offered under the “vintage- n ” mechanism is offered under the “vintage- $(n + 1)$ ” mechanism in the subsequent time period. However, we can allow for more general mechanisms. Informally, each mechanism features the following ingredients (see the Appendix for a formal description and analysis):

(i) the deviating firm partially or fully reveals information it has gathered concerning the car’s previous quality history;

¹²Recall that each r_n is determined by the indifference condition (6) involving the cutoff type θ_n ; in turn, the latter is determined by equations (2) and (5), and is easily seen to be decreasing and continuous in y .

¹³Assuming $R(0) > c$ implies that some production is viable. The assumption that $R(1) < c$ guarantees that a unit mass of firms is sufficient to exhaust industry profits. We also want to avoid dealing with the case where costs are so low that, under observable quality some qualities would be available at zero price. In this case, all qualities below some level would not be purchased by anybody.

(ii) upon entering the mechanism, and prior to receiving the car, the consumer pays a price r_0 and sends a message m_0 ;

(iii) at the end of each period k , the consumer sends a message m_k and pays a price r_k , which may depend on all messages sent up to and including time k (where time is indexed relative to the inception of the relationship between the consumer and the firm);

(iv) finally, the message m_k indicates (possibly among other things) whether or not the consumer desires to continue the relationship with the firm; similarly, after receiving the message m_k , the firm can indicate that it intends to terminate the mechanism. In the latter case, an additional terminal transfer \bar{r}_k will be effected.

The role of messages is twofold: first, they allow the firm to design type-dependent payments and consumption histories; second, they allow the firm to extract information about qualities. Of course, the rental contracts in the previous subsection do not use any of this additional structure: they are very simple contracts.

Our assumptions allow for considerable flexibility in designing mechanisms. The following examples illustrate possible specifications of the mechanism parameters.

(i) a firm may choose to fully reveal the quality history of the car, or nothing at all (so the consumer does not learn whether she is receiving a new or used car), or perhaps only reveal whether the current quality is above some threshold;

(ii) the consumer may be asked to report her type θ upon entering the mechanism, via the message m_0 ; on the other hand, no initial report may be required;

(iii) at the end of each period, the consumer may be asked to report whether a depreciation has occurred, and her period payment r_k might reflect the current quality of the car; on the other hand, there may be no communication and/or transfers until the consumer returns the car;

(iv) the mechanism may last until the car dies (as in a sales contract), or it may last for a pre-specified number of periods, or until the car has depreciated to some quality level.

In order to focus on the distortions caused by adverse selection, we abstract from issues related to lack of commitment; that is, we assume that each firm is bound to the menu of mechanisms it announces. We also emphasize that we make the strong informational assumption that consumers observe the entire contract terms offered by a firm. Note however that, it would be easy to enrich the model to show that, given that other firms publicize their contract terms, it is optimal for a firm to also do so, because hiding prior contract terms would be construed as a bad signal about the history of the unit that the firm rents.

Finally, we assume that firms expect consumers to best-respond to the mechanisms they offer, both on and off the equilibrium path.

Theorem 3 *The following constitutes a market equilibrium for any $\Delta > 0$:*

(i) *firms produce the first-best output y^* , and offer $N + 1$ vintage-dependent rental contracts at the instantaneous rental prices r_0, \dots, r_N determined by equation (6);*

(ii) *for every $n = 0, \dots, N$, consumer types $\theta \in [\theta_n^*, \theta_{n-1}^*]$ rent vintage- n cars and only keep cars of quality q_n , where the cutoffs $\theta_0^*, \dots, \theta_N^*$ are determined by equation (5).*

The proof of this result is in the Appendix; we now provide a brief sketch of the argument. Individual rationality implies that a consumer of type $\theta \in [\theta_n, \theta_{n-1}]$ will agree to transact with a

deviating firm under some mechanism M only if the value of participating in mechanism M , then reverting to renting vintage- n cars is at least as large as her payoff if she rents vintage- n cars forever. This provides an upper bound on the revenues that a deviating firm may obtain from mechanism M by transacting with type θ . We employ this bound to show that any deviation is dominated by a menu of one-period rental contracts, each targeted to a specific consumer type. The deviator fully reveals the quality history of the car, and chooses each rental price so as to leave the target type indifferent between (i) renting the deviator’s car in the current period, then continuing with her designated putative equilibrium rental contract, and (ii) employing her designated putative equilibrium contract in the current as well as in all subsequent periods.

We then show that consumer indifference implies that the rental prices charged by the deviator for each quality cannot exceed the rental prices for vintages defined by equation (6); hence, there can be no profitable deviation. Furthermore, since industry output equals y^* , which is determined by the zero-profit condition, no new entry can occur.

3.5 Discussion

We now provide some discussion of the results and of the assumptions of the stochastic depreciation model.

3.5.1 Contract characteristics

For rental contracts to implement the efficient allocation, it is important that contracts be of *indeterminate duration*; a rental contract that required that the good be returned after some fixed number of periods would not lead to the efficient allocation. To see this, recall that the key feature of the mechanism is that the vintage of the good is a perfect signal of quality. Thus, the trading behavior of consumers must signal the depreciation history. If a consumer were required to return the good after a fixed number of periods, then the fact that the good is returned would not convey any information regarding its quality—the good may or may not have depreciated by the time the good is returned.

A menu of *leasing contracts* would not lead to an efficient allocation either. A leasing contract consists of two prices: a rental price that the consumer pays for a pre-specified length of time, and a purchase price that the consumer pays at the end of the period if he chooses to purchase the unit. Thus, a leasing contract suffers from a combination of the shortcomings of resale markets and of the type of fixed-duration rental contracts just discussed. To see this, we need to consider two cases: (1) either the consumer returns the good at the termination of the lease for all depreciation histories, or, (2) for some depreciation histories, the consumer purchases the good at the termination of the lease. In case (1), it is not possible to infer the quality of the good from the behavior of the consumer. Therefore, at some point in the future, potential consumers of the unit are uncertain about the quality of the good. This means that, with positive probability, the good is allocated inefficiently for at least one period. In case (2), efficiency requires that the consumer who purchases the good sell it once it depreciates. However, the consumer now faces incentives that are similar to those she faces in a system of resale markets, and we have seen that efficiency cannot be obtained in that case either.

The inefficiency of leasing contracts is in contrast with the result obtained by Johnson and Waldman (2001), who show that leasing contracts can lead to efficiency. The difference is due to the fact that in Johnson and Waldman (2001) there are only two consumer types and, as in Hendel and Lizzeri (2002), the timing of depreciation is deterministic.

Finally, we observe that, *if firms could observe the quality of a used car*, and they could commit to credibly revealing it, then efficiency could be achieved via a sequence of resale transactions. Observe that, if quality is observable, by equation (7), consumers are indifferent between (1) renting the same vintage- n car at rental price r_n^* given by equation (6) until it depreciates, and (2) buying a vintage- n car at price p_n^* and reselling it to other consumers at price p_{n+1}^* as soon as it depreciates. Now suppose that firms can observe quality; then they can act as intermediaries in secondary markets by buying used goods and certifying their quality. Specifically, types $\theta \in [\theta_n^*, \theta_{n-1}^*]$ buy a vintage- n car at the price p_n^* , keep it until it depreciates, then resell it to a firm for a price of p_{n+1}^* and buy another vintage- n car; the firm then resells the car in the vintage- $(n+1)$ market, at a price of p_{n+1}^* . As long as firms certify that vintage- n cars are of quality q_n , consumer incentives are the same as under rental contracts so that efficient sorting will obtain in a consumer equilibrium. Our proof of Theorem 3 can be adapted to show that there is a market equilibrium in which firms truthfully certify quality and buy and sell cars at the prices p_n^* as indicated above.

Inefficiency of resale markets depends on the presence of asymmetric information. If firms, like consumers in our model, cannot observe the quality of a used car prior to owning or using it, the above transactions cannot occur in equilibrium: the argument is analogous to the proof of Theorem 1.

3.5.2 Strict incentives

In the model of rental contracts of Section 3.3, as long as the unit has not depreciated, consumers are indifferent between continuing to rent the same unit, and renting another unit of the same vintage. However, for the vintage of a good to serve its signaling role, it is important that a consumer choose to continue to rent the same unit as long as it does not depreciate. This may raise some concerns about the robustness of the mechanism. However, we can make a small modification of the contracts to guarantee that consumers have strict incentives to hold on to their units as long as they do not depreciate, while maintaining their incentives to get rid of the good once it depreciates. Assume that, for every vintage n , there are two (instantaneous) rental prices, r_n^0 and r_n^s , where r_n^0 is the rental price paid in the first period of rental of vintage n , and r_n^s is the rental price for subsequent periods. To guarantee strict incentives to keep the good when it does not depreciate, all we need is that $r_n^s < r_n^0$. It turns out that we can choose rental prices that satisfy this constraint without affecting any of the other incentives, i.e., (1) the incentives to choose the right vintage ex-ante, and (2) the incentives to return the good when it does depreciate.¹⁴

¹⁴For (2), we only need to ensure that $r_n^s > r_{n+1}^0$. To guarantee (1) we just need to choose the set of first-period rental prices $\{r_n^0\}$ so as to equalize the ex-ante values of marginal consumers of consuming vintage n and vintage $n+1$, given that they will be paying the slightly smaller prices r_n^s to keep renting the units. In other words, relative to r_n as set by equation 6, we lower r_n^s and raise r_n^0 to keep the values of the marginal types unchanged.

3.5.3 The depreciation process

Observe that Theorems 1 and 2 continue to hold if the special assumption that the rate of depreciation is constant through time does *not* hold. We could allow for a much more general depreciation process where the probability that a car depreciates is a function of the age of the unit. It is easy to see that a menu of rental contracts leads to efficient sorting also in this more complicated environment: the value to a consumer of following the efficient policy is unchanged relative to equation 14. Furthermore, the incentive to return a unit once it depreciates is essentially unchanged. Showing that resale markets fail to implement the efficient allocation is more complex since the value functions are now more complicated, but the logic is the same. Finally, while the structure of the proof of Theorem 3 relies on stationarity, we conjecture that the result would still hold when depreciation is time-dependent.

Note also that Theorem 2 does not rely on the depreciation rate being observable. Consumers just need to know that, by renting vintage n , they can obtain a quality- q_n good. Indeed, none of the calculations concerning consumer incentives in the discussion preceding (and proving) Theorem 2 depend on the probability of depreciation. Thus, the depreciation rate could even be private information to the consumers who have previously consumed the good (or to the firms producing them): Theorem 2 would continue to hold.¹⁵ Of course, observability of depreciation rates would matter for the case of resale markets. We conjecture that private information about the depreciation rates would lead to even larger distortions in this case.

On the other hand, the exact efficiency result of this section does rely in an important way on the fact that depreciation occurs one step at a time, since this implies that, in equilibrium, the quality of the good is known. If at any point in time the quality of the good can decrease by one or more than one step, then it is not possible to obtain first-best efficiency through rental contracts. However, we show in Section 4 that, even if depreciation can occur in more than one step, approximate efficiency obtains if the time between periods is small.

3.5.4 Noise traders

Suppose that there is a fraction of consumers who have to trade for exogenous reasons (e.g., moving to another country). This phenomenon has been explored by Greenwald (1986) in a modification of Akerlof's adverse selection model. In his model, the presence of such 'noise traders' has a positive welfare effect because it increases the volume of trade. Greenwald showed that noise traders generate a multiplier effect because they cause a price increase which induces some non-noise traders to sell generating more beneficial trades. In contrast, in our model, noise traders would have a negative welfare effect because they would make the vintage a less precise signal of quality. A car of vintage n may be returned by a noise trader prior to depreciation, when its quality is still q_n ; this car would now be of vintage $n + 1$, and it would therefore end up in the hands of a consumer with lower valuation who should be consuming quality q_{n+1} instead. Note however, that this distortion may not be too large. First, the good would go to some consumer in the interval $[\theta_{n+1}, \theta_n]$ instead

¹⁵Equations (1)–(4) must clearly be adapted to distinguish between cars characterized by the same quality, but different depreciation rates. However, by returning cars immediately upon depreciation, consumers can secure a constant stream of quality- q_n cars, regardless of depreciation rates. This suffices to extend Theorem 2.

of some consumer in $[\theta_n, \theta_{n-1}]$; these intervals are “close” if there are many qualities. Second, the good is misallocated only until it depreciates, because at that point the vintage- $(n + 1)$ consumer will keep the good, which is now the right match for him. Rental prices would have to be adjusted to reflect these misallocations, but the adjustment is minor as long as the fraction of noise traders is not too large.

3.5.5 Discreteness

We have assumed that there is a finite number of possible qualities. While we have not analyzed a version of the model with a continuum of qualities, we can consider what happens in the present setting when the discrete grid of qualities approaches a continuous interval $[q_N, q_0]$. More precisely, consider the following two distributions of qualities: the first is a distribution $\{q_0, \dots, q_N\}$ with quality q_n having mass λ_n ; the second is a distribution $\{q_{0,0}, q_{0,1}, \dots, q_{0,K}; q_{1,0}, q_{1,1}, \dots, q_{1,K}; \dots; q_{N,0}, q_{N,1}, \dots, q_{N,K}\}$, where the sum of the mass of qualities $q_{n,0}, \dots, q_{n,K}$ is equal to λ_n . Clearly, rental contracts implement the *ex-post* efficient allocation for both quality distributions; the only difference is the number of possible vintages.

In the case of resale markets, equation (12) may suggest that the incentives to keep the wrong-quality good become negligible as the quality grid becomes finer—so that one might conjecture that, if there is a continuum of qualities, there is no such incentive. However, consider the marginal type $\theta_{n,0}$, i.e. the consumer who is *ex-ante* indifferent between the vintages corresponding to qualities $q_{n,0}$ and $q_{n,1}$. While it is true that, for this consumer, the gains from keeping quality $q_{n,1}$ are very small if K is large, the gains from keeping quality $q_{n+1,0} = q_{n+1}$ are clearly of the same order of magnitude as they are in the model in which $K = 1$. Thus, the ineffectiveness of resale markets does not appear to depend on the discreteness of the quality distribution.

3.5.6 Equilibrium Under Selling: observable vs unobservable trading histories

Theorem 1 shows that, with resale markets and three or more qualities, there is no efficient equilibrium. In order to analyze the nature of the distortions generated by asymmetric information, we now fully characterize equilibrium in a setting with three qualities. Furthermore, to clarify the role of observability of trading histories under resale markets, we compare our framework with observable vintages to a scenario in which consumers can only distinguish new and used goods, but do not observe the number of times a good has been traded.

We construct equilibria wherein prices do not depend on calendar time and/or the age of the car. Of course, consumers *do* observe calendar time, and can also be assumed to observe the age of the car. Due to stationarity, given the current quality of the car, neither of these variables influences future realizations of the depreciation process. Thus, if prices are also independent of calendar time and age of the car, these variables cannot influence consumers’ optimal decisions; and, if this is the case, then equilibrium prices will in fact be independent of these variables. In other words, equilibrium behavior and prices can be independent of age and calendar time, even though these variables are observable. There may exist other equilibria that do not exhibit this property.

It is clear from our Theorem 2 that, under rental, observability of trading histories is beneficial. If consumers could not observe the vintage of a car, matching would, by necessity, be much coarser. However, under resale markets, the equilibrium allocation is inefficient, so it is not immediately clear that observability is similarly beneficial. For example, as mentioned in the Introduction, House and Leahy (2001), in a different setup, show that observing trading histories can have a negative welfare impact.

As we have seen in Theorem 1 three is the minimal number of qualities such that asymmetric information introduces distortions in resale markets. Going beyond three qualities is conceptually simple but tedious and adds no new insight.

Observable Vintages In a steady state equilibrium, the set of consumer types is partitioned into four intervals: (1) types in $[\underline{\theta}, \theta_2]$ never buy any car; (2) types in $[\theta_2, \theta_1]$ buy vintage 2; (3) types in $[\theta_1, \theta_{01}]$ buy vintage 1; and types in $[\theta_{01}, \bar{\theta}]$ buy vintage 0. Denote by q_n^e the average quality of cars that were vintage $n-1$ the previous date and just became vintage n ; in other words, q_n^e is the average quality of cars that were just traded. Clearly, $q_0^e = q_0$. The following proposition characterizes equilibrium.

Proposition 2 (i) *There exists a consumer equilibrium under resale with observable vintages. In this equilibrium:*

(ii) $q_2^e = q_2 \leq q_1^e < q_1$.

(iii) *Buyers of vintage-2 cars keep their cars until they die. Buyers of vintage-1 cars keep quality- q_1 cars and sell quality- q_2 cars. Finally, there exists $\theta_0 \in (\theta_{01}, \bar{\theta}]$ such that types $\theta \in [\theta_{01}, \theta_0]$ buy vintage 0, keep qualities q_0 and q_1 , and sell q_2 , whereas types $\theta \in [\theta_0, \bar{\theta}]$ buy vintage 0, keep q_0 only, and sell all other qualities.*

The proof of this proposition can be found in a “Web Appendix” available from the authors’ Web sites. A few observations are in order.

First, only quality- q_2 cars are sold as vintage 2 goods; for this reason, $q_2^e = q_2$. On the other hand, since $q_2 \leq q_1^e < q_1$, a positive mass of cars that are offered on the market as vintage-1 goods must be of quality q_2 . Part (iii) implies that no quality- q_0 car is ever offered for resale.

Note also that, while $\theta_0 > \theta_{01}$, there are parameter values for which $\theta_0 = \bar{\theta}$. That is, it is always the case that some high types buy vintage-0 cars and keep both qualities q_0 and q_1 ; however, it may be the case that *all* high types adopt this policy. In such cases, no cars of quality q_1 are offered for resale, and the equilibrium effectively features two vintages.

Unobservable Vintages Now assume that consumers cannot distinguish goods that have been sold only once from goods that have been sold more than once. In this environment, consumers are partitioned into three intervals (1) types in $[\underline{\theta}, \theta_2]$ who never buy any car; (2) types in $[\theta_2, \theta_u]$ who buy used cars; (3) types in $[\theta_u, \bar{\theta}]$ who buy who buy vintage 0. Denote by q_u the average quality of used cars. The following proposition characterizes equilibrium.

Proposition 3 (i) *There exists a consumer equilibrium under resale with unobservable vintages. In this equilibrium:*

(ii) $q_2 \leq q_u < q_1$.

(iii) *Buyers of used cars keep quality- q_1 cars and sell quality- q_2 cars. There exists $\theta_0 \in (\theta_u, \bar{\theta}]$ such that types $\theta \in [\theta_u, \theta_0]$ buy new cars, keep qualities q_0 and q_1 , and sell q_2 , whereas types $\theta \in [\theta_0, \bar{\theta}]$ buy new cars, only keep q_0 , and sell all other qualities.*

The proof of this proposition is available in the Web Appendix.

Thus, a positive mass of used cars is of quality q_2 . However, no quality- q_0 cars are offered for resale.

The behavior of new good consumers is qualitatively the same as that of vintage-zero consumers in the case of observable quality. The behavior of used good consumers reflects a form of arbitrage: specifically, a consumer who owns a used good of quality q_2 can sell it at a price p^u and buy another used good for the same price. If $q_u > q_2$, which is the case whenever $\theta_0 < \bar{\theta}$, all used goods consumers are better off selling quality- q_2 used goods and only keeping quality- q_1 cars.

Finally, as above, it is always the case that some high types keep both qualities q_0 and q_1 (i.e. $\theta_0 > \theta_u$); furthermore, for certain parameter values, *all* high types adopt this policy (i.e. $\theta_0 = \bar{\theta}$).

Comparison of the two scenarios and the role of observable vintages The differences in equilibrium outcomes between the two scenarios¹⁶ may be understood by focusing on two forces. Suppose first that the resale behavior of new goods consumers is the same in the two scenarios. Then, multiple secondary markets allow better sorting of used units. Loosely speaking, the ability to distinguish between two used-car vintages effectively unbundles goods sold by new goods consumers; within at most two periods, quality- q_1 goods are allocated to vintage-1 consumers, and quality- q_2 goods are allocated to vintage-2 consumers. Thus, higher-quality used cars are assigned to higher-valuation consumers. If instead vintages are unobservable, such unbundling is impossible and all consumers who are not new goods consumers end up consuming the same average quality over their lifetimes.

Second, in the case of unobservable vintages, cars that are sold by new goods consumers are pooled with quality- q_2 cars that are sold by used goods consumers.¹⁷ Such pooling will reduce the resale value of new goods, and this will induce a higher fraction of new goods consumers to keep quality- q_1 cars.

A direct comparison of the equilibrium allocations in the two scenarios cannot be provided because the overall equilibrium cannot be solved for in closed form. We therefore proceeded numerically. In our extensive numerical analysis, *we always found social surplus to be higher when vintages are observable.*

We now briefly describe the computations we carried out. The model parameters are the quality levels q_0, q_1, q_2 , per-period depreciation probabilities $\gamma_0, \gamma_1, \gamma_2$, the instantaneous discount factor ρ , the total mass of cars Y , and the distribution of consumer types F . As noted above, the length Δ of each period is immaterial to the analysis of the simple depreciation model, so we set $\Delta = 1$ for

¹⁶This discussion assumes that $\theta_0 \in (\theta_{01}, \bar{\theta})$ in the observable-vintages case, and $\theta_0 \in (\theta_u, \bar{\theta})$ when vintages are unobservable. When the interval $[\theta_0, \bar{\theta}]$ is empty in both scenarios, the allocations are the same.

¹⁷These include consumers who owned quality q_1 units that depreciated, and those who purchased a used good the previous period and found its quality to be q_2 .

simplicity. For reasons of numerical tractability, we assume that types are uniformly distributed on $[0, 1]$.¹⁸

Consider first the observable-vintage model. Once values for the above parameters have been specified, the cutoff types $\theta_{01}, \theta_1, \theta_2$ can be computed in closed form. Moreover, for every choice of $\theta_0 \in (\theta_{01}, \bar{\theta}]$, prices and value functions for each cutoff type can also be explicitly computed. Then, a simple line search algorithm is used to determine the value of θ_0 for which a consumer of type $\theta = \theta_0$ who does not own a car is just indifferent between (a) buying vintage 0 and keeping quality q_0 only, and (b) buying vintage 0 and keeping q_0 and q_1 . Finally, social surplus can be easily computed given the values of the cutoff types thus determined.

Calculations for the unobservable-vintage model are analogous: first, the cutoff types θ_u and θ_2 are computed from parameter values; then, $\theta_0 \in (\theta_u, \bar{\theta}]$ is determined via a line search; finally, social surplus can be calculated.

In addition to comparing social surplus in the two models, we computed two measures of the efficiency gain resulting from vintage observability, as follows. Let S_v and S_{nv} denote per-period surplus under observable and unobservable vintages respectively; also let S_{eff} denote per-period surplus under the efficient allocation. Finally, let S_1 denote per-period surplus in the absence of secondary markets, computed as follows: let $\theta_2 = F^{-1}(1 - Y)$ and

$$S_1 = \frac{1 - e^{-\rho\Delta}}{\rho} \int_{\theta_2}^{\bar{\theta}} E(q) \cdot \theta \, d\theta.$$

This represents the maximum surplus that can be achieved by opening a single market (i.e. the market for new goods): the $1 - Y$ consumers with low valuation for quality are excluded from consumption, but cars are randomly allocated among the Y higher-valuation consumers.

One possible measure of relative efficiency is then $\frac{S_v}{S_{eff}} - \frac{S_{nv}}{S_{eff}}$, the difference between the fraction of the efficient social surplus realized with and without vintage observability. Alternatively, we can measure realized efficiency as a fraction of $S_{eff} - S_1$, the *maximum possible efficiency gain* relative to a single-market environment; the quantities $\frac{S_v - S_1}{S_{eff} - S_1}$ and $\frac{S_{nv} - S_1}{S_{eff} - S_1}$ are the fractions of this gain actually realized under observable and unobservable vintages respectively, so another measure of relative efficiency is their difference $\frac{S_v - S_u}{S_e - S_1}$.

Notice that, for the purposes of comparing social surplus, the minimum and maximum quality levels can be chosen arbitrarily; we set $q_0 = 1$ and $q_2 = 0$. Hence, a full parameterization of both models involves choosing the values of $q_1 \in [q_2, q_0]$, $\gamma_0, \gamma_1, \gamma_2, \rho$ and Y .

In order to explore the parameter space, we first fixed values of ρ and Y ; we then generated values of $q_1 \in (q_2, q_0)$ lying on a grid of pre-specified size M . Finally, we generated depreciation probabilities by specifying minimum and maximum probabilities $\gamma^{\min}, \gamma^{\max}$, and then generating $\gamma_0, \gamma_1, \gamma_2 \in [\gamma^{\min}, \gamma^{\max}]$ on a grid of size M .

In one series of numerical experiments, we chose $e^{-\rho} = 0.9$ and $Y = 0.8$, and specified a grid size of $M = 20$ points; Table 1 summarizes some of our findings. The columns correspond to a different choice of minimum and maximum depreciation probabilities. In rows 4 and 5, we report

¹⁸We also ran some experiments with Beta-distributed consumer types. Again, we found social surplus to be higher when vintages are observable.

	High Depreciation		Low Depreciation	
	$\gamma^{\min} = 0.1, \gamma^{\max} = 0.9$		$\gamma^{\min} = 0.01, \gamma^{\max} = 0.1$	
	Max	Min	Max	Min
$\frac{S_v - S_{nv}}{S_{eff}}$	5.49%	0	8.31%	0
$\frac{S_v - S_{nv}}{S_{eff} - S_1}$	18.6%	0	41.87%	0
$\frac{S_v}{S_{eff}}$	–	89.87%	–	92.81%
$\frac{S_v - S_1}{S_{eff} - S_1}$	–	38.14%	–	42.53%
$\frac{S_1}{S_{eff}}$	95.90%	62.48%	96.20%	62.26%

Table 1: Efficiency Gains

the maximum and minimum efficiency gains due to vintage observability according to two different measures, calculated over the $20^4 = 160,000$ different parameterizations generated by the choice of $\gamma^{\min}, \gamma^{\max}$. As noted above, in all our calculations, vintage observability was always beneficial; however, for certain parameter values, the equilibrium was the same regardless of whether vintages are observable or not (i.e. no quality- q_1 cars are traded in either setting); in these cases the efficiency gain was zero. The figures in row 5 suggest that the efficiency gain from vintage observability, especially when measured relative to a single-market environment, can be substantial, and are larger when depreciations are less frequent.

The sixth and seventh rows report the *minimum* realized efficiency gain $\frac{S_v}{S_{eff}}$ under resale markets and observable vintages, and the minimum realized gain $\frac{S_v - S_1}{S_{eff} - S_1}$ relative to a single-market environment. Since rental contracts achieve the efficient allocation, these quantities measure the potential inefficiency associated with resale. Recall that, according to Theorem 1, efficiency obtains even with resale contracts if there are only two qualities; hence, the *maximum* realized efficiency gain can be made arbitrarily close to 1 by choosing q_1 very near q_0 or q_2 . For this reason, it is not explicitly indicated in Table 1.

We wish to emphasize that the numbers in row 6 correspond to “worst-case scenarios”, i.e. parameter values for which resale markets perform particularly poorly. For different parameter values, the gains from using rental contracts rather than selling contracts are not very large. This may help explain why rental contracts are not commonly observed in the car market; perhaps the gains are not large enough to justify the larger administrative costs, and the potential problems of moral hazard in maintenance that are likely to be associated with rental contracts. Furthermore, we note that leasing contracts (which constitute more than a third of transactions in the new car market) share some of the advantages of rental contracts—although, as pointed out in Section 3.5.1, the two are not perfect substitutes.

Finally, the last row reports the minimum and maximum surplus achievable in a single-market environment; these figures can be useful as a reference.

4 General Depreciation Model

We now consider the general environment described in Section 2.1. Recall that, in contrast with the model discussed in Section 3, we now assume that the initial quality of the new good is uncertain, and that the good may depreciate by an arbitrary number of quality steps (or die) in any time period.

As for the simple depreciation model, in the observable-quality benchmark, efficiency requires assortative matching of qualities to consumers. It is easy to verify that the system of instantaneous rental prices defined by equation (6) still sustain the efficient allocation.

4.1 Experimentation and the Trickle-Down Algorithm

Assume now that the quality of the good is not observable before purchase: it becomes observable only to the current user at the end of the first period of consumption. The key distinction between the present model and the simple one-step depreciation model presented in Section 3 is the fact that efficient sorting now requires *experimentation*: for instance, highest-valuation consumers need to try several units before finding one of quality q_0 .

Consequently, in the environment under consideration, it is impossible to obtain the first-best allocation: the first consumer of the good consumes the ‘wrong’ quality with positive probability. However, we will show that approximate payoff efficiency is possible when the length of the periods Δ shrinks to zero (i.e. trading becomes more and more frequent).

As in the simpler setting of Section 3, we assume that the *vintage* of each car is observable. For $n = 0, \dots, N$ and $m = n, \dots, N$, denote by v_m^n the mass of cars of vintage n and quality m . These quantities must satisfy the following equations:

$$\begin{aligned}
 v_0^0 &= \gamma_{0,0}(\Delta)v_0^0 + \chi_0 y & (15) \\
 v_m^0 &= \chi_m y \\
 v_n^n &= \gamma_{n,n}(\Delta)v_n^n + \gamma_{n-1,n}\Delta v_{n-1}^{n-1} + \gamma_{n,n}(\Delta)v_n^{n-1} \\
 v_m^n &= \sum_{\ell=n-1}^{m-1} \gamma_{\ell,m}\Delta v_\ell^{n-1} + \gamma_{m,m}(\Delta)v_m^{n-1}.
 \end{aligned}$$

That is: vintage-0 cars consists of quality- q_0 cars that have not depreciated, and newly-produced cars. Since vintage-0 cars worse than q_0 are immediately retraded, the stock of quality- q_m cars of vintage 0, for $m > 0$, consists solely of newly-produced cars. Vintage- n cars of quality q_n come from three sources: vintage- n , quality- q_n cars that have not depreciated, vintage- $(n-1)$ quality- q_{n-1} cars that have depreciated to q_n , and vintage- $(n-1)$ quality- q_n cars that have not depreciated. Vintage- n cars of quality worse than q_n all come from the stock of vintage- $(n-1)$ cars, and are immediately retraded.

Observe that the masses v_m^n , for $n \neq m$, measure the efficiency loss due to the fact that consumers need to experiment in order to receive a car of their designated quality.

Furthermore, the mass of new cars must correspond to the mass of cars dying in each period:

$$y = \sum_{\ell=0}^N \sum_{k=\ell}^N \gamma_{k,N+1} \Delta v_k^\ell. \quad (16)$$

Finally, there is a total of Y cars on the market at each given time:

$$\sum_{\ell=0}^N \sum_{k=\ell}^N v_k^\ell = Y. \quad (17)$$

We can now define cutoff types to identify consumers of each vintage. Let $\theta_{-1} = \bar{\theta}$ for notational convenience; then, for $n = 0, \dots, N$, let θ_n be defined by the condition

$$F(\theta_{n-1}) - F(\theta_n) = \sum_{m=n}^N v_m^n. \quad (18)$$

4.2 Rental and Experimentation under Asymmetric Information

We now turn to the analysis of consumer incentives. Each consumer is facing a stationary, infinite-horizon dynamic programming problem, with states \emptyset (corresponding to the event that the car just died), q_0, \dots, q_N . The possible actions (controls) are “rent your current car for an additional period” (not available in state \emptyset) and “return your current car, if any, and rent a vintage- n car”, for $n = 0, \dots, N$. The transition probabilities are determined by actions and depreciation probabilities in the obvious way. As in the simpler setting of Section 3, we represent rental fees by means of instantaneous rental prices r_0, \dots, r_N .

In order to describe the value functions, it is useful to introduce additional notation. The probability that a newly rented vintage- n car is of quality q_m equals

$$\begin{aligned} \lambda_m^0 &= \chi_m \\ \lambda_m^n &= \frac{\sum_{\ell=n-1}^{m-1} \gamma_{\ell,m} \Delta v_\ell^{n-1} + \gamma_{m,m}(\Delta) v_m^{n-1}}{\sum_{k=n}^N \left(\sum_{\ell=n-1}^{k-1} \gamma_{\ell,k} \Delta v_\ell^{n-1} + \gamma_{k,k}(\Delta) v_k^{n-1} \right)}, \end{aligned} \quad (19)$$

for $m = n, \dots, N$. These expressions are derived by looking at the supply of vintage- n cars of each quality, as it appears in Eqs. (15); for vintage 0, the supply consists solely of newly produced cars, whose quality is distributed according to the proportions χ_0, \dots, χ_N . For vintages $n > 0$, the supply consists of vintage- $(n-1)$ cars, and we keep track of the various ways a quality- q_m car might be offered in the vintage- n market according to the trickle-down algorithm.

We denote the expectation operator corresponding to the distribution $\lambda_n^n, \dots, \lambda_N^n$ by E^n ; for simplicity, we also define $L_m^n = \sum_{\ell=m+1}^N \lambda_\ell^n$. Note that, although the notation does not emphasize this fact, both λ_m^n and E^n are also a function of Δ .

By standard arguments, there exists a stationary policy that is optimal for the consumer. We now describe the set of stationary policies. If the current state is \emptyset (no car), the policy must specify which vintage to rent: thus, this portion of the policy can be represented by an integer $n \in \{0, \dots, N\}$. If the current state is instead q_n , for $n = 0, \dots, N$, the policy must specify whether to keep the current car (i.e. rent the currently rented unit for another time period), or return it and rent another car. A consumer who chooses to return her current car faces the same problem as a consumer whose car has just died: hence, it is without loss of generality to restrict attention to policies that prescribe that the *same* vintage be rented if the current car dies, or if it is returned. Such policies are thus fully specified by a pair (n, M) , where $n \in \{0, \dots, N\}$ is the vintage the consumer rents in state \emptyset , and $M \subset \{n, \dots, N\}$ is a (possibly empty) collection of quality indices corresponding to qualities the consumer keeps.¹⁹

Consider one such stationary policy (n, M) with $M \neq \emptyset$ (see Lemma 3 for the case $M = \emptyset$). The value function for consumer type θ in state \emptyset can be written as follows:

$$\begin{aligned}
V_M^n(\theta, \emptyset) &= \frac{1 - e^{-\rho\Delta}}{\rho} [E^n(q|q \leq q_n)\theta - r_n] + & (20) \\
&+ e^{-\rho\Delta} \left\{ \left[\sum_{m \geq n : m \notin M} \lambda_m^n + \sum_{m \geq n : m \in M} \lambda_m^n \sum_{\ell > m : \ell \notin M} \gamma_{m,\ell} \Delta \right] V_M^n(\theta, \emptyset) + \right. \\
&\left. + \sum_{m \in M} \lambda_m^n \left[\gamma_{m,m}(\Delta) V_M^n(\theta, q_m) + \sum_{\ell > n : \ell \in M \setminus \{m\}} \gamma_{m,\ell} \Delta V_M^n(\theta, q_\ell) \right] \right\}; & (21)
\end{aligned}$$

if instead the consumer is currently renting a quality- q_ℓ car,

$$\begin{aligned}
V_M^n(\theta, q_\ell) &= \frac{1 - e^{-\rho\Delta}}{\rho} [q_\ell\theta - r_n] + e^{-\rho\Delta} \left[\left(\sum_{k > \ell : k \notin M} \gamma_{\ell,k} \Delta \right) V_M^n(\theta, \emptyset) + \right. & (22) \\
&\left. + \left(\gamma_{\ell,\ell}(\Delta) V_M^n(\theta, q_\ell) + \sum_{k > \ell : k \in M} \gamma_{\ell,k} \Delta V_M^n(\theta, q_k) \right) \right].
\end{aligned}$$

We now define the instantaneous rental prices r_0, \dots, r_N so as to ensure that

$$V_{\{N\}}^N(\theta_N, \emptyset) = 0, \quad V_{\{n\}}^n(\theta_n, \emptyset) = V_{\{n+1\}}^{n+1}(\theta_n, \emptyset), \quad n = 0, \dots, N-1. \quad (23)$$

The main result of this paper can now be stated.

Theorem 4 *There exists $\Delta^* > 0$ such that, for all $\Delta \in (0, \Delta^*)$, there is a consumer equilibrium wherein cutoff types and instantaneous rental prices are determined by equations (18) and (23)*

¹⁹In general, one cannot guarantee *a priori* (without fixing rental prices and solving for the optimal policy) that restricting attention to cutoff policies—i.e. $M = \{n, \dots, m\}$ for some $m \geq n$ —will be w.l.o.g.. Intuitively, without restrictions on the depreciation probabilities $\gamma_{n,m}$, it may be the case that a car of *current* quality q_{n+1} yields a expected discounted utility than a car of current quality q_n (e.g. if quality q_{n+1} depreciates slowly, whereas q_n does not depreciate but dies with high probability).

respectively, and for every $n = 0, \dots, N$, consumer types $\theta \in [\theta_n, \theta_{n-1}]$ rent vintage- n cars and only keep cars of quality q_n .

Furthermore, as $\Delta \rightarrow 0$, cutoff types and instantaneous rental prices converge to their observable-quality counterparts: $\theta_n \rightarrow \theta_n^*$ and $r_n \rightarrow r_n^*$ for all $n = 0, \dots, N$.

We prove this result via several lemmas.

We first establish that the Eqs. (15), (16) and (17) uniquely determine masses in each stage of the trickle-down algorithm, and show that these masses converge to the appropriate efficient quantities as the length of each time period shrinks.

Lemma 1 *There is a unique solution to Eqs. (15), (16) and (17). Furthermore, as $\Delta \rightarrow 0$, for every $n = 0, \dots, N$, $m v_n^n \rightarrow v_n^*$ and $v_m^n \rightarrow 0$ for $m = n+1, \dots, N$. Consequently, $\theta_n \rightarrow \theta_n^*$ as $\Delta \rightarrow 0$.*

Proof. See Appendix. ■

Turn now to consumer incentives. The following Lemma provides the key step in the proof of Theorem 4: it shows that the value functions $V_{\{n\}}^n(\theta, \emptyset)$ can be written as a weighted average of a “long-run” component, which corresponds to the net payoff from renting a car of known quality q_n in each period, and an “experimentation” component; furthermore, the weight on the latter vanishes as $\Delta \rightarrow 0$. This is then shown to imply that rental prices, as defined above, converge to their observable-quality counterparts as $\Delta \rightarrow 0$.

Lemma 2 *For every $n = 0, \dots, N$,*

$$V_{\{n\}}^n(\theta, \emptyset) = (1 - w_n) \frac{E^n(q|q \leq q_n)\theta - r_n}{\rho} + w_n \frac{q_n\theta - r_n}{\rho}$$

where $w_n = \frac{e^{-\rho\Delta}\lambda_n^n(1-G_n\Delta)}{1-e^{-\rho\Delta}(1-G_n\Delta)+e^{-\rho\Delta}\lambda_n^n(1-G_n\Delta)} \in (0, 1)$, and $w_n \rightarrow 1$ as $\Delta \rightarrow 0$. Therefore, the rental prices defined in equation (23) satisfy $r_n \rightarrow r_n^*$ as $\Delta \rightarrow 0$.

Proof. See Appendix. ■

We now employ the decomposition of payoffs provided in Lemma 2 to show that experimentation policies of the form (n, M) with $M \neq \{n\}$ cannot be optimal for *any* type, and “pure-consumption” policies ($M = \emptyset$) can be disregarded w.l.o.g.

Lemma 3 *There exists $\Delta^* > 0$ such that, for $\Delta \in (0, \Delta^*)$, and for all θ , the policies (n, M) with $M \neq \{n\}$ are suboptimal. Furthermore, for every $n = 0, \dots, N$, every Δ , and every θ , in each state $\emptyset, q_0, \dots, q_N$, the policy (n, \emptyset) is not strictly better than the policy $(n, \{n\})$.*

Proof. Consider type $\underline{\theta}$ and policy (n, M) , and let $m = \max M$; by Lemma 2, there exists $\Delta_{n,m} > 0$ such that, for $\Delta \in (0, \Delta_{n,m})$, $\frac{q_m\underline{\theta} - r_n}{\rho} < (1 - w_n) \frac{E^n(q|q \leq q_n)\underline{\theta} - r_n}{\rho} + w_n \frac{q_n\underline{\theta} - r_n}{\rho} = V_{\{n\}}^n(\underline{\theta}, \emptyset)$ and $q_m \leq w_n q_n \leq (1 - w_n)E^n(q|q \leq q_n) + w_n q_n = \frac{\partial V_{\{n\}}^n(\underline{\theta}, \emptyset)}{\partial \theta} (1 - e^{-\rho\Delta})$. This implies that, for

$\Delta \in (0, \Delta_{n,m})$, $\frac{q_m \theta - r_n}{\rho} < V_{\{n\}}^n(\theta, \emptyset)$ for all types θ . Now consider one such Δ , and suppose that (n, M) is optimal for some type θ . Then, in particular, $V_M^n(\theta, \emptyset) \geq V_{\{n\}}^n(\theta, \emptyset) > \frac{q_m \theta - r_n}{\rho}$. Moreover, by equation (22), since $m = \max M$ (so $k > m$ implies $k \notin M$) and $\gamma_{m,m}(\Delta) = 1 - G_m \Delta$,

$$\begin{aligned} V_M^n(\theta, q_m) &= \frac{1 - e^{-\rho \Delta}}{\rho} [q_m \theta - r_n] + e^{-\rho \Delta} [G_m \Delta V_M^n(\theta, \emptyset) + \gamma_{m,m}(\Delta) V_M^n(\theta, q_m)] = \\ &= \frac{1 - e^{-\rho \Delta}}{1 - e^{-\rho \Delta}(1 - G_m \Delta)} \frac{q_m \theta - r_n}{\rho} + \frac{e^{-\rho \Delta} G_m \Delta}{1 - e^{-\rho \Delta}(1 - G_m \Delta)} V_m^n(\theta, \emptyset); \end{aligned}$$

it follows that $V_M^n(\theta, q_m) < V_M^n(\theta, \emptyset)$: that is, renting another car of vintage n and then reverting to (n, M) is more profitable for type θ than following the policy (n, M) if her current car is of quality q_m , i.e. the lowest quality the consumer is supposed to keep under policy (n, M) .

Choosing $\Delta^* = \min_{n,m} \Delta_{n,m}$ completes the proof of the first claim. As for the second, for any type θ and any $n = 0, \dots, N$, the policy (n, \emptyset) yields $\frac{E^n(q|q \leq q_n) \theta - r_n}{\rho}$ in any state; by Lemma 2, $V_{\{n\}}^n(\theta, \emptyset) = (1 - w_n) \frac{E^n(q|q \leq q_n) \theta - r_n}{\rho} + w_n \frac{q_n \theta - r_n}{\rho}$, with $w_n \in [0, 1]$. Since $E^n(q|q \leq q_n) \leq q_n$ for all $n = 0, \dots, N$, it follows that, in any state $\emptyset, q_0, \dots, q_N$, the consumer is at least as well off returning the current car and switching to the policy $(n, \{n\})$. ■

The argument can now be concluded. An optimal policy of the form (n, M) exists for every $\theta \in [\theta_N, \bar{\theta}]$; by Lemma 3, for $\Delta \in (0, \Delta^*)$, we can restrict attention to policies of this class with $M = \{n\}$. But rental prices are defined so as to ensure that, for all types $\theta \in [\theta_n, \theta_{n-1}]$, it is optimal to adopt policy $(n, \{n\})$ in state \emptyset and adhere to it in the continuation; hence, $(n, \{n\})$ must be an optimal policy for these types.

4.3 Supply Side in the General Depreciation Model

The analysis of producers' incentives is more delicate than in the simple depreciation environment. Since initial quality is uncertain and depreciation by more than one quality level is possible, experimentation is necessary in the trickle-down mechanism for consumers to obtain the 'right' car quality. This implies that some delay is inevitable before the right match between cars and consumers is achieved. When the time between periods Δ is small, this delay is short; however, the presence of this delay raises the possibility that producers may choose to deviate from the menu of rental contracts to offer a mechanism that *accelerates experimentation*. For instance, a firm might require consumers to report the current quality of the car when they return it, then offer it to the 'right' consumer type in the following period. Even in the absence of such reports, it can be shown that profitable deviations from the menu of rental contracts defined above are possible.

For instance, learning the current quality of the car from its previous consumer is beneficial to a firm for two reasons: (i) one or more steps in the trickle-down mechanism may be bypassed, and (ii) the next consumer will not need to experiment in order to find the right quality for her; therefore, she will be willing to pay a higher per-period rental price to the deviating firm. However, if the time between periods is small, *the gain from such deviations is also small*: if Δ is small, (i) allocating a car via the trickle-down mechanism only imposes a short delay, and (ii) the cost of experimentation is small.

These intuitive observations can be formalized and developed in two directions. In the setting of Section 3.4, it is possible to adapt the proof of Theorem 3 to establish the following approximate equilibrium result.

Theorem 5 *For every $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that, for all $\Delta \in (0, \Delta_\varepsilon)$, the following constitutes a market ε -equilibrium:*

(i) *firms produce the first-best output, and offer $N + 1$ vintage-dependent rental contracts at the instantaneous rental prices r_0, \dots, r_N determined by equation (23);*

(ii) *for every $n = 0, \dots, N$, consumer types $\theta \in [\theta_n, \theta_{n-1}]$ rent vintage- n cars and only keep cars of quality q_n , where the cutoffs $\theta_0, \dots, \theta_N$ are determined by equation (18).*

Furthermore, as $\Delta \rightarrow 0$, for every $n = 0, \dots, N$, $\theta_n \rightarrow \theta_n^$ and $r_n \rightarrow r_n^*$.*

The proof of Theorem 5 can be found in the Appendix; here we provide a brief sketch. As explained in Section 3.4, each deviation can be shown to be dominated by a menu of one-period rental contracts; the argument is independent of the specific features of the depreciation process. Prices in the dominating menu of rental contracts are set so as to ensure that each target consumer is indifferent between renting a car from the deviating firm, then reverting to experimentation with her designated vintage, and experimenting with that vintage forever.

As suggested by the above intuitive discussion, under general depreciation, the rental prices charged in each period by the deviating firm for a car of quality q_n can be larger than $\frac{1-e^{-\rho\Delta}}{\rho}r_n$, because the latter is determined taking into account the cost of experimentation borne by consumers. We also noted above that the gain from such a one-period deviation is “small” if Δ is not too large; however, we must quantify gains from deviations *per unit of calendar time*, because as Δ becomes smaller, the expected number of periods until the car dies grows larger. We thus show that gains per time unit vanish as $\Delta \rightarrow 0$, which completes the proof.

Theorem 5 may be interpreted as stating that, if Δ is small, then there is an *approximate* equilibrium wherein *all* firms offer the rental contracts described in Section 4.2. The previous version of this paper (Hendel, Lizzeri and Siniscalchi, 2002) considered a model characterized by initial uncertainty about quality, no depreciation, and a positive probability that the car dies in each period; these assumptions correspond to $\gamma_{n,m} = 0$ for $m = n + 1, \dots, N$ and $\gamma_{n,N+1} > 0$. In this environment, we established a complementary result: when Δ is small, there is an *exact* equilibrium wherein *almost all* firms offer the rental contracts described above, but a small mass of firms offer other types of contracts.

Specifically, we first showed that producers may profitably deviate from the menu of rental contracts by offering a leasing contract, so that the consumer pays a rental price for experimenting with the good and a keeping price to purchase the good if it is of the right quality. In particular, a profitable deviation consists of a contract tailored to a marginal type θ_n , who is just indifferent between experimenting with vintages n and $n + 1$ under the rental contracts described above. Prices in the deviating contract are chosen so as to induce this consumer to keep *both qualities q_n and q_{n+1}* . This accelerates experimentation: if the quality of the car is q_{n+2} or worse, the firm learns this exactly one period earlier.

We then construct an equilibrium in the class of leasing contracts, wherein a (small) fraction of firms offer “accelerating” contracts of this kind, and all other firms offer rental contracts as

described in the preceding section. Since a positive mass of “accelerating” contracts is offered, misallocations occur with positive probability and tend to reduce revenues from these contracts; indeed, in equilibrium, the costs resulting from misallocation exactly offsets the benefits from accelerated experimentation, so that firms are just indifferent between the two types of contracts. But this implies that, as $\Delta \rightarrow 0$, the fraction of firms offering rental contracts converges to 1, so that the equilibrium allocation converges to the efficient allocation. Intuitively, as the time between transactions becomes negligible, so do benefits from accelerated experimentation; in equilibrium, these equal the cost of misallocation, so the equilibrium allocation must be asymptotically efficient.

5 Conclusions

We presented a model of adverse selection durable goods market in which (approximately) efficient sorting can be obtained through smoothly functioning secondary markets. We first discussed a simple depreciation model, in which the quality of new goods is known, but goods may depreciate by one quality level with positive probability in each period; thus, there is asymmetric information in secondary markets. We showed that resale markets do not lead to efficient allocations, but menus of rental contracts replicate the observable-quality outcome. We also showed that competitive firms have the incentive to provide the efficient amount of output via the menu of rental contracts that implements the efficient allocation. We then considered a generalization of the simple model in which initial quality is uncertain and depreciation by more than one step can occur in a given period. For the second model, the observable quality allocation cannot be achieved by any mechanism. However, we showed that rental contracts lead to approximate efficiency if retrading is frequent.

Throughout this paper, we have assumed that the consumer learns the quality of a unit as soon as she uses it for the first time. This leads to rather strong efficiency results. If instead one were to assume that the quality discovery process may take a minimal amount of time, and that such learning may be imperfect, then new effects would arise out of the interaction between asymmetric information and slow learning. In particular, we conjecture that in such a model, a consumer may get rid of a high quality unit incorrectly believing it to be low quality. Once the high quality good is in the hands of a low valuation consumer, it may become impossible to get it back in the hands of high valuation consumers. Thus, some degree of misallocation may be inescapable.

It would be interesting to extend the model to study matching under asymmetric information in the labor market, so as to understand the relation between job mobility and wage growth. To this end, two additional key features should be incorporated in the model. First, both sides in the labor market can take actions after learning the quality of a match: both employers and workers can in principle choose to dissolve a match, whereas in our model the car cannot decide to get rid of the consumer. Second, idiosyncratic components are likely to be a more important feature of match quality in labor markets than in markets for durable goods.

Finally, it may be instructive to contrast our results with those from the literature on the Coase conjecture. This literature deals with a monopolistic producer of a durable good of *known* quality. In that setting, a monopolist may prefer a rental contract over a sale contract, because the former avoids the commitment problem.²⁰ In the context of a durable-goods monopoly, if consumers

²⁰See Bulow (1982) for this argument.

are patient, the stationary subgame-perfect equilibrium outcome under selling is approximately efficient.²¹ In contrast, under rental, the equilibrium outcome involves the monopolist producing too little output. Thus, the consequences for efficiency of these alternative contractual arrangements are the opposite of those we find in our model.

6 Appendix

6.1 Proof of Lemma 1

Note first that, by Eqs. (15), one can write

$$v^n = \begin{bmatrix} \gamma_{n,n}(\Delta) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} v^n + A_{n-1,n} v^{n-1}$$

for all $n = 1, \dots, N$, where $v^n = [v_n^n, \dots, v_N^n]'$ and

$$A_{n-1,n} = \begin{bmatrix} \gamma_{n-1,n}\Delta & \gamma_{n,n}(\Delta) & 0 & \dots & \dots & 0 \\ \gamma_{n-1,n+1}\Delta & \gamma_{n,n+1}\Delta & \gamma_{n+1,n+1}(\Delta) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n-1,N}\Delta & \gamma_{n,N}\Delta & \gamma_{n+1,N}\Delta & \dots & \dots & \gamma_{N,N}(\Delta) \end{bmatrix}.$$

Next, since $\gamma_{n,n}(\Delta) = 1 - G_n\Delta$,

$$\begin{aligned} 1' A_{n-1,n} v^{n-1} &= [G_{n-1}\Delta - \gamma_{n-1,N+1}\Delta \quad 1 - \gamma_{n,N+1}\Delta \quad \dots \quad 1 - \gamma_{N,N+1}\Delta] v^{n-1} = \\ &= -\gamma_{n-1,n-1}(\Delta) v_{n-1}^{n-1} + \sum_{m=n-1}^N (1 - \gamma_{m,N+1}\Delta) v_m^{n-1} = \\ &= -\gamma_{n-1,n-1}(\Delta) v_{n-1}^{n-1} + 1' v^{n-1} - \sum_{m=n-1}^N \gamma_{m,N+1}\Delta v_m^{n-1} \end{aligned}$$

and hence

$$\begin{aligned} 1' v^n - \gamma_{n,n}(\Delta) v_n^n &= 1' v^{n-1} - \gamma_{n-1,n-1}(\Delta) v_{n-1}^{n-1} - \sum_{m=n-1}^N \gamma_{m,N+1}\Delta v_m^{n-1} = \\ &= 1' v^0 - \gamma_{0,0}(\Delta) v_0^0 - \sum_{\ell=0}^{n-1} \sum_{k=\ell}^N \gamma_{k,N+1}\Delta v_k^\ell. \end{aligned}$$

In particular, since vintage- N cars can only have quality q_N , $1' v^N = v_N^N$ and $G_N = \gamma_{N,N+1}$ and therefore also $\gamma_{N,N}(\Delta) = 1 - \gamma_{N,N+1}\Delta$. Hence

$$\gamma_{N,N+1}\Delta v^N = 1' v^N - \gamma_{N,N}(\Delta) v_N^N = 1' v^0 - \gamma_{0,0}(\Delta) v_0^0 - \sum_{\ell=0}^{N-1} \sum_{k=\ell}^N \gamma_{k,N+1}\Delta v_k^\ell$$

²¹See Gul, Sonnenschein, and Wilson (1986).

and therefore

$$\sum_{\ell=0}^N \sum_{k=\ell}^N \gamma_{k,N+1} \Delta v_k^\ell = 1'v^0 - \gamma_{0,0}(\Delta)v_0^0 = y.$$

This shows that, if the quantities v_m^n are defined via equation (15), they automatically satisfy equation (16). It is also easy to see that $v_n^n \geq (\prod_{\ell=1}^n \gamma_{\ell-1,\ell}) \chi_0 y$; furthermore, $v_m^n = 0$ if $y = 0$. Hence, as long as $\chi_0 > 0$, there exists y^* such that equation (17), too, is satisfied.

To prove the second part of the claim, note first that all quantities v_m^n are bounded, so $y \rightarrow 0$ as $\Delta \rightarrow 0$. This immediately implies that $v_m^0 \rightarrow 0$ for $m > 0$; proceeding by induction, assume that we have shown $v_m^{n-1} \rightarrow 0$ for $m > n - 1$: then the last line of equation (15) implies that $v_m^n \rightarrow 0$ as well for $m > n$ (in particular, the terms in the summation corresponding to $\ell = n - 1$ vanish because v_{n-1}^{n-1} is bounded). Furthermore, it is clear that $v_n^* = \sum_{\ell=0}^n v_n^\ell$ for all Δ ; therefore, $|v_n^* - v_n^n| = |v_n^* - \sum_{\ell=0}^n v_n^\ell + \sum_{\ell=0}^n v_n^\ell - v_n^n| = \left| 0 + \sum_{\ell=0}^{n-1} v_n^\ell \right| \rightarrow 0$ as $\Delta \rightarrow 0$.

6.2 Proof of Lemma 2

A preliminary result is needed for this and other proofs.

Lemma 4 For all $n = 0, \dots, N$, $\liminf_{\Delta \rightarrow 0} \lambda_n^n > 0$.

Proof. By equation (19), the claim is clearly true for $n = 0$. For $n > 0$, note first that the denominator of λ_m^n can be rewritten as follows:

$$\begin{aligned} \sum_{k=n}^N \left(\sum_{\ell=n-1}^{k-1} \gamma_{\ell,k} \Delta v_\ell^{n-1} + \gamma_{k,k}(\Delta) v_k^{n-1} \right) &= \sum_{\ell=n-1}^{N-1} \sum_{k=\ell}^N \gamma_{\ell,k} \Delta v_\ell^{n-1} + \sum_{\ell=n}^N \gamma_{\ell,\ell}(\Delta) v_\ell^{n-1} \\ &= \sum_{\ell=n-1}^{N-1} G_\ell \Delta v_\ell^{n-1} + \sum_{\ell=n}^N \gamma_{\ell,\ell}(\Delta) v_\ell^{n-1} \\ &= G_{n-1} \Delta v_{n-1}^{n-1} + \sum_{\ell=n}^N v_\ell^{n-1}. \end{aligned}$$

Accordingly, rewrite λ_n^n as follows:

$$\begin{aligned} \lambda_n^n &= \frac{\gamma_{n-1,n} \Delta v_{n-1}^{n-1} + \gamma_{n,n}(\Delta) v_n^{n-1}}{G_{n-1} \Delta v_{n-1}^{n-1} + \sum_{\ell=n}^N v_\ell^{n-1}} \\ &= \frac{\gamma_{n-1,n} v_{n-1}^{n-1} + \gamma_{n,n}(\Delta) \frac{v_n^{n-1}}{\Delta}}{G_{n-1} v_{n-1}^{n-1} + \sum_{\ell=n}^N \frac{v_\ell^{n-1}}{\Delta}}. \end{aligned}$$

Now Lemma 1 shows that $v_{n-1}^{n-1} \rightarrow v_{n-1}^* > 0$ as $\Delta \rightarrow 0$. Furthermore, we claim that $\sup_{\Delta > 0} \frac{v_m^n}{\Delta} < \infty$ for all n and $m > n$. To see this, observe first that, from equations (16) and (17),

$$\frac{y}{\Delta} = \sum_{\ell=0}^N \sum_{k=\ell}^N \gamma_{k,N+1} v_k^\ell \leq \sum_{\ell=0}^N \sum_{k=\ell}^N v_k^\ell = Y < 1;$$

since $v_m^0 = \chi_m y$ for $m > 0$, this immediately implies that the claim is true for $n = 0$. Assuming that it is true for $n - 1 \geq 0$, for $m > n$, equation (15) implies that

$$\frac{v_m^n}{\Delta} = \sum_{\ell=n-1}^{m-1} \gamma_{\ell,m} v_\ell^{n-1} + \gamma_{m,m}(\Delta) \frac{v_m^{n-1}}{\Delta}$$

and the induction hypothesis implies that $\sup_{\Delta>0} \frac{v_m^{n-1}}{\Delta} < \infty$; since $\gamma_{m,m}(\Delta) \rightarrow 1$, the claim is true for n as well.

The proof of the Lemma can now be completed: we have

$$\begin{aligned} \lambda_n^n &\geq \frac{\gamma_{n-1,n} v_{n-1}^{n-1}}{G_{n-1} v_{n-1}^{n-1} + \sum_{\ell=n}^N \frac{v_\ell^{n-1}}{\Delta}} \\ &\geq \frac{\gamma_{n-1,n} v_{n-1}^{n-1}}{G_{n-1} v_{n-1}^{n-1} + \sum_{\ell=n}^N \sup_{\Delta>0} \frac{v_\ell^{n-1}}{\Delta}} \\ &\rightarrow \frac{\gamma_{n-1,n} v_{n-1}^*}{G_{n-1} v_{n-1}^* + \sum_{\ell=n}^N \sup_{\Delta>0} \frac{v_\ell^{n-1}}{\Delta}} > 0, \end{aligned}$$

and the claim follows. ■

Turn now to the proof of Lemma 2. Note that, for $M = \{n\}$, the functions $V_{\{n\}}^n$ can be rewritten in the following simpler form:

$$\begin{aligned} V_{\{n\}}^n(\theta, \emptyset) &= \frac{1 - e^{-\rho\Delta}}{\rho} [E^n(q|q \leq q_n)\theta - r_n] + \\ &+ e^{-\rho\Delta} \left\{ (L_{n+1}^n + \lambda_n^n G_n \Delta) V_{\{n\}}^n(\theta, \emptyset) + \lambda_n^n \gamma_{n,n}(\Delta) V_{\{n\}}^n(\theta, q_n) \right\}, \end{aligned} \quad (24)$$

$$\begin{aligned} V_{\{n\}}^n(\theta, q_n) &= \frac{1 - e^{-\rho\Delta}}{\rho} [q_n \theta - r_n] + e^{-\rho\Delta} \left\{ G_n \Delta V_{\{n\}}^n(\theta, \emptyset) + \gamma_{n,n}(\Delta) V_{\{n\}}^n(\theta, q_n) \right\} = \\ &= \frac{(1 - e^{-\rho\Delta}) \frac{q_n \theta - r_n}{\rho} + e^{-\rho\Delta} G_n \Delta V_{\{n\}}^n(\theta, \emptyset)}{1 - e^{-\rho\Delta} (1 - G_n \Delta)}. \end{aligned} \quad (25)$$

Plugging back into equation (24) yields

$$\begin{aligned} V_{\{n\}}^n(\theta, \emptyset) &= \frac{1 - e^{-\rho\Delta}}{\rho} [E^n(q|q \leq q_n)\theta - r_n] + e^{-\rho\Delta} (L_{n+1}^n + \lambda_n^n G_n \Delta) V_{\{n\}}^n(\theta, \emptyset) + \\ &+ e^{-\rho\Delta} \lambda_n^n \gamma_{n,n}(\Delta) \frac{(1 - e^{-\rho\Delta}) \frac{q_n \theta - r_n}{\rho} + e^{-\rho\Delta} G_n \Delta V_{\{n\}}^n(\theta, \emptyset)}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} = \\ &= \frac{(1 - e^{-\rho\Delta}) \frac{E^n(q|q \leq q_n)\theta - r_n}{\rho} + e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta) \frac{(1 - e^{-\rho\Delta}) \frac{q_n \theta - r_n}{\rho}}{\rho}}{1 - e^{-\rho\Delta} \left[L_{n+1}^n + \lambda_n^n G_n \Delta + \lambda_n^n (1 - G_n \Delta) \frac{e^{-\rho\Delta} G_n \Delta}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} \right]}. \end{aligned}$$

Rewrite the denominator as follows:

$$\begin{aligned}
& 1 - e^{-\rho\Delta} \left[1 - \lambda_n^n + \lambda_n^n G_n \Delta + \lambda_n^n (1 - G_n \Delta) \frac{e^{-\rho\Delta} G_n \Delta}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} \right] = \\
& = 1 - e^{-\rho\Delta} \left[1 - \lambda_n^n (1 - G_n \Delta) + \lambda_n^n (1 - G_n \Delta) \frac{e^{-\rho\Delta} G_n \Delta}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} \right] = \\
& = 1 - e^{-\rho\Delta} \left[1 - \lambda_n^n (1 - G_n \Delta) \left(1 - \frac{e^{-\rho\Delta} G_n \Delta}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} \right) \right] = \\
& = 1 - e^{-\rho\Delta} \left[1 - \lambda_n^n (1 - G_n \Delta) \frac{1 - e^{-\rho\Delta}}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} \right] = \\
& = (1 - e^{-\rho\Delta}) + \frac{e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta) (1 - e^{-\rho\Delta})}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} = \\
& = (1 - e^{-\rho\Delta}) \left\{ 1 + \frac{e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta)}{1 - e^{-\rho\Delta} (1 - G_n \Delta)} \right\}
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
V_{\{n\}}^n(\theta, \emptyset) &= (1 - w_n) \frac{E^n(q|q \leq q_n)\theta - r_n}{\rho} + w_n \frac{q_n \theta - r_n}{\rho} = \\
&= \frac{[(1 - w_n)E^n(q|q \leq q_n) + w_n q_n] \theta - r_n}{\rho},
\end{aligned}$$

where

$$\begin{aligned}
w_n &= \frac{\frac{e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta)}{1 - e^{-\rho\Delta} (1 - G_n \Delta)}}{1 + \frac{e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta)}{1 - e^{-\rho\Delta} (1 - G_n \Delta)}} = \\
&= \frac{e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta)}{1 - e^{-\rho\Delta} (1 - G_n \Delta) + e^{-\rho\Delta} \lambda_n^n (1 - G_n \Delta)}.
\end{aligned}$$

By Lemma 4, $\liminf_{\Delta \rightarrow 0} \lambda_n^n \equiv \Lambda_n > 0$, so

$$w_n \geq \frac{e^{-\rho\Delta} \Lambda_n (1 - G_n \Delta)}{1 - e^{-\rho\Delta} (1 - G_n \Delta) + e^{-\rho\Delta} \Lambda_n (1 - G_n \Delta)} \rightarrow \frac{\Lambda_n}{0 + \Lambda_n} = 1,$$

i.e. $w_n \rightarrow 1$. Now consider rental prices. Clearly, $r_N = r_N^*$; thus, assume that $r_{n+1} \rightarrow r_{n+1}^*$ for $n < N$; then $V_{\{n\}}^n(\theta_n, \emptyset) = V_{\{n+1\}}^{n+1}(\theta_n, \emptyset)$ iff

$$\begin{aligned}
r_n &= r_{n+1} + [(1 - w_n)E^n(q|q \leq q_n) + w_n q_n - (1 - w_{n+1})E^{n+1}(q|q \leq q_{n+1}) + w_{n+1} q_{n+1}] \theta_n \rightarrow \\
&\rightarrow r_{n+1}^* + (q_n - q_{n+1}) \theta_n^* = r_n^*,
\end{aligned}$$

because $w_n \rightarrow 1$ and $\theta_n \rightarrow \theta_n^*$.

6.3 Supply Side under Simple and General Depreciation

The proofs of Theorems 3 and 5 are very similar, except that an additional step is required for the latter. It is thus convenient to present them together.

We begin by describing the realizations of the quality process, or *quality histories*. Recall that a car of quality q_n that depreciates becomes a car of quality q_m with probability $\gamma_{n,m}\Delta$. Also, when a car of quality q_N depreciates, it disappears (“dies”). Thus, we are led to consider quality histories of the form $(q_0, \dots, q_0, q_1, \dots, q_1, q_2, \dots, q_n, \dots, q_N, \dots, q_N, 0)$, where 0 denotes that the car has died. Formally, let \mathcal{Q} be the set of all finite sequences $\{q^0, \dots, q^J\}$ such that (i) if $q^0 = q_n$, then $\chi_n > 0$, and (ii) $q^j \geq q^{j+1}$ for all $j = 0, \dots, J - 1$; also, let $\bar{\mathcal{Q}}$ the set of all complete quality histories: that is, $(q^0, \dots, q^J) \in \bar{\mathcal{Q}}$ iff $(q^0, \dots, q^J) \in \mathcal{Q}$ and $q^J = 0$.

Let Q denote the set of all qualities: that is, $Q = \{q_0, \dots, q_N, 0\}$; then, for all integers m , Q^m denotes the Cartesian product of m copies of Q (in particular, $Q^0 = \emptyset$).

Recall that depreciation events occur *at the end of the period*. Therefore, the history (q^0, \dots, q^J) should be interpreted as follows: q^0 is the initial quality of the car; then, for $j > 0$, q^j is the quality of the car in period j , which is determined by the realization of the depreciation process at the end of period $j - 1$.

We now describe deviations from the putative equilibrium rental contracts. A deviation consists of a collection of mechanisms, each targeted to a specific type of consumer. We begin by analyzing single mechanisms.

It turns out that, in order to assess whether a deviation is profitable, only certain elements of a mechanism need to be explicitly described. In particular, below we derive upper bounds on the revenues of a deviating firm. These bounds are determined solely by individual rationality considerations, taking into account the fact that equilibrium contracts offered by other firms are always available to consumers. Therefore, we only need a representation of a mechanism that allows us to compute consumers’ utility and payment flows.

Moreover, it is technically convenient to analyze a larger set of deviations than would be feasible for a firm operating in the environment described in the main text; in particular, we assume that (i) the deviating firm knows the initial quality of its newly-produced car, and (ii) the firm can ascertain the type of any consumer it transacts with. Since the firm can commit to the contracts it offers, having access to such information can only have a positive effect on revenues; therefore, our upper bound will be valid a fortiori when all informational constraints are taken into account.

We first specify under what circumstances a car may be offered via the mechanism; we do so by indicating a set of *initial quality histories*. The interpretation is that the consumer who enters the mechanism does not necessarily know the previous history of the car, but knows that it belongs to the specified initial set. It is up to the deviating firm to decide how much to reveal to consumers.

Second, we must be able to establish when the car exits the mechanism—either because it dies, or because it is returned to the firm. A specific mechanism will prescribe that certain actions be taken (e.g. the consumer is supposed to keep the car for 3 periods, then return it if the car has depreciated at least once, and otherwise keep it for 2 more periods). These prescriptions and actions determine a set of *final quality histories*; our minimalistic description of a deviation only requires the specification of the latter.

Third, we define revenues. Again, a specific mechanism will prescribe that certain transfers be effected, possibly contingent on the actions taken by the consumer (e.g. the consumer pays a price p upon entering the mechanism; then, if she keeps the car for more than 3 periods, she pays a rental price r for each additional period.) And, again, such specifics are irrelevant for our purposes; we only define a *revenue function* that indicates, for every continuation history that is consistent with some initial history and leads to a final history, the transfer effected by the consumer to the firm.

Finally, we specify a set of *target consumer types* that are allowed to enter the mechanism. We need not describe the specifics of the mechanism that result in only certain types entering the mechanism; as noted above, for the purposes of the present analysis, we simply assume that firms can decide whether or not to transact with her.

Definition 1 *A (reduced-form) mechanism is a tuple $M = (I, F, R, \Theta)$, where:*

- $I, F \subset \mathcal{Q}$ and both sets are nonempty.
- If $(q^0, \dots, q^J) \in F$, then there exists $j_0 < J$ such that $(q^0, \dots, q^{j_0}) \in I$;
- If $(q^0, \dots, q^{j_0}) \in I$, then there exist $J > j_0$ and $\{q^{j_0+1}, \dots, q^J\} \in \mathcal{Q}^{J-(j_0+1)}$ such that $(q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^J) \in F$;
- If $(q^0, \dots, q^{j_0}) \in I$, then there does not exist $j_1 > 0$ and $\{q^{j_0+1}, \dots, q^{j_1}\} \in \mathcal{Q}^{j_1-(j_0+1)}$ such that $(q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^{j_1}) \in I$.

Now define the set of continuation histories

$$H = \{(q^{j_0}, \dots, q^J) \in \mathcal{Q}^{J-j_0+1} : (q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^J) \in F \text{ for some } (q^0, \dots, q^{j_0}) \in I\}$$

- If $(q^{j_0}, \dots, q^J) \in H$, then there is no $\{q^{J+1}, \dots, q^K\} \in \mathcal{Q}^{K-J}$ such that $(q^{j_0}, \dots, q^J, q^{J+1}, \dots, q^K) \in H$.
- $R : H \rightarrow \mathbb{R}$
- $\Theta \subset [\underline{\theta}, \bar{\theta}]$.

Suppose that, in period j_0 , a consumer enters the mechanism and receives a car characterized by the initial quality history $(q^0, \dots, q^{j_0}) \in I$. The consumer does not observe the entire history; however, as soon as she receives the car, she learns q^{j_0} . She then keeps the car until its realized partial history is one of the elements of the set F —say, $(q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^J)$. The consumer then returns the car at the end of period J , and her total payments to the firm from period j_0 through time J are given by $e^{\rho\Delta(J-j_0)}R(q^{j_0}, \dots, q^J)$.²² That is, $R(q^{j_0}, \dots, q^J)$ is the discounted value of the transfer at the beginning of period j_0 .

²²If (q^{j_0}, \dots, q^J) is a feasible intermediate partial history, then in particular q^{j_0} is one of the possible initial qualities of the car, i.e. $(q^0, \dots, q^{j_0}) \in I$. In other words, the very first observation the consumer makes is the initial quality of the car (which was realized in period $j_0 - 1$).

Also note that, as a consequence of the definition, intermediate partial histories have length at least 2: they contain the initial quality of the car, and the quality resulting from the realization of the depreciation process at the end of the first period of the mechanism. Thus, initial histories can never be complete histories.

The last restriction on initial histories rules out the possibility that both a history and one of its subhistories be elements of I . For instance, the set $\{(0, 0), (0, 0, 1)\}$ violates this restriction. The intuition is that, if $(0, 0)$ is an initial history, then the consumer enters the mechanism in period 2, so $(0, 0, 1)$ could not also be an initial history.

The restriction on continuation histories is a definiteness requirement: the consumer must be able to tell whether a final history has obtained based on what she observes. If (q^{j_0}, \dots, q^J) and $(q^{j_0}, \dots, q^J, q^{J+1}, \dots, q^K)$ were both possible continuation histories, the consumer would not be able to decide whether or not to exit in period J . Thus, we eliminate this possibility.

For example, under *simple* depreciation, the equilibrium mechanism for vintage-1 cars is defined as follows: I consists of all partial histories (q^0, \dots, q^{j_0}) such that $q^{j_0} = q_1$ and $q^j = q_0$ for all $j < j_0$; F contains all histories (q^0, \dots, q^J) such that $q^J = q_2$ and $q^j > q_2$ for $j < J$; H is a set containing all partial histories of the form (q_1, \dots, q_1, q_2) (any number of repetitions of q_1) and $R(q^{j_0}, \dots, q^J)$ equals $\sum_{j=j_0}^{J-1} \frac{1-e^{-\rho\Delta}}{\rho} r_1 e^{-\rho\Delta(j-j_0)} = \frac{1-e^{-\rho\Delta[(J-1)-j_0]}}{\rho} r_1^*$.

We emphasize that this is *not* a complete description of a mechanism and/or of the consumer's optimizing behavior conditional upon entering the mechanism; it is merely a reduced-form representation of those elements that are essential to the analysis.

The initial quality distribution $(q_n, \chi_n : n = 0, \dots, N+1)$ and depreciation probabilities $\gamma_{n,m}\Delta$ determine a probability distribution $\Pr[\cdot]$ over the set of complete histories $\bar{\mathcal{Q}}$. Certain derived probabilities will now be obtained. First, for any partial history (q^0, \dots, q^j) ,

$$\Pr[(q^0, \dots, q^j)] = \Pr[\{(\bar{q}^0, \dots, \bar{q}^j) \in \bar{\mathcal{Q}} : \forall j' = 0, \dots, j, \bar{q}^{j'} = q^{j'}\}].$$

It also makes sense to define conditional probabilities of the following type:

$$\Pr[(q^{j_0+1}, \dots, q^j) | (q^0, \dots, q^{j_0})] = \frac{\Pr[(q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^j)]}{\Pr[(q^0, \dots, q^j)]}.$$

Finally, fix a mechanism $M = (I, F, R, \Theta)$. We are interested in the conditional probability of reaching a mechanism by way of a specific initial history $(q^0, \dots, q^{j_0}) \in I$, given that the mechanism is reached in period j_0 . Assuming throughout there are histories of such length in I , this probability can be computed as follows:

$$\Pr[(q^0, \dots, q^{j_0}) | I, j_0] = \frac{\Pr[(q^0, \dots, q^{j_0})]}{\Pr[\{(\hat{q}^0, \dots, \hat{q}^{j_0}) \in I\}]}$$

A collection of mechanisms that constitute a deviation must be internally consistent. To motivate, consider the following two mechanisms: M_0 is such that $I_0 = \{(q_0)\}$, and the consumer is supposed to keep the car for exactly 3 periods, then return it to the firm regardless of the realization of the depreciation process; M_1 is such that $I_1 = \{(q_0, q_1, q_2)\}$, and the consumer keeps the car until it dies. The pair (M_0, M_1) is not a well-defined deviation, because it does not specify what to do if the car does not depreciate each period. This motivates the following definition.

Definition 2 A *menu* is a collection \mathcal{M} of (reduced-form) mechanisms such that, for every partial history (q^0, \dots, q^J) , there is a unique mechanism $M = (I, F, R, \Theta) \in \mathcal{M}$ and period $j_0 \leq J$ such that

1. $(q^0, \dots, q^{j_0}) \in I$;
2. for some $J' > J$ and $(q^{J+1}, \dots, q^{J'}) \in \mathcal{Q}^{J'-(J+1)}$, $(q^0, \dots, q^J, q^{J+1}, \dots, q^{J'}) \in F$.

That is: at any point in the course of the car's quality history, it is always clear which mechanism must be used (or is being used). As noted above, we allow for menus that distinguish between different initial qualities of the deviator's car (more precisely, between histories that differ in their period-zero component, which corresponds to the initial quality of a new car).

Throughout the remainder of this proof, let $V^e(\theta)$ denote the expected payoff to type θ if she uses the putative equilibrium rental contracts and follows the appropriate policy for her; under simple depreciation, for $\theta \in [\theta_n^*, \theta_{n-1}^*]$, $V^e(\theta) = \frac{q_n \theta - r_n^*}{\rho}$; under general depreciation, $\theta \in [\theta_n, \theta_{n-1}]$, $V^e(\theta) = V_{n, \{n\}}(\theta, \emptyset)$. For the sake of notational uniformity, we denote cutoff types for the simple depreciation model by θ_n , etc, suppressing stars.

Suppose that the deviating firm offers a menu \mathcal{M} , and consider an arbitrary mechanism $M = (I, F, R, \Theta) \in \mathcal{M}$; the firm's expected revenues from M , if a consumer enters it in period j_0 , can be expressed as follows:

$$\begin{aligned}
R_{M, j_0} &= \sum_{(q^0, \dots, q^{j_0}) \in I} \Pr[(q^0, \dots, q^{j_0}) | I, j_0] \\
&\quad \times \sum_{\substack{(q^{j_0+1}, \dots, q^j): \\ (q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^j) \in F}} \Pr[(q^{j_0+1}, \dots, q^j) | (q^0, \dots, q^{j_0})] R(q^{j_0}, \dots, q^j).
\end{aligned}$$

Suppose that a consumer $\theta \in [\theta_n, \theta_{n-1}]$ is targeted by mechanism M , so $\theta \in \Theta$. Individual rationality then determines an upper bound on her willingness to pay for M in period j_0 . Specifically, let $V(\theta)$ denote the value from type θ 's best strategy not involving participation in M when she does not have a car; note that this strategy may prescribe participating (at a later date) in some other mechanism offered by the deviator—i.e. it is not necessarily confined to the putative equilibrium rental contracts. Hence, in general, $V(\theta) \geq V^e(\theta)$. In any case, in period j_0 , type θ only accepts to participate in the mechanism M if her expected payment does not exceed the difference between (i) the consumption value of entering M in period j_0 , then following her best continuation policy when the mechanism terminates, and (ii) the value of adopting her best continuation policy at j_0 .

This determines the following upper bound on expected revenues from type θ to the deviator:

$$\begin{aligned}
\bar{R}_{M,j_0}(\theta) &= -V(\theta) + \sum_{(q^0, \dots, q^{j_0}) \in I} \Pr[(q^0, \dots, q^{j_0}) | I, j_0] \sum_{\substack{(q^{j_0+1}, \dots, q^j): \\ (q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^j) \in F}} \Pr[(q^{j_0+1}, \dots, q^j) | (q^0, \dots, q^{j_0})] \\
&\quad \times \left(\sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} \frac{1 - e^{-\rho\Delta}}{\rho} q^k \theta + e^{-\rho\Delta j} V(\theta) \right) \\
&= \sum_{(q^0, \dots, q^{j_0}) \in I} \Pr[(q^0, \dots, q^{j_0}) | I, j_0] \sum_{\substack{(q^{j_0+1}, \dots, q^j): \\ (q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^j) \in F}} \Pr[(q^{j_0+1}, \dots, q^j) | (q^0, \dots, q^{j_0})] \\
&\quad \times \left(\sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} \frac{1 - e^{-\rho\Delta}}{\rho} q^k \theta - (1 - e^{-\rho\Delta j}) V(\theta) \right). \tag{26}
\end{aligned}$$

We now construct a new menu \mathcal{M}' that still satisfies each target consumer's individual rationality constraint, and yields at least as much revenues as \mathcal{M} to the deviating firm. The new menu consists of one-period "rental" contracts, targeted to a single consumer type, wherein the firm fully discloses the history of the car up to the current period; each mechanism in the original menu is replaced by a collection of such one-period rental contracts, and payments are defined so as to leave the target consumer indifferent between taking up the contract (for one period), then reverting to her designated putative equilibrium rental contract, and choosing the latter right away. This implies that any policy involving her designated putative equilibrium contract, as well as any contract made available to her by the deviator, yields exactly the same expected payoff, so the new menu consists of individually rational mechanisms.

Formally, consider an arbitrary $M = (I, F, R, \Theta) \in \mathcal{M}$; for every $\theta \in \Theta$ and partial history $(q^0, \dots, q^{j_0}, \dots, q^j)$ such that (i) $(q^0, \dots, q^{j_0}) \in I$ and (ii) for some $(q^{j+1}, \dots, q^J) \in \mathcal{Q}^{J-(j+1)}$, $(q^0, \dots, q^j, \dots, q^J) \in F$, define a mechanism $M(\theta, q^0, \dots, q^j)$ with (q^0, \dots, q^j) as unique unique initial history,

$$\{(q^0, \dots, q^j, q^{j+1}) : \Pr[(q^0, \dots, q^j, q^{j+1}) | (q^0, \dots, q^j)] > 0\}$$

as set of final histories, θ as unique target type, and

$$R_\theta(q^j, q^{j+1}) = \frac{1 - e^{-\rho\Delta}}{\rho} q^j \theta - (1 - e^{-\rho\Delta}) V^e(\theta) \tag{27}$$

as revenue function. It is clear that the collection of mechanisms thus obtained is a menu; note that, in particular, this menu prescribes different contracts for a newly produced car, depending on its initial quality.

To verify individual rationality, observe that, by construction, if a consumer of type θ enters the mechanism $M(\theta, q^0, \dots, q^{j_0})$ [which can happen only in period j_0 , following the partial history (q^0, \dots, q^{j_0}) of the car offered by the deviator], her per-period payoff is $(1 - e^{-\rho\Delta}) V^e(\theta)$. Hence, her per-period payoff from *any* mechanism offered by the deviator is the same, and of course it coincides with the expected per-period payoff from her designated putative equilibrium rental contracts.

We now verify that the deviator does not lose by offering the menu \mathcal{M}' in lieu of \mathcal{M} . Consider any mechanism $M = (I, F, R, \Theta) \in \mathcal{M}$, any initial history $(q^0, \dots, q^{j_0}) \in I$, any final history $(q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^j) \in F$ consistent with (q^0, \dots, q^{j_0}) , and any type θ . Under the menu \mathcal{M}' , the firm receives

$$\begin{aligned}
& \sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} R_\theta(q^k, q^{k+1}) \\
&= \sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} \left[\frac{1 - e^{-\rho\Delta}}{\rho} q^k \theta - (1 - e^{-\rho\Delta}) V^e(\theta) \right] \\
&= \sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} \frac{1 - e^{-\rho\Delta}}{\rho} q^k \theta - (1 - e^{-\rho\Delta}) V^e(\theta) \sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} \\
&= \sum_{k=j_0}^{j-1} e^{-\rho\Delta(k-j_0)} \frac{1 - e^{-\rho\Delta}}{\rho} q^k \theta - (1 - e^{-\rho\Delta(j-j_0)}) V^e(\theta)
\end{aligned}$$

Notice that this quantity appears in the last line of equation (26); hence, taking conditional expectations over all histories $(q^0, \dots, q^{j_0}, q^{j_0+1}, \dots, q^j) \in F$ consistent with (q^0, \dots, q^{j_0}) , and then over all $(q^0, \dots, q^{j_0}) \in I$, yields precisely the upper bound $\bar{R}_{M, j_0}(\theta)$ on revenues accruing to the deviating firm from mechanism M if it transacts with type θ beginning in period j_0 . Since this is true for all target types and all mechanisms, the new menu \mathcal{M}' yields at least as much revenues as the initial one.

Recall that, in the case of simple depreciation, for $\theta \in [\theta_n, \theta_{n-1}]$, $V^e(\theta) = \frac{q_n \theta - r_n}{\rho}$. In the general depreciation case, Lemma 2 shows that, for every $n = 0, \dots, N$, $V^e(\theta) = V_{n, \{n\}}^\rho(\theta, \emptyset) = (1 - w_n) \frac{E^n(q|q \leq q_n) \theta - r_n}{\rho} + w_n \frac{q_n \theta - r_n}{\rho}$; write this as $\frac{\tilde{q}_n \theta - r_n}{\rho}$, where

$$\tilde{q}_n = (1 - w_n) E^n(q|q \leq q_n) + w_n q_n, \quad n = 0, \dots, N.$$

It is then possible to rewrite equation (23), which determines the putative equilibrium rental prices r_0, \dots, r_N , as follows: $\tilde{q}_N \theta_N - r_N = 0$, $\tilde{q}_n \theta_n - r_n = \tilde{q}_{n+1} \theta_n - r_{n+1}$ for $n = 0, \dots, N-1$. Notice that this is analogous to equation (6), except that the ‘‘experimentation-corrected’’ quantities \tilde{q}_n are used in lieu of the actual ones (note however that $\tilde{q}_N = q_N$). In other words, the simple depreciation case corresponds to setting $w_n = 1$ independently of Δ .

For $n < N$, $r_n = r_{n+1} + \theta_n(\tilde{q}_n - \tilde{q}_{n+1})$, and hence

$$r_n = \tilde{q}_N \theta_N + \sum_{m=n}^{N-1} \theta_m (\tilde{q}_m - \tilde{q}_{m+1}). \quad (28)$$

Considering $\theta \in [\theta_n, \theta_{n-1}]$ and substituting for $V^e(\theta)$ in equation (27) then yields

$$R_\theta(q^j, q^{j+1}) = \frac{1 - e^{-\rho\Delta}}{\rho} q^j \theta - (1 - e^{-\rho\Delta}) \frac{\tilde{q}_n \theta - r_n}{\rho} = \frac{1 - e^{-\rho\Delta}}{\rho} [(q^j - \tilde{q}_n) \theta + r_n].$$

Assume for concreteness that $q^j = q_\ell$, and rewrite the above as

$$R_\theta(q^j, q^{j+1}) = \frac{1 - e^{-\rho\Delta}}{\rho} [(\tilde{q}_\ell - \tilde{q}_n)\theta + r_n] + \frac{1 - e^{-\rho\Delta}}{\rho} (q_\ell - \tilde{q}_\ell)\theta.$$

We claim first that $(\tilde{q}_\ell - \tilde{q}_n)\theta + r_n \leq r_\ell$. Suppose first that $\ell \leq n$: from equation (28),

$$r_\ell - r_n = \sum_{m=\ell}^{n-1} \theta_m (\tilde{q}_m - \tilde{q}_{m+1}) \geq \theta \sum_{m=\ell}^{n-1} (\tilde{q}_m - \tilde{q}_{m+1}) = (\tilde{q}_\ell - \tilde{q}_n)\theta,$$

because, for $m = \ell, \dots, n-1$, $\theta_m \geq \theta_{n-1} \geq \theta \in [\theta_n, \theta_{n-1}]$. If instead $\ell > n$,

$$r_n - r_\ell = \sum_{m=n}^{\ell-1} \theta_m (\tilde{q}_m - \tilde{q}_{m+1}) \leq \theta \sum_{m=n}^{\ell-1} (\tilde{q}_m - \tilde{q}_{m+1}) = (\tilde{q}_n - \tilde{q}_\ell)\theta,$$

because, for $m = n, \dots, \ell-1$, $\theta_m \leq \theta_n \leq \theta \in [\theta_n, \theta_{n-1}]$. Therefore, if $q^j = q_\ell$,

$$R_\theta(q^j, q^{j+1}) \leq \frac{1 - e^{-\rho\Delta}}{\rho} r_\ell + \frac{1 - e^{-\rho\Delta}}{\rho} (q_\ell - \tilde{q}_\ell)\theta \leq \frac{1 - e^{-\rho\Delta}}{\rho} r_\ell + \frac{1 - e^{-\rho\Delta}}{\rho} (q_\ell - \tilde{q}_\ell)\bar{\theta}.$$

Therefore, the menu \mathcal{M}' (hence, the original menu \mathcal{M}) cannot improve upon the menu consisting of the putative equilibrium rental contracts by more than $\frac{1 - e^{-\rho\Delta}}{\rho} \max_n (q_n - \tilde{q}_n)\bar{\theta}$ per period. Under simple depreciation, $\tilde{q}_n = q_n$ for all n , which concludes the proof of Theorem 3.

To complete the proof of Theorem 5, note that the gains from deviating from the putative equilibrium rental contracts cannot exceed the maximum per-period gain times the expected lifetime of the car (i.e. the expected number of periods until the car dies). For every $n = 0, \dots, N$, the number of periods until a car of quality q_n depreciates is a geometric random variable with parameter $G_n\Delta$, so the expected number of periods until depreciation is $\frac{1}{G_n\Delta}$ (recall that a car depreciates at the end of the period, so if depreciation occurs in the first time period, this means that the car has remained at quality level q_n for one period). We can then argue inductively as follows. Let L_n be the expected lifetime (in periods) of a car of quality q_n , where $n = N+1$ signifies death. Then $L_{N+1} = 0$ and

$$L_n = \frac{1}{G_n\Delta} + \sum_{m=n+1}^{N+1} \frac{\gamma_{n,m}}{G_n} L_m.$$

Thus, if the deviating firm has a car of quality q_n , she cannot improve upon the putative equilibrium menu by more than $\frac{1 - e^{-\rho\Delta}}{\rho} \max_\ell (q_\ell - \tilde{q}_\ell)\bar{\theta} \cdot L_n$, where again we let $n = N+1$ signify that the car has already died. We argue that, for all n , $\frac{1 - e^{-\rho\Delta}}{\rho} \max_\ell (q_\ell - \tilde{q}_\ell)\bar{\theta} \cdot L_n \rightarrow 0$ as $\Delta \rightarrow 0$. This is

trivially true for $n = N + 1$. Thus, assume it is true for $n + 1$. Then

$$\begin{aligned}
& \lim_{\Delta \rightarrow 0} \frac{1 - e^{-\rho\Delta}}{\rho} \max_{\ell} (q_{\ell} - \tilde{q}_{\ell}) \bar{\theta} \cdot L_n \\
= & \lim_{\Delta \rightarrow 0} \left[\frac{1 - e^{-\rho\Delta}}{\rho} \max_{\ell} (q_{\ell} - \tilde{q}_{\ell}) \bar{\theta} \cdot \frac{1}{G_n \Delta} + \sum_{m=n+1}^{N+1} \frac{\gamma_{n,m}}{G_n} \frac{1 - e^{-\rho\Delta}}{\rho} \max_{\ell} (q_{\ell} - \tilde{q}_{\ell}) \bar{\theta} \cdot L_m \right] \\
= & \lim_{\Delta \rightarrow 0} \frac{1 - e^{-\rho\Delta}}{\rho G_n \Delta} \max_{\ell} (q_{\ell} - \tilde{q}_{\ell}) \bar{\theta} \\
= & \lim_{\Delta \rightarrow 0} \frac{1 - e^{-\rho\Delta}}{\rho G_n \Delta} \cdot \lim_{\Delta \rightarrow 0} \max_{\ell} (q_{\ell} - \tilde{q}_{\ell}) \bar{\theta} = \frac{1}{G_n} \cdot \lim_{\Delta \rightarrow 0} \max_{\ell} (q_{\ell} - \tilde{q}_{\ell}) \bar{\theta} = 0,
\end{aligned}$$

because $\tilde{q}_{\ell} \rightarrow q_{\ell}$ for all ℓ as $\Delta \rightarrow 0$. This proves Theorem 5.

7 References

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WebAppendix

NOTE: This section contains material we will make available for download from our Web pages; it is *not* part of the actual paper. We construct equilibria under selling with both observable and unobservable vintages (see Section 3.5.6 in the paper).

Equilibrium Under Selling with Unobservable Vintages

There are only “new” and “old” cars; prices are p_n and p_u . Types $\theta \in [\theta_0, \bar{\theta}]$ buy new cars and keep only quality q_0 (i.e. sell as soon as car depreciates). Types $\theta \in [\theta_{01}, \theta_0]$ buy new cars and keep qualities q_0 and q_1 . Types $\theta \in [\theta_1, \theta_{01}]$ buy used cars, keep q_1 , and sell q_2 . It is convenient to denote masses of buyers as follows: $v_0 = 1 - F(\theta_0)$ is the mass of new car buyers who only keep q_0 ; $v_{01,0}$ (resp. $v_{01,1}$) is the mass of new car buyers who keep q_0 and q_1 and, in any given period (in steady state), happen to own a quality- q_0 (resp. q_1); it must be the case that $v_{01,0} + v_{01,1} = F(\theta_{01}) - F(\theta_0)$. Furthermore, let $v_{1,1}$ be the mass of types who buy used and happen to own a quality- q_1 car; finally, $v_{1,2}$ is the mass of buyers who buy used and happen to own a quality- q_2 ; it must be the case that $v_{1,1} + v_{1,2} = F(\theta_{01}) - F(\theta_1)$ and $1 - F(\theta_1) = Y$.

Now let φ denote the fraction of new cars that are bought by types who then only keep quality q_0 . We have

$$\begin{aligned}
 v_0 &= (1 - \gamma_0 \Delta)v_0 + v_{1,2} \gamma_2 \Delta \varphi \\
 v_{01,0} &= (1 - \gamma_0 \Delta)v_{01,0} + v_{1,2} \gamma_2 \Delta (1 - \varphi) \\
 v_{01,1} &= (1 - \gamma_1 \Delta)v_{01,1} + \gamma_0 \Delta v_{01,0} \\
 v_{1,1} &= (1 - \gamma_1 \Delta)v_{1,1} + \gamma_0 \Delta v_0 \\
 v_{1,2} &= (1 - \gamma_2 \Delta)v_{1,2} + \gamma_1 \Delta (v_{01,1} + v_{1,1}).
 \end{aligned}$$

To clarify: in steady state, the total mass of types θ who buy new and keep q_0 equals the mass of such individuals whose car did not die in the previous period, plus the mass of such individuals whose cars died in the previous period and was replaced by a new car; in particular, the steady-state flow of replacement cars equals the mass of cars that were in the hands of buyers who buy used, happened to hold a car of quality q_2 , and whose car died. An identical interpretation holds for $v_{01,0}$. The interpretation of $v_{01,1}$ is similar, but now the interpretation of the second term is different: the inflow of buyers into this category equals the mass of buyers who are also buying new and keeping q_0 and q_1 , who had a car of quality q_0 in the previous period, which however depreciated. For $v_{1,1}$, the second term represents the mass of cars held by consumers who buy new and keep only q_0 (these are the only used cars that enter the market at quality level q_1). Finally, for $v_{1,2}$, the second term has an analogous interpretation; the first is more noteworthy. Recall that, in the equilibrium we are trying to construct, cars of quality q_2 that do not depreciate (hence, die) are immediately sold; however, until they depreciate, they remain part of the pool of used cars. Hence the first term.

Rearranging terms and noting that $(v_0 + v_{01,0})\gamma_0 \Delta = (v_{01,1} + v_{1,1})\gamma_1 \Delta = v_{1,2}\gamma_2 \Delta$ and $(v_0 +$

$v_{01,0}) + (v_{01,1} + v_1) + v_{1,2} = Y$, we get $v_{1,2}\gamma_2(\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}) = Y$ and therefore

$$\begin{aligned} v_0 + v_{01,0} &= \frac{\gamma_0^{-1}}{\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}} Y \equiv \lambda_0 Y \\ v_{01,1} + v_{1,1} &= \frac{\gamma_1^{-1}}{\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}} Y \equiv \lambda_1 Y \\ v_{1,2} &= \frac{\gamma_2^{-1}}{\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}} Y \equiv \lambda_2 Y. \end{aligned}$$

Finally, taking into account the way each quality is split among each group,

$$\begin{aligned} v_0 &= \lambda_0 Y \varphi \\ v_{01,0} &= \lambda_0 Y (1 - \varphi) \\ v_{01,1} &= \lambda_1 Y (1 - \varphi) \\ v_{1,1} &= \lambda_1 Y \varphi \\ v_{1,2} &= \lambda_2 Y. \end{aligned}$$

Note that *these quantities are independent of Δ* . Moreover, the fraction of quality- q_0 cars owned by buyers who keep q_0 and q_1 is $\frac{\lambda_0}{\lambda_0 + \lambda_1} = \frac{\gamma_0^{-1}}{\gamma_0^{-1} + \gamma_1^{-1}} \equiv \varphi_0$, the fraction of quality- q_0 cars in the new market is $\frac{\lambda_0}{\lambda_0 + \lambda_1(1 - \varphi)} = \frac{\gamma_0^{-1}}{\gamma_0^{-1} + \gamma_1^{-1}(1 - \varphi)} \equiv \lambda_0^n$, and the fraction of quality- q_1 cars in the used market is $\frac{\lambda_1 \varphi}{\lambda_1 \varphi + \lambda_2} = \varphi_1$. All these quantities are also independent of Δ .

Turn now to the value functions. Consider buyers who participate in the used-car market. Recall they must keep q_1 and sell q_2 immediately (because there are some quality- q_1 cars in the used-car market, and the price they get for their car equals the price they pay for another used car). We must determine the fraction of quality- q_1 cars that are supplied in every period. Types $\theta \in [\theta_0, \bar{\theta}]$ sell quality- q_1 cars, so the fresh supply of this quality equals $v_0 \gamma_0 \Delta = \lambda_0 Y \varphi \gamma_0 \Delta$; on the other hand, the $v_{01,1}$ types $\theta \in [\theta_{01}, \theta_0]$ who held a quality- q_1 car in the previous period, which then depreciated, sell a mass $v_{01,1} \gamma_1 \Delta = \lambda_1 Y (1 - \varphi) \gamma_1 \Delta$ of quality- q_2 cars. Furthermore, the $v_{1,2}$ types $\theta \in [\theta_1, \theta_{01}]$ who had a bad draw in the previous period, as well as the $v_{1,1}$ types in the same interval who had a quality- q_1 car in the previous period, which then depreciated, are also reselling their cars on the used market. This adds $v_{1,1} \gamma_1 \Delta + v_{1,2} (1 - \gamma_2 \Delta) = \lambda_1 Y \varphi \gamma_1 \Delta + \lambda_2 Y (1 - \gamma_2 \Delta)$ cars (note that we must make sure that the cars offered do not die). Hence, the fraction of quality- q_1 used cars offered each period in vintage 1 is

$$\begin{aligned} & \frac{\lambda_0 Y \varphi \gamma_0 \Delta}{\lambda_0 Y \varphi \gamma_0 \Delta + \lambda_1 Y (1 - \varphi) \gamma_1 \Delta + \lambda_1 Y \varphi \gamma_1 \Delta + \lambda_2 Y (1 - \gamma_2 \Delta)} \\ &= \frac{\lambda_0 \varphi \gamma_0 \Delta}{\lambda_0 \varphi \gamma_0 \Delta + \lambda_1 \gamma_1 \Delta + \lambda_2 (1 - \gamma_2 \Delta)} \\ &= \frac{\lambda_0 \varphi \gamma_0 \Delta}{\lambda_0 \varphi \gamma_0 \Delta + \lambda_2} \equiv \varphi^u, \end{aligned}$$

using the fact that $\lambda_1\gamma_1 = \lambda_2\gamma_2$. Observe that φ^u *does* depend upon Δ . Note that the fraction of quality- q_1 used cars of at any point in time, φ_1 , will in general be different from φ^u , because cars of quality q_2 accumulate in the used-car market. Hence

$$\begin{aligned} V_u(\theta) &= -p_u + \varphi^u \left\{ \frac{1 - e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta} [(1 - \gamma_1\Delta)W_{u,1}(\theta) + \gamma_1\Delta(p_u + V_u(\theta))] \right\} \\ &\quad + (1 - \varphi^u) \left\{ \frac{1 - e^{-\rho\Delta}}{\rho} q_2\theta + e^{-\rho\Delta} [V_u(\theta) + (1 - \gamma_2\Delta)p_u] \right\}, \\ W_{u,1}(\theta) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta} \{(1 - \gamma_1\Delta)W_{u,1}(\theta) + \gamma_1\Delta[p_u + V_u(\theta)]\}. \end{aligned}$$

To clarify: if the used car is q_1 , then buyers enjoy it for one period; then, if it does not depreciate, they get the continuation value $W_{u,1}(\theta)$ determined by the assumption that the car is sold as soon as it depreciates. If the used car is q_2 , it is sold immediately, but one must take into account the fact that the car may still die (hence the buyer may be unable to resell it).

Next, consider $\theta \in [\theta_{01}, \theta_0]$. These buyers buy a new car, and sell it when it depreciates to q_2 . We must still keep track of the continuation values; however, now a new car is guaranteed to be of quality q_0 .

$$\begin{aligned} V_{n,01}(\theta) &= -p_n + \frac{1 - e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta} [(1 - \gamma_0\Delta)W_{n,01,0}(\theta) + \gamma_0\Delta W_{n,01,1}(\theta)] \\ W_{n,01,1}(\theta) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta} \{(1 - \gamma_1\Delta)W_{n,01,1}(\theta) + \gamma_1\Delta[p_u + V_{n,01}(\theta)]\} \\ W_{n,01,0}(\theta) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta} \{(1 - \gamma_0\Delta)W_{n,01,0}(\theta) + \gamma_0\Delta W_{n,01,1}(\theta)\}. \end{aligned}$$

Finally, we consider buyers who buy new cars and keep q_0 .

$$\begin{aligned} V_{n,0}(\theta) &= -p_n + \frac{1 - e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta} \{(1 - \gamma_0\Delta)W_{n,0}(\theta) + \gamma_0\Delta[p_u + V_{n,0}(\theta)]\} \\ W_{n,0}(\theta) &= V_{n,0}(\theta) + p_n. \end{aligned}$$

To construct an equilibrium, consider an arbitrary $\varphi \in [0, 1]$: $\varphi = 1$ cannot yield an equilibrium, because it would induce an efficient allocation, which, by Theorems 1 and 1, is impossible. For each such φ , it is possible to choose prices p_u and p_n such that

$$V_u(\theta_1) = 0, \quad V_{n,01}(\theta_{01}) = V_u(\theta_{01}).$$

We now consider three cases. (1) If $V_{n,0}(\theta_0) < V_{n,01}(\theta_0)$ for all φ (recall that φ determines the cutoff θ_0), then in particular this is true for $\varphi = 0$, where $\theta_0 = \bar{\theta}$; this implies that, for $\varphi = 0$, $V_{n,0}(\theta) < V_{n,01}(\theta)$ for all θ (the argument requires a decomposition analogous to the one in the proof of Lemma 2). Hence $\varphi = 0$ yields the right incentives to all consumers types when they do not own a car.

(2) If $V_{n,0}(\theta_0) > V_{n,01}(\theta_0)$ for all values of φ , then in particular this is the case at $\varphi = 1$, where $\theta_0 = \theta_{01}$; for this value of φ , it is then the case that $V_{n,0}(\theta_0) > V_{n,01}(\theta_0) = V_u(\theta_0)$. Hence there exists a price $p'_n > p_n$ such that $V_{n,0}(\theta_0) = V_u(\theta_0) > V_{n,01}(\theta_0)$; that is, at the prices p'_n, p_u , all types $\theta \in [\theta_0 = \theta_{01}, \bar{\theta}]$ buy new cars and keep only q_0 , and types $\theta \in [\theta_1, \theta_0]$ buy used cars.

(3) If, finally, there exist φ, φ' such that at the corresponding prices and cutoff types θ_0, θ'_0 , $V_{n,0}(\theta_0) < V_{n,01}(\theta_0)$ and $V_{n,0}(\theta'_0) > V_{n,01}(\theta'_0)$, then by continuity there exists φ'' such that equality obtains.

Thus, in all three cases, for an appropriate choice of prices and $\varphi \in [0, 1]$, consumers follow the policies described above when they do not own a car; to complete the argument, we now show that they also do so when they already own a car (i.e. they adopt the “right” keeping policies).

The argument for consumers who experiment with used cars is straightforward: if they currently own quality q_1 (resp. q_2) given that their best continuation policy is to buy another used car, they can only do worse (resp. better) in expectation by selling their current car. Thus, turn to type θ_{01} , assuming that $\varphi < 1$ (otherwise this case is irrelevant). It is clear that this type should not keep a car of quality q_2 . If her current car instead is of quality q_1 , her continuation value is

$$\begin{aligned} W_{n,01,1}(\theta_{01}) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_1 \theta_{01} + e^{-\rho\Delta} \{(1 - \gamma_1 \Delta) W_{n,01,1}(\theta_{01}) + \gamma_1 \Delta [p_u + V_{n,01}(\theta_{01})]\} = \\ &= \frac{1 - e^{-\rho\Delta}}{\rho} q_1 \theta_{01} + e^{-\rho\Delta} \{(1 - \gamma_1 \Delta) W_{n,01,1}(\theta_{01}) + \gamma_1 \Delta [p_u + V_u(\theta_{01})]\} = \\ &= W_u(\theta_{01}); \end{aligned}$$

if she instead sells her car, then she can get at most $p_u + V_{n,01}(\theta_{01}) = p_u + V_u(\theta_{01}) \leq W_u(\theta_{01})$. The inequality follows because the l.h.s. is the value of receiving a used car, which may be of quality q_1 or q_2 , and following the optimal keeping policy for used cars, whereas the r.h.s. is the value of receiving a car of quality q_1 , then following the same optimal keeping policy. Hence, type θ_{01} should keep a car of quality q_1 , and consequently she should also keep a car of quality q_0 . This implies that all other types in $[\theta_{01}, \theta_0]$ also have the correct incentives.

Finally, consider type θ_0 , assuming $\varphi > 0$ (otherwise this case is irrelevant). We must ensure that this type will be willing to sell quality q_1 . If she does, she obtains

$$p_u + V_{n,0}(\theta_0) = p_u + V_{n,01}(\theta_0) \geq W_{n,01,1}(\theta_{01}),$$

where the inequality follows because $p_u + V_{n,01}(\theta_0)$ is the value of receiving a car of quality q_0 , and keeping it until it depreciates to q_2 , then buying a new car and continuing with the same keeping policy. Since $V_{n,0}(\theta_{01}) = V_{n,01}(\theta_0)$, $W_{n,01,1}(\theta_{01})$ can equivalently be viewed as the value of keeping a car of quality q_1 until it depreciates, then reverting to the designated policy for type θ_0 . This shows that keeping q_1 is not a profitable deviation for type θ_0 , and concludes the proof.

Equilibrium Under Selling with Observable Vintages

Notation is approximately as above. Now types in $[\theta_0, \bar{\theta}]$ buy new (i.e. vintage 0) cars and keep only q_0 ; their mass is v_0 . Types in $[\theta_{01}, \theta_0]$ buy vintage 0 and keep q_0 and q_1 ; $v_{01,0}$ is the mass of such types who happen to own a quality- q_0 car, and $v_{01,1}$ is the mass of such types who own quality

q_1 . Types in $[\theta_1, \theta_{01}]$ buy vintage 1 and keep only q_1 ; $v_{1,1}$ and $v_{1,2}$ denote the masses of such types who own qualities q_1 and q_2 respectively. Finally, types in $[\theta_2, \theta_1]$ buy vintage 2 and keep quality q_2 ; their mass is v_2 . We thus have, in steady state,

$$\begin{aligned}
v_0 &= (1 - \gamma_0)v_0 + (v_2 + v_{1,2})\gamma_2\Delta\varphi \\
v_{01,0} &= (1 - \gamma_0\Delta)v_{01,0} + (v_2 + v_{1,2})\gamma_2\Delta(1 - \varphi) \\
v_{01,1} &= (1 - \gamma_1\Delta)v_{01,1} + v_{01,0}\gamma_0\Delta \\
v_{1,1} &= (1 - \gamma_1\Delta)v_{1,1} + v_0\gamma_0\Delta \\
v_{1,2} &= v_{01,1}\gamma_1\Delta \\
v_2 &= (1 - \gamma_2\Delta)v_2 + v_{1,1}\gamma_1\Delta + v_{1,2}(1 - \gamma_2\Delta).
\end{aligned}$$

To clarify, quality- q_1 cars of vintage 1 are cars previously owned by types in $[\theta_0, \bar{\theta}]$ that have just depreciated; quality- q_2 cars of vintage 1 instead are cars that were discarded by types in $[\theta_{01}, \theta_0]$. The latter cars are immediately resold, and hence become of vintage 2, provided they do not die: this explains the third term in the r.h.s. of the last equation. The remaining cars of vintage 2 are either surviving vintage-2 cars or vintage-1 cars that have just depreciated from q_1 to q_2 .

We solve as above. In particular, $(v_0 + v_{01,0})\gamma_0\Delta = (v_2 + v_{1,2})\gamma_2\Delta$ and $(v_{01,1} + v_{1,1})\gamma_1\Delta = (v_2 + v_{1,2})\gamma_2\Delta$, and we obtain $v_0 + v_{01,0} = \lambda_0 Y$, $v_{01,1} + v_{1,1} = \lambda_1 Y$ and $v_{1,2} + v_2 = \lambda_2 Y$, with λ_i as above. Therefore

$$\begin{aligned}
v_0 &= \lambda_0 Y \varphi \\
v_{01,0} &= \lambda_0 Y (1 - \varphi) \\
v_{01,1} &= \lambda_1 Y (1 - \varphi) \\
v_{1,1} &= \lambda_1 Y \varphi \\
v_{1,2} &= \lambda_1 Y (1 - \varphi) \gamma_1 \\
v_2 &= \lambda_2 Y - \lambda_2 Y (1 - \varphi) \gamma_2
\end{aligned}$$

(note that $\lambda_i \gamma_i = \lambda_j \gamma_j$ for all $i, j = 0, \dots, 2$). The fraction of quality- q_1 cars of vintage 1 is $\varphi_1 = \frac{\varphi}{\varphi + (1 - \varphi)\gamma_1\Delta}$, and the fraction of quality- q_0 cars in the hands of types $\theta \in [\theta_{01}, \theta_0]$ is $\varphi_0 = \frac{\lambda_0}{\lambda_0 + \lambda_1}$.

Turn now to value functions and prices. For types $\theta \in [\theta_2, \theta_1]$,

$$V_2(\theta) = -p_2 + \frac{1 - e^{-\rho\Delta}}{\rho} q_2 \theta + e^{-\rho\Delta} \{(1 - \gamma_2\Delta)[V_2(\theta) + p_2] + \gamma_2\Delta V_2(\theta)\}.$$

Next, consider $\theta \in [\theta_1, \theta_{01}]$. We must distinguish between buyers who currently own a quality- q_1 car, and those who currently own q_2 (and hence will immediately dispose of it). The key issue here is the composition of the supply of vintage-1 cars: $\lambda_0 Y \varphi \gamma_0 \Delta$ come from types $\theta \in [\theta_0, 1]$, and hence are of quality q_1 ; $\lambda_1 Y (1 - \varphi) \gamma_1 \Delta$ come from types $\theta \in [\theta_{01}, \theta_0]$, and hence are of quality q_2 . Therefore, the fraction of quality- q_1 cars supplied is

$$\frac{\lambda_0 Y \varphi \gamma_0 \Delta}{\lambda_0 Y \varphi \gamma_0 \Delta + \lambda_1 Y (1 - \varphi) \gamma_1 \Delta} = \frac{\varphi}{\varphi + 1 - \varphi} = \varphi,$$

where we use the fact that $\lambda_0\gamma_0 = \lambda_1\gamma_1$. Hence we can write

$$\begin{aligned} V_1(\theta) &= -p_1 + \varphi \left\{ \frac{1 - e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta} [(1 - \gamma_1\Delta)W_{1,1}(\theta) + \gamma_1\Delta(V_1(\theta) + p_2)] \right\} \\ &\quad + (1 - \varphi) \left\{ \frac{1 - e^{-\rho\Delta}}{\rho} q_2\theta + e^{-\rho\Delta} [V_1(\theta) + (1 - \gamma_2\Delta)p_2] \right\} \\ W_{1,1}(\theta) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta} \{ (1 - \gamma_1\Delta)W_{1,1}(\theta) + \gamma_1\Delta[V_1(\theta) + p_2] \}. \end{aligned}$$

To clarify: consider a buyer who currently has no car. If she buys a vintage-1 car, with probability φ she gets q_1 ; $W_{1,1}(\theta)$ represents her continuation payoff, assuming the car does not depreciate at the end of the period. With probability $1 - \varphi$, she gets q_2 , in which case she sells the car immediately, provided the car does not die at the end of the period; note that, in any case, the buyer will purchase a vintage-1 car in the next period if her current car is of quality q_2 .

Now consider $\theta \in [\theta_{01}, \theta_0]$. Recall that these buyers sell their cars only when it depreciates to q_2 .

$$\begin{aligned} V_{01}(\theta) &= -p_0 + \frac{1 - e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta} [(1 - \gamma_0\Delta)W_{01,0}(\theta) + \gamma_0\Delta W_{01,1}(\theta)] \\ W_{01,1}(\theta) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta} \{ (1 - \gamma_1\Delta)W_{01,1}(\theta) + \gamma_1\Delta[p_1 + V_{01}(\theta)] \} \\ W_{01,0}(\theta) &= \frac{1 - e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta} [(1 - \gamma_0\Delta)W_{01,0}(\theta) + \gamma_0\Delta W_{01,1}(\theta)] \end{aligned}$$

Note that the problem is exactly the same as the problem faced by consumers $\theta \in [\theta_{01}, \theta_0]$ in the no-vintages case: simply let $p_0 = p_n$ and $p_1 = p_u$.

Finally, $V_0(\theta)$ is exactly like $V_{n,0}$ in the no-vintage case:

$$\begin{aligned} V_0(\theta) &= -p_0 + \frac{1 - e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta} \{ (1 - \gamma_0\Delta)W_0(\theta) + \gamma_0\Delta[p_1 + V_0(\theta)] \} \\ W_0(\theta) &= V_0(\theta) + p_1. \end{aligned}$$

To establish the existence of an equilibrium, we proceed as in the case of unobservable vintages. For every value of $\varphi \in [0, 1]$, we can determine p_0, p_1, p_2 via the indifference conditions

$$V_2(\theta_2) = 0, \quad V_1(\theta_1) = V_2(\theta_1), \quad V_{01}(\theta_{01}) = V_1(\theta_{01}).$$

However, Theorems 1 and 1 imply that $\varphi = 1$ cannot correspond to an equilibrium, because it implies efficiency. Furthermore, the proof of Theorem 1 shows that, at $\varphi = 1$, type $\theta_{01} = \theta_0$ will strictly prefer to keep quality q_1 rather than resell her current car and buy another vintage-0 car. This easily implies that, for $\varphi = 1$, $V_0(\theta_0) < V_{01}(\theta_0)$. Hence, we only need to consider two cases: if $V_0(\theta_0) > V_{01}(\theta_0)$ for all φ , then $\varphi = 0$ yields the right incentives when consumers do not own a car; otherwise, $V_0(\theta_0) = V_{01}(\theta_0)$ for some $\varphi \in [0, 1)$.

Incentives when consumers already own a car are verified as in the case of unobservable vintages, so the proof is omitted.