Dynamic Matching,Two-sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition

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March 13, 2006

Abstract

Consider a decentralized, dynamic market with an infinite horizon and participation costs in which both buyers and sellers have private information concerning their values for the indivisible traded good. Time is discrete, each period has length δ, and each unit of time continuums of new buyers and sellers consider entry. Traders whose expected utility is negative choose not to enter. Within a period each buyer is matched anonymously with a seller and each seller is matched with zero, one, or more buyers. Every seller runs a first price auction with a reservation price and, if trade occurs, both the seller and winning buyer exit the market with their realized utility. Traders who fail to trade continue in the market to be rematched. We characterize the steady-state equilibria that are perfect Bayesian. We show that, as δ converges to zero, equilibrium prices at which trades occur converge to the Walrasian price and the realized allocations converge to the competitive allocation. We also show existence of equilibria for δ sufficiently small, provided the discount rate is small relative to the participation costs.

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†We owe special thanks to Zvika Neeman who originally devised a proof showing the strict monotonicity of strategies. We also thank Hector Chade, Patrick Francois, Paul Milgrom, Dale Mortensen, Peter Norman, Mike Peters, Jeroen Swinkels, Asher Wolinsky, Jianjun Wu, Okan Yilankaya, and two very perceptive anonymous referees; the participants at the "Electronic Market Design Meeting" (June 2002, Schloss Dagstuhl), the 13th Annual International Conference on Game Theory at Stony Brook, the 2003 NSF Decentralization Conference held at Purdue University, the 2003 General Equilibrium Conference held at Washington University, the 2003 Summer Econometric Society Meeting held at Northwestern University, the 2004 Canadian Economic Theory Conference held in Montreal, and Games 2004 held at Luminy in Marseille; seminar participants at Carnegie-Mellon, Washington University, Northwestern University, University of Michigan, Harvard and MIT, UBC, and Stanford; and the members of the collaborative research group on "Foundations of Electronic Marketplaces" for their constructive comments. Adam Wong provided excellent research assistance. Finally, both of us acknowledge gratefully that this material is based on work supported by the National Science Foundation under Grant IIS-0121541. Artyom Shneyerov also acknowledges support from the Canadian SSHRC Grant 410-2003-1366.

1 Introduction

The frictions of asymmetric information, search costs, and strategic behavior interfere with efficient trade. Nevertheless economists have long believed that for private goods' economies the presence of many traders overcomes these imperfections and results in convergence to perfect competition. Two classes of models demonstrate this. First, static double auction models in which traders' costs and values are private exhibit rapid convergence to the competitive price and the efficient allocation within a one-shot centralized market. Second, dynamic matching and bargaining models in which traders' costs and values are common knowledge also converge to the competitive equilibrium. The former models are unrealistic in that they assume traders who fail to trade now can not trade later. Tomorrow (almost) always exists for economic agents. The latter models are unrealistic in assuming traders have no private information. Information about a trader's cost/value (almost) always contains a component that is private to him. This paper's contribution is to formulate a natural model of dynamic matching and bargaining with two-sided incomplete information and to show that it converges to the competitive allocation and price as frictions vanish.

An informal description of our model and result is this. An indivisible good is traded in a market in which time progresses in discrete periods of length δ and generations of traders overlap. Each unit of time traders who are active in the market incur a participation cost κ and a discount rate β. Thus the per period participation cost is $\delta\kappa$, the per period discount factor is $e^{-\beta\delta}$, and they both vanish as the period length converges to zero. Each period every active buyer randomly matches with an active seller. Depending on the luck of the draw, a seller may end up being matched with several buyers, a single buyer, or no buyers. Each seller solicits a bid from each buyer with whom she is matched and, if the highest of the bids is satisfactory to her, she sells her single unit of the good and both she and the successful buyer exit the market. A buyer or seller who fails to trade remains in the market, is rematched the next period, and tries again to trade.

Each unit of time a large number of potential sellers (formally, measure 1 of sellers) considers entry into the market along with a large number of potential buyers (formally, measure a of buyers). Each potential seller independently draws a cost c in the unit interval from a distribution G_S and each potential buyer draws independently a value v in the unit interval from a distribution G_B. Individuals' costs and values are private to them. A potential trader only enters the market if, conditional on his private cost or value, his equilibrium expected utility of entry is at least zero. Potential traders whose discounted expected utilities are negative elect not to participate.

If in period t trade occurs between a buyer and seller at price p, then they exit with their gains from trade, $v - p$ and $p - c$ respectively, less their participation costs accumulated at the discount rate β from their times of entry onward. If δ is large (i.e., periods are long), then participation costs accumulate in a short number of periods and a trader who chooses to enter must be confident that he can obtain a profitable trade without much search. If, however, δ is small, then a trader can wait through many matches looking for a good price with little concern about participation costs and discounting offsetting his gains from trade. This option value effect drives convergence and puts pressure on traders on the opposite side of the market to offer competitive terms. As δ becomes small the market for each trader becomes, in effect, large.

We characterize steady state equilibria for this market in which each agent maximizes his expected utility going forward. We show that, as the period length δ goes to zero, all such equilibria converge to the Walrasian price and the competitive allocation. The Walrasian price p_W in this market is the solution to the equation

$$G_S(p_W) = a(1 - G_B(p_W)), \tag{1}$$

i.e., it is the price at which the measure of entering sellers with costs less than p_W equals the measure of entering buyers with values greater than p_W. If the market were completely centralized with every active buyer and seller participating in an exchange that cleared each period's bids and offers simultaneously, then p_W would be the market clearing price each period. Our precise result is this. Among active traders, let $\bar{c}_\delta$ and $\underline{v}_\delta$ be the maximal seller's type and minimal buyer's type respectively and let $[\underline{p}_\delta, \bar{p}_\delta]$ be the range of prices at which trades occur. Also let $\underline{c}_\delta$ be the smallest bid acceptable to any active seller. As $\delta \to 0$, then $\underline{c}_\delta$, $\bar{c}_\delta$, $\underline{v}_\delta$, $\underline{p}_\delta$, and $\bar{p}_\delta$ all converge to the same limit p. In the steady state, the only way for the market to clear is for this limit p to be equal to the competitive price p_W. That the resulting allocations give traders the expected utility they would realize in a perfectly competitive market follows. Finally, we show that if the period length δ and, relative to the level of participation costs κ, the discount rate β are both sufficiently small, then a full trade equilibrium exists. Full trade equilibria are a special class of equilibria in which active sellers immediately trade upon being matched with at least one buyer.

This is a step towards a theory of how a completely decentralized, dynamic market with two-sided incomplete information and participation costs implements, increasingly well, an almost efficient allocation as the speed with which traders' are able to seek out potential trading partners increases. In making this step we assume independent private values, which means that all traders a priori know the underlying Walrasian price in the market. In the theory that we ultimately seek traders' would have less restrictive preferences (e.g., correlated private values or interdependent values) in which the Walrasian price follows some stochastic process. Efficient trade would therefore require that traders' equilibrium strategies reveal sufficient information not only to identify the most valuable currently feasible trades, but also to reveal the underlying, changing Walrasian price even as it simultaneously facilitates trade at that price. A complete theory would both identify sufficient conditions for which convergence to an efficient allocation and Walrasian price is guaranteed and show how, when those conditions are not met, the equilibrium may fail to converge to efficiency.

We hope that the insights and results here will contribute to the development of such a theory. This paper first shows that convergence to one price occurs and is driven by option value: as δ, the length of a period decreases, each trader becomes more willing to decline a merely decent offer so as to preserve the option to accept a really excellent offer in the future. With both buyers and sellers doing this the price distribution in the market rapidly narrows as δ becomes small. Second, it shows that the price distribution must converge to the Walrasian price because if an equilibrium exists in which it does not, then too many traders accumulate on the long side of the market, which creates incentives for these traders to deviate from their equilibrium strategies by bidding more aggressively. We would be surprised if either of these insights fail to carry over to models with less restrictive processes for generating valuations.

This progression has been true for static double auctions. Almost all the early papers assumed independent private values, e.g., Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989a), Satterthwaite and Williams (1989b), Williams (1991), and Rustichini, Satterthwaite, and Williams (1994). Recently Cripps and Swinkels (2004) and Fudenberg, Mobius, and Szeidl (2003) have generalized rates of convergence results from the independent private values environment to the correlated private value case. Further, Reny and Perry (2003) in a carefully constructed model with interdependent valuations have shown that the static double auction equilibrium exists and converges to a rational expectations equilibrium as the number of traders on both sides of the market becomes large.

A substantial literature exists that investigates the non-cooperative foundations of perfect competition using dynamic matching and bargaining games.[1] Most of the work of which we are aware has assumed complete information in that each participant knows every other participant's values (or costs) for the traded good. The books of Osborne and Rubinstein (1990) and Gale (2000) contain excellent discussions of both their own and others' contributions to this literature. Papers that have been particularly influential include Mortensen (1982), Rubinstein and Wolinsky (1985, 1990), Gale (1986, 1987) and Mortensen and Wright (2002). Of these, our paper is most closely related to the models and results of Gale (1987) and Mortensen and Wright (2002). The two main differences between their work and ours are that (i) when two traders meet they reciprocally observe the other's cost/value rather than remaining uninformed and (ii) the terms of trade are determined as the outcome of a full information bargaining game rather than an auction. The first difference—full versus incomplete information—is fundamental, for the purpose of our paper is to determine if a decentralized market can elicit private valuation information at the same time it uses that information to assign the available supply almost efficiently. The second difference is natural given our focus on incomplete information.

The most important dynamic bargaining and matching models that incorporate incomplete information are Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano

[1]There is a related literature that we do not discuss here concerning the micro-structure of intermediaries in markets, e.g., Spulber (1999) and Rust and Hall (2002). These models allow entry of an intermediary who posts fixed ask and offer prices and is assumed to be large enough to honor any size buy or sell order without exhausting its inventory or financial resources.

(2002).[2,3] To understand how our paper relates to these papers, consider the following problem as the baseline. Each unit of time fixed measures of sellers and buyers enter the market, each of whom has a private cost/value for a single unit of the homogeneous good. The sellers' units of supply need to be reallocated to those traders who most highly value them. Whatever mechanism that is employed must induce the traders to reveal sufficient information about their costs/valuations in order to carry out the reallocation. The static double auction results of Satterthwaite and Williams (1989a) and Rustichini, Satterthwaite and Williams (1994) show that even moderately-sized centralized double auctions held once per unit time solves this problem essentially perfectly by closely approximating the Walrasian price and then using that price to mediate trade.[4]

Given this definition of the problem, the reason why Wolinsky (1988), Serrano (2002), and De Fraja and Sákovics (2001) do not obtain competitive outcomes as the frictions in their models vanish is clear: the problems their models address are different and, as their results establish, not intrinsically perfectly competitive even when the market becomes almost frictionless. Thus Wolinsky's model relaxes the homogeneous good assumption and does not fully analyze the effects of entry/exit dynamics. Serrano's model embeds a discrete-price double auction mechanism in a dynamic matching framework. There are, however, no entering cohorts of traders. Consequently, the option-value effects become progressively smaller as the most avid buyers and sellers leave the market through trading and are not replaced. As the market runs down and becomes small, necessarily it becomes less and less competitive. Not surprisingly Serrano finds that "equilibria with Walrasian and non-Walrasian features persist."

Closest to our model is De Fraja and Sákovics' model. Traders search for the best price in a market similar to the market we study. Option value drives the market to one price, much as in our model. Its entry/exit specification, however, is quite different than our specification in that it does not specify that fixed measures of buyers and sellers enter the market each unit of time as in our baseline problem. Instead, whenever two

[2] Butters in an unfinished manuscript (circa 1979) that was well before its time considers convergence in a dynamic matching and bargaining problem. The main differences between our model and his are (i) he assumes an exogenous exit rate instead of a participation cost, (ii) traders who have zero probability of trade participate in the market until they exit stochastically due to the exogenous exit rate, and (iii) the matching is one-to-one and the matching probabilities do not depend on the ratio of buyers and sellers in the market. We thank Asher Wolinsky for bringing Butters' manuscript to our attention after we had completed an earlier version of this paper.

[3] In a companion paper we (Satterthwaite and Shneyerov, 2003) consider a dynamic matching and bargaining model that has no participation costs, but instead has the alternative friction of a fixed, exogenous, per unit time rate of exit among active traders. For this model we show convergence to p_W as the period length becomes short, but have not been able to show existence. There are two main effects of substituting an exit rate for participation costs. First, traders who enter with postitive expected utility do not necessarily trade; they may spontaneously exit with zero utility prior to making a successful match. This makes market clearing more subtle and, as a consequence, demonstrates more clearly than this paper's model the power of supply and demand to force price to converge to p_W. Second, the structure of equilibrium strategies is different than in the participation cost case. In particular, full trade equilibria, which play a leading role in this paper's existence proof, are easily shown not to exist.

[4] Another example of a centralized trading institution is the system of simultaneous ascending-price auctions, studied in Peters and Severinov (2002). They also find robust convergence to the competitive outcome.

traders consummate a trade and exit the market, then two traders of identical types replace them. This means that, no matter what limiting price the market converges to as search becomes cheap, the distribution of traders' valuations in the market remains constant. Consequently, if the limiting price is above the Walrasian price, then sellers do not accumulate in the market and, unlike in our model, sellers have no incentive to reduce the prices that they ask. Supply and demand does not affect price in De Fraja and Sákovics' model. Instead the exogenously specified balance of bargaining power determines the limiting price.

The next section formally states the model and our convergence and existence results. Section 3 derives basic properties of equilibria. Section 4 proves convergence for all equilibria and Section 5 proves that, for sufficiently small δ and β, the special class of equilibria—full trade equilibria—exist. Section 6 concludes.

2 Model and Theorems

We study the steady state of a market with two-sided incomplete information and an infinite horizon. In it heterogeneous buyers and sellers meet once per period ($t = \ldots, -1, 0, 1, \ldots$) and trade an indivisible, homogeneous good. Every seller is endowed with one unit of the traded good and has cost $c \in [0,1]$. This cost is private information to her; to other traders it is an independent random variable with distribution G_S and density g_S. Similarly, every buyer seeks to purchase one unit of the good and has value $v \in [0,1]$. This value is private; to others it is an independent random variable with distribution G_B and density g_B. Our model is therefore the standard independent private values model. We assume that the two densities are bounded away from zero: a $\underline{g} > 0$ exists such that, for all $c, v \in [0,1]$, $g_S(c) > \underline{g}$ and $g_B(v) > \underline{g}$.

The strategy of a seller, $S : [0,1] \to R \cup \{\mathcal{N}\}$, maps her cost c into either a decision $\mathcal{N}$ not to enter or a minimal bid that she is willing to accept. Similarly the strategy of a seller, $B : [0,1] \to R \cup \{\mathcal{N}\}$, maps his value v into either a decision $\mathcal{N}$ not to enter or the bid that he places when he is matched with a seller. A trader only enters if his expected discounted utility from doing so is non-negative; if he elects not to enter he receives utility zero.

The length of each period is $\delta > 0$. Each unit of time measure 1 of potential sellers and measure a of potential buyers consider entry where $a > 0$. This means that each period measure δ of potential sellers and measure δa of potential buyers consider entry. A period consists of five steps:

1. Entry occurs. A type v potential buyer becomes active only if $B(v) \neq \mathcal{N}$ and a type c potential seller becomes active only if $S(c) \neq \mathcal{N}$.

2. Every active seller and buyer incurs participation cost $\delta\kappa$ where $\kappa > 0$ is the cost per unit time of being active.

3. Each active buyer is randomly matched with one active seller. Consequently every seller is equally likely to end up matched with any active buyer. The probability

π_k that a seller is matched with $k \in \{0, 1, 2, \ldots\}$ buyers is therefore Poisson:

$$\pi_k(\zeta) = \frac{\zeta^k}{k!\, e^{\zeta}} \tag{2}$$

where ζ is the endogenous ratio of active buyers to active sellers.[5] Consequently a seller may end up being matched with zero buyers, one buyer, two buyers, etc. These matches are anonymous, i.e., no trader knows the history of any trader with whom he or she happens to be matched.

4. Each buyer simultaneously announces a bid $B(v)$ to the seller with whom he is matched. We assume that, at the time he submits his bid, each buyer only knows the endogenous steady state probability distribution of how many buyers with whom he is competing. After receiving the bids, the seller either accepts or rejects the highest bid. Denote by $S(c)$ the minimal bid (i.e., reservation price) acceptable to a type c seller. If two or more buyers tie with the highest bid, then the seller uses a fair lottery to choose between them. If a type v buyer trades in period t, then he leaves the market with utility $v - B(v)$. If a type c seller trades at price p, then she leaves the market with utility $p - c$ where p is the bid she accepts.

5. All remaining traders carry over to the next period.

Traders discount their expected utility at the rate $\beta \geq 0$ per unit time; $e^{-\beta\delta}$ is therefore the factor by which each trader discounts his utility per period of time.

To formalize the fact that the distribution of trader types within the market's steady state is endogenous, let T_S be the measure of active sellers in the market at the beginning of each period, T_B be the measure of active buyers, F_S be the distribution of active seller types, and F_B be the distribution of active buyer types. The corresponding densities are f_S and f_B and, establishing useful notation, the right-hand distributions are $\bar{F}_S \equiv 1 - F_S$ and $\bar{F}_B \equiv 1 - F_B$. The ratio ζ is therefore equal to T_B/T_S.

By a steady state equilibrium we mean one in which every seller in every period plays a time invariant strategy $S(\cdot)$, every buyer plays a time invariant strategy $B(\cdot)$, and both these strategies are always optimal. Let $W_S(c)$ and $W_B(v)$ be the sellers and buyers' interim utilities for sellers of type c and the buyers of type v respectively, i.e., they are beginning-of-period, steady state, equilibrium net payoffs conditional on their types. Given the friction δ, a market equilibrium M_δ consists of strategies $\{S, B\}$, traders' masses $\{T_S, T_B\}$, and distributions $\{F_S, F_B\}$ such that (i) $\{S, B\}$, $\{T_S, T_B\}$, and $\{F_S, F_B\}$ generate $\{T_S, T_B\}$ and $\{F_S, F_B\}$ as their steady state and (ii) no type of trader can increase his or her expected utility (including the continuation payoff from matching in future periods if trade fails) by a unilateral deviation from the strategies $\{S, B\}$, and (iii) equilibrium strategies $\{S, B\}$, masses $\{T_S, T_B\}$, and distributions $\{F_S, F_B\}$ are

[5] In a market with M sellers and ζM buyers, the probability that a seller is matched with k buyers is $\pi_k^M = \binom{\zeta M}{k}\left(\frac{1}{M}\right)^k\left(1 - \frac{1}{M}\right)^{\zeta M - k}$. Poisson's theorem (see, for example, Shiryaev, 1995) shows that $\lim_{M\to\infty} \pi_k^M = \pi_k$.

common knowledge among all active and potential traders. We study perfect Bayesian equilibria of this model [6]

Four points need emphasis concerning this setup. First, since within a given match buyers announce their bids simultaneously and only then does the seller decide to accept or reject the highest of the bids, the subgame perfection aspect of perfect Bayesian equilibria implies that a seller whose highest received bid is above her dynamic opportunity cost of $c + e^{-\beta\delta} W_S(c)$ accepts that bid. In other words, a seller's strategy is her dynamic opportunity cost,

$$S(c) = c + e^{-\beta\delta} W_S(c),$$

and is independent of the number of buyers who are bidding, i.e., $S(c)$ is her reservation price. Second, beliefs are simple to handle because our assumptions that there are continuums of traders, that all matching is anonymous, and that traders' values and costs conform to the standard independent private values model imply that off-the-equilibrium path actions do not cause inference ambiguities. Third, step 5 within each period requires every trader who enters to stay in the market until he eventually succeeds in trading. Obviously, given that our goal is modeling a decentralized market, this is inappropriate; traders should be free to exit. However, given independent private values and a steady state equilibrium, forbidding exit has no loss of generality. The reason is that a trader only enters the market if his expected utility is non-negative. Being in a steady state implies that if he had non-negative expected utility when he entered, then at the beginning of any subsequent period after failing to trade he has the same non-negative utility going forward. Even if exit without trade were permitted, he would not do so. Fourth, an uninteresting no-trade equilibrium always exists in which all potential buyers and sellers decline to enter. We analyze equilibria in which positive trade occurs, i.e., equilibria in which each period positive measures of buyers and sellers enter and ultimately trade.

Let $\bar{c}$ and $\underline{v}$ to be the maximal seller and minimal buyer types that choose to enter. These are the marginal participation types. Let $\underline{c}, \underline{p}$, and $\bar{p}$ be respectively the lowest bid that is acceptable to the cost zero seller, the lowest bid any active buyer makes, and the highest bid that any active buyer makes. Formally, define $A_S \subset [0,1]$ and $A_B \subset [0,1]$ to be the sets of active sellers and buyers' types respectively. Then:

$$\begin{aligned}
\bar{c} &\equiv \sup\{c \,|\, c \in A_S\} \qquad \text{(maximum active seller type)},\\
\underline{v} &\equiv \inf\{v \,|\, v \in A_B\} \qquad \text{(minimum active buyer type)},\\
\underline{c} &= \inf\{S(c) \mid c \in A_S\} \qquad \text{(minimum acceptable bid)},\\
\underline{p} &= \inf\{B(v) \mid v \in A_B\} \qquad \text{(maximum bid)},\\
\bar{p} &= \sup\{B(v) \mid v \in A_B\} \qquad \text{(minimum bid)}.
\end{aligned} \tag{3}$$

Figures 1 and 3 below, among other purposes, illustrate how these descriptors summarize an equilibrium's structure.

[6] See definition 8.2 (page 333) in Fudenberg and Tirole, 1991

Given an equilibrium M_δ, we index with δ its components S_δ, B_δ, $F_{S\delta}$, $F_{B\delta}$, $T_{S\delta}$, $T_{B\delta}$, and ζ_δ and its descriptors $\bar{c}_\delta$, $\underline{v}_\delta$, $\underline{c}_\delta$, $\underline{p}_\delta$, and $\bar{p}_\delta$. This notation allows us to state our convergence result:

Theorem 1 *Fix $\kappa > 0$ and $\beta \geq 0$. Suppose that a $\bar{\delta} > 0$ exists such that for all $\delta \in (0, \bar{\delta})$ a market equilibrium M_δ exists in which positive trade occurs. Let $\{\bar{c}_\delta, \underline{v}_\delta, \underline{c}_\delta, \underline{p}_\delta, \bar{p}_\delta\}$ be the descriptors of these equilibria and let $W_{S\delta}(c)$ and $W_{B\delta}(v)$ be traders' interim expected utilities. Then*

$$\lim_{\delta \to 0} \bar{c}_\delta = \lim_{\delta \to 0} \underline{v}_\delta = \lim_{\delta \to 0} \underline{c}_\delta = \lim \underline{p}_\delta = \lim_{\delta \to 0} \bar{p}_\delta = p_W. \tag{4}$$

In addition, each trader's interim expected utility converges to the utility he would realize if the market were perfectly competitive:

$$\lim_{\delta \to 0} W_{S\delta}(c) = \max[0, p_W - c] \tag{5}$$

and

$$\lim_{\delta \to 0} W_{B\delta}(v) = \max[0, v - p_W]. \tag{6}$$

Sections 3 and 4 below prove this.

In section 5, given $\kappa > 0$, we prove, for sufficiently small β and δ, that a special class of equilibria—full trade equilibria—exist and result in positive trade. Formally a full trade equilibrium is an equilibrium in which $\bar{c}_\delta = \underline{v}_\delta$. Figure 1 diagrams such an equilibrium and shows each of the descriptors $\bar{c}_\delta$, $\underline{v}_\delta$, $\underline{c}_\delta$, $\underline{p}_\delta$, and $\bar{p}_\delta$. We call these equilibria full trade because if in a particular period a seller is matched with at least one buyer, then that seller for sure trades no matter what her cost c_i is and what the matched buyer(s) value v_j is. In a full trade equilibrium trade occurs as fast as possible among active sellers and buyers. The theorem is this:

Theorem 2 *For given $\kappa > 0$, a neighborhood X of the point $(0,0)$ exists such that for all non-negative β and positive δ in X an equilibrium M_δ exists in which positive trade occurs.*

Two points deserve emphasis. First, the restriction that β must be small relative to κ implies that our existence theorem concerns situations in which participation costs, not delay per se, are the issue. Section 5.3, which discusses the existence result, develops intuition why the β-κ ratio is important. Second, we do not know if all equilibria are full trade or not. Conceivably for some parameter values, a sequence of full trade equilibria may not exist, but a sequence of non-full trade equilibrium in which $\bar{c}_\delta > \underline{v}_\delta$ may exist. If so, convergence of price to p_W is guaranteed because Theorem 1 applies to all sequences of equilibria.

The intuition for our convergence result can be understood through the following logic. In a match in which a buyer is bidding for an object, the "type" that is relevant is not his static type v, but rather his dynamic opportunity value

$$I_{B\delta}(v) = v - e^{-\beta\delta} W_{B\delta}(v).$$

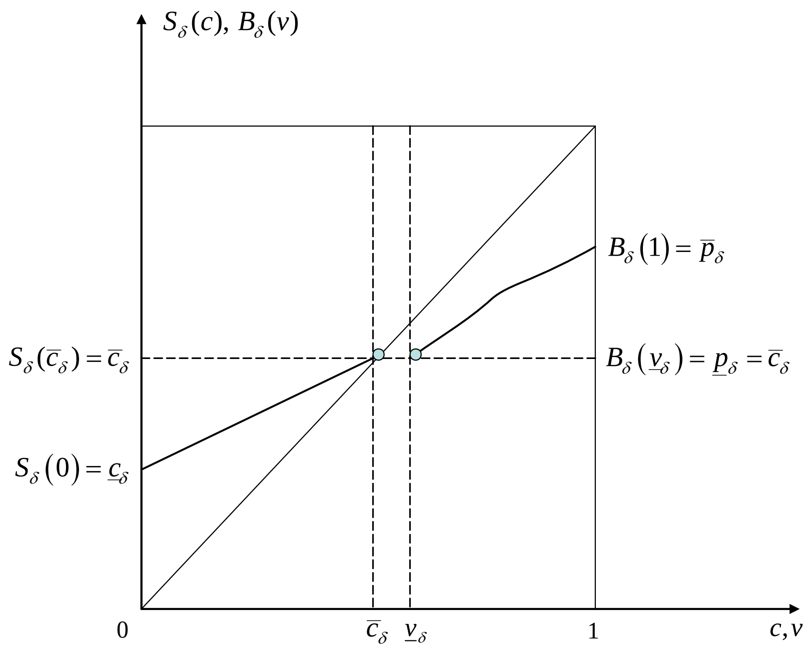

Figure 1: Strategies in full trade equilibrium

Similarly the cost that is relevant to a type c seller is not c, but her dynamic opportunity cost

$$I_{S\delta}(c) = c + e^{-\beta\delta} W_{S\delta}(c).$$

When the buyers bid in the auction, they act as if their types were drawn from the density $h_{B\delta}(\cdot)$ of $I_{B\delta}(v)$ and the sellers' types were drawn from the density $h_{S\delta}(\cdot)$ of $I_{S\delta}(c)$. Since $\lim_{\delta\to 0} W_{S\delta}(c) = \max[0, p_W - c]$ and $\lim_{\delta\to 0} W_{B\delta}(v) = \max[0, v - p_W]$, the convergence theorem indicates that, as the time period length $\delta \to 0$, the distributions of the "dynamic types" $I_{S\delta}(c)$ and $I_{B\delta}(v)$ approach p_W and become degenerate: the distribution of sellers' dynamic opportunity costs concentrates just below $\bar{c}_\delta$ and the distribution of buyers' dynamic opportunity costs concentrates just above $\underline{v}_\delta$. Viewed this way, as $\delta \to 0$, the dynamic matching and bargaining market in equilibrium progressively exhibits less and less heterogeneity among buyers and sellers until there is none and the incomplete information vanishes. The underlying driver causing the heterogeneity to vanish as $\delta \to 0$ is the option value that each trader's optimal search generates. This is the same pathway that drives convergence in the full information matching and bargaining models of Gale (1987) and Mortensen and Wright (2002). In their models, as in our model, the option value that optimal search creates causes the distributions of buyers and sellers' dynamic opportunity costs to become degenerate as the friction goes to zero. Once this is understood, our result that incomplete information does not disrupt convergence is natural.

Figure 2 is a table of graphs illustrating the general character of these equilibria and the manner in which they converge. These computed examples assume that the primitive distributions G_S and G_B are uniform on $[0, 1]$, equal masses of buyers and sellers consider entry each unit of time (i.e., $a = 1$), the participation cost is $\kappa = 1$, and discount rate is $\beta = 1$.[7] The left column shows an equilibrium for $\delta = 0.10$ while the right column shows an equilibrium for $\delta = 0.02$. Traders' costs and values, c and v, are on each graph's abscissa. The top graph in each column shows strategies: sellers' strategies $S(c)$ are to the left and above the diagonal while buyers' strategies $B(v)$ are to the right and below the diagonal. Because masses of entering traders are equal and their cost/value distributions are uniform, the Walrasian price is 0.5; this is the horizontal line cutting the center of the graph. Observe also that $B(v)$ and $S(c)$ are not defined for non-entering types. Comparison of the strategies in the top two graphs illustrates the convergence of all the descriptors $\{\bar{c}_\delta, \underline{v}_\delta, \underline{c}_\delta, \underline{p}_\delta, \bar{p}_\delta\}$ toward p_W as δ decreases from 0.10 to 0.02.

The middle graph in each column shows the endogenous densities, f_S and f_B, of active traders in the equilibrium. The density f_S for active sellers is on the left and the density for active buyers on the right. Note that to the right of $\bar{c}$ the density f_S is zero because sellers with $c > \bar{c}$ choose not to enter. Similarly, to the left of $\underline{v}$ the density f_B is zero. These two densities show that both high value, active sellers and low value, active buyers often have to wait before trading and therefore accumulate in the

[7] In computing these equilibria we numerically solve equations (44) and (45) from Section 5.1 that characterize a full trade equilibrium. See also the discussion in Section 5.3. The Mathematica program that was used is available upon request.

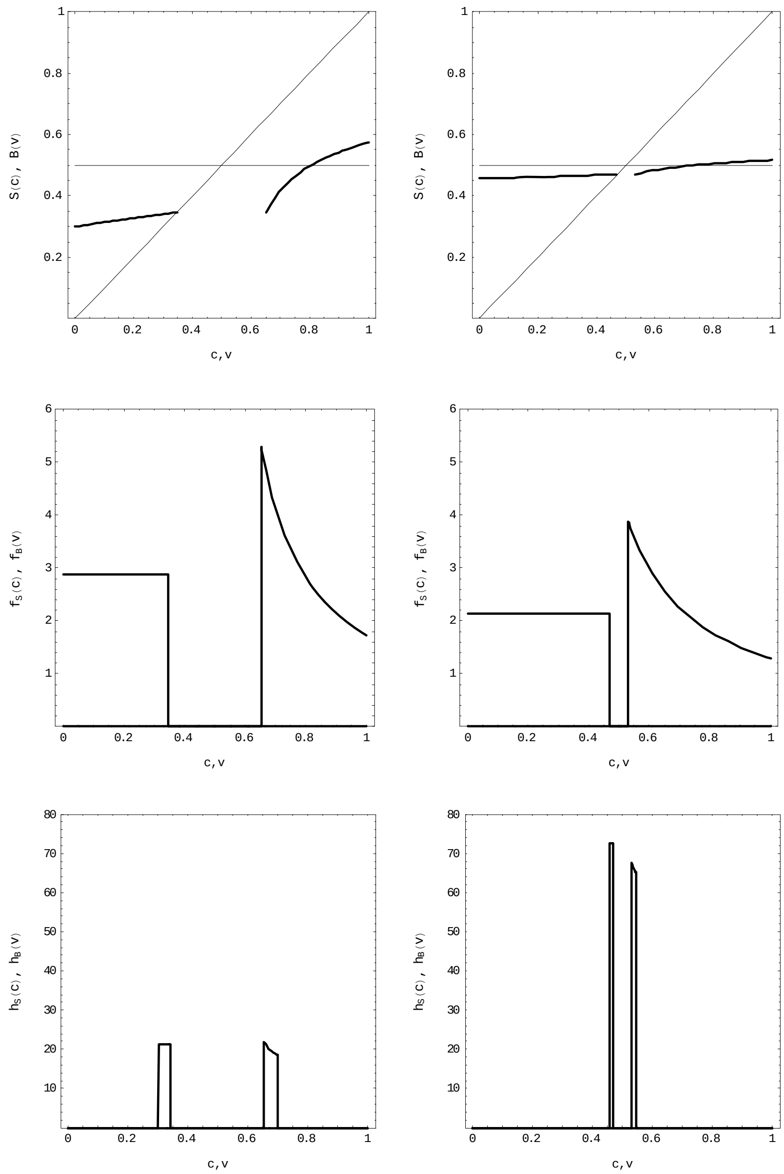

Figure 2: Equilibrium strategies and associated steady state densities. The left column of graphs is for $\delta = 0.10$ and the right column is for $\delta = 0.02$. The top row shows buyer and seller strategies S and B. The middle row graphs the densities f_S and f_B of the traders' types, and the bottom row graphs the densities h_S and h_B of the traders' dynamic opportunity costs and values.

market. The bottom graph in each column shows the equilibrium densities h_S and h_B of the dynamic opportunity costs and values and dramatically demonstrates how, as δ decreases, the heterogeneity of traders's dynamic opportunity costs and values narrows.

The ex ante expected utility of a trader is

$$W_\delta = \frac{1}{1+a}\int_0^1 W_{S\delta}(c)\, g_S(c)\, dc + \frac{a}{1+a}\int_0^1 W_{B\delta}(v)\, g_B(v)\, dv$$

where the weights on sellers' and buyers' ex ante utilities depends on the measure a of potential buyers who consider entry each period relative to the measure 1 of potential sellers who consider entry each period. For the two equilibria graphed in Figure 1 $W_{0.1} = 0.0564$ and $W_{0.02} = 0.1088$.In the limit, when $\delta \to 0$ and the market is perfectly competitive, $W_{0.0} = 0.1250$. Define the relative inefficiency per trader of an equilibrium M_δ to be $(W_0 - W_\delta)/W_0$. The relative inefficiency of the graphed $\delta = 0.10$ equilibrium is 0.548 and the relative inefficiency of the graphed $\delta = 0.02$ equilibrium is 0.129. This convergence towards 0 is driven by two distinct mechanisms. First, the direct effect of cutting δ, the period length, from 0.10 to 0.02 is that, even if traders' strategies remained unchanged, then the decrease in δ reduces by a factor of five the wait before trading for all traders who are not lucky enough to trade immediately upon entry. Second, the gap between buyers and sellers' strategies reduces, resulting in fewer traders who should trade—sellers for whom $c < 0.5$ and buyers for whom $v > 0.5$—but do not trade. Thus, when $\delta = 0.10$, sellers whose cost c is in the interval $(0.347, 0.500)$ do not trade even though they would in the competitive limit. When $\delta = 0.02$, however, this interval shrinks in length by approximately a factor of five to $(0.470, 0.5000)$, which results in a second efficiency gain.

One final comment concerning the model and theorems is important. In setting up the model we assume that traders use symmetric pure strategies. We do this for simplicity of exposition. At a cost in notation we could define trader-specific and mixed strategies and then prove that they in fact must be symmetric and (essentially) pure because of independence, anonymity in matching, and the strict monotonicity of strategies. To see this, first consider the implication of independence and anonymous matching for buyers. Even if different traders follow distinct strategies, every buyer would still independently draw his opponents from the same population of active traders.[8] Therefore, for a given value v, every buyer will have the identical best-response correspondence. Second, we show below that every selection from this correspondence is strictly increasing; consequently, the best-response is pure apart from a measure zero set of values where jumps occur. These jump points are the only points where mixing can occur, but because their measure is zero, the mixing has no consequence for the maximization problems of the other traders.

[8] This is strictly true because we assume a continuum of traders.

3 Basic properties of equilibria

In this section we derive formulas for probabilities of trade and establish the strict monotonicity of strategies. These facts are inputs into the next two sections' proofs. We separate them out because they apply for all $\delta > 0$. We emphasize that they apply to all equilibria, not just full trade equilibria.

3.1 Discounted ultimate probability of trade and participation cost

An essential construct for the analysis of our model is the discounted ultimate probability of trade. It allows a trader's expected gains from participating in the market to be written as simply as possible. Let, in the steady state, $\rho_S(\lambda)$ be the probability of trade in a given period of a seller who chooses reservation price λ and, similarly, let $\rho_B(\lambda)$ be a the probability of trade in a give period of a buyer who chooses bid λ. Also, let $\bar{\rho}_S(\lambda) = 1 - \rho_S(\lambda)$ and $\bar{\rho}_B(\lambda) = 1 - \rho_B(\lambda)$.

Define recursively $P_B(\lambda)$ to be a buyer's discounted ultimate probability of trade if he bids λ:

$$P_B(\lambda) = \rho_B(\lambda) + \bar{\rho}_B(\lambda) e^{-\beta\delta} P_B(\lambda).$$

Therefore

$$P_B(\lambda) = \frac{\rho_B(\lambda)}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_B(\lambda)}. \tag{7}$$

Observe that this is in fact a discount factor because every active trader ultimately trades. Its interpretation as a probability follows from formula (9) immediately below. The parallel recursion for sellers implies that

$$P_S(\lambda) = \frac{\rho_S(\lambda)}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_S(\lambda)}. \tag{8}$$

This construct is useful within a steady state equilibrium because it converts the buyer's dynamic decision problem into a static decision problem. Specifically, the discounted expected utility W^B of a type v buyer who follows the stationary strategy of bidding λ is

$$W^B(\lambda, v) = \rho_B(\lambda)(v - \lambda) - \kappa\delta + \bar{\rho}_B(\lambda) e^{-\beta\delta} W^B(\lambda, v).$$

Solving this recursion gives the explicit formula:

$$W^B(\lambda, v) = P_B(\lambda)(v - \lambda) - K_B(\lambda), \tag{9}$$

where

$$\begin{aligned} K_B(\lambda) &= \frac{\kappa\delta}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_B(\lambda)} \\ &= \kappa\delta \frac{P_B(\lambda)}{\rho_B(\lambda)} \end{aligned} \tag{10}$$

is the the buyer's expected discounted participation costs over his lifetime in the market. To get intuition for the last equation, note that $1/\rho_B(\lambda)$ is the expected lifetime of a trader in the market, so that $\kappa\delta/\rho_B(\lambda)$ is the expected participation cost over the lifetime. The discounted participation cost $K_B(\lambda)$ equals the the expected participation cost over the lifetime times the discount factor $P_B(\lambda)$. Similarly, the discounted expected utility W^S of a type c seller who follows the stationary strategy of accepting bids of at least λ is

$$W^S(\lambda, c) = P_S(\lambda)(\lambda - c) - K_S(\lambda) \tag{11}$$

where

$$K_S(\lambda) = \frac{\kappa\delta}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_S(\lambda)}. \tag{12}$$

is the discounted participation cost of a seller who asks λ. In accord with our convention for non-entering types, we assume that

$$\rho_B(\mathcal{N}) = \rho_S(\mathcal{N}) = K_B(\mathcal{N}) = K_S(\mathcal{N}) = 0.$$

In section 3.3 we derive explicit formulas for $\rho_B(\cdot)$ and $\rho_S(\cdot)$

3.2 Strategies are strictly increasing

This subsection demonstrates the most basic property that our equilibria satisfy: strategies are strictly increasing. We need the following preliminary result.

Lemma 3 *In equilibrium, $P_B[B(\cdot)]$ is non-decreasing and $P_S[S(\cdot)]$ is non-increasing over $[0,1]$. The buyers for whom $v > \underline{v}$ elect to enter, while the buyers for whom $v < \underline{v}$ do not:*

$$(\underline{v}, 1] \subset A_B, \quad [0, \overline{v}) \subset \bar{A}_B.$$

The type $\underline{v}$ is indifferent between entering and not entering. Similarly,

$$[0, \bar{c}) \subset A_S, \quad (\bar{c}, 1] \subset \bar{A}_S$$

and the type $\bar{c}$ is indifferent between entering or not.

Equation (3) and the paragraph above define the sets A_B and A_S and the descriptors $\underline{v}$ and $\bar{c}$.

Proof. The buyer's interim utility,

$$W_B(v) = \sup_{\lambda \in R \cup \{\mathcal{N}\}} (v - \lambda) P_B(\lambda) - K_B(\lambda) = (v - B(v)) P_B(B(v)) - K_B(B(v)),$$

is the upper envelope of a set of affine functions. It follows by the Envelope Theorem that $W_B(\cdot)$ is a continuous, increasing and convex function. Because W_B is continuous, the definition of $\underline{v} = \inf\{v : v \in A_B\}$ implies that (i) $W_B(\underline{v}) = 0$ and $\underline{v}$ is indifferent between entering or not, and (ii) the types $v < \underline{v}$ prefer not to enter. Further, convexity implies that $W_B'(\cdot)$ is non-decreasing. By the Envelope Theorem $W_B'(\cdot) = P_B[B(\cdot)]$;

$P_B[B(\cdot)]$ is therefore non-decreasing at all differentiable points. Milgrom and Segal's (2002) Theorem 1 implies that at non-differentiable points $v' \in [0,1]$

$$\lim_{v \to v'-} W_B'(v) \le P_B\left(B\left(v'\right)\right) \le \lim_{v \to v'+} W_B'(v).$$

Thus $P_B[B(\cdot)]$ is everywhere non-decreasing for any best-response B. Further, Milgrom and Segal's Theorem 2 implies that

$$W_B(v) = W_B(\underline{v}) + \int_{\underline{v}}^{v} P_B[B(x)]\,dx \qquad \text{for } v \ge \underline{v}. \tag{13}$$

Since $\underline{v}$ is indifferent between entering or not, we can choose a best-response $\widetilde{B}$ in which $\underline{v}$ is active, while $\widetilde{B}(v) = B(v)$ for $v \neq \underline{v}$. $\widetilde{B}$ may different from B at $v = \underline{v}$, since in $\widetilde{B}$, the type $\underline{v}$ is active, while in B he may not be. Importantly, the function $W_B(\cdot)$ is the same for both B and $\widetilde{B}$, since by Milgrom and Segal's Theorem 2, the envelope condition (13) holds for any selection from the best-response correspondence. Now $P_B\left[\widetilde{B}(\underline{v})\right] > 0$ since otherwise the active buyer $\underline{v}$ would not be able to recover his positive participation cost. Since $P_B\left[\widetilde{B}(\cdot)\right]$ is non-decreasing, $P_B\left[\widetilde{B}(v)\right] > 0$ for all $v \ge \underline{v}$, and the envelope condition (13) then implies that the buyers for whom $v > \underline{v}$ elect to enter. The argument for the sellers is parallel and is omitted.■

Recall that we assume if a potential trader's expected utility from entering is at least zero, then he or she enters. Thus types $\underline{v}$ and $\bar{c}$ enter, which keeps our notation simple. Since $\{\underline{v}\}$ and $\{\bar{c}\}$ have measure 0, all our results would hold in substance under the alternative assumption that entry only occurs if expected utility is positive.

Lemma 4 *B is strictly increasing on $[\underline{v}, 1]$.*

Proof. Pick any v, $v' \in [\underline{v}, 1]$ such that $v < v'$. Since $P_B[B(\cdot)]$ is non-decreasing, $P_B[B(v)] \le P_B[B(v')]$ necessarily. We first show that B is non-decreasing on $[\underline{v}, 1]$. Suppose, to the contrary, that $B(v) > B(v')$. The auction rules imply that $P_B(\cdot)$ is non-decreasing; therefore $P_B[B(v)] \ge P_B[B(v')]$. Consequently $P_B[B(v)] = P_B[B(v')] > 0$. But this gives v incentive to lower his bid to $B(v')$, since by doing so he will buy with the same positive probability but pay a lower price. This contradicts B being an optimal strategy and establishes that B is non-decreasing. If $B(v') = B(v)$ $(= \lambda)$ because B is not strictly increasing, then any buyer with $v'' \in (v, v')$ will raise his bid infinitesimally from λ to $\lambda' > \lambda$ to avoid the rationing that results from a tie. This proves that B is strictly increasing on $[\underline{v}, 1]$.■[9]

Lemma 5 *S is continuous and strictly increasing on $[0, \bar{c}]$.*

Proof. Any active seller will accept the highest bid she receives, provided it is above her dynamic opportunity cost:

$$S(c) = c + e^{-\beta\delta} W_S(c) \tag{14}$$

[9]This proof uses the same argument that Satterthwaite and Williams (1989a) used to prove their Theorem 2.2.

Milgrom and Segal's Theorem 2 implies that $W_S(\cdot)$ is continuous and can be written, for any active seller type c as

$$\begin{aligned} W_S(c) &= W_S(\bar{c}) + \int_c^{\bar{c}} P_S(S(x))dx \qquad (15) \\ &= \int_c^{\bar{c}} P_S(S(x))dx \end{aligned}$$

where the second line follows from the definition of $\bar{c}$ and the continuity of $W_S(\cdot)$. Combining (14) and (15) we see that

$$S(c) = c + e^{-\beta\delta} \int_c^{\bar{c}} P_S(S(x))dx$$

for all sellers that are active. This also implies that $S(\cdot)$ is continuous. Therefore, for almost all active sellers $c \in [0, \bar{c}]$,

$$S'(c) = 1 - e^{-\beta\delta} P_S[S(c)] > 0 \qquad (16)$$

because $W_S'(c) = -P_S[S(c)]$. Since $S(\cdot)$ is continuous, this is sufficient to establish that $S(\cdot)$ is strictly increasing for all active sellers $c \in [0, \bar{c}]$.■

Lemma 6 *$\underline{c} < B(\underline{v}) < \underline{v}$, $S(\bar{c}) = \bar{c} < \bar{p}$, and $B(\underline{v}) \leq \bar{c}$.*

Proof. Given that S is strictly increasing, $S(0) = \underline{c}$ is the lowest reservation price any seller ever has. A buyer with valuation $v < \underline{c}$ does not enter the market since he can only hope to trade by submitting a bid at or above $\underline{c}$, i.e. above his valuation. In equilibrium, any buyer who enters the market must submit a bid below his valuation and above $\underline{c}$, since otherwise he is unable to recover a positive participation cost. It follows that $\underline{c} < B(\underline{v}) < \underline{v}$. Similarly, a seller who is only willing to accept a bid at or above $\bar{p}$ never enters the market, since she is unable to recover her participation cost. This implies $S(\bar{c}) < \bar{p}$. Any active seller has acceptance strategy given by (14), so in particular $S(\bar{c}) = \bar{c}$.

Finally, suppose that $B(\underline{v}) > \bar{c}$. Then the buyer for whom $v = \underline{v}$ bids more than necessary to win the object: he can only be successful if there are no rival buyers, and when this is the case, bidding $\bar{c}$ is sufficient to secure acceptance of the bid by the seller.■

All these findings are summarized as follows. In reading the theorem, recall that the descriptors $(\underline{c}, \bar{c}, \underline{v}, \underline{p}, \bar{p})$ are defined in equation (3).

Theorem 7 *Suppose that $\{B, S\}$ is a stationary equilibrium. Then, over $[\underline{v}, 1]$ and $[0, \bar{c}]$, B and S are strictly increasing, S is continuous and, almost everywhere on $[0, \bar{c}]$, has derivative*

$$S'(c) = 1 - e^{-\beta\delta} P_S[S(c)].$$

Finally, B and S have the properties that $\underline{c} < \underline{p} < \underline{v}$, $S(\bar{c}) = \bar{c} < \bar{p}$, and $\underline{p} = B(\underline{v}) \leq S(\bar{c}) = \bar{c}$.

The strict monotonicity of B on $[\underline{v},1]$ and S on $[0,\bar{c}]$ allows us to define V and C, their inverses over $[B(\underline{v}), B(1)]$ and $[S(0), S(\bar{c})]$:

$$\begin{aligned} V(\lambda) &= \inf\{v \in [0,1] : B(v) > \lambda\}, \\ C(\lambda) &= \inf\{c \in [0,1] : S(c) > \lambda\}. \end{aligned}$$

Finally, that $B(\underline{v}) \leq S(\bar{c})$ is a weak inequality, not a strong inequality, makes possible the existence of full trade equilibria.

3.3 Explicit formulas for the probabilities of trading

Focus on a seller of type c who in equilibrium has a positive probability of trade. In a given period she is matched with zero buyers with probability π_0 and with one or more buyers with probability $\bar{\pi}_0 = 1 - \pi_0$. Suppose she is matched and v^* is the highest type buyer with whom she is matched. Since by Theorem 7 each buyer's bid function $B(\cdot)$ is increasing, she accepts his bid if and only if $B(v^*) \geq \lambda$ where λ is her reservation price. The distribution from which v^* is drawn is $F_B^*(\cdot)$: for $v \in [\underline{v}, 1]$,

$$F_B^*(v) = \frac{1}{\bar{\pi}_0(\zeta)} \sum_{i=1}^{\infty} \pi_i(\zeta) [F_B(v)]^i \tag{17}$$

where $F_B(\cdot)$ is the steady state distribution of buyer types and $\{\pi_0, \pi_1, \pi_2, \ldots\}$ are the probabilities with which each seller is matched with zero, one, two, or more buyers. Note that this distribution is conditional on the seller being matched. Thus if a seller has reservation price λ, her probability of trading in a given period is

$$\rho_S(\lambda) = \bar{\pi}_0 [1 - F_B^*(V(\lambda))]. \tag{18}$$

This formula takes into account the probability that she is not matched in the period.

A similar expression obtains for $\rho_B(\lambda)$, the probability that a buyer submitting bid λ successfully trades in any given period. In order to derive this expression, we need a formula for $\omega_k(\zeta)$, the probability that the buyer is matched with k rival buyers. If T_B is the mass of active buyers and T_S is the mass of active sellers, then $\omega_k(\zeta) T_B$, the mass of buyers participating in matches with k rival buyers, equals $k+1$ times $\pi_{k+1}(\zeta) T_S$, the mass of sellers matched with $k+1$ buyers:

$$\omega_k(\zeta) T_B = (k+1) \pi_{k+1}(\zeta) T_S.$$

Solving, substituting in the formula for $\pi_{k+1}(\zeta)$, and recalling that $\zeta = T_B/T_S$ shows that $\omega_k(\zeta)$ and $\pi_k(\zeta)$ are identical:

$$\omega_k(\zeta) = \frac{(k+1)}{\zeta} \pi_{k+1}(\zeta) = \frac{(k+1)}{\zeta} \frac{\zeta^{k+1}}{(k+1)!\, e^{\zeta}} = \pi_k(\zeta). \tag{19}$$

The striking implication of this, which follows from the number of buyers in a given meeting being Poisson, is that the distribution of bids that a buyer must beat is exactly the same distribution of bids that each seller receives when she is matched with at least one buyer.[10]

Turning back to ρ_B, a buyer who bids λ and is the highest bidder has probability $F_S(C(\lambda))$ of having his bid accepted. This is just the probability that the seller with whom the buyer is matched will have a low enough reservation price so as to accept his bid. If a total of $j+1$ buyers are matched with the seller with whom the buyer is matched, then he has j competitors and the probability that all j competitors will bid less than λ is $[F_B(V(\lambda))]^j$. Therefore the probability that the bid λ is successful in a particular period is

$$\begin{aligned}\rho_B(\lambda) &= F_S(C(\lambda))\sum\nolimits_{j=0}^{\infty}\omega_j(\zeta)\ \ [F_B(V(\lambda))]^j \\ &= F_S(C(\lambda))\sum\nolimits_{j=0}^{\infty}\pi_j(\zeta)\ \ [F_B(V(\lambda))]^j \\ &= F_S(C(\lambda))\left[\pi_0+\bar{\pi}_0F^*(V(\lambda))\right].\end{aligned} \tag{20}$$

We are now in position to prove Theorem 1 on convergence and Theorem 2 on existence. The next section proves convergence to p_W, the Walrasian price, in two steps. Convergence to one price follows from each trader's search for a better price becoming cheaper as δ, the period length, becomes shorter. Traders who do not offer a good price to the opposite side of the market fail to trade and therefore revise their price, narrowing the range of prices realized in the market. That convergence is to p_W is a consequence of supply and demand. Traders who enter stay in the market until they trade. If the one price to which the market converges is not Walrasian, then either more buyers will enter than sellers or more sellers will enter than buyers. Either way the market will not clear, traders on one side of the market will accumulate without bound, and the market will not be in a steady state equilibrium. Therefore, if a sequence of steady state equilibria exists as $\delta\to 0$, the equilibria must converge to p_W .

Section 5 proves existence in three steps. In the first step we identify a class of equilibria—full trade equilibria—and show that a necessary condition for a full trade equilibria to exist is that it satisfy a system of two equations that β, the discount rate, and δ, the period length parameterize. In step 2 we show that at $(\beta,\delta)=(0,0)$ a solution to these equations always exists and apply the implicit function theorem to establish that a unique solution always exists for all (β,δ) in a neighborhood around $(0,0)$. Finally, in step 3 we show that if β is sufficiently small relative to κ,the per unit time participation cost, then the solution to the two equation system—which exists—defines an equilibrium, i.e., a solution to the system is sufficient for an equilibrium to exist. Section 5 then concludes with a discussion of two issues: why is it important that β be small relative to κ and what are we able to say about the rate at which equilibria approach full efficiency as $\delta\to 0$.

[10] Myerson (1998) studies games with population uncertainty and shows that the Poisson assumption is both necessary and sufficient for players' beliefs about the number of other players to be equal to the external observer's beliefs.

4 Proof of convergence

Theorem 1 consists of two parts: "the law of one price" part, which given the characterization in Theorem 7, reduces to

$$\lim_{\delta\to0}\underline{c}_\delta=\lim_{\delta\to0}\bar{p}_\delta=\lim_{\delta\to0}\underline{v}_\delta=p_W,$$

and the efficiency part

$$\lim_{\delta\to0}W_{S\delta}\left(c\right)=\max\left[0,p_W-c\right],\ \lim_{\delta\to0}W_{B\delta}\left(v\right)=\max\left[0,v-p_W\right]$$

These are dealt with separately in Theorems 8 and 12 below. All the proofs in this Section apply to all equilibria, not only to full trade equilibria. Figure 3 shows the structure of a non-full trade equilibrium. The difference between this figure and Figure 1 of a full trade equilibrium is that the equality $\underline{p}=B\left(\underline{v}\right)=S\left(\bar{c}\right)=\bar{c}$, which defines an equilibrium to be full trade, is changed to the inequality, $\underline{p}=B\left(\underline{v}\right)\leq S\left(\bar{c}\right)=\bar{c}$, which allows sellers and buyers' strategies to overlap.

Theorem 8 $\lim_{\delta\to0}\underline{c}_\delta=\lim_{\delta\to0}\bar{p}_\delta=\lim_{\delta\to0}\underline{v}_\delta=p_W$.

The proof of this Theorem relies on three Lemmas.

Lemma 9 $\lim_{\delta\to0}\left(\bar{p}_\delta-\bar{c}_\delta\right)=0$.

Proof. Suppose not, i.e., there exists an $\varepsilon>0$ such that $\bar{p}_\delta-\bar{c}_\delta>\varepsilon$ along a subsequence. Let

$$\begin{aligned}b_\delta&=\bar{p}_\delta-\varepsilon/2,\\ v_\delta&=\sup\left\{v:B_\delta\left(v\right)\leq b_\delta\right\}.\end{aligned}$$

Let the probability γ_δ be the seller's equilibrium belief that the maximum bid in a given period is greater than or equal to b_δ. If $\lim_{\delta\to0}\gamma_\delta=\underline{\gamma}>0$ along a subsequence, the seller for whom $c_\delta=\left(\bar{c}_\delta+b_\delta\right)/2$ would prefer to enter for small enough δ. The reason is this. By definition $b_\delta-\bar{c}_\delta>\varepsilon/2$ and, therefore, $b_\delta-c_\delta>\varepsilon/4$. Consequently, if seller c_δ sets her offer to be $\lambda=b_\delta$, then the gain $b_\delta-c_\delta$ she realizes if she trades is at least $\varepsilon/4$ and her per period probability of trade is $\rho_S\left(\lambda\right)>\underline{\gamma}$. Inspection of formulas (8) and (12) establishes that as $\delta\to0$ her discounted probability of trade goes to 1 and her discounted participation costs goes to 0. Therefore her expected utility, as given by (11), is

$$W^S\left(b_\delta,c_\delta\right)=P_S\left(\lambda\right)\left(b_\delta-c_\delta\right)-K_S\left(\lambda\right)>\frac{\varepsilon}{4}>0$$

as $\delta\to0$, a contradiction.

If, on the other hand, $\lim_{\delta\to0}\gamma_\delta=0$ along all subsequences, then the buyer for whom $v=1$ would prefer a deviation to b_δ. If he deviates, then in the limit, as $\delta\to0$, his probability of trading in a given period, $\rho_B\left(b_\delta\right)$, approaches 1. This is an immediate implication of the observation that follows (19): γ_δ is not only the probability that the

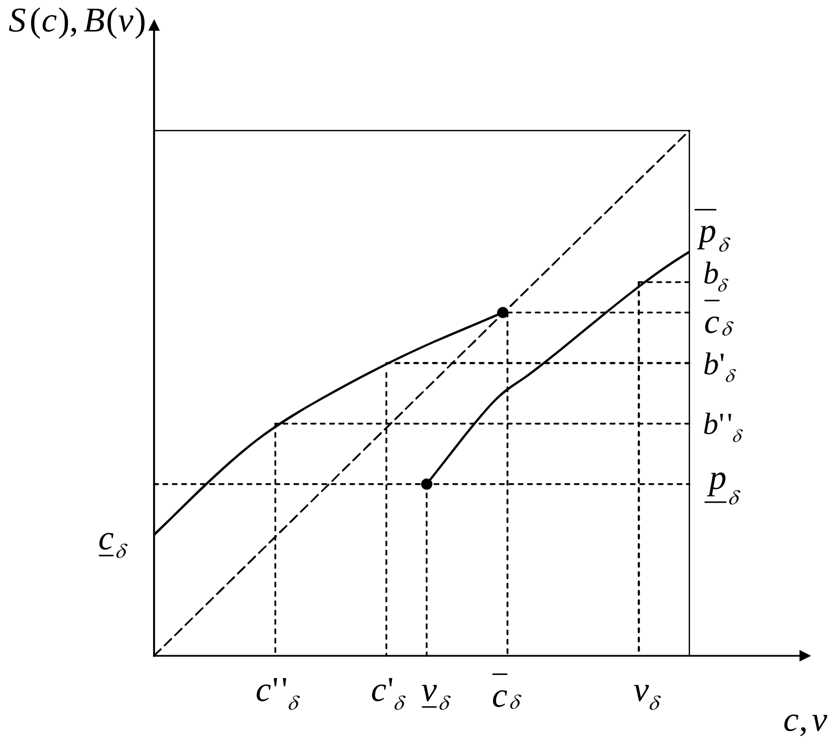

Figure 3: Structure of an equilibrium that is not full trade. The figure also shows the construction of b', b'', c'_δ and c''_δ that are used in the proof of Lemma 10.

maximum bid a seller receives in a given period is greater than or equal to b_δ, but it is also the probability that the maximum competing bid the type 1 buyer must beat is greater than or equal to b_δ. Therefore $\gamma_\delta \to 0$ implies that deviating to b_δ results in his discounted probability approaching 1 and discounted participation cost approaching 0. Consequently, this buyer deviates and secures the lower price b_δ, which completes the proof.■

Lemma 10 $\lim_{\delta \to 0} \left(\bar{p}_\delta - \underline{p}_\delta \right) = 0.$

Proof. The proof is by contradiction: pick a small ε, suppose $\bar{p}_\delta - \underline{p}_\delta > \varepsilon > 0$ along a subsequence, and define

$$b'_\delta = \bar{p}_\delta - \frac{1}{3}\varepsilon, \tag{21}$$

$$b''_\delta = \bar{p}_\delta - \frac{2}{3}\varepsilon.$$

Note that $b''_\delta - \underline{p}_\delta > \frac{\varepsilon}{3}$. Select a buyer and let

$$\phi_\delta = F_{S\delta}\left(C_\delta\left(b'_\delta\right)\right)$$

be the equilibrium probability that the seller with whom he is matched in a given period would accept a bid that is less than or equal to b'_δ. Lemma 9 guarantees that the seller for whom $S(c) = b'_\delta$ exists, at least for small enough δ. Select a seller and let

$$\begin{aligned} \psi_\delta &= \sum_{k=0}^{\infty} \pi_k \left[F_{B\delta}\left(V_\delta\left(b'_\delta\right)\right)\right]^k \\ &= \pi_0 + \bar{\pi}_0 F^*_{B\delta}\left(V_\delta\left(b'_\delta\right)\right). \end{aligned}$$

be the equilibrium probability that, in a given period, she receives either no bid or the highest bid she receives is less or equal to b'_δ. Observe that ψ_δ is the equilibrium probability that a buyer's bid b'_δ is maximal in a given match; this follows directly from formula (20) for $\rho_B(\lambda)$. Given these definitions, this Lemma's proof consists of three steps.

Step 1. *The fraction of sellers for whom $S_\delta(c) \le b'_\delta$ does not vanish as $\delta \to 0$, i.e., $\underline{\phi} \equiv \underline{\lim}_{\delta \to 0} \phi_\delta > 0$.*

Suppose not. Then $\phi_\delta \to 0$ along a subsequence. Fix this subsequence and fix some period, say period 0. Let N_δ be a sequence of integers whose values are chosen later in the proof. Define, without loss of generality, the time segment Υ_δ of length N_δ periods that begins with period 0 and ends with period N_δ. Define three masses of sellers :

- $m^+_{N\delta}$ is the mass of sellers who enter the market within time segment Υ_δ and for whom $b''_\delta \le S_\delta(c) \le b'$.
- $m^-_{N\delta}$ is the mass of sellers who both enter and exit the market within time segment Υ_δ and for whom $b''_\delta \le S_\delta(c) \le b'_\delta$.

- m_δ the *steady state* mass of active sellers for whom $b''_\delta \leq S_\delta(c) \leq b'_\delta$.

The assumption that $\phi_\delta \to 0$ implies that $m_\delta \to 0$. We show next that $m_\delta \to 0$ entails $\underline{c}_\delta \to b''_\delta$. This establishes a contradiction because Theorem 7 states that $\underline{c}_\delta < \underline{p}_\delta$ and by construction $\underline{p}_\delta + \frac{\varepsilon}{3} < b''_\delta$.

The fraction of sellers in the mass $m^+_{N\delta}$ that do not exit during the time segment Υ_δ is

$$\frac{m^+_{N\delta} - m^-_{N\delta}}{m^+_{N\delta}} \leq \frac{m_\delta}{m^+_{N\delta}}$$

because the surviving mass $m^+_{N\delta} - m^-_{N\delta}$ of sellers who entered in time segment Υ_δ cannot exceed the total, steady state of the mass m_δ of sellers with reservation prices in the interval $[b''_\delta, b'_\delta]$. Therefore the fraction of the sellers in $m^+_{N\delta}$ that have traded within time segment Υ_δ is at least

$$1 - \frac{m_\delta}{m^+_{N\delta}}. \tag{22}$$

In the mass $m^+_{N\delta}$, pick a seller c''_δ who enters in period 0 and for whom

$$S_\delta\left(c''_\delta\right) = b''_\delta.$$

Such a seller c''_δ always exists because S_δ is continuous (see Theorem 7) and $\underline{g}$ is a lower bound on the density of entering sellers. This seller's reservation price is as low as any other seller in $m^+_{N\delta}$ and has the full time segment Υ_δ in which to consummate a trade. Her probability of trading within Υ_δ is therefore as high as any other seller in $m^+_{N\delta}$. Let r_δ be her probability of trading within the time segment Υ_δ. It is therefore at least as great as the average probability of trading across all sellers in $m^+_{N\delta}$:

$$r_\delta \geq 1 - \frac{m_\delta}{m^+_{N\delta}}. \tag{23}$$

Now, since the slope of S_δ is at most one (see the formula in Theorem 7), it follows that

$$m^+_{N\delta} \geq \frac{\varepsilon}{3}\underline{g}\delta N_\delta, \tag{24}$$

because $m^+_{N\delta}$ is minimized when the slope of S_δ is the largest (i.e., equal to 1) and the density g_S is minimal. Substituting this lower bound on $m^+_{N\delta}$ into (23) gives

$$r_\delta \geq 1 - \frac{m_\delta}{\frac{\varepsilon}{3}\underline{g}\delta N_\delta}.$$

For seller c''_δ her discounted probability of trading $P_{S\delta}\left(b''_\delta\right)$ from setting reservation price b''_δ is bounded from below by

$$P_{S\delta}\left(b''_\delta\right) \geq e^{-\beta\delta N_\delta}\left(1 - \frac{m_\delta}{\frac{\varepsilon}{3}\underline{g}\delta N_\delta}\right). \tag{25}$$

The right-hand side understates the discounted probability of trade because, literally, the lower bound is the discounted probability of trader c''_δ waiting the full N_δ periods before attempting to trade, having only probability $1 - m_\delta \big/ (\frac{\varepsilon}{3}\underline{g}\delta N_\delta)$ of succeeding in that period, and then never trying again.

Set the period length to be

$$N_\delta = \min\left\{k : k \text{ is integer, } k \geq \frac{\sqrt{m_\delta}}{\delta}\right\}.$$

Substitution of this choice into (25) and taking the limit as $\delta \to 0$ shows that discounted probability of seller c''_δ trading approaches 1 from below because $m_\delta \to 0$:

$$\lim_{\delta\to 0} P_{S\delta}\left(S_\delta\left(c''_\delta\right)\right) \geq \lim_{\delta\to 0} \exp\left(-\beta\sqrt{m_\delta}\right)\left(1 - \frac{\sqrt{m_\delta}}{\frac{\varepsilon}{3}\underline{g}}\right) = 1.$$

Recall from Theorem 7 that, for almost all $c \in [0, \bar{c})$,

$$S'_\delta(c) = 1 - e^{-\beta\delta} P_{S\delta}\left[S_\delta(c)\right].$$

Since $P_{S\delta}(S_\delta(c)) \geq P_{S\delta}(b''_\delta)$ for $c \leq c''_\delta$ and S_δ is increasing on $[0, \bar{c})$, it follows that, for all seller types $c \in [0, c''_\delta]$, $P_{S\delta}\left[S_\delta(c)\right] \to 1$ and

$$\lim_{\delta\to 0} S'_\delta(c) = 0.$$

Consequently, since S_δ is continuous,

$$\underline{c}_\delta = S_\delta(0) \to b''_\delta.$$

This is in contradiction to $\underline{c}_\delta < \underline{p}_\delta < b''_\delta - \varepsilon/3$. Therefore it cannot be that $\phi_\delta \to 0$.

Step 2. *If the ratio of buyers to sellers ζ_δ is bounded away from 0, then the probability ψ_δ that the highest bid in a given meeting is less than b'_δ is also bounded away from 0.* Proof of this step stands alone and is not based on the result in step 1 of this proof.

Formally, if $\underline{\lim}_{\delta\to 0}\zeta_\delta > 0$, then $\underline{\psi} \equiv \underline{\lim}_{\delta\to 0}\, \psi_\delta > 0$. Suppose not. Then $\psi_\delta \to 0$ and $\zeta_\delta \to \underline{\zeta} > 0$ along a subsequence. Fix this subsequence and recall that by construction

$$b''_\delta > \underline{p}_\delta + \frac{\varepsilon}{3}. \tag{26}$$

First, we show that the seller with cost c''_δ such that $S\left(c''_\delta\right) = b''_\delta$ prefers to enter. Since $\zeta_\delta \to \underline{\zeta}$ and $\psi_\delta \to 0$, for all δ sufficiently small, the probability that he meets a buyer for whom $B(v) \geq b'_\delta = b''_\delta + \varepsilon/3$ is at least $\frac{1}{2}\left(1 - e^{-\underline{\zeta}}\right)$. This is because, with $\psi_\delta \to 0$, (i) almost every bid she receives is greater than b'_δ and (ii) her probability of getting at least one bid is approaching $1 - e^{-\underline{\zeta}}$. Therefore, as $\delta \to 0$ her discounted probability of trading with a buyer for whom $B_\delta(v) \geq b'_\delta$ approaches 1 even as her discounted participation costs, given by formula (10), approach 0. Consequently, the profit of the c''_δ seller, in the limit as $\delta \to 0$, is at least $\varepsilon/3$, and she will choose to enter.

Second, since she chooses to enter, it must be that $c''_\delta \leq \bar{c}_\delta$. Therefore the slope of S for $c \in [0, c''_\delta)$ satisfies

$$S'(c) = 1 - e^{-\beta\delta} P_{S\delta}(c) \to 0$$

since $P_{S\delta}(S_\delta(c)) \geq P_{S\delta}(S_\delta(c''_\delta))$ and $P_{S\delta}(S_\delta(c''_\delta)) \to 1$. Therefore $\underline{c}_\delta \to b''_\delta$, a contradiction of (26) and Theorem 7's requirement that $\underline{c}_\delta < \underline{p}_\delta$.

Step 3. *For small enough δ, a buyer for whom $v = 1$ prefers to deviate to bidding b'_δ instead of $\bar{p}_\delta$.* There are two cases to consider.

Case 1. $\underline{\lim}_{\delta\to 0}\zeta_\delta > 0$. We show, using both steps 1 and 2 of this proof, that bidding $\bar{p}_\delta$ cannot be equilibrium behavior for a type 1 buyer. Recall that ϕ_δ is the probability that a seller will accept a bid less than b'_δ and that, according to step 1, $\underline{\phi} = \lim_{\delta\to 0}\phi_\delta > 0$. Additionally, recall that ψ_δ is the probability that the maximal rival bid a buyer faces in a given period is no greater than b'_δ and that, according to step 2, $\lim_{\delta\to 0}\psi_\delta = \underline{\psi} > 0$. For small enough $\delta > 0$, this second probability is bounded from below by $(1/2)\,\underline{\psi}$. It follows that, for small enough δ, the buyer who bids b'_δ (i) wins over all his rival buyers with probability greater than $(1/2)\,\underline{\psi}$, and (ii) has his bid accepted by the seller with probability greater than $(1/2)\,\underline{\phi}$. Therefore, as $\delta \to 0$, the buyer who bids b'_δ trades with a discounted probability approaching 1 and a discounted participation cost approaching 0. Consequently deviating to b'_δ gives him a profit of at least $1 - b'_\delta$, which is greater than $1 - \bar{p}_\delta$, that profit he would make with his equilibrium bid $B(1) = \bar{p}_\delta$. Therefore deviation to b'_δ is profitable for him.

Case 2. $\underline{\lim}_{\delta\to 0}\zeta_\delta = 0$. Fix a subsequence such that $\zeta_\delta \to 0$. The proof of this case relies only on the result in step 1 of this proof. The probability of meeting no rival buyers in a given period is $e^{-\zeta_\delta}$ and, since $\zeta_\delta \to 0$, this probability is at least $1/2$ for sufficiently small δ. In any given period, for a type 1 buyer and for all small δ, (i) the probability of meeting no rivals is at least $1/2$ and (ii) the probability of meeting a seller who would accept the bid b'_δ is at least $(1/2)\,\underline{\phi}$. It follows that as $\delta \to 0$, his discounted probability of trading approaches 1 and his discounted participation cost approaches 0. Therefore deviating to b'_δ gives him a profit of at least $1 - b'_\delta > 1 - \bar{p}_\delta$, which proves that a deviation to b'_δ is profitable for him.

Step 3 completes the Lemma's proof because it contradicts the hypothesis that, $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = \varepsilon > 0$.■

Lemma 11 $\lim_{\delta\to 0}\left(\underline{v}_\delta - \underline{p}_\delta\right) = 0$.

Proof. Suppose not. Recall that $B(\underline{v}_\delta) \equiv \underline{p}_\delta$ and that Theorem 7 states that $\underline{v}_\delta > \underline{p}_\delta$. Pick a subsequence such that $\underline{v}_\delta - \underline{p}_\delta \geq \eta > 0$ along it. Define $\xi_\delta = \frac{1}{2}\left(\underline{p}_\delta + \underline{v}_\delta\right)$ and observe that $\xi_\delta - \underline{p}_\delta \geq \eta/2$ and $\xi_\delta < \underline{v}_\delta$. The latter inequality implies that a type ξ_δ buyer does not enter the market because his expected utility is non-positive. But suppose to the contrary that a type ξ_δ buyer enters and bids $\bar{p}_\delta$. Bidding $\bar{p}_\delta$ guarantees that he wins the auction in whatever match he finds himself, i.e., $\rho_{B\delta}(\bar{p}_\delta) = 1$. Therefore in the first period after he enters he earns profit of

$$\begin{aligned} & \xi_\delta - \bar{p}_\delta - \kappa\delta \\ = & \xi_\delta - \underline{p}_\delta + \underline{p}_\delta - \bar{p}_\delta - \kappa\delta \\ \geq & \frac{\eta}{2} + \underline{p}_\delta - \bar{p}_\delta - \kappa\delta \\ \to & \frac{\eta}{2} \end{aligned}$$

because Lemma 10 states that, as $\delta \to 0$, $\bar{p}_\delta - \underline{p}_\delta \to 0$. This contradicts the equilibrium decision of the type ξ_δ buyer not to enter.■

Proof of Theorem 8. Consider any sequence of equilibria $\delta_n \to 0$. The descriptors $\bar{p}_\delta$ and $\underline{v}_\delta$ converge because

$$\begin{aligned} \lim_{\delta\to 0}(\bar{p}_\delta - \underline{v}_\delta) &= \lim_{\delta\to 0}(\bar{p}_\delta - \underline{v}_\delta) - \lim_{\delta\to 0}(\underline{p}_\delta - \underline{v}_\delta) \\ &= \lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) \\ &= 0 \end{aligned} \tag{27}$$

where $\lim_{\delta\to 0}(\underline{p}_\delta - \underline{v}_\delta) = 0$ (from Lemma 11) implies the first equality and $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = 0$ (from Lemma 10) implies the third equality. Theorem 7 establishes that $\bar{c}_\delta \in [\underline{p}_\delta, \bar{p}_\delta)$; therefore Lemma 10 implies

$$\lim_{\delta\to 0}(\bar{p}_\delta - \bar{c}_\delta) = 0. \tag{28}$$

Pick a convergent subsequence of $\left(\underline{v}_\delta, \bar{p}_\delta, \bar{c}_\delta, \underline{p}_\delta\right)$ and denote its limit as (p_*, p_*, p_*, p_*).

Traders who choose to become active in the market exit only by trading. Therefore in the steady state the mass of sellers entering each period must equal the mass of buyers entering each period:

$$G_S\left(\bar{c}_\delta\right) = a\ \bar{G}_B(\underline{v}_\delta). \tag{29}$$

Taking the limit in (29) along the convergent subsequence as $\delta \to 0$, we get

$$G_S\left(p_*\right) = a\ \bar{G}_B(p_*).$$

This is just equation (1) that defines the Walrasian price; therefore $p_* = p_W$. Since p_W is the common limit of all convergent subsequences, it follows that the original sequence $\left(\underline{v}_\delta, \bar{p}_\delta, \bar{c}_\delta, \underline{p}_\delta\right)$ converges to the same limit:

$$\lim_{\delta\to 0}\bar{p}_\delta = \lim_{\delta\to 0}\underline{p}_\delta = \lim_{\delta\to 0}\bar{c}_\delta = \lim_{\delta\to 0}\underline{v}_\delta = p_W. \tag{30}$$

All that remains is to show that $\underline{c}_\delta$ also converges to p_W. The type $\bar{c}_\delta$ seller who is on the margin between participating and not participating must in expectation be just recovering his participation cost each period. Recall that $S_\delta\left(\bar{c}_\delta\right) = \bar{c}_\delta$. Since the price this seller receives is no more than the highest bid, $\bar{p}_\delta$, it follows that

$$\rho_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right]\left(\bar{p}_\delta - \bar{c}_\delta\right) \geq \kappa\delta.$$

Therefore

$$\frac{\rho_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right]}{\delta} \geq \frac{\kappa}{\bar{p}_\delta - \bar{c}_\delta} \to \infty$$

by (28). The discounted probability of trade may be written as

$$\begin{aligned} P_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right] &= \frac{\rho_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right]}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_S\left[S_\delta\left(\bar{c}_\delta\right)\right]} \qquad (31) \\ &= \frac{1}{\frac{\frac{1-e^{-\beta\delta}}{\delta}}{\frac{\rho_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right]}{\delta}} + e^{-\beta\delta}}. \end{aligned}$$

It follows that $\lim_{\delta\to 0} P_S\left[S_\delta\left(\bar{c}_\delta\right)\right] = 1$ because $\lim_{\delta\to 0}\frac{1-e^{-\beta\delta}}{\delta} = \beta$ and $\lim_{\delta\to 0}\frac{\rho_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right]}{\delta} = \infty$. Further, for all $c \in [0, \bar{c}_\delta]$,

$$\lim_{\delta\to 0} P_{S\delta}\left[S_\delta\left(c_\delta\right)\right] = 1 \qquad (32)$$

because $P_{S\delta}\left[S_\delta\left(\cdot\right)\right]$ is decreasing. Therefore

$$S'_\delta\left(c\right) = 1 - e^{-\beta\delta}P_{S\delta}\left(S_\delta\left(c\right)\right),$$

the slope of S_δ on $[0, \bar{c}_\delta]$, converges to 0. Together with the continuity of S_δ this implies that $\underline{c}_\delta \to \bar{c}_\delta$, which completes the proof.■

Next we prove the second part of Theorem 1.

Theorem 12 $\lim_{\delta\to 0} W_{S\delta}\left(c\right) = \max\left[0, p_W - c\right]$ *and* $\lim_{\delta\to 0} W_{B\delta}\left(v\right) = \max\left[0, v - p_W\right]$.

Proof. Equation (32) establishes that, for all $c \in [0, \bar{c}_\delta]$, $\lim_{\delta\to 0} P_{S\delta}\left[S_\delta\left(\bar{c}_\delta\right)\right] = 1$. The same argument, slightly adapted, shows that, for all $v \in [\underline{v}_\delta, 1]$, $\lim_{\delta\to 0} P_{B\delta}\left[B_\delta\left(v\right)\right] = 1$. Thus the buyer for whom $v = \underline{v}_\delta$ must just recover its participation cost each period:

$$\rho_{B\delta}\left[B_\delta\left(\underline{v}_\delta\right)\right]\left(\underline{v}_\delta - B_\delta\left(\underline{v}_\delta\right)\right) = \rho_{B\delta}\left(\underline{p}_\delta\right)\left(\underline{v}_\delta - \underline{p}_\delta\right) = \kappa\delta.$$

Therefore

$$\frac{\rho_{B\delta}\left(\underline{p}_\delta\right)}{\delta} = \frac{\kappa}{\underline{v}_\delta - \underline{p}_\delta} \to \infty$$

by Lemma 11 and, exactly as with (31),

$$\lim_{\delta\to 0} P_{B\delta}\left(\underline{p}_\delta\right) = \lim_{\delta\to 0}\frac{\rho_{B\delta}\left(\underline{p}_\delta\right)}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_{B\delta}\left(\underline{p}_\delta\right)} = 1$$

Since $P_{B\delta}\left(\cdot\right)$ is increasing, this establishes that $\lim_{\delta\to 0} P_{B\delta}\left[B_\delta\left(v_\delta\right)\right] = 1$ for all $v_\delta \in [\underline{v}_\delta, 1]$.

The envelope theorem—see equations (13) and (15)—implies that

$$\begin{aligned} W_{S\delta}(c) &= \int_{c}^{\bar{c}_\delta} P_{S\delta}\left[S_\delta(x)\right]dx, \\ W_{B\delta}(v) &= \int_{\underline{v}_\delta}^{v} P_{B\delta}\left[B_\delta(x)\right]dx. \end{aligned}$$

Passing to the limit as $\delta \to 0$ gives $\lim_{\delta\to 0} W_{S\delta}(c) = \max[0, p_W - c]$ and $\lim_{\delta\to 0} W_{B\delta}(v) = \max[0, v - p_W]$ because $\bar{c}_\delta \to p_W$ and $\underline{v}_\delta \to p_W$.■

5 Existence of full trade equilibria

Recall that Theorem 7 shows that every equilibrium must satisfy $B(\underline{v}) \leq S(\bar{c})$. The intuition for this is that the type $\underline{v}$ buyer can only trade if there is no rival buyer. Consequently he should certainly not bid more than $S(\bar{c})$, the lowest bid that every seller accepts. In a full trade equilibrium $B(\underline{v}) = S(\bar{c})$ and the type $\bar{c}$ seller, who is the highest cost active seller, always trades if she is matched with at least one buyer, even if he is the lowest value active buyer. This, of course, means that any seller with cost less than $\bar{c}$ also trades if she is matched and that a buyer fails to trade only because he is beaten in the bidding by another buyer. In this section we characterize these equilibria and, given $\kappa > 0$, prove their existence for each sufficiently small pair of non-negative β and positive δ. Specifically, Lemma 13 proves that, given $\kappa > 0$ and $\beta \geq 0$, then for each sufficiently small β and δ the vector of equilibrium descriptors $(\bar{c}_\delta, \underline{v}_\delta, \zeta_\delta)$ exists and is unique. Theorem 14 then shows that, given $\kappa > 0$, there is a neighborhood X of $(0,0)$ such that if $(\beta, \delta) \in X$, then a unique full trade equilibrium exists that the vector $(\bar{c}_\delta, \underline{v}_\delta, \zeta_\delta)$ characterizes. Theorem 2 then follows immediately as a corollary.

5.1 Preliminaries

Before introducing the equations that determine $(\bar{c}, \underline{v}, \zeta)$, we derive sellers and buyers' probabilities of trade as a function of their types c and v and the buyer-seller ratio ζ. As a consequence of the equilibrium being full trade, buyers' trade probabilities are independent of sellers' equilibrium strategy S. That the sellers' strategy does not feed back and affect the buyers' trade probabilities and strategy implies that the market fundamentals—$G_S, G_B, a, \kappa, \beta$, and δ—fully determine the equilibrium. This fact drives both the uniqueness and existence results of this section.

Given that the market is in a steady state, within every period the cohort of buyers who has the highest valuations in their matches and therefore trades is replaced by an entering cohort of equal size and composition. Therefore F_B^*, the distribution function of the maximal valuation within a match, is equal to the distribution of v in the entering cohort conditional on $v \geq \underline{v}$:

$$F_B^*(v) = \frac{G_B(v) - G_B(\underline{v})}{1 - G_B(\underline{v})}. \tag{33}$$

Let $\hat{\rho}_B(v)$ be the equilibrium probability that a type v buyer trades in any given period. It, as reference back to equation (20) and its derivation explains, is equal to the probability that he bids against no rival buyers $\left(\omega_0 = e^{-\zeta}\right)$ plus the complementary probability $\left(\bar{\omega}_0 = 1 - e^{-\zeta}\right)$ times the probability that the maximal value among the rival buyers in his match is no greater than v:[11]

$$\hat{\rho}_B(v) = e^{-\zeta} + \left(1 - e^{-\zeta}\right) F_B^*(v). \tag{34}$$

The discounted equilibrium trading probability for a type v buyer is therefore

$$\hat{P}_B(v) = \frac{\hat{\rho}_B(v)}{1 - e^{-\beta\delta} + e^{-\beta\delta}\hat{\rho}_B(v)}. \tag{35}$$

The hats on $\hat{\rho}_B(\cdot)$ and $\hat{P}_B(\cdot)$ emphasize that these equilibrium probabilities are functions of the buyer's value v, not of his bid $B(v)$.

With this notation in place we can introduce the equations that determine $(\bar{c}, \underline{v}, \zeta)$ in a full trade equilibrium. First, since every meeting results in a trade, the mass of entering buyers must equal the mass of entering sellers in the steady state:

$$\begin{aligned} G_S(\bar{c}) &= a\left[1 - G_B(\underline{v})\right] \\ &= a\bar{G}_B(\underline{v}). \end{aligned} \tag{36}$$

Second, the buyer for whom $v = \underline{v}$ must be indifferent between being active and staying out of the market. The type $\underline{v}$ buyer only trades in a period when there are no rival buyers; his probability of trading is $\omega_0 = e^{-\zeta}$. Since in a full trade equilibrium $B(\underline{v}) = S(\bar{c})$ and Theorem 7 states that $S(\bar{c}) = \bar{c}$, indifference necessarily implies that his expected gains from trade in any period, $(\underline{v} - \bar{c})e^{-\zeta}$ equals his per period participation cost:

$$(\underline{v} - \bar{c})e^{-\zeta} = \kappa\delta. \tag{37}$$

Third, parallel logic applies to any seller for whom $c = \bar{c}$. This seller always trades in any period in which he is matched with at least one buyer; the probability of this event is $1 - \pi_0 = \bar{\pi}_0 = 1 - e^{-\zeta}$. Denote the expected price that any seller receives as p. Note that this expected price is not a function of the sellers' type c; it is the same for all active sellers in a full trade equilibrium. Then, since the $\bar{c}$ seller is indifferent between trading and staying out of the market, it must be that

$$(p - \bar{c})\left(1 - e^{-\zeta}\right) = \kappa\delta. \tag{38}$$

[11] Formula (20),

$$\rho_B(\lambda) = F_S(C(\lambda))\left[\pi_0 + \bar{\pi}_0 F^*(V(\lambda))\right],$$

illustrates why the full trade case is different than the general case. In the full trade case the factor $F_S(C(\lambda))$ is degenerate: for all $\lambda \in \left[\underline{p}, \bar{p}\right]$, $F_S(C(\lambda)) = 1$. As a consequence the seller's inverse strategy $C(\lambda)$ does not affect $\rho_B(\cdot)$. In the general case, an interval $[\underline{p}, \underline{\lambda}\,] \subset \left[\underline{p}, \bar{p}\right]$ may exist such that, for all $\lambda \in [\underline{p}, \underline{\lambda}\,]$, $F_S(C(\lambda)) < 1$ and $C(\lambda)$ does affect $\rho_B(\cdot)$.

In order to find the price p, we use the envelope theorem to solve for the bidding strategies of the active buyers (i.e., those buyers for whom $v \geq \underline{v}$) as follows:

$$\begin{aligned} W_B(v) &= (v - B(v))\,\hat{P}_B(v) - K_0(v) \\ &= \int_{\underline{v}}^{v} \hat{P}_B(x)\,dx, \end{aligned} \tag{39}$$

where $\hat{P}_B(v)$ is the type v buyer's discounted probability of trading and

$$K_0(v) = \frac{\kappa\delta}{1 - e^{-\beta\delta} + e^{-\beta\delta}\hat{\rho}_B(v)}$$

is his discounted participation cost. Solving equation (39) for $B(v)$ gives

$$B(v) = v - \frac{\kappa\delta}{\hat{\rho}_B(v)} - \frac{1}{\hat{P}_B(v)} \int_{\underline{v}}^{v} \hat{P}_B(x)dx. \tag{40}$$

Observe that this formula calculates $B(v)$ directly; it is not a fixed point condition. The expected price p that a seller receives is the expected value of $B(v)$ for that buyer who has the highest valuation:

$$p = \int_{\underline{v}}^{1} B(v)\,dF_B^*(v) = \frac{1}{1 - G_B(\underline{v})} \int_{\underline{v}}^{1} B(v)\,dG_B(v) \tag{41}$$

where the second equality follows from equation (33).

Equations (36-38) form a system of three equations in the three unknowns $(\bar{c}, \underline{v}, \zeta)$ that, for given κ, β, and δ, must hold in any full trade equilibrium. In Theorem 14 below we prove that the converse claim is also true: given $\kappa > 0$, if β is non-negative, δ is positive, and they are in a sufficiently small neighborhood of $(0, 0)$, then a unique full trade equilibrium exists that corresponds to a solution $(\bar{c}, \underline{v}, \zeta)$ of the system of equations. The three characterizing equations (36-38) therefore identify a full trade equilibrium.

It is useful to reduce (36-38) to two equations. Substitute

$$\bar{c} = \underline{v} - \kappa\delta e^{\zeta} \tag{42}$$

from equation (38) into equation (36) to obtain

$$G_S\left(\underline{v} - \kappa\delta e^{\zeta}\right) - a\left(1 - G_B(\underline{v})\right) = 0. \tag{43}$$

This eliminates $\bar{c}$. Equation (36) can be re-written as

$$\begin{aligned} p &= \bar{c} + \frac{\kappa\delta}{1 - e^{-\zeta}} \\ &= \underline{v} - \kappa\delta e^{\zeta} + \frac{\kappa\delta}{1 - e^{-\zeta}}. \end{aligned}$$

Given κ, the new, two equation system in the two variables $(\underline{v}, \zeta)$ and the two parameters (β, δ) is then

$$
\begin{aligned}
G_S\left(\underline{v} - \kappa\delta e^{\zeta}\right) - a\left(1 - G_B\left(\underline{v}\right)\right) &= 0 \qquad (44)\\
p - \underline{v} + \kappa\delta\frac{e^{\zeta} - 2}{1 - e^{-\zeta}} &= 0. \qquad (45)
\end{aligned}
$$

where equations $(41), (40)$, and (35) together imply that p is a function of β and δ.

5.2 Proof of Theorem 2

The method of proof we use has five steps. First, we fix $\kappa > 0$ and tediously substitute (41) into the system (44–45) to eliminate p. Second, we change the domain of the system from the economically meaningful set $(\underline{v}, \zeta, \beta, \delta) \in D = (0, 1) \times (0, \infty) \times [0, 1) \times (0, \infty)$ to the mathematically more convenient set $D_1 = (0, 1) \times (0, 2) \times (-0.1, 1) \times (-0.1, 1)$. Third, we prove that at $\delta = 0$ the system has a unique solution: $(\underline{v}, \zeta) = (p_W, \zeta^*)$, where $\zeta^* = 1.14619$ is the unique positive solution of the equation $e^{\zeta} - \zeta - 2 = 0$. Fourth, we apply the implicit function theorem in a neighborhood of $(\underline{v}, \zeta, \beta, \delta) = (p_W, \zeta^*, 0, 0)$ to establish the existence of a unique, differentiable solution $(\underline{v}(\delta), \zeta(\delta))$ for the system of equations. Fifth, by construction, we show that, if both β and δ are sufficiently small, then the solution $(\underline{v}(\delta), \zeta(\delta))$ characterizes the unique, full trade equilibrium of the market. Lemma 13 accomplishes the first four steps and Theorem 14 accomplishes the last step.

Lemma 13 *Given $\kappa > 0$, a neighborhood X of the point $(0, 0)$ exists such that the system (36)-(38) has a unique, differentiable solution $(\bar{c}(\beta, \delta), \underline{v}(\beta, \delta), \zeta(\beta, \delta))$ for all $(\beta, \delta) \in X$. At $\delta = 0$, $\underline{v}(\beta, 0) = p_W$ and $\zeta(0, 0) = \zeta^* = 1.14619$ where ζ^* is the unique positive solution of the equation $e^{\zeta} - \zeta - 2 = 0$.*

Proof. By (41),

$$
p - \underline{v} = \frac{1}{1 - G_B(\underline{v})} \int_{\underline{v}}^{1} \left(B(v) - \underline{v}\right) dG_B(v) \qquad (46)
$$

and, by (40),

$$
\begin{aligned}
B(v) - \underline{v} &= v - \underline{v} - \int_{\underline{v}}^{v} \frac{\hat{P}_B(x)}{\hat{P}_B(v)} dx - \frac{\kappa\delta}{\hat{\rho}_B(v)} \\
&= \int_{\underline{v}}^{v} \left(1 - \frac{\hat{P}_B(x)}{\hat{P}_B(v)}\right) dx - \frac{\kappa\delta}{\hat{\rho}_B(v)}.
\end{aligned}
$$

Algebra shows that

$$
1 - \frac{\hat{P}_B(x)}{\hat{P}_B(v)} = \beta\delta\frac{1 - e^{-\beta\delta}}{\beta\delta} \frac{\hat{\rho}_B(v) - \hat{\rho}_B(x)}{\hat{\rho}_B(v)\left(1 - e^{-\beta\delta} + e^{-\beta\delta}\hat{\rho}_B(x)\right)}; \qquad (47)
$$

substituting this into (40) gives

$$B(v)-\underline{v}=\delta\left(\beta\frac{1-e^{-\beta\delta}}{\beta\delta}\int_{\underline{v}}^{v}\frac{\hat{\rho}_B(v)-\hat{\rho}_B(x)}{\hat{\rho}_B(v)\left(1-e^{-\beta\delta}+e^{-\beta\delta}\hat{\rho}_B(x)\right)}dx-\frac{\kappa}{\hat{\rho}_B(v)}\right).$$

Inserting this expression for $B(v)-\underline{v}$ into (46) and substituting the resulting expression for $p-\underline{v}$ into (45) gives

$$\delta L(\underline{v},\zeta,\beta,\delta)=0, \tag{48}$$

where

$$\begin{aligned}
L(\underline{v},\zeta,\beta,\delta) &= \frac{1}{1-G_B(\underline{v})}\int_{\underline{v}}^{1}\frac{\beta}{\kappa}\frac{1-e^{-\beta\delta}}{\beta\delta}\int_{\underline{v}}^{v}\frac{\hat{\rho}_B(v)-\hat{\rho}_B(x)}{\hat{\rho}_B(v)\left(1-e^{-\beta\delta}+e^{-\beta\delta}\hat{\rho}_B(x)\right)}dx\,dG_B(v)\\
&\quad-\frac{1}{1-G_B(\underline{v})}\int_{\underline{v}}^{1}\frac{1}{\hat{\rho}_B(v)}dG_B(v)\\
&\quad+\frac{e^{\zeta}-2}{1-e^{-\zeta}}.
\end{aligned}\tag{49}$$

The second term of L can be written as

$$\begin{aligned}
&\frac{1}{1-G_B(\underline{v})}\int_{\underline{v}}^{1}\frac{1}{\hat{\rho}_B(v)}dG_B(v)\\
=&\ \frac{1}{1-G_B(\underline{v})}\int_{\underline{v}}^{1}\frac{1}{e^{-\zeta}+\left(1-e^{-\zeta}\right)\frac{G_B(v)-G_B(\underline{v})}{1-G_B(\underline{v})}}dG_B(v)\\
=&\ \frac{1}{1-e^{-\zeta}}\log\frac{1}{e^{-\zeta}}\\
=&\ \frac{\zeta}{1-e^{-\zeta}};
\end{aligned}$$

substituting this into (49) results in

$$\begin{aligned}
L(\underline{v},\zeta,\beta,\delta) &= \frac{1}{1-G_B(\underline{v})}\int_{\underline{v}}^{1}\frac{\beta}{\kappa}\frac{1-e^{-\beta\delta}}{\beta\delta}\int_{\underline{v}}^{v}\frac{\hat{\rho}_B(v)-\hat{\rho}_B(x)}{\hat{\rho}_B(v)\left(1-e^{-\beta\delta}+e^{-\beta\delta}\hat{\rho}_B(x)\right)}dx\,dG_B(v)\\
&\quad+\frac{e^{\zeta}-\zeta-2}{1-e^{-\zeta}}
\end{aligned}$$

Simplifying further, $L(\underline{v},\zeta,\beta,\delta)$ becomes:

$$\begin{aligned}
L(\underline{v},\zeta,\beta,\delta) &= \frac{1}{\left(1-G_B(\underline{v})\right)^2}\int_{\underline{v}}^{1}\frac{\beta}{\kappa}\frac{1-e^{-\beta\delta}}{\beta\delta}\int_{\underline{v}}^{v}\frac{\left(1-e^{-\zeta}\right)\left(G_B(v)-G_B(x)\right)}{\hat{\rho}_B(v)\left(1-e^{-\beta\delta}+e^{-\beta\delta}\hat{\rho}_B(x)\right)}dx\,dG_B(v)\\
&\quad+\frac{e^{\zeta}-\zeta-2}{1-e^{-\zeta}}.
\end{aligned}\tag{50}$$

Given this work, the system (44-45) is, for $\delta>0$, equivalent to the system

$$\begin{aligned}
G_S\left(\underline{v}(\beta,\delta)-\kappa\delta e^{\zeta(\beta,\delta)}\right)-a\left(1-G_B\left(\underline{v}(\beta,\delta)\right)\right) &= 0 \qquad (51)\\
L\left(\underline{v}(\beta,\delta),\zeta(\beta,\delta),\beta,\delta\right) &= 0 \qquad (52)
\end{aligned}$$

where we index $\underline{v}$ and ζ with $\beta$and δ to indicate that we solve this system for them as functions of β and δ. Note that we have divided the second equation through by δ; this is essential in order to ensure a non-zero Jacobian.

We apply the implicit function theorem to this system in a neighborhood of the point $(\underline{v}, \zeta, \beta, \delta) = (p_W, \zeta^*, 0, 0)$. To do so the system must be continuously differentiable in a neighborhood of $(p_W, \zeta^*, 0, 0)$. Therefore change the domain of the system from $D = (0,1) \times (0, \infty) \times [0,1) \times (0, \infty)$ to $D_1 = (0,1) \times (0,2) \times (-0.1, 1) \times (-0.1, 1)$. Equation (51) is obviously continuously differentiable on D_1. To see that this is also so for (52) recall formula (34) for $\hat{\rho}_B$ and observe that $\hat{\rho}_B(v), \hat{\rho}_B(x)$, and $\left(1 - e^{-\beta\delta}\right)/\beta\delta$ are continuously differentiable functions of $\underline{v}, \zeta$, β, and δ on D_1.[12] The function L, as a composition of these functions, is continuously differentiable on D_1 provided the factor $\left(1 - e^{-\beta\delta} + e^{-\beta\delta}\hat{\rho}_B(x)\right)$ in the denominator of the inner integral is always positive. This in fact is the case because $\hat{\rho}_B(x)$ is increasing for $x \geq \underline{v}$ and $\hat{\rho}_B(\underline{v}) = \omega_0 = e^{-\zeta}$ (see equations 33 and 34). Therefore

$$\begin{aligned} 1 - e^{-\beta\delta} + e^{-\beta\delta}\hat{\rho}_B(x) &\geq 1 - e^{-\beta\delta} + e^{-\beta\delta}e^{-\zeta} \\ &= e^{-\beta\delta}(e^{\beta\delta} - 1 + e^{-\zeta}). \end{aligned}$$

This last expression is positive if $e^{\beta\delta} - 1 + e^{-\zeta} > 0$ or, equivalently, whenever

$$\beta\delta > \ln(1 - e^{-\zeta}). \tag{53}$$

Inspection establishes that all $(\underline{v}, \zeta, \beta, \delta) \in D_1$ satisfy (53) because $\sup_{\zeta \in (0,2)} \ln\left(1 - e^{-\zeta}\right) = -.145$. Therefore $L(\cdot)$ is continuously differentiable on D_1. Finally, also note that within its interior D_1 includes all points in $\{(\underline{v}, \zeta, \beta, \delta) \in D_1 : \beta = 0 \text{ and } \delta = 0\}$.

At $(\beta, \delta) = (0,0)$, $L(\underline{v}, \zeta, 0, 0)$ takes the simple form

$$L(\underline{v}, \zeta, 0, 0) = \frac{e^{\zeta} - \zeta - 2}{1 - e^{-\zeta}}.$$

This is a function increasing in ζ:

$$\frac{d}{d\zeta}\frac{e^{\zeta} - \zeta - 2}{1 - e^{-\zeta}} = \frac{e^{\zeta}\left(\zeta + e^{2\zeta} - 3e^{\zeta} + 3\right)}{\left(1 - e^{-\zeta}\right)^2} > 0$$

because $e^{2\zeta} - 3e^{\zeta} + 3 = \left(e^{\zeta} - \frac{3}{2}\right)^2 + 3 - \left(\frac{3}{2}\right)^2 > 0$. Therefore

$$\frac{\partial L(\underline{v}, \zeta, 0, 0)}{\partial \zeta} > 0. \tag{54}$$

We now claim that $(\underline{v}, \zeta) = (p_W, \zeta^*)$ is the unique solution to the system (51-52) at $(\beta, \delta) = (0,0)$. This is seen in two steps. First, if $\delta = 0$, equation (51) reduces to $G_S(\underline{v}(0)) - a(1 - G_B(\underline{v}(0))) = 0$, which is just equation (1) defining p_W, the Walrasian price. Therefore, at $(\beta, \delta) = (0,0)$, p_W is the unique solution to (51), the system's first

[12] The expression $\left(1 - e^{-\beta\delta}\right)/\beta\delta$ is indeterminate at $\delta = 0$, but selecting $\lim_{\delta \to 0}\left(1 - e^{-\beta\delta}\right)/\beta\delta$ to be its value at that point makes it continuous and differentiable over all of D_1.

equation. Second, inspection shows that the equation $L(p_W, \zeta, 0, 0) = 0$ reduces to $e^{\zeta} - \zeta - 2 = 0$ at $(\beta, \delta) = (0, 0)$; its unique, positive solution is $\zeta = \zeta^* = 1.14619$. Thus, as claimed, $(\underline{v}, \zeta) = (p_W, \zeta^*)$ is the unique solution to the system at $(\beta, \delta) = (0, 0)$.

The Jacobian J of the system (51-52) at $(\underline{v}, \zeta, \beta, \delta) = (p_W, \zeta^*, 0, 0)$ is not zero:

$$\begin{aligned} J &= \begin{vmatrix} g_S(p_W) + a g_B(p_W) & 0 \\ * & \frac{\partial L(p_W, \zeta^*, 0, 0)}{\partial \zeta} \end{vmatrix} \\ &= (g_S(p_W) + a g_B(p_W)) \frac{\partial L(p_W, \zeta^*, 0)}{\partial \zeta} > 0, \end{aligned}$$

where $*$ denotes some expression and the last line follows from inequality (54). Consequently, the implicit function theorem applies: in some neighborhood X of $(\beta, \delta) = (0, 0)$ a unique solution $(\underline{v}(\beta, \delta), \zeta(\beta, \delta))$ to the system (44-45) exists and is differentiable in β and δ. The remaining function $\bar{c}(\delta)$ then is recovered from formula (42).■

It remains to show that the equilibrium's characterizing equations are also sufficient; this is accomplished in Theorem 14 below. We have already shown that equations (36-38) must hold in any full trade equilibrium, and that there exists a unique solution to these equations when β and δ are sufficiently small. We start with the solution values $(\bar{c}(\beta, \delta), \underline{v}(\beta, \delta), \zeta(\beta, \delta))$, construct the unique steady state densities f_B and f_S and strategies B and S, compute the masses T_B and T_S that these densities and strategies imply, and check that the ratio of these masses equals the solution value ζ. The key insight to our construction is the observation that the strategy for buyers can be constructed separately from the strategy for sellers; the solution $(\bar{c}(\beta, \delta), \underline{v}(\beta, \delta), \zeta(\beta, \delta))$ is a sufficient link between the two.

A difficulty in the construction of the buyers' strategy B is that the lowest active buyer type $\underline{v}(\beta, \delta)$ (and hence also the neighboring types) may have an incentive to deviate, by lowering his bid into the support $[\underline{c}, \bar{c})$ of the distribution of the sellers' "offers."[13] We show that such a deviation is not profitable if β is sufficiently small.

Theorem 14 *Given $\kappa > 0$, a neighborhood X of $(0, 0)$ exists such that, for all pairs of non-negative β and positive δ in X, a unique full trade equilibrium exists.*

Proof. Consider sellers first. We already know the marginal participation type, $\bar{c}$, among sellers; it is is a component of $(\bar{c}, \underline{v}, \zeta)$. A seller trades in any period in which she is matched with at least one buyer; this probability is $\bar{\pi}_0 = 1 - \pi_0 = 1 - e^{-\zeta}$ and is independent of her type. Using formula (7), her discounted probability of trade is

$$\hat{P}_S = \frac{1 - e^{-\zeta}}{1 - e^{-\beta\delta} + e^{-\beta\delta}(1 - e^{-\zeta})}. \tag{55}$$

For active sellers—those with costs $c \leq \bar{c}$—formula (15) gives their continuation values,

$$\begin{aligned} W_S(c) &= \int_c^{\bar{c}} \hat{P}_S dx \\ &= \hat{P}_S(\bar{c} - c), \end{aligned} \tag{56}$$

[13] We are grateful to a referee for drawing our attention to the possibiity of such a deviation, thereby discovering a flaw in the previous version's proof.

and formula (14) gives their equilibrium strategies,

$$\begin{aligned} S(c) &= c + e^{-\beta\delta} W_S(c) \\ &= c + e^{-\beta\delta} \hat{P}_S(\bar{c} - c). \end{aligned} \tag{57}$$

Note that this is a linear function of c with slope $1 - e^{-\beta\delta}\hat{P}_S$. Sellers for whom $c > \bar{c}$ best-respond by not entering.

To construct the unique, steady state distribution of seller types, F_S, observe that the seller's best-response strategy S is increasing on $[0, \bar{c}]$ and satisfies $S(\bar{c}) = \bar{c}$. Since all active sellers trade with the same probability in any period, the distribution of their types in the market F_S is just the distribution of the entering cohort conditional on $c \leq \bar{c}$:

$$F_S(c) = \frac{G_S(c)}{G_S(\bar{c})}. \tag{58}$$

To complete the construction of the seller's part of equilibrium, we must find the steady state mass of active sellers T_S. Mass balance holds every period in a steady state; therefore $T_S\left(1 - e^{-\zeta}\right) = \delta G_S(\bar{c})$ and, solving,

$$T_S = \frac{\delta G_S(\bar{c})}{1 - e^{-\zeta}}. \tag{59}$$

Turning to the buyers, our first step is to show that buyers' unique, symmetric, mutual best-response strategy for $v \geq \underline{v}$ is in fact given by (40):

$$B(v) = v - \frac{\kappa\delta}{\hat{\rho}_B(v)} - \frac{1}{\hat{P}_B(v)} \int_{\underline{v}}^{v} \hat{P}_B(x)dx \tag{60}$$

where $\underline{v}$ denotes $\underline{v}(\beta, \delta)$, a component of the characterizing equations' solution of the $(36-38)$. For $v < \underline{v}$, the best-response is not to enter. Observe that the formula implies, as it should, that $B(\cdot)$ is an increasing function and that $B(\underline{v}) = \bar{c}$.

Standard auction-theoretic arguments (e.g., the constraint simplification theorem in Milgrom (2004, page 105)) imply that the envelope condition (39) is sufficient to rule out any deviation from $B(v)$ to a bid $\lambda \geq \bar{c}$. For deviations from $B(v)$ downward into the region $[\underline{c}, \bar{c})$ the restriction that (β, δ) is contained in a sufficiently small neighborhood of 0 is sufficient. To see this, fix $\kappa > 0$, restrict $(\beta, \delta) \in [0,1] \times (0, \ln 2)$, and observe that this restriction implies $e^{-\beta\delta} \in \left(\frac{1}{2}, 1\right)$. Recall that the buyer's payoff function (equation 9) may be written as $W^{B\delta}(\lambda, v) = (v - \lambda)P_B(\lambda) - K_B(\lambda)$ where $K_B(\lambda)$ is his expected discounted participation costs from following the stationary strategy of bidding λ. Since $W^{B\delta}(\lambda, v)$ is continuous at $\lambda = \bar{c}$, a sufficient condition for ruling out profitable downward deviations is, for all λ and v such that $v \geq \bar{c} > \lambda$,[14]

$$\frac{\partial W^{B\delta}(\lambda, v)}{\partial \lambda} = (v - \lambda)P_B'(\lambda) - P_B(\lambda) - K_B'(\lambda) > 0. \tag{61}$$

[14] $W^{B\delta}(\lambda, v)$, however ,is not a differentiable function at $\lambda = \bar{c}$.

This condition is met if $-K_B'(\lambda) > 1$ because $(v - \lambda) > 0$, $P_B'(\lambda) > 0$, $P_B(\lambda) < 1$, and $K_B'(\lambda) < 0$.

Differentiate $-K_B(\lambda)$:

$$
\begin{aligned}
-K_B'(\lambda) &= \kappa\delta \frac{e^{-\beta\delta}}{\left(1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_B(\lambda)\right)^2} \rho_B'(\lambda) \\
&\geq \frac{1}{2}\kappa\delta\rho_B'(\lambda) \qquad (62)
\end{aligned}
$$

because $e^{-\beta\delta} \in \left[\frac{1}{2}, 1\right]$, $\rho_B(\lambda) \in [0,1]$, and consequently $e^{-\beta\delta} / \left(1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_B(\lambda)\right)^2 \geq \frac{1}{2}$. The probability $\rho_B(\lambda)$ of a seller successfully trading is given by formula (20):

$$
\rho_B(\lambda) = F_S(C(\lambda))\left[\pi_0 + \bar{\pi}_0 F^*(V(\lambda))\right] = \pi_0 F_S(C(\lambda))
$$

where the second equality follows from the fact that, for deviations $\lambda \in [\underline{c}, \overline{c})$, $F^*(V(\lambda)) = 0$ because the deviating buyer's bid λ loses whenever he faces a rival buyer.

Differentiation gives

$$
\rho_B'(\lambda) = e^{-\zeta} f_S(C(\lambda)) \frac{1 - e^{-\beta\delta} + e^{-\beta\delta}\left(1 - e^{-\zeta}\right)}{1 - e^{-\beta\delta}} \qquad (63)
$$

where $e^{-\zeta} = \pi_0$, $f_S(C(\lambda))$ is the density of sellers' types at $C(\lambda)$, and

$$
\frac{1 - e^{-\beta\delta} + e^{-\beta\delta}\left(1 - e^{-\zeta}\right)}{1 - e^{-\beta\delta}} = C'(\lambda) = \frac{1}{S'(C(\lambda))}
$$

from equation (57). Recall the mathematical inequality that, for all β and δ, $\beta\delta \geq 1 - e^{-\beta\delta}$. Also recall that $\underline{g} > 0$ bounds the density g_S from below on $[0,1]$, which implies that $f_S(c) = g_S(c) / G_S(\overline{c}) \geq \underline{g}$. Therefore, again using $e^{-\beta\delta} \in \left[\frac{1}{2}, 1\right]$,

$$
\rho_B'(\lambda) \geq e^{-\zeta}\underline{g}\frac{\left(1 - e^{-\zeta}\right)}{\beta\delta}.
$$

Then

$$
\begin{aligned}
-K_B'(\lambda) &\geq \frac{1}{2}e^{-\zeta}\underline{g}\left(1 - e^{-\zeta}\right)\frac{\kappa}{\beta} \\
&\geq \frac{1}{4}\underline{g}e^{-\zeta^*}\left(1 - e^{-\zeta^*}\right)\frac{\kappa}{\beta} \\
&> 1
\end{aligned}
$$

where the first inequality follows from substituting the bound for $\rho_B'(\lambda)$ into (62), the second inequality follows from writing Lemma 13's result as $\zeta(\beta, \delta) = \zeta^* + o(||(\beta, \delta)||)$ for some neighborhood of $(0,0)$, and the third inequality follows from choosing β sufficiently small relative to κ. This shows that, for (β, δ) in a sufficiently small neighborhood of $(0,0)$ that is contained in $[0,1] \times (0, \ln 2)$, $W_\lambda^\delta(\lambda, v) > 0$ for $\lambda \in (\underline{c}, \overline{c})$ and therefore no deviation to $\lambda < \overline{c}$ is profitable for active buyers. Buyers' unique symmetric mutual best-response strategy for $v \geq \underline{v}$ is therefore given by (40). For $v < \underline{v}$, the best response is not to enter. Observe that the formula implies, as it should, that B is an increasing function and that $B(\underline{v}) = \bar{c}$.

The steady state distribution, F_B^*, of the maximal rival buyer's type is given by formula (33). The distribution F_B is uniquely recoverable from F_B^*, the distribution of the maximal value. Equations (34) and (33) imply that in the steady state

$$\hat{\rho}_B(v) = e^{-\zeta} + \left(1 - e^{-\zeta}\right) \frac{G_B(v) - G_B(\underline{v})}{1 - G_B(\underline{v})}. \tag{64}$$

On the other hand, direct computation shows that

$$\begin{aligned} \hat{\rho}_B(v) &= \sum_{k=0}^{\infty} e^{-\zeta} \frac{\zeta^k}{k!} F_B(v)^k \\ &= e^{-\zeta[1-F_B(v)]} \sum_{k=0}^{\infty} e^{-\zeta F_B(v)} \frac{[\zeta F_B(v)]^k}{k!} \\ &= e^{-\zeta[1-F_B(v)]}. \end{aligned} \tag{65}$$

Equating the right-hand sides of (64) and (65) and then solving, we obtain the unique steady state distribution F_B that corresponds to ζ and $\underline{v}$ in the characterizing equations' solution:

$$F_B(v) = 1 + \frac{1}{\zeta} \log \left[e^{-\zeta} + \left(1 - e^{-\zeta}\right) \frac{G_B(v) - G_B(\underline{v})}{1 - G_B(\underline{v})} \right]. \tag{66}$$

To complete the construction of the buyer's part of equilibrium, we must compute T_B, the steady state mass of buyers. Mass balance of buyers implies

$$T_B F_B'(v)\, \hat{\rho}_B(v) = a\delta g_B(v). \tag{67}$$

Substitution of (64) and the derivative of (66) into this and solving gives the formula:

$$T_B = \zeta \frac{a\delta(1 - G_B(\underline{v}))}{1 - e^{-\zeta}} = \zeta \frac{a\delta \bar{G}_B(\underline{v})}{1 - e^{-\zeta}}. \tag{68}$$

A review of this construction shows that strategy B, the distribution F_B^*, and the mass T_B depend only on the fundamentals G_B, a, δ, and the solution $(\bar{c}, \underline{v}, \zeta)$ to the characterizing equations, but not the sellers' strategy S. This insulation of the buyers' optimal actions from the sellers' actions is, we again emphasize, the key insight behind this construction and the proof's overall design.

We need to check that, besides being mutual best-responses and inducing their own steady state distributions F_B and F_S, the strategies result in steady state masses of buyers and sellers that have the required ratio ζ that was computed as a component of the solution to the characterizing equations. This is confirmed by dividing equation (68) by (59):

$$\begin{aligned} \frac{T_B}{T_S} &= \zeta \frac{a(1 - G_B(\underline{v}))}{G_S(\bar{c})} = \zeta \frac{a\bar{G}_B(\underline{v})}{G_S(\bar{c})} \\ &= \zeta, \end{aligned}$$

where the last line follows from market clearing, equation (36).■

5.3 Discussion

Lemma 13, Theorem 14, and their proofs raise two issues. First, even if δ is small, why does β have to be small relative to κ in order to insure that a solution $(\bar{c}, \underline{v}, \zeta)$ to the equations (36-38) in fact characterizes a full trade equilibrium. Second, our theorems do not discuss the rate at which these markets converge to full efficiency as δ approaches zero. Why not?

The reason why an active, type v buyer might consider deviating from the bid $B(v)$ downward with a bid λ into the interval $[\underline{c}, \bar{c})$ is that the better price, if he successfully trades, more than offsets his lower per period probability of trading and increased participation costs. It is immediately clear why if β is small relative to κ such a deviation is unprofitable. Consider the extreme case of infinitessimal discounting: β is arbitrarily small. In this case, the equilibrium strategy of sellers (see equation 14) is essentially flat: $S(\bar{c}) = \bar{c}$ and $S(0) = \underline{c} = \bar{c} - \varepsilon$ for some infinitesimal $\varepsilon > 0$. Therefore deviating into $[\underline{c}, \bar{c})$ results in an infinitesimal gain in price at the cost of a non-infinitessimal reduction in the per period probability of trade—a bad deal and therefore not an equilibrium deal.

If β is large, the logic reverses: deviating into $[\underline{c}, \bar{c})$ results in an improved expected discounted margin that more than offsets the increase in expected discounted participation costs. To illustrate, take as a baseline the second of the two full trade equilibria that we computed in Section 2 (see Figure 2). This equilibrium, for which $\beta = 1$ and $\delta = 0.02$, has $\bar{c} = 0.470, \underline{v} = 0.530$, and $\zeta = 1.103$ as the solution to its characterizing equations. A check that this is in fact is an equilibrium is to calculate that if the type $\underline{v} = 0.530$ buyer deviates from his equilibrium bid $B(\underline{v}) = 0.470$ to a bid of $\lambda = 0.465 \in [\underline{c}, \bar{c}) = [0.456, 0.470)$,then the change in his payoff is negative. Doing so causes his per period probability of trade ρ_B to decrease by one-third from 0.33 to 0.22, his discounted probability of trade P_B to decreases from 0.96 to 0.93, his margin $(v - \lambda)$ to increase from 0.060 to 0.065, his expected discounted margin $P_B(\lambda)(v-\lambda)$ to increase from 0.058 to 0.061, and his discounted expected participation costs K_B to increase from 0.058 to 0.087. Overall the 0.029 increase in participation costs overwhelms the 0.003 increase in the expected discounted margin, so the net effect of the deviation is negative—as it must be in equilibrium.

Alter the baseline case by increasing β from 1 to 10 while keeping δ fixed at 0.02. Solving the characteristic equations for this modified situation yields $\bar{c} = 0.476$, $\underline{v} = 0.524$, and $\zeta = 0.879$. Construct the bidding function $B(\cdot)$ and the offer function $S(\cdot)$ that are associated with this vector of descriptors using equations (60) and (57). Does this (S, B) constitute an equilibrium? That it is not may be seen by checking if the type $\underline{v} = 0.524$ buyer can profit by deviating from the prescribed bid $B(\underline{v}) = 0.476$ to a bid $\lambda = 0.430 \in [\underline{c}, \bar{c}) = [0.345, 0.476)$. Doing so decreases his per period probability of trade ρ_B by one-third from 0.42 to 0.27, decreases his discounted probability of trade P_B from 0.80 to 0.67, increases his margin $(v - \lambda)$ from 0.048 to 0.094, increases his expected discounted margin $P_B(\lambda)(v - \lambda)$ from 0.038 to 0.063, and increases his discounted expected participation costs K_B from 0.038 to 0.050. Overall the 0.012 increase in participation costs does not offset 0.025 increase in the expected discounted margin, so the net effect of the deviation is positive. Thus (S, B) is not an equilibrium.

Inspection of these calculations makes clear the effect of an increase in β. In both the $\beta = 1$ case and the $\beta = 10$ case the type $\underline{v}$ buyer deviates down into the interval $[\underline{c}, \bar{c})$ so as to decrease his per period probability of trade ρ_B by one-third. In the $\beta = 1$ case this causes his expected discounted margin $P_B(\lambda)(v - \lambda)$ to increase only 5% while in the $\beta = 10$ case this causes his expected discounted margin to increase fully 66%. This dramatic difference follows directly from formula (57) for the full trade offer function,

$$S(c) = c + e^{-\beta\delta} \hat{P}_S (\bar{c} - c),$$

which implies that the slope of $S(\cdot)$ is quite flat for $\beta = 1$ but pretty steep for $\beta = 10$. Offsetting the change in expected discounted margin is the increase in expected discounted participation costs. This increase results from the lower per period trading probability ρ_B causing the type $\underline{v}$ buyer to wait in expectation longer before trading. But in both cases the percentage increases are 53%: the deviation causes his expected wait to increase from 3.0 periods to 4.6 in the $\beta = 1$ case and from 2.4 to 3.7 periods in the $\beta = 10$ case. Therefore, to summarize, increasing β causes nonexistence of full trade equilibria because for the type $\underline{v}$ buyer it greatly increases the elasticity of $P_B(\lambda)(v - \lambda)$ with respect to $\rho_B(\lambda)$, but leaves the elasticity of K_B with respect to $\rho_B(\lambda)$ unchanged.

Theorem 14 emphasizes that if β and δ are sufficiently small, then a *unique* full trade equilibrium exists. Nevertheless Theorem 2, our main existence theorem, makes no mention of uniqueness. The reason is that we do not know whether non-full trade equilibria do or do not exist. We note, however, that these existence issues are delicate and depend on details of the model. Formula (57) for $S(\cdot)$, with its sensitivity to β, is a direct consequence of our assumption that sellers' can not commit to a reservation price until after she receives her bids. For full trade equilibria, if the model permitted all sellers with types $c \in [0, \bar{c}]$ to commit to the reservation price $B(\underline{v}) = \bar{c}$, then equilibria would exist for large β provided δ were sufficiently small.[15]

Turn now to the second issue: the rate of convergence. Let β be sufficiently small relative to κ so that a full trade equilibrium exists. Differentiability of the solution to the characterizing equations in a neighborhood of $(\beta, \delta) = (0, 0)$ immediately implies that the descriptors of the full trade equilibrium all converge to their limiting values at a linear rate: $\bar{c}(\delta), \underline{c}(\delta), \bar{v}(\delta)\, \underline{v}(\delta), \bar{p}(\delta), \underline{p}(\delta) = p_W + o(\delta)$ and $\zeta(\delta) = \zeta^* + o(\delta)$. Further, convergence of interim and ex ante welfare also converge linearly, e.g., $W_{S\delta}(c)$ $= \max[0, p_W - c] + o(\delta)$ and $W_{B\delta}(v) = \max[0, v - p_W] + o(\delta)$.

The mechanics underlying these linear rates can be seen by breaking the inefficiency of these equilibria into its two sources: delay costs and exclusion costs. First, given the matching technology of the market, a full trade equilibrium minimizes delay because if a seller is matched with at least one buyer, then trade occurs. Delay with its associated costs of κ per unit time occurs because the matching technology each period fails to match $\pi_0 = e^{-\zeta}$ proportion of the sellers to one or more sellers and, as a result, π_0

[15] Ability to commit on the part of sellers would make it straightforward to compute examples of non-full-trade equilibria. It would not, however, immediately resolve our inability to prove existence of non-full-trade equilibria.

proportion of sellers fail to trade each period. The welfare losses due to delay therefore decreases with δ; this is most easily seen from equation (12) for seller's discounted expected participation costs coupled with the observation that $\rho_S(S(\bar{c})) = \bar{\pi}_0(\delta) = 1 - \pi_0(\delta) = e^{-\zeta^*} + o(\delta)$. Second, exclusion costs arise from the "wedge" between $\bar{c}(\delta)$, the highest cost seller who enters, and $\underline{v}(\delta)$, the lowest value buyer who enters. These traders, not being active, do not trade in the full trade equilibrium, yet roughly half would enter and trade if δ approached zero and the market approached full efficiency. The ex ante welfare loss that this exclusion causes is proportional to $(\bar{v}(\delta) - \underline{c}(\delta))^2$, i.e., it is quadratic in the thickness of the wedge. Observe that $\bar{v}(\delta) - \underline{c}(\delta)$ approaches zero linearly because $\bar{v}(\delta)$ and $\underline{c}(\delta)$ each approach p_W at a linear rate. Given this linear shrinkage of the wedge, ex ante welfare shrinks quadratically because each time the wedge is halved, (i) only half as many traders are excluded and (ii) the traders who are excluded have only half the potential gains from trade that the traders who are no longer excluded expect to realize.[16] Finally, summing these two rates gives an overall linear rate because, for small δ, the linear convergence of the delay costs dominates the quadratic convergence of the exclusion costs.

This rate of convergence for full trade equilibria is nice, yet we make no mention of it in either Theorem 1 or Theorem 2. The reason is that, for a particular value of δ, the existence of a unique full trade equilibrium does not rule out another equilibrium that is not full trade, except for the special case of no time discounting. Proving a theorem that establishes a linear rate of convergence for all equilibria is formidable because every step in our proofs above becomes much more complex whether it be in deriving a set of characterizing equations, applying the implicit function theorem, or showing that the characterizing equations identify all equilibria.

6 Conclusions

In this paper we have shown that convergence to the competitive price and allocation can be achieved with a decentralized matching and bargaining market in which all traders have private information about their values/costs. The significance of this contribution is that it directly addresses a critical shortcoming in each of two literatures it combines. Existing matching and bargaining models that demonstrate robust convergence ignore the ubiquity of incomplete information. Existing double auction models robustly demonstrate convergence in the presence of incomplete information, but ignore the equally ubiquitous future opportunities for trade that exist in almost all real markets. Our model and results cure both these shortcomings in the independent

[16] The quadratic rate of convergence to efficiency that Satterthwaite and Williams (1989a) and Rustichini, Satterthwaite, and Williams (1994) obtain for the static double auction with independent private values follows analogous logic. Traders in bidding misrepresent their true cost/value. This difference between the bid and the cost/value creates a wedge that excludes trades that would be efficient. The thickness of the wedge goes to zero linearly in the number of traders and, as in the case here, this causes the expected ineficiency in per trader terms to go to zero quadratically. In the static double auction, by construction, no delay costs exist, so the overall rate of convergence to efficiency is the quadratic rate for the exclusion costs.

private values case.

Of the many open questions that remain, we mention two. First, no constrained optimal benchmark has been derived for the dynamic matching and bargaining model with incomplete information. Presumably mechanism design techniques can be used to establish such a benchmark.[17] Then it would be possible to compare the realized efficiency of models such as ours that have specific matching and bargaining protocols against the efficiency of the constrained optimal mechanism. Second, we assume an independent private values environment. Relaxing this assumption to allow costs/values to be correlated or interdependent would, in order to be interesting, involve letting the underlying Walrasian price vary over time according to some stochastic process. The mechanism then would have the demanding task of converging, as the period length becomes short, to this non-stationary price. Such a model would give insight into how robustly decentralized matching markets can "discover" price in a dynamic market just as the recent papers of Fudenberg, Mobius, and Szeidl (2003), Reny and Perry (2003), and Cripps and Swinkels (2006) have shown that the static double auction can discover price in environments more general than the independent private values with unit demand and supply environment.

If these and other questions can be answered in future work, then this theory may become useful in designing decentralized markets with incomplete information in much the same way auction theory has become useful in designing specific auctions for real allocation problems. The ubiquity of the Internet with its capability for facilitating matches and reducing period length makes pursuit of this end attractive.

References

[1] Butters, G. Circa 1979. Equilibrium price distributions in a random meetings market. Working paper.

[2] Chatterjee, K. and W. Samuelson. 1983. Bargaining under incomplete information. *Operations Research* 31: 835-51.

[3] Cripps, M. and J. Swinkels. 2006. Efficiency in large double auctions. *Econometrica* 74, 47-92.

[4] De Fraja, G. and J. Sákovics. 2001. Walras retrouvé: Decentralized trading mechanisms and the competitive price. *J. of Political Economy* 109 (August): 842-63.

[5] Fudenberg, D., M. Mobius, and A. Szeidl. 2003. Existence of equilibrium in large double auctions. Working paper, April.

[6] Fudenberg, D. and J. Tirole. 1991. *Game Theory.* Cambridge, MA: The MIT Press.

[7] Gale, D. 1986. Bargaining and competition Part I: Characterization. *Econometrica* 54 (July): 785-806.

[17] Satterthwaite and Williams (2002) have done this for static double auctions.

[8] Gale, D. 1987. Limit theorems for markets with sequential bargaining. *J. of Economic Theory* 43 (October): 20-54.

[9] Gale, D. 2000. *Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games.* Cambridge: Cambridge University Press.

[10] Gresik, T. and M. Satterthwaite. 1989. The rate at which a simple market converges to efficiency as the number of traders increases: An asymptotic result for optimal trading mechanisms. *Journal of Economic Theory* 48(1): 304-32.

[11] Milgrom, P. 2004. *Putting Auction Theory to Work*. Cambridge: University Press.

[12] Milgrom, P. and I. Segal. 2002. Envelope theorems for arbitrary choice sets. *Econometrica* 70 (March): 583-601.

[13] Mortensen, D. 1982. The matching process as a noncooperative bargaining game. Chapter 7 in *The Economics of Information and Uncertainty* (J. McCall, Ed.). Chicago, University of Chicago Press.

[14] Mortensen, D. and R. Wright. 2002. Competitive pricing and efficiency in search equilibrium. *International Economic Review* 43 (February): 1-20.

[15] Myerson, R. 1998. Population uncertainty and Poisson games. International Journal of Game Theory 27: 375-392.

[16] Myerson, R. and M. Satterthwaite. 1983. Efficient mechanisms for bilateral trading. *Journal of Economic Theory* 29: 265-81.

[17] Osborne, M. and A. Rubinstein. 1990. *Bargaining and Markets.* San Diego: Academic Press.

[18] Peters, M. and S. Severinov. 200. Internet Auctions with Many Traders. Forthcoming *Journal of Economic Theory.*

[19] Reny, P. and M. Perry. 2003. Toward a strategic foundation of rational expectations equilibrium. Working paper, June.

[20] Rubinstein, A. and A. Wolinsky. 1985. Equilibrium in a market with sequential bargaining. *Econometrica* 53 (September): 1133-50.

[21] Rubinstein, A. and A. Wolinsky. 1990. Decentralized trading, strategic behavior, and the Walrasian outcome. *Review of Economic Studies* 57 (January): 63-78.

[22] Rust, J. and G. Hall. 2002. Middlemen versus market makers: A theory of competitive exchange. *J. of Political Economy* 111: 353-403.

[23] Rustichini, A., M. Satterthwaite, and S. Williams. 1994. Convergence to efficiency in a simple market with incomplete information. *Econometrica* 62(5): 1041-63.

[24] Satterthwaite, M. and S. Williams. 1989a. The rate of convergence to efficiency in the buyer's bid double auction as the market becomes large. *Review of Economic Studies* 56: 477-98.

[25] Satterthwaite, M. and S. Williams. 1989b. Bilateral trade with the sealed bid double auction: Existence and efficiency." *Journal of Economic Theory* 48(1): 107-33.

[26] Satterthwaite, M. and S. Williams. 2002. The optimality of a simple market mechanism. *Econometrica* 70: 1841-63.

[27] Satterthwaite, M. and A. Shneyerov. 2003. Convergence of a dynamic matching and bargaining market with two-sided incomplete information to perfect competition. CMS-EMS Working paper 1384, Northwestern University.

[28] Serrano, R. 2002. Decentralized information and the Walrasian outcome: A pairwise meetings market with private values. *Journal of Mathematical Economics* 38: 65-89.

[29] Shiryaev, A. 1995. *Probability. Graduate texts in Mathematics No. 95*, 2nd edition. New York: Springer-Verlag.

[30] Spulber, D. 1999. *Market Microstructure: Intermediaries and the Theory of the Firm.* Cambridge: Cambridge University Press.

[31] Williams, S. 1991. Existence and convergence of equilibria in the buyer's bid double auction. *Review of Economic Studies* 58: 351-74.

[32] Wolinsky, A. 1988. Dynamic markets with competitive bidding. *Review of Economic Studies* 55 (January): 71-84.